Pacific Journal of Mathematics

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Volume 251 No. 2 June 2011

PACIFIC JOURNAL OF MATHEMATICS
Vol. 251, No. 2, 2011

QUIVER GRASSMANNIANS, QUIVER VARIETIES AND THE PREPROJECTIVE ALGEBRA

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Quivers play an important role in the representation theory of algebras, with a key ingredient being the path algebra and the preprojective algebra. Quiver grassmannians are varieties of submodules of a fixed module of the path or preprojective algebra. In the current paper, we study these objects in detail. We show that the quiver grassmannians corresponding to submodules of certain injective modules are homeomorphic to the lagrangian quiver varieties of Nakajima which have been well studied in the context of geometric representation theory. We then refine this result by finding quiver grassmannians which are homeomorphic to the Demazure quiver varieties introduced by the first author, and others which are homeomorphic to the graded/cyclic quiver varieties defined by Nakajima. The Demazure quiver grassmannians allow us to describe injective objects in the category of locally nilpotent modules of the preprojective algebra. We conclude by relating our construction to a similar one of Lusztig using projectives in place of injectives. In an appendix added after the first version of the current paper was released, we show how subsequent results of Shipman imply that the above homeomorphisms are in fact isomorphisms of algebraic varieties.

Savage's research was supported by the Natural Sciences and Engineering Research Council of Canada. Tingley was supported by Australian Research Council grant DP0879951 and NSF grant DMS-0902649.

MSC2010: primary 16G20; secondary 17B10.

Keywords: quiver, preprojective algebra, quiver grassmannian, quiver variety, Kac–Moody algebra, Demazure module.

Introduction

Quivers play a fundamental role in the theory of associative algebras and their representations. Gabriel's theorem, which states a precise relationship between indecomposable representations of certain quivers and root systems of associated Lie algebras, indicated that the representation theory of quivers was also intimately connected to the representation theory of Kac–Moody algebras. This eventually lead to the Ringel–Hall construction of quantum groups and the quiver variety constructions of Lusztig and Nakajima.

Fix a quiver (directed graph) $Q = (Q_0, Q_1)$ with vertex set Q_0 and arrow set Q_1. The corresponding path algebra $\mathbb{C}Q$ is the algebra spanned by the set of directed paths, with multiplication given by concatenation. There is a natural grading $\mathbb{C}Q = \bigoplus_n (\mathbb{C}Q)_n$ of the path algebra by length of paths. Representations of a quiver are equivalent to representations (or modules) of its path algebra. Note that $(\mathbb{C}Q)_0$-modules are simply Q_0-graded vector spaces, and in particular all $\mathbb{C}Q$-modules are Q_0-graded. For a $\mathbb{C}Q$-module V and $u \in \mathbb{N}Q_0$, the associated *quiver grassmannian* is the variety $\mathrm{Gr}_Q(u, V)$ of all $\mathbb{C}Q$-submodules of V of graded dimension u. These natural objects (or closely related ones) can be found in several places in the literature. For instance, they appear in [Crawley-Boevey 1996; Schofield 1992] in the study of spaces of morphisms of $\mathbb{C}Q$-modules and in [Caldero and Chapoton 2006; Caldero and Keller 2006; Derksen et al. 2009] in connection with the theory of cluster algebras. Geometric properties have been studied in [Caldero and Reineke 2008; Szántó 2009; Wolf 2009] and representation theoretic properties in [Fedotov 2010; Geiss et al. 2006; Lusztig 1998; 2000; Nakajima 2003; Reineke 2008].

Let $\mathfrak{g}$ be the Kac–Moody algebra whose Dynkin diagram is the underlying graph of Q (the graph obtained by forgetting the orientation of all arrows) and let $\tilde{Q}$ be the *double quiver* obtained from Q by adding an oppositely oriented arrow $\bar{a}$ for every $a \in Q_1$. One is often interested in modules of the preprojective algebra $\mathcal{P} = \mathcal{P}(Q)$, which is a certain natural quotient of the path algebra $\mathbb{C}\tilde{Q}$ and inherits the grading. In particular, $\mathcal{P}$-modules are also $\mathbb{C}\tilde{Q}$-modules. To each vertex $i \in Q_0$, we have an associated one-dimensional simple $\mathcal{P}$-module s^i. For

$$w = \sum_i w_i i \in \mathbb{N}Q_0,$$

we let $s^w = \bigoplus_i (s^i)^{\oplus w_i}$ be the corresponding semisimple module. By Baer's Theorem, the category of $\mathcal{P}$-modules has enough injectives, so we can define q^w to be the injective hull of s^w. One of the main results of the current paper is that the quiver grassmannian $\mathrm{Gr}_{\tilde{Q}}(v, q^w)$ is homeomorphic to the lagrangian Nakajima quiver variety $\mathfrak{L}(v, w)$ used to give a geometric realization of irreducible highest weight representations of $\mathfrak{g}$; [Nakajima 1994; 1998]. In addition, for each σ in the Weyl group of $\mathfrak{g}$, there is a natural finite-dimensional submodule $q^{w,\sigma}$ of q^w such that the quiver grassmannian $\mathrm{Gr}_{\tilde{Q}}(v, q^{w,\sigma})$ is homeomorphic to the Demazure quiver variety $\mathfrak{L}_\sigma(v, w)$ defined in [Savage 2006d]. Since Nakajima's realization of highest weight representations and the first author's realization of Demazure modules depend only on the topological information of the spaces involved, such homeomorphisms allow one to replace quiver varieties by quiver grassmannians in the constructions. This change of setting affords some advantages. In particular, it avoids the description as a moduli space. One can view it as a uniform way of picking a representative from each orbit in the original moduli space descriptions.

Quiver grassmannians admit natural group actions. We describe these actions and show that certain special cases agree, under the homeomorphisms described above, with well-studied groups actions on Nakajima quiver varieties. In this way, we are able to give a quiver grassmannian realization of the cyclic/graded quiver varieties used by Nakajima [2004] to define t-analogs of q-characters of quantum affine algebras.

The injective modules q^w are locally nilpotent if and only if the quiver Q is of finite or affine type. However, it turns out that the submodules $q^{w,\sigma}$ are always nilpotent. The limit $\tilde{q}^w$ of these submodules is the injective hull of the semisimple module s^w in the category of locally nilpotent $\mathcal{P}$-modules, giving us a description of the indecomposable injectives in this category.

Lusztig has previously presented a canonical bijection between the points of the lagrangian Nakajima quiver variety and the points of a type of quiver grassmannian inside a projective (as opposed to injective) object. In finite type, the projective objects are also injective. It turns out that, on the level of geometric realizations of representations of finite type $\mathfrak{g}$, the two constructions are related by the Chevalley involution. Outside of finite type, there are some other subtle yet important differences between the two constructions. In particular, the description in terms of projective objects requires one to impose a nilpotency condition in the definitions. However, the description in terms of injectives given in the current paper requires no such condition and is in this way simpler. Furthermore, through the use of the distinguished modules $q^{w,\sigma}$ mentioned above, one can always consider quiver grassmannians of submodules of a fixed *finite-dimensional* module of the preprojective algebra. Thus, one can avoid working with infinite-dimensional objects.

Motivated by an earlier version of the current paper [Savage and Tingley 2009], I. Shipman [2010] has recently proven that the canonical bijection given by Lusztig and mentioned above is, in fact, an isomorphism of algebraic varieties. We have added an Appendix explaining how this result allows us to conclude that the maps between quiver grassmannians and lagrangian Nakajima quiver varieties described in the current paper are also isomorphisms of algebraic varieties.

Throughout this paper, we work over the field $\mathbb{C}$ of complex numbers. While many results hold in more generality, this assumption will streamline the exposition and several results we quote in the literature are stated over $\mathbb{C}$. We will always use the Zariski topology and do not assume that algebraic varieties are irreducible. We let $\mathbb{N} = \mathbb{Z}_{\geq 0}$ and denote the fundamental weights and simple roots of a Kac–Moody algebra by ω_i and α_i respectively.

This paper is organized as follows. In Section 1 we review some results on quivers, path algebras and preprojective algebras. In Section 2 we discuss various module categories of these objects and introduce our main object of study, the quiver grassmannian. We review the definition of the quiver varieties of Lusztig and Nakajima in Section 3 and realize these as quiver grassmannians in Section 4. In Section 5 we introduce a natural group action and show how it can be used to recover group actions typically constructed on quiver varieties. We also define graded/cyclic versions of quiver grassmannians. In Section 6 we use quiver grassmannians to give a geometric realization of integrable highest weight representations of a symmetric Kac–Moody algebra and discuss the compatibility of this construction with the natural nesting of quiver grassmannians. Finally, in Section 7 we discuss a precise relationship between our construction and a similar one due to Lusztig. The Appendix, added after the appearance of [Shipman 2010], provides a proof that the maps between quiver grassmannians and quiver varieties described in the current paper are isomorphisms of algebraic varieties.

1. Quivers, path algebras, and preprojective algebras

We briefly review the relevant definitions concerning quivers. We refer the reader to [Deng et al. 2008; Ringel 1998; Savage 2006a] for further details.

