MULTIGRADED FUJITA APPROXIMATION

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The original Fujita approximation theorem states that the volume of a big divisor $D$ on a projective variety $X$ can always be approximated arbitrarily closely by the self-intersection number of an ample divisor on a birational modification of $X$. One can also formulate it in terms of graded linear series as follows: Let $W_\bullet = \{W_k\}$ be the complete graded linear series associated to a big divisor $D$, where

$$W_k = H^0(X, \mathcal{O}_X(kD)).$$

For each fixed positive integer $p$, define $W_\bullet^{(p)}$ to be the graded linear sub-series of $W_\bullet$ generated by $W_p$:

$$W_m^{(p)} = \begin{cases} 0 & \text{if } p \nmid m, \\ \text{Image}(S^kW_p \to W_{kp}) & \text{if } m = kp. \end{cases}$$

Then the volume of $W_\bullet^{(p)}$ approaches the volume of $W_\bullet$ as $p \to \infty$. We will show that, under this formulation, the Fujita approximation theorem can be generalized to the case of multigraded linear series.

1. Introduction

Let $X$ be an irreducible variety of dimension $d$ over an algebraically closed field $K$, and let $D$ be a (Cartier) divisor on $X$. When $X$ is projective, the following limit, which measures how fast the dimension of the section space $H^0(X, \mathcal{O}_X(mD))$ grows, is called the volume of $D$:

$$\text{vol}(D) = \text{vol}_X(D) = \lim_{m \to \infty} \frac{h^0(X, \mathcal{O}_X(mD))}{m^d/d!}.$$ 

One says that $D$ is big if $\text{vol}(D) > 0$. It turns out that the volume is an interesting numerical invariant of a big divisor [Lazarsfeld 2004a, Section 2.2.C], and it plays a key role in several recent works in birational geometry [Tsuji 2000; Boucksom et al. 2004; Hacon and McKernan 2006; Takayama 2006].

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When $D$ is ample, one can show that $\text{vol}(D) = D^d$, the self-intersection number of $D$. This is no longer true for a general big divisor $D$, since $D^d$ may even be negative. However, Fujita [1994] showed that the volume of a big divisor can always be approximated arbitrarily closely by the self-intersection number of an ample divisor on a birational modification of $X$. This theorem, known as Fujita approximation, has several implications for the properties of volumes, and is also a crucial ingredient in [Boucksom et al. 2004] (see [Lazarsfeld 2004b, Section 11.4] for more details).

Lazarsfeld and Mustaţă [2009] (henceforth [LM]) recently obtained, among other things, a generalization of Fujita approximation to graded linear series. Recall that a graded linear series $W_\bullet = \{W_k\}$ on a (not necessarily projective) variety $X$ associated to a divisor $D$ consists of finite dimensional vector subspaces $W_k \subseteq H^0(X, \mathcal{O}_X(kD))$ for each $k \geq 0$, with $W_0 = K$, such that

$$W_k \cdot W_\ell \subseteq W_{k+\ell}$$

for all $k, \ell \geq 0$. Here the product on the left denotes the image of $W_k \otimes W_\ell$ under the multiplication map $H^0(X, \mathcal{O}_X(kD)) \otimes H^0(X, \mathcal{O}_X(\ell D)) \rightarrow H^0(X, \mathcal{O}_X((k + \ell)D))$. In order to state the Fujita approximation for $W_\bullet$, they defined, for each fixed positive integer $p$, a graded linear series $W_\bullet^{(p)}$ which is the subgraded linear series of $W_\bullet$ generated by $W_p$:

$$W_\bullet^{(p)}_m = \begin{cases} 0 & \text{if } p \nmid m, \\ \text{Im}(S^k W_p \rightarrow W_{kp}) & \text{if } m = kp. \end{cases}$$

Then under mild hypotheses, they showed that the volume of $W_\bullet^{(p)}$ approaches the volume of $W_\bullet$ as $p \rightarrow \infty$. See [LM, Theorem 3.5] for the precise statement, as well as [LM, Remark 3.4] for how this is equivalent to the original statement of Fujita when $X$ is projective and $W_\bullet$ is the complete graded linear series associated to a big divisor $D$ (that is, $W_k = H^0(X, \mathcal{O}_X(kD))$ for all $k \geq 0$).

The goal of this note is to generalize the Fujita approximation theorem to multigraded linear series. We will adopt the following notation from [LM, Section 4.3]: Let $D_1, \ldots, D_r$ be divisors on $X$. For $\vec{m} = (m_1, \ldots, m_r) \in \mathbb{N}^r$, write $\vec{m} D = \sum m_i D_i$, and put $|\vec{m}| = \sum |m_i|$.