A *quiver* is a directed graph. That is, it is a quadruple $Q = (Q_0, Q_1, s, t)$ where Q_0 and Q_1 are sets and s and t are maps from Q_1 to Q_0. We call Q_0 and Q_1 the sets of *vertices* and *directed edges* (or *arrows*) respectively. For an arrow $a \in Q_1$, we call $s(a)$ the *source* of a and $t(a)$ the *target* of a. Usually we will write $Q = (Q_0, Q_1)$, leaving the maps s and t implied. The quiver Q is said to be *finite* if Q_0 and Q_1 are finite. A *loop* is an arrow a with $s(a) = t(a)$. In this paper, all quivers will be assumed to be finite and without loops. A quiver is said to be of *finite type* if the underlying graph of Q (i.e the graph obtained from Q by forgetting the orientation of the edges) is a Dynkin diagram of finite ADE type. Similarly, it is of *affine* (or *tame*) *type* if the underlying graph is a Dynkin diagram of affine type and of *indefinite* (or *wild*) *type* if the underlying graph is a Dynkin diagram of indefinite type.

A *path* in Q is a sequence $\beta = a_l a_{l-1} \cdots a_1$ of arrows such that $t(a_i) = s(a_{i+1})$ for $1 \leq i \leq l-1$. We call l the *length* of the path. We let $s(\beta) = s(a_1)$ and

$t(\beta) = t(a_l)$ denote the initial and final vertices of the path β. For each vertex $i \in I$, we have a trivial path e_i with $s(e_i) = t(e_i) = i$.

The *path algebra* $\mathbb{C}Q$ associated to a quiver Q is the $\mathbb{C}$-algebra whose underlying vector space has basis the set of paths in Q, and with the product of paths given by concatenation. More precisely, if $\beta = a_l \cdots a_1$ and $\beta' = b_m \cdots b_1$ are two paths in Q, then $\beta\beta' = a_l \cdots a_1 b_m \cdots b_1$ if $t(\beta') = s(\beta)$ and $\beta\beta' = 0$ otherwise. This multiplication is associative. There is a natural grading

$$\mathbb{C}Q = \bigoplus_{n \geq 0} (\mathbb{C}Q)_n$$

where $(\mathbb{C}Q)_n$ is the span of the paths of length n.

Given a quiver $Q = (Q_0, Q_1)$, we define the *double quiver* associated to Q to be the quiver $\tilde{Q} = (Q_0, \tilde{Q}_1)$ where

$$\tilde{Q}_1 = \bigcup_{a \in Q_1} \{a, \bar{a}\}, \quad \text{where} \quad s(\bar{a}) = t(a),\ t(\bar{a}) = s(a).$$

We then have a natural involution $\tilde{Q}_1 \to \tilde{Q}_1$ given by $a \mapsto \bar{a}$ (where $\bar{\bar{a}} = a$). The algebra

$$\mathcal{P} = \mathcal{P}(Q) = \mathbb{C}\tilde{Q} / \sum_{a \in Q_1} (a\bar{a} - \bar{a}a)$$

is called the *preprojective algebra* associated to Q. It inherits a grading

$$\mathcal{P} = \bigoplus_{n \geq 0} \mathcal{P}_n$$

from the grading on $\mathbb{C}Q$. Up to isomorphism, the preprojective algebra $\mathcal{P}(Q)$ depends only on the underlying graph of Q. See [Lusztig 1991, §12.15] for details.

2. Modules of the path algebra and quiver grassmannians

2A. ***Module categories.*** For an associative algebra A, let A-Mod denote the category of A-modules and A-mod the category of finite-dimensional A-modules. We will use the notation $V \in A$-Mod (resp. $V \in A$-mod) to indicate that V is an object in the category A-Mod (resp. A-mod). Note that $\mathcal{P}_0$-mod is equivalent to the category of finite-dimensional Q_0-graded vector spaces whose morphisms are linear maps preserving the grading, and we will often blur the distinction between these two categories. Up to isomorphism, the objects of $\mathcal{P}_0$-mod are classified by their graded dimension. We denote the graded dimension of a module V by $\dim_{Q_0} V = \sum_i (\dim V_i) i \in \mathbb{N}Q_0$ and let $\dim_{\mathbb{C}} V = \sum_{i \in Q_0} \dim V_i \in \mathbb{N}$. We will sometimes view the graded dimension $\dim_{Q_0} V$ of V as its isomorphism class.

For $V, W \in \mathcal{P}_0$-mod, we denote the set of $\mathcal{P}_0$-module morphisms from V to W by $\mathrm{Hom}_{\mathcal{P}_0}(V, W)$. Under the equivalence of categories above, $\mathrm{Hom}_{\mathcal{P}_0}(V, W)$ is identified with $\bigoplus_{i\in Q_0}\operatorname{Hom}_{\mathbb{C}}(V_i, W_i)$. We define $\operatorname{End}_{\mathcal{P}_0} V$ to be $\operatorname{Hom}_{\mathcal{P}_0}(V, V)$ and $\mathrm{GL}_V = \prod_{i\in Q_0} GL(V_i)$ to be group of invertible elements of $\operatorname{End}_{\mathcal{P}_0} V$. For $V \in \mathcal{P}_0$-mod, we will write $U \subseteq V$ to mean that U is a $\mathcal{P}_0$-submodule of V. This is the same as a Q_0-graded subspace. Note that any $\mathcal{P}$-module becomes a $\mathcal{P}_0$-module by restriction, and thus can be thought of as a Q_0-graded vector space.

Suppose $A = \bigoplus_{n\geq 0} A_n$ is a graded algebra and V is an A-module. Then V is *nilpotent* if there exists an $n \in \mathbb{N}$ such that $A_k \cdot V = 0$ for all $k \geq n$. We say V is *locally nilpotent* if for all $v \in V$, there exists $n \in \mathbb{N}$ such that $A_k \cdot v = 0$ for all $k \geq n$. We denote by A-lnMod the category of locally nilpotent A-modules. For $n \geq 0$, we define $A_{\geq n} = \bigoplus_{k\geq n} A_k$ and we let $A_+ = A_{\geq 1}$.

Proposition 2.1. *For a quiver Q, the following are equivalent*:

(i) *$\mathcal{P}(Q)$ is finite-dimensional*,

(ii) *all finite-dimensional $\mathcal{P}(Q)$-modules are nilpotent*,

(iii) *all finite-dimensional $\mathcal{P}(Q)$-modules are locally nilpotent*, *and*

(iv) *Q is of finite type.*

Proof. The equivalence of (i) and (iv) is well-known; see [Reiten 1997], for example. That (ii) implies (iv) was proven in [Crawley-Boevey 2001] and the converse was proven by Lusztig [Lusztig 1991, Proposition 14.2]. Since a finite-dimensional module is nilpotent if and only if it is locally nilpotent, (ii) is equivalent to (iii). □

2B. ***Simple objects.*** For each $i \in Q_0$, let s^i be the simple $\mathbb{C}\tilde{Q}$-module given by $s^i_i = \mathbb{C}$ and $s^i_j = 0$ for $i \neq j$. Then s^i is also naturally a $\mathcal{P}$-module which we also denote by s^i.

Lemma 2.2. *The set $\{s^i\}_{i\in Q_0}$ is a set of representatives of the isomorphism classes of simple objects of $\mathbb{C}\tilde{Q}$-lnMod and $\mathcal{P}$-lnMod. In particular*, *if Q is of finite type*, *then $\{s^i\}_{i\in Q_0}$ is a set of representatives of the isomorphism classes of simple objects of $\mathbb{C}\tilde{Q}$-mod and $\mathcal{P}$-mod.*

Proof. Any nonzero element of a simple locally nilpotent module M generates a finite-dimensional module which must be all of M. Therefore M is finite-dimensional and hence nilpotent. Then $(\mathbb{C}\tilde{Q})_+$ and $\mathcal{P}_+$ are two-sided ideals of $\mathbb{C}\tilde{Q}$ and $\mathcal{P}$ respectively that act nilpotently on any nilpotent module. Therefore, simple nilpotent $\mathbb{C}\tilde{Q}$-modules and $\mathcal{P}$-modules are the same as simple $\mathbb{C}\tilde{Q}/(\mathbb{C}\tilde{Q})_+$-modules and $\mathcal{P}/\mathcal{P}_+$-modules respectively. Since

$$\mathbb{C}\tilde{Q}/(\mathbb{C}\tilde{Q})_+ \cong \mathcal{P}/\mathcal{P}_+ \cong \bigoplus_{i\in I} \mathbb{C}e_i,$$

the first statement follows. The second statement then follows from Proposition 2.1. □

Lemma 2.3. *Fix a quiver Q and let A be either $\mathbb{C}\tilde{Q}$ or $\mathcal{P}(Q)$. If $V \in A$-lnMod, then the socle of V is* $\{v \in V \mid A_+ \cdot v = 0\}$.

Proof. It is clear that $\{v \in V \mid A_+ \cdot v = 0\}$ is a sum of simple subrepresentations of V and is thus contained in the socle of V. Similarly, by Lemma 2.2, any simple subrepresentation of (V, x) is contained in $\{v \in V \mid A_+ \cdot v = 0\}$. □

2C. *Projective covers.* Recall that if A is an associative algebra and V is an A-module, then a *projective cover* of V is a pair (P, f) such that P is a projective A-module and $f : P \to V$ is a superfluous epimorphism of A-modules. This means that $f(P) = V$ and $f(P') \neq V$ for all proper submodules P' of P. We often omit the homomorphism f and simply call P a projective cover of V.

Definition 2.4. For $i \in Q_0$, let $p^i = \mathcal{P}e_i$.

Lemma 2.5. *Assume Q is a quiver of finite type. For $i \in Q_0$, $\{p^i\}_{i \in Q_0}$ is a set of representatives of the isomorphism classes of indecomposable projective $\mathcal{P}$-modules. Furthermore, p^i is a projective cover of s^i.*

Proof. This follows from [Auslander et al. 1995, Proposition 4.8]. □

Lemma 2.6. *Assume Q is a quiver of affine* (*tame*) *or indefinite* (*wild*) *type. Then there exist $i \in Q_0$ for which the simple module s^i does not have a projective cover.*

Proof. Since the module s^i is obviously cyclic, by [Anderson and Fuller 1992, Lemma 27.3] it has a projective cover if and only if $s^i \cong \mathcal{P}e/Ie$ for some idempotent $e \in \mathcal{P}$ and some left ideal I contained in the Jacobson radical of $\mathcal{P}$. Assume this is true for some idempotent e and ideal I. Then we must have $e = e_i$ and then I would have to contain $\mathcal{P}_+e_i$, the ideal consisting of all paths of length at least one starting at vertex i. We identify $\mathbb{Z}Q_0$ with the root lattice via $\sum v_j j \leftrightarrow \sum v_j \alpha_j$. Let β be a minimal positive imaginary root and let i be in the support of β (i.e., $\beta = \sum \beta_j \alpha_j$ with $\beta_i > 0$). By [Crawley-Boevey 2001, Theorem 1.2], there is a simple module T of $\mathcal{P}$ whose dimension vector is β and so, in particular, $\dim T_i \neq 0$. Since the simple module T cannot be killed by $\mathcal{P}_+e_i$ (since then T_i would be a proper submodule), $\mathcal{P}_+e_i$ is not contained in the Jacobson radical of $\mathcal{P}$. This contradicts the fact that I is contained in the Jacobson radical. □

2D. *Injective hulls.* Recall that if A is an associative algebra and V is an A-module, then an *injective hull* of V is an injective A-module E that is an essential extension of V (that is, V is a submodule of E and any nonzero submodule of E intersects V nontrivially). By Baer's Theorem [1940], the category $\mathcal{P}$-Mod has enough injectives. In particular, the simple modules s^i have injective hulls. Here we give an explicit description of these injective hulls in the finite type case, and study some of their properties in the more general case.

Definition 2.7. Assume Q is a quiver of finite type. For $i \in Q_0$, let

$$q^i = \mathrm{Hom}_{\mathbb{C}}(e_i \mathcal{P}, \mathbb{C})$$

be the dual space of the right $\mathcal{P}$-module $e_i\mathcal{P}$. Define a left $\mathcal{P}$-module structure on q^i by setting $a \cdot f(x) = f(xa)$, for $a \in \mathcal{P}$, $f \in q^i$, and $x \in e_i\mathcal{P}$.

Lemma 2.8. *If Q is a quiver of finite type, then $\{q^i\}_{i \in Q_0}$ is a set of representatives of the isomorphism classes of indecomposable injective $\mathcal{P}$-modules. Furthermore, q^i is an injective hull of s^i.*

Proof. If Q is of finite type, then $\mathcal{P}$ is finite-dimensional by Proposition 2.1. The result then follows from Lemma 2.5 and a well-known fact about modules over finite-dimensional algebras; see, for example, [Lam 1999, Corollary 3.66]. $\square$

For $w = \sum_i w_i i \in \mathbb{N}Q_0$, define the semisimple $\mathcal{P}$-module

$$s^w = \bigoplus_{i \in Q_0} (s^i)^{\oplus w_i}.$$

Let q^i be the injective hull of s^i in the category $\mathcal{P}$-Mod (if Q is a quiver of finite type, this agrees with the notation of Definition 2.7). Then

$$q^w = \bigoplus_{i \in I} (q^i)^{\oplus w_i}$$

is the injective hull of s^w.

Lemma 2.9. *For $w \in \mathbb{N}Q_0$, any finite-dimensional submodule of q^w is nilpotent.*

Proof. Let V be a finite-dimensional submodule of q^w. Then we have the chain of submodules $V = \mathcal{P}_{\geq 0} V \supseteq \mathcal{P}_{\geq 1} V \supseteq \mathcal{P}_{\geq 2} V \supseteq \cdots$. Since q^w is an essential extension of s^w, we have $s^w \cap \mathcal{P}_{\geq n} V \neq 0$ for all $n \in \mathbb{N}$ such that $\mathcal{P}_{\geq n} V \neq 0$. Because $\mathcal{P}_1$ acts trivially on s^w, we have $\dim \mathcal{P}_{\geq n+1} V < \dim \mathcal{P}_{\geq n} V$ for all $n \in \mathbb{N}$ such that $\mathcal{P}_{\geq n} V \neq 0$. Thus $\mathcal{P}_{\geq n} V = 0$ for n large enough. $\square$

Remark 2.10. It follows from Lemma 2.9 and Proposition 7.10 that if Q is a quiver of finite type, then p^w (and q^w) is nilpotent. However, in general the p^w are not nilpotent.

Proposition 2.11. *If Q is of affine (tame) type, then q^w is locally nilpotent for all $w \in \mathbb{N}Q_0$. If Q is connected and of indefinite (wild) type, then q^w is not locally nilpotent for any $w \in \mathbb{N}Q_0$, $w \neq 0$.*

The following proof was explained to us by W. Crawley-Boevey.

Proof. It suffices to consider the case where $w = i$ for some $i \in Q_0$. We identify $\mathbb{Z}Q_0$ with the root lattice via $\sum v_j j \leftrightarrow \sum v_j \alpha_j$. We first assume that Q is connected of wild type. Let β be a minimal positive imaginary root. Thus $(\beta, j) \leq 0$

for all $j \in Q_0$. Suppose the support of β is all of Q_0. Since Q is wild, β cannot be a radical vector (see [Kac 1990, Theorem 4.3]), so $(\beta, j) < 0$ for some $j \in Q_0$. If, on the other hand, the support of β is not all of Q_0, we take $j \in Q_0$ to be a vertex not in the support of β but connected to it by an arrow and we again have $(\beta, j) < 0$. By [Crawley-Boevey 2001, Theorem 1.2], there is a simple module T for the preprojective algebra of dimension β. By [Crawley-Boevey 2000, Lemma 1], $\mathrm{Ext}^1(T, s^j)$ is nonzero. Let V be a nontrivial extension of T by s^j. This module must embed in the injective hull q^j of s^j and thus q^j cannot be locally nilpotent. Thus the result holds whenever $(\beta, i) < 0$. For general i, choose a shortest path from i to some j with $(\beta, j) < 0$ and consider the corresponding nilpotent module U with head s^j and socle s^i. Then, as above, there is a nontrivial extension of T by U, which must embed into q^i. So q^i is not locally nilpotent.

Now assume that Q is of tame type. Since the preprojective algebra of a tame quiver is a finitely generated $\mathbb{C}$-algebra, noetherian, and a polynomial identity ring [Baer et al. 1987, Theorem 6.5] (see [Ringel 1998] for a proof that the preprojective algebra considered there is the same as the one considered here), any simple module is finite-dimensional; see [McConnell and Robson 2001, Theorem 13.10.3]. By [Jategaonkar 1976, Theorem 2], the injective hull of a simple $\mathcal{P}$-module is artinian. In particular, finitely generated submodules of injective hulls of simple modules are artinian and noetherian. Thus they are of finite length and hence finite-dimensional. Now, the dimension vectors of simple $\mathcal{P}$-modules are the coordinate vectors $i \in Q_0$ and the minimal imaginary root δ. Since $(\delta, i) = 0$ for all $i \in Q_0$, there are no nontrivial extensions between simples of dimension δ and the one-dimensional simples. Therefore, the composition factors of the finite-dimensional submodules of the injective hull q^i of s^i are all one-dimensional simple modules. Thus q^i is locally nilpotent. □

Remark 2.12. In types A and D, there exist simple and explicit descriptions of the representations q^i, $i \in Q_0$, in terms of classical combinatorial objects such as Young diagrams; see [Frenkel and Savage 2003; Savage 2006b; 2006c]. This allows one to give simple and explicit descriptions of the injective modules q^w for any $w \in \mathbb{N}Q_0$ when the underlying graph of the corresponding quiver is of type A or D.

2E. *Quiver grassmannians.*

Definition 2.13 (quiver grassmannian). For a $\mathbb{C}Q$-module V, let $\mathrm{Gr}_Q(V)$ be the variety of all $\mathbb{C}Q$-submodules of V. We have a natural decomposition

$$\mathrm{Gr}_Q(V) = \bigsqcup_{u \in \mathbb{N}Q_0} \mathrm{Gr}_Q(u, V), \quad \mathrm{Gr}_Q(u, V) = \{U \in \mathrm{Gr}_Q(V) \mid \dim U = u\}.$$

We call $\mathrm{Gr}_Q(u, V)$ a *quiver grassmannian.* Note that $\mathrm{Gr}_Q(u, V)$ is a closed subset of the usual grassmannian of dimension u subspaces of V and thus is a projective variety. If V is a $\mathcal{P}$-module, then $\mathcal{P}$-submodules of V are the same as $\mathbb{C}\tilde{Q}$-submodules of V. Hence one can think of $\mathrm{Gr}_{\tilde{Q}}(V)$ as the variety of all $\mathcal{P}$-submodules of V. Therefore, we will often write $\mathrm{Gr}_{\mathcal{P}}(V)$ and $\mathrm{Gr}_{\mathcal{P}}(u, V)$ for $\mathrm{Gr}_{\tilde{Q}}(V)$ and $\mathrm{Gr}_{\tilde{Q}}(u, V)$ when V is a $\mathcal{P}$-module.