**Definition.** A multigraded linear series $W_\bullet$ on $X$ associated to the $D_i$ consists of finite-dimensional vector subspaces

$$W_\vec{k} \subseteq H^0(X, \mathcal{O}_X(\vec{k}D))$$
for each $\tilde{k} \in \mathbb{N}^r$, with $W_{\tilde{0}} = K$, such that

$$W_{\tilde{k}} \cdot W_{\tilde{m}} \subseteq W_{\tilde{k} + \tilde{m}},$$

where the multiplication on the left denotes the image of $W_{\tilde{k}} \otimes W_{\tilde{m}}$ under the natural map

$$H^0(X, \mathcal{O}_X(\tilde{k}D)) \otimes H^0(X, \mathcal{O}_X(\tilde{m}D)) \to H^0(X, \mathcal{O}_X((\tilde{k} + \tilde{m})D)).$$

Given $\tilde{a} \in \mathbb{N}^r$, denote by $W_{\tilde{a}}$ the singly graded linear series associated to the divisor $\tilde{a}D$ given by the subspaces $W_{k\tilde{a}} \subseteq H^0(X, \mathcal{O}_X(k\tilde{a}D))$. Then put

$$\text{vol}_{W_{\tilde{a}}} (\tilde{a}) = \text{vol}(W_{\tilde{a}}, \cdot)$$

(assuming that this quantity is finite). It will also be convenient for us to consider $W_{\tilde{a}}$ when $\tilde{a} \in \mathbb{Q}^r_{\geq 0}$, given by

$$W_{\tilde{a},\tilde{k}} = \begin{cases} W_{k\tilde{a}} & \text{if } k\tilde{a} \in \mathbb{N}^r, \\ 0 & \text{otherwise.} \end{cases}$$

Our multigraded Fujita approximation, similar to the singly graded version, is going to state that (under suitable conditions) the volume of $W_{\cdot}$ can be approximated by the volume of the following finitely generated submultigraded linear series of $W_{\cdot}$:

**Definition.** Given a multigraded linear series $W_{\cdot}$ and a positive integer $p$, define $W_{\tilde{m}}^{(p)}$ to be the submultigraded linear series of $W_{\cdot}$ generated by all $W_{\tilde{m}_i}$ with $|\tilde{m}_i| = p$, or concretely,

$$W_{\tilde{m}}^{(p)} = \begin{cases} 0 & \text{if } p \nmid |\tilde{m}|, \\ \sum_{|\tilde{m}_1| = p} W_{\tilde{m}_1} \cdots W_{\tilde{m}_k} & \text{if } |\tilde{m}| = kp. \end{cases}$$

We now state our multigraded Fujita approximation when $W_{\cdot}$ is a complete multigraded linear series, since this is the case of most interest and allows for a more streamlined statement. The Remark on page 335 points out what assumptions on $W_{\cdot}$ are actually needed in the proof.

**Theorem.** Let $X$ be an irreducible projective variety of dimension $d$, and let $D_1, D_2, \ldots, D_r$ be big divisors on $X$. Let $W_{\cdot}$ be the complete multigraded linear series associated to the $D_i$, namely

$$W_{\tilde{m}} = H^0(X, \mathcal{O}_X(\tilde{m}D))$$
for each \( \vec{m} \in \mathbb{N}^r \). Then given any \( \varepsilon > 0 \), there exists an integer \( p_0 = p_0(\varepsilon) \) having the property that if \( p \geq p_0 \), then

\[
\left| 1 - \frac{\text{vol}_{W_0\left(p\right)}(\vec{a})}{\text{vol}_{W_0}(\vec{a})} \right| < \varepsilon
\]

for all \( \vec{a} \in \mathbb{N}^r \).

2. Proof of the Theorem

The main tool in our proof is the theory of Okounkov bodies developed systematically in [Lazarsfeld and Mustaţă 2009]. Given a graded linear series \( W_0 \) on a \( d \)-dimensional variety \( X \), its Okounkov body \( \Delta(W_0) \) is a convex body in \( \mathbb{R}^d \) that encodes many asymptotic invariants of \( W_0 \), the most prominent one being the volume of \( W_0 \), which is precisely \( d! \) times the Euclidean volume of \( \Delta(W_0) \). The idea first appeared in Okounkov’s papers [1996; 2003] in the case of complete linear series of ample line bundles on a projective variety. Later it was further developed and applied to much more general graded linear series by Lazarsfeld and Mustaţă [2009] and also independently by Kaveh and Khovanskii [2008; 2009].

**Proof of the Theorem.** Let \( T = \{(a_1, \ldots, a_r) \in \mathbb{R}^r_+ \mid a_1 + \cdots + a_r = 1\} \), and let \( T_Q \) be the set of all points in \( T \) with rational coordinates. The fraction inside (1) is invariant under scaling of \( \vec{a} \) due to homogeneity, hence it is enough to prove (1) for \( \vec{a} \in T_Q \).