Example 2.14 (grassmannians). If Q is the quiver with a single vertex and no arrows, then $\mathcal{P} = \mathbb{C}$ and $\mathcal{P}$-modules are simply vector spaces. Then $\mathrm{Gr}_{\mathcal{P}}(u, V) = \mathrm{Gr}(u, V)$ is the usual grassmannian of dimension u subspaces of V.

Example 2.15 (partial flag varieties). Let Q be the quiver with $Q_0 = \{1, 2, \dots, n\}$ and $Q_1 = \{a_1, \dots, a_{n-1}\}$, where $s(a_i) = i$, $t(a_i) = i+1$ for all $i = 1, \dots, n-1$. Fix a positive integer d and set $V_i = \mathbb{C}^d$ for all $i = 1, \dots, n$. For each $1 \le i \le n-1$, let a_i act by the identification $V_i \cong V_{i+1}$. Then for $u \in \mathbb{N}Q_0$ with $u_1 \le u_2 \le \cdots \le u_n \le d$, the quiver grassmannian $\mathrm{Gr}_{\mathcal{P}}(u, V)$ is isomorphic to the partial flag variety

$$\{0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n \subseteq \mathbb{C}^d \mid \dim F_i = u_i\}.$$

Definition 2.16. For $V \in \mathcal{P}$-Mod, we define a natural action of $\mathrm{Aut}_{\mathcal{P}} V$ on $\mathrm{Gr}_{\mathcal{P}}(u, V)$ by

$$(g, U) \mapsto g(U), \quad g \in \mathrm{Aut}_{\mathcal{P}} V, \quad U \in \mathrm{Gr}_{\mathcal{P}}(u, V).$$

3. Quiver varieties

We briefly recall certain quiver varieties defined by Lusztig and Nakajima, referring the reader to [Lusztig 1991; Nakajima 1994; 1998] for further details, as well as the Demazure quiver varieties introduced in [Savage 2006d]. We fix a quiver $Q = (Q_0, Q_1)$ and let $\mathcal{P} = \mathcal{P}(Q)$ denote its preprojective algebra.

3A. ***Lusztig and Nakajima quiver varieties.*** For $V \in \mathcal{P}_0$-mod, define

$$\mathrm{Rep}_{\tilde{Q}} V = \bigoplus_{a \in \tilde{Q}_1} \mathrm{Hom}_{\mathbb{C}}(V_{s(a),t(a)}).$$

For a path $\beta = a_l \cdots a_1$ in Q and $x = (x_a)_{a \in \tilde{Q}_1} \in \mathrm{Rep}_{\tilde{Q}} V$, we define $x_\beta = x_{a_l} \cdots x_{a_1}$. For an element $\sum_j c_j \beta_j \in \mathbb{C}Q$, we define

$$x_{\sum_j c_j \beta_j} = \sum_j c_j x_{\beta_j}.$$

Thus each $x \in \mathrm{Rep}_{\tilde{Q}} V$ defines a representation $\mathbb{C}\tilde{Q} \to \mathrm{End}_{\mathbb{C}} V$ of graded dimension $\dim_{Q_0} V$ (i.e., whose induced representation of $(\mathbb{C}Q)_0$ is in the isomorphism class determined by $\dim_{Q_0} V$). Furthermore, each such representation comes from an element of $x \in \mathrm{Rep}_{\tilde{Q}} V$. These two statements are simply the equivalence of categories between the representations of the quiver and of the path algebra. We say that x is *nilpotent* if there exists $N > 0$ such that $x_\beta = 0$ for all paths β of length greater than N.

Definition 3.1 (Lusztig nilpotent variety). For $V \in \mathcal{P}_0$-mod, define $\Lambda(V) = \Lambda_Q(V)$ to be the set of all nilpotent $\mathcal{P}$-module structures on V compatible with its $\mathcal{P}_0$-module structure. More precisely,

$$\Lambda(V) = \left\{ x \in \operatorname{Rep}_{\tilde{Q}} V \;\middle|\; \sum_{\substack{a \in Q_1,\\ t(a)=i}} x_a x_{\bar{a}} - \sum_{\substack{a \in Q_1,\\ s(a)=i}} x_{\bar{a}} x_a = 0 \;\; \forall\, i \in Q_0,\ x \text{ nilpotent} \right\}.$$

We call $\Lambda(V)$ a *Lusztig nilpotent variety*.

As above, elements of $\Lambda(V)$ are in natural one-to-one correspondence with nilpotent representations $\mathcal{P} \to \operatorname{End}_{\mathbb{C}} V$ of graded dimension $\dim_{Q_0} V$.

For $V, W \in \mathcal{P}_0$-mod, let $\Lambda(V, W) = \Lambda(V) \times \operatorname{Hom}_{\mathcal{P}_0}(V, W)$. We say that $(x, t) \in \Lambda(V, W)$ is *stable* if there exists no nontrivial x-invariant $\mathcal{P}_0$-submodule of V contained in $\ker t$. This is equivalent to the condition that $\ker((x, t)|_{V_i}) = 0$ for all $i \in Q_0$ (see [Frenkel and Savage 2003, Lemma 3.4] — while the statement there is for type A, the proof carries over to the more general case). We denote the set of stable elements by $\Lambda(V, W)^{\mathrm{st}}$. There is a natural action of GL_V on $\Lambda(V, W)$ and the restriction to $\Lambda(V, W)^{\mathrm{st}}$ is free; see [Nakajima 1994; 1998]. We denote the GL_V-orbit through a point (x, t) by $[x, t]$.

Definition 3.2 (lagrangian Nakajima quiver variety). For $V, W \in \mathcal{P}_0$-mod, let $\mathfrak{L}(V, W) = \Lambda(V, W)^{\mathrm{st}}/\mathrm{GL}_V$. We call $\mathfrak{L}(V, W)$ a *lagrangian Nakajima quiver variety*. Up to isomorphism, this variety depends only on $v = \dim_{Q_0} V$ and $w = \dim_{Q_0} W$ and so we will sometimes denote it by $\mathfrak{L}(v, w)$.

Remark 3.3. The quiver varieties defined above are lagrangian subvarieties of what are usually called the Nakajima quiver varieties [Nakajima 1994; 1998].

3B. ***Group actions.*** Let $G_{\mathcal{P}}$ be the group of algebra automorphisms of $\mathcal{P}$ that fix $\mathcal{P}_0$. The group GL_W acts naturally on $\operatorname{Hom}_{\mathcal{P}_0}(V, W)$. As above, we identify elements of $\Lambda(V)$ with nilpotent representations $\mathcal{P} \to \operatorname{End}_{\mathbb{C}} V$ of graded dimension $\dim_{Q_0} V$. Then

$$(h, (x, t)) \mapsto (h \star x, t), \quad h \star x = x \circ h^{-1}, \quad h \in G_{\mathcal{P}},$$

defines a $G_{\mathcal{P}}$-action on $\Lambda(V, W)$. The actions of GL_W and $G_{\mathcal{P}}$ commute and both commute with the GL_V-action. Since they also preserve the stability condition, they define a $\mathrm{GL}_W \times G_{\mathcal{P}}$-action on $\mathfrak{L}(v, w)$.

We can use this action to define $\mathrm{GL}_W \times \mathbb{C}^*$-actions on $\mathfrak{L}(v, w)$ as follows. Suppose a function $m : \tilde{Q}_1 \to \mathbb{Z}$ is given such that $m(a) = -m(\bar{a})$ for all $a \in \tilde{Q}_1$.

Then the map $a \mapsto z^{m(a)+1}a$, $z \in \mathbb{C}^*$, extends to an automorphism of $\mathcal{P}$ fixing $\mathcal{P}_0$. We denote this automorphism by $h_m(z)$. Thus h_m defines a group homomorphism $\mathbb{C}^* \to G_{\mathcal{P}}$. Then the homomorphism

$$\mathrm{GL}_W \times \mathbb{C}^* \to \mathrm{GL}_W \times G_{\mathcal{P}}, \quad (g, z) \mapsto (zg, h_m(z)) \tag{3-1}$$

defines a $\mathrm{GL}_W \times \mathbb{C}^*$-action on $\mathfrak{L}(v, w)$ which we denote by $\star_m$.

We give two important examples of this action [Nakajima 2001, §2.7; 2004]. First, for each pair $i, j \in Q_0$ connected by at least one edge, let b_{ij} denote the number of arrows in Q_1 joining i and j. We fix a numbering $a_1, \dots, a_{b_{ij}}$ of these arrows, which induces a numbering $\bar{a}_1, \dots, \bar{a}_{b_{ij}}$ of the corresponding arrows in $\bar{Q}_1$. Define $m_1 : H \to \mathbb{Z}$ by

$$m_1(a_p) = b_{ij} + 1 - 2p, \quad m_1(\bar{a}_p) = -b_{ij} - 1 + 2p.$$

For the second action, we define $m_2(a) = 0$ for all $a \in Q_1$.

3C. *Demazure quiver varieties.* Let $\mathfrak{g}$ be the Kac–Moody algebra corresponding to the underlying graph of Q (the one whose Dynkin diagram is this graph) and let $\mathcal{W}$ be its Weyl group. Recall that $\mathcal{W}$ acts naturally on the weight lattice of $\mathfrak{g}$. For $u \in \mathbb{Z}Q_0$, we define elements of the weight and root lattice by

$$\omega_u = \sum_{i \in Q_0} u_i \omega_i, \quad \alpha_u = \sum_{i \in Q_0} u_i \alpha_i.$$

Proposition/Definition 3.4 [Savage 2006d, Proposition 5.1]. The lagrangian Nakajima quiver variety $\mathfrak{L}(v, w)$ is a point if and only if $\omega_w - \alpha_v = \sigma(\omega_w)$ for some $\sigma \in \mathcal{W}$ (i.e., $\omega_w - \alpha_v$ is an extremal weight of the irreducible representation of highest weight ω_w, equivalently v is w-extremal in the sense of Definition 4.7). In this case, we let $(x^{w,\sigma}, t^{w,\sigma})$ be a representative (unique up to isomorphism) of the GL_V-orbit corresponding to this point. So $\mathfrak{L}(v, w) = \{[x^{w,\sigma}, t^{w,\sigma}]\}$ when $\omega_w - \alpha_v = \sigma(\omega_w)$.

Definition 3.5 (Demazure quiver variety). For $\sigma \in \mathcal{W}$ and $v, w \in \mathbb{N}Q_0$, let $\mathfrak{L}_\sigma(v, w)$ be the subvariety consisting of all $[x, t] \in \mathfrak{L}(v, w)$ such that (x, t) is isomorphic to a subrepresentation of $(x^{w,\sigma}, t^{w,\sigma})$. We call $\mathfrak{L}_\sigma(v, w)$ a *Demazure quiver variety*.

Remark 3.6. It follows from the uniqueness assertion in Proposition/Definition 3.4 that the $\mathrm{GL}_W \times G_{\mathcal{P}}$-action on $\mathfrak{L}(v, w)$ fixes $\mathfrak{L}_\sigma(v, w)$ for all $\sigma \in \mathcal{W}$. Thus we have an induced $\mathrm{GL}_W \times G_{\mathcal{P}}$-action on the Demazure quiver varieties.

4. Quiver varieties as quiver grassmannians

4A. *Lagrangian Nakajima quiver varieties as quiver grassmannians.* We will now show that certain quiver grassmannians are homeomorphic to the lagrangian Nakajima quiver varieties. We begin with a key technical proposition.

Proposition 4.1. *Suppose $A = \bigoplus_{n\geq 0} A_n$ is a graded algebra and V is a locally nilpotent A-module. Furthermore, suppose S is a semisimple locally nilpotent A-module with injective hull E.*

(i) *Let $\pi : E \to S$ be an A_0-linear retract for the canonical embedding $\iota : S \to E$ (that is, an A_0-linear map such that $\pi\iota = \mathrm{id}$) and let $\tau : V \to S$ be a homomorphism of A_0-modules. Then there exists a unique A-module homomorphism $\gamma : V \to E$ such that the following diagram commutes:*

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ V \arrow[ur, "\gamma"] \arrow[r, "\tau"'] & S \end{tikzcd}$$

Furthermore, the map γ is injective if and only if $\tau|_{\mathrm{socle}\, V}$ is injective.

(ii) *Suppose $\pi_1, \pi_2 : E \to S$ are A_0-linear retracts for the canonical embedding $\iota : S \to E$. Then there exists a unique $\gamma \in \mathrm{Aut}_A E$ such that $\pi_2 = \pi_1\gamma$. The map γ fixes S pointwise. Conversely, given an A_0-linear retract $\pi : E \to S$ and any $\gamma \in \mathrm{Aut}_A E$ fixing S pointwise, $\pi\gamma : E \to S$ is also a A_0-linear retract.*

Proof. Since V is locally nilpotent, we have a filtration

$$0 = V^{(0)} \subseteq V^{(1)} = \mathrm{socle}\, V \subseteq V^{(2)} \subseteq V^{(3)} \subseteq \cdots$$

of V where $V^{(n)} = \{m \in V \mid A_{\geq n} \cdot m = 0\}$. We prove by induction on n that there exists a unique homomorphism $\gamma_n : V^{(n)} \to E$ such that the diagram

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ V^{(n)} \arrow[ur, "\gamma_n"] \arrow[r, "\tau_n"'] & S \end{tikzcd} \tag{4-1}$$

commutes, where $\tau_n = \tau|_{V^{(n)}}$. Since $V^{(1)} = \mathrm{socle}\, V$ and $A_+ \cdot \mathrm{socle}\, V = 0$, we must have $\gamma_1(V^{(1)}) \subseteq S$ and so the unique choice for γ_1 is τ_1. Suppose the statement holds for $n = k$. Since E is injective, there exists an A-module homomorphism $\hat{\gamma}_{k+1}$ such that the following diagram commutes:

$$\begin{tikzcd} V^{(k+1)} \arrow[r, "\hat{\gamma}_{k+1}"] & E \\ V^{(k)} \arrow[u, hook] \arrow[ur, "\gamma_k"'] & \end{tikzcd}$$

Define γ_{k+1} by

$$\gamma_{k+1} = \hat{\gamma}_{k+1} - \pi \circ \hat{\gamma}_{k+1} + \tau.$$

It is then clear that the diagram (4-1) commutes (with $n = k+1$). Note also that $\gamma_{k+1}|_{V^{(k)}} = \gamma_k$. We claim that γ_{k+1} is a homomorphism of A-modules. Since it is an A_0-module homomorphism by definition, it suffices to show it commutes with the action of A_+.

For $r \in A_+$ and $m \in V^{(k+1)}$, we have $r \cdot m \in V^{(k)}$. Also, $A_+ \cdot S = 0$. Then

$$\begin{aligned} r \cdot \gamma_{k+1}(m) &= r \cdot (\hat{\gamma}_{k+1}(m) - \pi \circ \hat{\gamma}_{k+1}(m) + \tau(m)) \\ &= r \cdot \hat{\gamma}_{k+1}(m) = \hat{\gamma}_{k+1}(r \cdot m) = \gamma_k(r \cdot m) \\ &= \gamma_{k+1}(r \cdot m), \end{aligned}$$

as desired.

Now suppose that γ'_{k+1} is another $\mathcal{P}$-module homomorphism making (4-1) commute (with $n = k+1$). By the inductive hypothesis, we have $\gamma_{k+1}|_{V^{(k)}} = \gamma'_{k+1}|_{V^{(k)}}$. For all $r \in A_+$ and $m \in V^{(k+1)}$, we have

$$r \cdot \gamma_{k+1}(m) = \gamma_{k+1}(r \cdot m) = \gamma'_{k+1}(r \cdot m) = r \cdot \gamma'_{k+1}(m).$$

Thus $\gamma_{k+1}(m) - \gamma'_{k+1}(m)$ lies in S. Therefore

$$\begin{aligned} \gamma_{k+1}(m) - \gamma'_{k+1}(m) &= \pi(\gamma_{k+1}(m) - \gamma'_{k+1}(m)) \\ &= \pi(\gamma_{k+1}(m)) - \pi(\gamma'_{k+1}(m)) = \tau(m) - \tau(m) = 0. \end{aligned}$$

The induction is complete and we obtain the desired map γ by taking the limit.

Note that $\gamma|_{\operatorname{socle} V} = \tau|_{\operatorname{socle} V}$. Since a homomorphism of modules is injective if and only if its restriction to the socle is injective, it follows that γ is injective if and only if $\tau|_{\operatorname{socle} V}$ is injective.

We now prove (ii). By (i), there exists a unique A-module homomorphism $\gamma : E \to E$ such that $\pi_2 = \pi_1 \gamma$. Similarly, there exists a unique A-module automorphism $\tilde{\gamma} : E \to E$ such that $\pi_1 = \pi_2 \tilde{\gamma}$ and $\gamma\tilde{\gamma} = \tilde{\gamma}\gamma = \mathrm{id}$ by the uniqueness assertion in (i). Thus γ is an A-automorphism of E. The converse statement is trivial. □

Remark 4.2. The retract $\pi : E \to S$ in Proposition 4.1 is equivalent to choosing an A_0-module decomposition $E = S \oplus T$. The second part of the proposition states that any two such decompositions are related by a unique A-module automorphism of E fixing S.

Definition 4.3. Let V be a $\mathcal{P}_0$-module of graded dimension v. Define $\widehat{\mathrm{Gr}}_{\mathcal{P}}(v, q^w)$ to be the variety of injective $\mathcal{P}_0$-module homomorphisms $\gamma : V \to q^w$ whose image is a $\mathcal{P}$-submodule of q^w.

Theorem 4.4. *Fix $v, w \in \mathbb{N}Q_0$. Then there is a bijective* GL_V*-equivariant algebraic map from $\widehat{\mathrm{Gr}}_{\mathcal{P}}(v, q^w)$ to $\Lambda(v, w)^{\mathrm{st}}$ and a bijective algebraic map from*

$\mathrm{Gr}_{\mathcal{P}}(v, q^w)$ *to* $\mathfrak{L}(v, w)$. *In particular*, $\widehat{\mathrm{Gr}}_{\mathcal{P}}(v, q^w)$ *is homeomorphic to* $\Lambda(v, w)^{\mathrm{st}}$ *and* $\mathrm{Gr}_{\mathcal{P}}(v, q^w)$ *is homeomorphic to* $\mathfrak{L}(v, w)$.

Remark 4.5. Lusztig [1998; 2000] has described a canonical bijection between the lagrangian Nakajima quiver varieties and grassmannian type varieties inside the projective modules p^w (see Section 7). In several places in the literature, it was claimed that the varieties defined by Lusztig are isomorphic (as algebraic varieties) to the lagrangian Nakajima quiver varieties. However, the authors were not aware of a proof existing in the literature. Most references for this statement were to [Lusztig 1998; 2000], where the points of the two varieties are shown to be in canonical bijection (similar to the situation in the current paper). Lusztig informed the authors that he was not aware of a proof that the varieties are isomorphic. After the appearance of an earlier version of the current paper [Savage and Tingley 2009], Shipman [2010] proved that the varieties are indeed isomorphic. From now on, we will incorporate Shipman's work, as it allows us to strengthen several results; in particular (see Corollary A.6 in the Appendix) the map $\bar{\iota}$ in the proof below is an isomorphism of algebraic varieties.