Let \( \Delta(W_0) \subseteq \mathbb{R}^d \times \mathbb{R}^r \) be the global Okounkov cone of \( W_0 \) as in [LM, Theorem 4.19], and let \( \pi : \Delta(W_0) \to \mathbb{R}^r \) be the projection map. For each \( \vec{a} \in T \), write \( \Delta(W_0)_{\vec{a}} \) for the fiber \( \pi^{-1}(\vec{a}) \). Define in a similar fashion the convex cone \( \Delta(W_0^{(p)}) \) and the convex bodies \( \Delta(W_0^{(p)})_{\vec{a}} \). By [LM, Theorem 4.19],

\[
\Delta(W_0)_{\vec{a}} = \Delta(W_{0\vec{a},0}) \quad \text{for all } \vec{a} \in T_Q.
\]

Although [LM, Theorem 4.19] requires \( \vec{a} \) to be in the relative interior of \( T \), here we know that (2) holds even for those \( \vec{a} \) in the boundary of \( T \) because the big cone of \( X \) is open and \( W_0 \) was assumed to be the complete multigraded linear series. By the singly graded Fujita approximation, \( \text{vol}(W_{0\vec{a},0}) \) can be approximated arbitrarily closely by \( \text{vol}(W_{0\vec{a}^{(p)}}) \) if \( p \) is sufficiently large. (Here by \( W_{0\vec{a}^{(p)}} \) we mean \( W_{0\vec{a},0}^{(p)} \) restricted to the \( \vec{a} \) direction, which certainly contains \( (W_{0\vec{a},0})^{(p)} \).) Hence given any finite subset \( S \subset T_Q \) and any \( \varepsilon' > 0 \), we have

\[
\text{vol}(\Delta(W_0^{(p)})_{\vec{a}}) \geq \text{vol}(\Delta(W_0)_{\vec{a}}) - \varepsilon' \quad \text{for all } \vec{a} \in S
\]
as soon as \( p \) is sufficiently large.

Because the function \( \vec{a} \mapsto \text{vol}(\Delta(W_0)_{\vec{a}}) \) is uniformly continuous on \( T \), given any \( \varepsilon' > 0 \), we can partition \( T \) into a union of polytopes with disjoint interiors
$T = \bigcup T_i$, in such a way that the vertices of each $T_i$ all have rational coordinates, and on each $T_i$ we have a constant $M_i$ such that

$$M_i \leq \text{vol}(\Delta(W_\bullet(\bar{a})) \leq M_i + \varepsilon' \quad \text{for all } \bar{a} \in T_i.$$  

Let $S$ be the set of vertices of all the $T_i$. Then as we saw in the end of the previous paragraph, as soon as $p$ is sufficiently large we have

$$\text{vol}(\Delta(W_\bullet(p))_{\bar{a}}) \geq \text{vol}(\Delta(W_\bullet(\bar{a})) - \varepsilon' \quad \text{for all } \bar{a} \in S.$$  

We claim that this implies

$$\text{vol}(\Delta(W_\bullet(p))_{\bar{a}}) \geq \text{vol}(\Delta(W_\bullet(\bar{a}))-2\varepsilon' \quad \text{for all } \bar{a} \in T_Q.$$  

To show this, it suffices to verify it on each of the $T_i$. Let $\bar{v}_1, \ldots, \bar{v}_k$ be the vertices of $T_i$. Then each $\bar{a} \in T_i$ can be written as a convex combination of the vertices: $\bar{a} = \sum t_j \bar{v}_j$ where each $t_j \geq 0$ and $\sum t_j = 1$. Since $\Delta(W_\bullet(p))$ is convex, we have

$$\Delta(W_\bullet(p))_{\bar{a}} \supseteq \sum t_j \Delta(W_\bullet(p))_{\bar{v}_j},$$

where the sum on the right means the Minkowski sum. By (3) and (4), the volume of each $\Delta(W_\bullet(p))_{\bar{v}_j}$ is at least $M_i - \varepsilon'$, hence by the Brunn–Minkowski inequality [Kaveh and Khovanskii 2008, Theorem 5.4], we have

$$\text{vol}(\Delta(W_\bullet(p))_{\bar{a}}) \geq M_i - \varepsilon' \quad \text{for all } \bar{a} \in T_i \cap T_Q.$$  

This combined with (3) shows that (5) is true on $T_i \cap T_Q$, hence it is true on $T_Q$ since the $T_i$ cover $T$.

Since (1) follows from (5) by choosing a suitable $\varepsilon'$, the proof is complete. □

**Remark.** In the statement of the Theorem we assume that $W_\bullet$ is the complete multigraded linear series associated to big divisors. But in fact since the main tool we used in the proof is the theory of Okounkov bodies established in [Lazarsfeld and Mustaţă 2009], in particular [LM, Theorem 4.19], the really indispensable assumptions on $W_\bullet$ are the same as those in [LM] (which they called Conditions (A') and (B'), or (C')). The only place in the proof where we invoke that we are working with a complete multigraded linear series is the sentence right after (2), where we want to say that (2) holds not only in the relative interior of $T$ but also in its boundary. Hence if $W_\bullet$ is only assumed to satisfy Conditions (A') and (B'), or (C'), then given any $\varepsilon > 0$ and any compact set $C$ contained in $T \cap \text{int}(\text{supp}(W_\bullet))$, there exists an integer $p_0 = p_0(C, \varepsilon)$ such that if $p \geq p_0$ then

$$\text{vol}_{W_\bullet(p)}(\bar{a}) > \text{vol}_{W_\bullet}(\bar{a}) - \varepsilon$$

for all $\bar{a} \in C \cap T_Q$. 

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References


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