Proof of Theorem 4.4. Fix $V \in \mathcal{P}_0$-mod of graded dimension v and a $\mathcal{P}_0$-module homomorphism $\pi : q^w \to s^w$ that is the identity on s^w. We identify s^w with the W appearing in the definition of the quiver varieties. A point $\gamma \in \widehat{\mathrm{Gr}}_{\mathcal{P}}(v, q^w)$ defines an embedding of V into q^w, hence a $\mathcal{P}$-module structure on V satisfying the stability condition and so a point of $\Lambda(v, w)^{\mathrm{st}}$. More precisely, $\gamma \in \widehat{\mathrm{Gr}}_{\mathcal{P}}(v, q^w)$ corresponds to the point $(\gamma^{-1} x^w \gamma, \pi\gamma) \in \Lambda(v, w)^{\mathrm{st}}$, where x^w is the element of $\mathrm{Rep}_{\bar{Q}}\, q^w$ corresponding to the $\mathcal{P}$-module q^w. Thus we have a map

$$\iota : \widehat{\mathrm{Gr}}_{\mathcal{P}}(v, q^w) \to \Lambda(V, W)^{\mathrm{st}},$$

which is clearly algebraic and GL_V-equivariant. By Proposition 4.1, ι is bijective. Passing to the quotient by GL_V we also obtain a bijective algebraic map $\bar{\iota}$ from $\mathrm{Gr}_{\mathcal{P}}(v, q^w)$ to $\mathfrak{L}(v, w)$.

Now, $\mathrm{Gr}_{\mathcal{P}}(v, q^w)$ and $\mathfrak{L}(v, w)$ are both projective. By, for example, [Hartshorne 1977, Theorem 4.9 and Exercise 4.4], the image of a projective variety under an algebraic map is always closed, so $\bar{\iota}$ takes closed subsets to closed subsets. Since $\bar{\iota}$ is a bijection, this implies that $\bar{\iota}^{-1}$ is continuous. Hence $\bar{\iota}$ is a homeomorphism. Since $\widehat{\mathrm{Gr}}_{\mathcal{P}}(v, q^w)$ and $\Lambda(v, w)^{\mathrm{st}}$ are principal G-bundles over $\mathrm{Gr}_{\mathcal{P}}(v, q^w)$ and $\mathfrak{L}(v, w)$, the map ι also induces a homeomorphism. □

Remark 4.6.

(i) The role of the retract π in Proposition 4.1 is to ensure the uniqueness of γ.

(ii) When Q is of finite type, the injective module q^w is also projective (see Proposition 7.10) and thus Theorem 4.4 follows from [Lusztig 2000, §2.1].

(iii) The isomorphisms of Theorem 4.4 depend on the choice of the retract $\pi : q^w \to s^w$. By Proposition 4.1(ii), isomorphisms coming from different retracts are related by an automorphism of q^w fixing s^w.

(iv) In Lusztig's grassmannian type realization of the lagrangian Nakajima quiver varieties [Lusztig 1998; 2000], one must require that the submodules contain all paths of large enough length (this corresponds to the nilpotency condition in the definition of the quiver varieties). In the current approach using injective modules, no such condition is required due to Lemma 2.9.

4B. ***Demazure quiver grassmannians.*** As before, let $\mathfrak{g}$ be the Kac–Moody algebra corresponding to the underlying graph of Q and let $\mathcal{W}$ be its Weyl group with Bruhat order $\preceq$.

Definition 4.7. For each $w \in \mathbb{N}Q_0$, we define an action of $\mathcal{W}$ on $\mathbb{Z}Q_0$ as follows. For $v \in \mathbb{Z}Q_0$ and $\sigma \in \mathcal{W}$, define $\sigma \cdot_w v = u$ where u is the unique element of $\mathbb{Z}Q_0$ satisfying

$$\sigma(\omega_w - \alpha_v) = \omega_w - \alpha_u.$$

We say that $v \in \mathbb{N}Q_0$ is *w-extremal* if $v \in \mathcal{W} \cdot_w 0$.

Lemma 4.8. *If $v, w \in \mathbb{N}Q_0$ and $\omega_w - \alpha_v$ is a weight of the irreducible highest weight representation of $\mathfrak{g}$ of highest weight ω_w (i.e the corresponding weight space is nonzero), then $\sigma \cdot_w v \in \mathbb{N}Q_0$ for all $\sigma \in \mathcal{W}$. In particular $\mathcal{W} \cdot_w 0 \subseteq \mathbb{N}Q_0$.*

Proof. This follows easily from the fact that $\mathcal{W}$ acts on the weights of highest weight irreducible representations and the weight multiplicities are invariant under this action. □

Proposition 4.9. *For $v \in \mathbb{N}Q_0$, the following statements are equivalent*:

(i) *v is w-extremal*,

(ii) *$\mathfrak{L}(v, w)$ consists of a single point*,

(iii) $\mathrm{Gr}_{\mathcal{P}}(v, q^w)$ *consists of a single point*, *and*

(iv) *there is a unique submodule of q^w of graded dimension v.*

Proof. The equivalence of (i) and (ii) is given in [Savage 2006d, Proposition 5.1]. The equivalence of (ii), (iii) and (iv) follows from Theorem 4.4. □

Definition 4.10 (Demazure quiver grassmannian). For $\sigma \in \mathcal{W}$, we let $q^{w,\sigma}$ denote the unique submodule of q^w of graded dimension $\sigma \cdot_w 0$. We call $\mathrm{Gr}_{\mathcal{P}}(v, q^{w,\sigma})$ a *Demazure quiver grassmannian.*

Proposition 4.11. *If $\sigma_1, \sigma_2 \in \mathcal{W}$ with $\sigma_1 \preceq \sigma_2$, then q^{w,σ_2} has a unique submodule of graded dimension $\sigma_1 \cdot_w 0$ and this submodule is isomorphic to q^{w,σ_1}.*

Proof. Since $\sigma_1 \preceq \sigma_2$, we have $L_{\omega_w,\sigma_1} \subseteq L_{\omega_w,\sigma_2}$, where L_{ω_w,σ_i} is the Demazure module corresponding to L_{ω_w} (the irreducible integrable highest weight $\mathfrak{g}$-module with highest weight ω_w) and σ_i. It then follows from [Savage 2006d, Theorem 7.1] that q^{w,σ_1} is (isomorphic to) a submodule of q^{w,σ_2}. Since any submodule of q^{w,σ_2} is also a submodule of q^w, uniqueness follows directly from Proposition 4.9. □

Proposition 4.12. *Fix $\sigma \in \mathcal{W}$ and $v, w \in \mathbb{N}Q_0$. Then $\mathrm{Gr}_{\mathcal{P}}(v, q^{w,\sigma})$ is isomorphic (as an algebraic variety) to the Demazure quiver variety $\mathfrak{L}_\sigma(v, w)$.*

Proof. This follows immediately from Definitions 3.5 and 4.10 and the description of the homeomorphism $\mathrm{Gr}_{\mathcal{P}}(v, q^w) \cong \mathfrak{L}(v, w)$ given in Theorem 4.4, which is actually an isomorphism of algebraic varieties by Corollary A.6. □

Remark 4.13. Note that if Q is a quiver of finite type and σ_0 is the longest element of $\mathcal{W}$, then $\mathfrak{L}_{\sigma_0}(v, w) = \mathfrak{L}(v, w)$ and $\mathrm{Gr}(v, q^{w,\sigma_0}) = \mathrm{Gr}(v, q^w)$ for all $v, w \in \mathbb{N}Q_0$.

The $(q^{w,\sigma})_{\sigma\in\mathcal{W}}$ form a directed system under the Bruhat order. Let $\tilde{q}^w$ be the direct limit of this system.

Lemma 4.14. *Any locally nilpotent submodule V of q^w is contained in $\tilde{q}^w$.*

Proof. First note that for $n \in \mathbb{N}$, the submodule $(q^w)^{(n)} = \{v \in q^w : \mathcal{P}_{\geq n} \cdot v = 0\}$ of q^w is finite-dimensional. This follows from the fact that q^i is a submodule of $\mathrm{Hom}_{\mathbb{C}}(e_i\mathcal{P}, \mathbb{C})$ (since this is an injective module containing s^i), which has this property, and $q^w = \bigoplus_{i\in I}(q^i)^{\oplus w_i}$.

Since V is locally nilpotent, we have a filtration

$$0 = V^{(0)} \subseteq V^{(1)} = \operatorname{socle} V \subseteq V^{(2)} \subseteq \cdots$$

where $V^{(n)} = \{v \in V : \mathcal{P}_{\geq n} \cdot v = 0\}$. Local nilpotency of V ensures that $\bigcup_n V^{(n)} = V$. It suffices to show that each $V^{(n)}$ is contained in $\tilde{q}^w$. Since $V^{(n)} \subseteq (q^w)^{(n)}$, it follows that $V^{(n)}$ is finite-dimensional. Choose a linear retract $\pi : q^w \to s^w$. By Theorem 4.4, V corresponds to a point of $\mathfrak{L}(v, w)$. Choose $\sigma \in \mathcal{W}$ sufficiently large so that the $(\omega_w - \alpha_v)$-weight space of the representation L_{ω_w} is contained in the Demazure module $L_{\omega_w,\sigma}$ (we can always do this since the weight space is finite-dimensional). Then by Proposition 4.12, we have that $V \subseteq q^{w,\sigma} \subseteq \tilde{q}^w$. □

Theorem 4.15. *We have that $\tilde{q}^w$ is the injective hull of s^w in the category $\mathcal{P}$-lnMod.*

Proof. Since each $q^{w,\sigma}$ is nilpotent, it follows that $\tilde{q}^w$ is locally nilpotent and thus belongs to the category $\mathcal{P}$-lnMod. Furthermore, it is clear that $\tilde{q}^w$ has socle s^w and that it is an essential extension of s^w. It remains to show that $\tilde{q}^w$ is an injective object of $\mathcal{P}$-lnMod. Suppose M and N are locally nilpotent $\mathcal{P}$-modules and we have a homomorphism $M \to \tilde{q}^w$ and an injection $M \hookrightarrow N$. Since q^w is injective in the category of $\mathcal{P}$-modules, there exists a homomorphism $h : N \to q^w$ such that the following diagram commutes:

$$\begin{tikzcd} N \arrow[drr, "h"] & & \\ M \arrow[u, hook] \arrow[r] & \tilde{q}^w \arrow[r, hook] & q^w \end{tikzcd}$$

Since N is locally nilpotent, $h(N)$ is a locally nilpotent submodule of q^w. Therefore the map h factors through $\tilde{q}^w$ by Lemma 4.14. □

Corollary 4.16. *We have that $\tilde{q}^w \cong q^w$ if and only if Q is of finite or affine* (*tame*) *type.*

Proof. This follows immediately from Theorem 4.15 and Proposition 2.11. □

We see from the above that $\{q^{w,\sigma}\}_{\sigma\in\mathcal{W}}$ is a "rigid" filtration of $\tilde{q}^w$ (rigid in the sense of the uniqueness of submodules of the given w-extremal graded dimensions). Proposition 4.12 can be seen as a representation theoretic interpretation of this filtration. It corresponds to the filtration by Demazure modules of the irreducible highest-weight representation of $\mathfrak{g}$ of highest weight ω_w. If the quiver Q is of finite type, the Weyl group $\mathcal{W}$, and hence this filtration, is finite. Otherwise they are infinite. In the infinite case, we have a filtration of the infinite-dimensional $\tilde{q}^w$ by finite-dimensional submodules $q^{w,\sigma}$, $\sigma \in \mathcal{W}$.

5. Group actions and graded quiver grassmannians

We now define a natural $\mathrm{GL}_W \times G_{\mathcal{P}}$-action on the quiver grassmannians and show that the maps of Theorem 4.4 are equivariant. We then define graded/cyclic quiver grassmannians and show they are isomorphic to the graded/cyclic quiver varieties of Nakajima [2001, §4.1; 2004, §4].

5A. $\mathbf{GL}_w \times G_{\mathcal{P}}$*-action and equivariance.* Let $\mathrm{GL}_w = \mathrm{GL}_{s^w}$ and recall that $G_{\mathcal{P}}$ is the group of algebra automorphisms of $\mathcal{P}$ that fix $\mathcal{P}_0$ pointwise. For a $\mathcal{P}$-module V and $h \in G_{\mathcal{P}}$, denote by ${}^hV$ the $\mathcal{P}$-module with action given by $(a, v) \mapsto h^{-1}(a) \cdot v$. Now, fix $(g, h) \in \mathrm{GL}_w \times G_{\mathcal{P}}$ and a $\mathcal{P}_0$-module retract $\pi : q^w \to s^w$. By Proposition 4.1, there exists a unique $\mathcal{P}$-module homomorphism $\gamma_{(g,h)} : {}^hq^w \to q^w$ such that the following diagram commutes:

$$\begin{tikzcd} {}^hq^w \arrow[r, "\gamma_{(g,h)}"] \arrow[d, "\pi"'] & q^w \arrow[d, "\pi"] \\ s^w \arrow[r, "g"] & s^w \end{tikzcd}$$

The uniqueness assertion of Proposition 4.1 ensures that $\gamma_{(g,h)}$ is bijective with inverse $\gamma_{(g^{-1},h^{-1})}$. Note that since the action of $\mathcal{P}_0$ on ${}^h q^w$ and q^w is the same, $\gamma_{(g,h)}$ can be considered as a $\mathcal{P}_0$-automorphism of q^w. This defines a group homomorphism $\mathrm{GL}_w \times G_{\mathcal{P}} \to \mathrm{GL}_{q^w}$, $(g,h) \mapsto \gamma_{(g,h)}$. In other words, it defines an action of $\mathrm{GL}_w \times G_{\mathcal{P}}$ on q^w by $\mathcal{P}_0$-module automorphisms. This in turn defines an action on $\widehat{\mathrm{Gr}}_{\mathcal{P}}(v, q^w)$ and $\mathrm{Gr}_{\mathcal{P}}(v, q^w)$ given by

$$\begin{aligned}(g,h) \star \gamma &= \gamma_{(g,h)}\gamma, &\quad \gamma &\in \widehat{\mathrm{Gr}}_{\mathcal{P}}(v, q^w)\\ (g,h) \star U &= \gamma_{(g,h)}(U), &\quad U &\in \mathrm{Gr}_{\mathcal{P}}(v, q^w).\end{aligned}$$

Proposition 5.1. *The isomorphisms of Theorem 4.4 are* $\mathrm{GL}_w \times G_{\mathcal{P}}$*-equivariant.*

Proof. Let $(x,t) \mapsto \gamma(x,t)$ be the map $\Lambda(v,w)^{\mathrm{st}} \xrightarrow{\cong} \widehat{\mathrm{Gr}}_{\mathcal{P}}(v, q^w)$ of Theorem 4.4. Fix $(x,t) \in \Lambda(v,w)^{\mathrm{st}}$. Recall that for $(g,h) \in \mathrm{GL}_w \times G_{\mathcal{P}}$, we have $(g,h)\star(x,t) = (h\star x, gt)$. Let V^x be the $\mathcal{P}$-module corresponding to x. Then ${}^h V^x$ is the $\mathcal{P}$-module corresponding to $h \star x$. We have the commutative diagram

$$\begin{tikzcd} & q^w \arrow[d, "\pi"] \\ V^x \arrow[ur, "\gamma(x,t)"] \arrow[r, "t"'] & s^w \end{tikzcd}$$

It follows that the diagram

$$\begin{tikzcd} & {}^h q^w \arrow[r, "\gamma_{(g,h)}"] \arrow[d, "\pi"] & q^w \arrow[d, "\pi"] \\ {}^h V^x \arrow[ur, "\gamma(x,t)"] \arrow[r, "t"'] & s^w \arrow[r, "g"'] & s^w \end{tikzcd}$$

commutes. By the uniqueness statement in Proposition 4.1, we have

$$\gamma((g,h)\star(x,t)) = \gamma(h\star x, gt) = \gamma_{(g,h)}\gamma(x,t) = (g,h)\star\gamma(x,t),$$

which proves that the map $\Lambda(v,w)^{\mathrm{st}} \cong \widehat{\mathrm{Gr}}_{\mathcal{P}}(v, q^w)$ is equivariant. The remaining claim follows from the fact that the isomorphism $\mathfrak{L}(v,w) \cong \mathrm{Gr}_{\mathcal{P}}(v, q^w)$ is obtained from the map $\Lambda(v,w)^{\mathrm{st}} \xrightarrow{\cong} \widehat{\mathrm{Gr}}_{\mathcal{P}}(v, q^w)$ by taking quotients by GL_V. $\square$

5B. ***Graded/cyclic quiver grassmannians.*** Fix an abelian reductive subgroup A and a group homomorphism $\rho : A \to \mathrm{GL}_w \times G_{\mathcal{P}}$, defining an action of A on q^w by $\mathcal{P}_0$-module automorphisms. The weight space corresponding to $\lambda \in \mathrm{Hom}(A, \mathbb{C}^*)$ is

$$q^w(\lambda) \overset{\mathrm{def}}{=} \{v \in q^w \mid \rho(a)(v) = \lambda(a)v \ \ \forall a \in A\}. \tag{5-1}$$

We define

$$\begin{aligned}\mathrm{Gr}_{\mathcal{P}}(q^w)^A &= \{U \in \mathrm{Gr}_{\mathcal{P}}(q^w) \mid \rho(a) \star U = U \ \ \forall\, a \in A\},\\ \mathrm{Gr}_{\mathcal{P}}(u, q^w)^A &= \mathrm{Gr}_{\mathcal{P}}(q^w)^A \cap \mathrm{Gr}_{\mathcal{P}}(u, q^w).\end{aligned}$$

Then for all $U \in \mathrm{Gr}_{\mathcal{P}}(q^w)^A$, we have the map $\rho_U : A \to \mathrm{GL}_U$, $a \mapsto \rho(a)|_U$. In other words, ρ_U is a representation of A in the category of $\mathcal{P}_0$-modules. If ρ_1 and ρ_2 are two such representations, we write $\rho_1 \cong \rho_2$ when ρ_1 and ρ_2 are isomorphic. That is, $\rho_1 \cong \rho_2$ for $\rho_i : A \to \mathrm{GL}_{U_i}$, if there exists a $\mathcal{P}_0$-module isomorphism $\xi : U_1 \to U_2$ such that $\rho_2 = \xi\rho_1\xi^{-1}$, where $\xi\rho_U\xi^{-1}$ denotes the homomorphism $a \mapsto \xi\rho_U(a)\xi^{-1}$. Then, for $\rho_1 : A \to \mathrm{GL}_U$, U a $\mathcal{P}_0$-module, we define

$$\mathrm{Gr}_{\mathcal{P}}(\rho_1, q^w)^A = \{U' \in \mathrm{Gr}_{\mathcal{P}}(q^w)^A \mid \rho_{U'} \cong \rho_1\}.$$

Note that $\mathrm{Gr}_{\mathcal{P}}(\rho_1, q^w)^A$ depends only on the isomorphism class of ρ_1.

Recall the action of $\mathrm{GL}_w \times G_{\mathcal{P}}$ on $\Lambda(V, W)^{\mathrm{st}}$ and $\mathfrak{L}(v, w)$ described in Section 3B (where we now identify W with s^w, $w = \dim_{Q_0} W$). Define

$$\begin{aligned}\mathfrak{L}(w)^A &= \{[x, t] \in \mathfrak{L}(v, w) \mid \rho(a) \star [x, t] = [x, t] \ \forall\, a \in A\},\\ \mathfrak{L}(v, w)^A &= \mathfrak{L}(w)^A \cap \mathfrak{L}(v, w).\end{aligned}$$

Fix a point $[x, t] \in \mathfrak{L}(v, w)^A$. For every $a \in A$, there exists a unique $\rho_1(a) \in \mathrm{GL}_V$ such that

$$\rho(a) \star (x, t) = \rho_1^{-1}(a) \cdot (x, t), \tag{5-2}$$

and the map $\rho_1 : A \to \mathrm{GL}_V$ is a homomorphism. Let $\mathfrak{L}(\rho_1, w)^A \subseteq \mathfrak{L}(v, w)^A$ be the set of A-fixed points y such that (5-2) holds for some representative (x, t) of y.

Theorem 5.2. *Let V be a $\mathcal{P}_0$-module and $\rho_1 : A \to \mathrm{GL}_V$ a group homomorphism. Then $\mathrm{Gr}_{\mathcal{P}}(\rho_1, q^w)^A$ is isomorphic to $\mathfrak{L}(\rho_1, w)^A$ as an algebraic variety.*

Proof. Choose $[x, t] \in \mathfrak{L}(\rho_1, w)^A$. Let $U = \gamma(x, t)(V)$ be the corresponding point of $\mathrm{Gr}_{\mathcal{P}}(v, q^w)^A$. We want to show that $\rho_1 \cong \rho_U$. Let $(g, h) \in A$ and consider the commutative diagram

$$\begin{tikzcd} & {}^h q^w \arrow[r, "\gamma_{(g,h)}"] \arrow[d, "\pi"] & q^w \arrow[d, "\pi"'] & \\ {}^h V^x \arrow[ur, "\gamma(x,t)"] \arrow[r, "t"'] & s^w \arrow[r, "g"'] & s^w & V \arrow[ul, "\gamma(x,t)"'] \arrow[l, "t"] \end{tikzcd}.$$

Then $\rho_U(g, h) = \gamma_{(g,h)}|_U$. Note that $\gamma(x, t)$ is an isomorphism when its codomain is restricted to U and we denote by $\gamma(x, t)^{-1}$ the inverse of this restriction. We claim that $\rho_1 = \tilde{\rho} \overset{\text{def}}{=} \gamma(x,t)^{-1}\left(\gamma_{(g,h)}|_U\right)\gamma(x,t)$. It suffices to show that

$$(h \star x, gt) = (g,h) \star (x,t) = \tilde{\rho}^{-1} \cdot (x,t) = (\tilde{\rho}^{-1} x \tilde{\rho}, t\tilde{\rho}).$$

We have

$$\begin{aligned}\tilde{\rho}^{-1} x &= \gamma(x,t)^{-1}(\gamma_{(g,h)}|_U)^{-1}\gamma(x,t)x \\ &= \gamma(x,t)^{-1}(\gamma_{(g,h)}|_U)^{-1}x\gamma(x,t) \\ &= \gamma(x,t)^{-1}(h \star x)(\gamma_{(g,z)}|_U)^{-1}\gamma(x,t) \\ &= (h \star x)\gamma(x,t)^{-1}(\gamma_{(g,z)}|_U)^{-1}\gamma(x,t) \\ &= (h \star x)\tilde{\rho}^{-1},\end{aligned}$$

so $\tilde{\rho}^{-1} x \tilde{\rho} = h \star x$. Similarly, $t\tilde{\rho} = t\gamma(x,t)^{-1}\left(\gamma_{(g,h)}|_U\right)\gamma(x,t) = gt$ and we are done. □

We now restrict to a special case of this construction that has been studied by Nakajima. In particular, we define $\mathrm{GL}_w \times \mathbb{C}^*$-actions on the quiver grassmannians corresponding to the actions on quiver varieties described in Section 3B.

For any function $m : \tilde{Q}_1 \to \mathbb{Z}$ such that $m(a) = -m(\bar{a})$ for all $a \in \tilde{Q}_1$, the group homomorphism (3-1) defines a $\mathrm{GL}_w \times \mathbb{C}^*$-action on q^w, $\widehat{\mathrm{Gr}}_{\mathcal{P}}(v, q^w)$ and $\mathrm{Gr}_{\mathcal{P}}(v, q^w)$ which we again denote by $\star_m$. If A is any abelian reductive subgroup of $\mathrm{GL}_w \times \mathbb{C}^*$, we can consider the weight decompositions as above. For the remainder of this section, we fix $m = m_2$ (see Section 3B). That is, $m(a) = 0$ for all $a \in Q_1$. We also write $\star$ for $\star_m$. Recall the definition (5-1) of $q^w(\lambda)$. For $x \in \mathcal{P}_n$, $v \in q^w(\lambda)$ and $(g,z) \in A$, we have

$$\rho(g,z)(x \cdot v) = \gamma_{(zg, h_m(z))}(x \cdot v) = z^{-n} x \cdot \gamma_{(zg, h_m(z))}(v) = z^{-n}\lambda(g,z)v.$$

Thus $\mathcal{P}_n : q^w(\lambda) \to q^w(l^{-n}\lambda)$, where we write $l^{-n}\lambda$ for the element $L(-n) \otimes \lambda$ of $\mathrm{Hom}(A, \mathbb{C}^*)$ and $L(-n) = \mathbb{C}$ with $\mathbb{C}^*$-module structure given by $z \cdot v = z^{-n} v$.

Now let (g,z) be a semisimple element of A and define

$$\begin{aligned}\mathrm{Gr}_{\mathcal{P}}(q^w)^{(g,z)} &= \{U \in \mathrm{Gr}_{\mathcal{P}}(q^w) \mid (g,z) \star U = U\}, \\ \mathrm{Gr}_{\mathcal{P}}(u, q^w)^{(g,z)} &= \mathrm{Gr}_{\mathcal{P}}(q^w)^{(g,z)} \cap \mathrm{Gr}_{\mathcal{P}}(u, q^w).\end{aligned}$$

The module q^w has an eigenspace decomposition with respect to the action of (g,z) given by

$$q^w = \bigoplus_{a \in \mathbb{C}^*} q^w(a), \quad q^w(a) = \{v \in q^w \mid (g,z) \star v = av\}.$$

Then $\mathrm{Gr}_{\mathcal{P}}(q^w)^{(g,z)}$ consists of those $U \in \mathrm{Gr}_{\mathcal{P}}(q^w)$ that are direct sums of subspaces of the weight spaces $q^w(a)$, $a \in \mathbb{C}^*$. Thus, each $U \in \mathrm{Gr}_{\mathcal{P}}(q^w)^{(g,z)}$ inherits a weight space decomposition, or $\mathbb{C}^*$-grading,

$$U = \bigoplus_{a\in\mathbb{C}^*} U(a), \quad U(a) = \{v \in U \mid (g,z)\star v = av\}.$$

As above we see that $\mathcal{P}_n : q^w(a) \to q^w(az^{-n})$ and $\mathcal{P}_n : U(a) \to U(az^{-n})$. We also regard s^w as an A-module via the composition

$$A \hookrightarrow \mathrm{GL}_w \times \mathbb{C}^* \xrightarrow{\text{projection}} \mathrm{GL}_w = \mathrm{GL}_{s^w}.$$

Thus s^w also inherits a $\mathbb{C}^*$-grading as above. For a $Q_0 \times \mathbb{C}^*$-graded vector space $V = \bigoplus_{i\in Q_0,\, a\in\mathbb{C}^*} V_{i,a}$, define the graded dimension (or character)

$$\operatorname{char} V = \sum_{\substack{i\in Q_0,\\ a\in\mathbb{C}*}} (\dim V_{i,a}) X_{i,a} \in \mathbb{N}[X_{i,a}]_{i\in Q_0,\, a\in\mathbb{C}^*}.$$

Recall that a $\mathcal{P}_0$-module is equivalent to an Q_0-graded vector space. Thus q^w, s^w, and elements of $\mathrm{Gr}_{\mathcal{P}}(q^w)^{(g,z)}$ have natural $Q_0 \times \mathbb{C}^*$-gradings and we can consider their graded dimensions.

Definition 5.3 (graded/cyclic quiver grassmannian). For a graded dimension $\mathbf{d} \in \mathbb{N}[X_{i,a}]_{i\in Q_0,\, a\in\mathbb{C}^*}$, define

$$\mathrm{Gr}_{\mathcal{P}}(\mathbf{d}, q^w)^{(g,z)} = \{U \in \mathrm{Gr}_{\mathcal{P}}(q^w)^{(g,z)} \mid \operatorname{char} U = \mathbf{d}\}.$$

We call $\mathrm{Gr}_{\mathcal{P}}(\mathbf{d}, q^w)^{(g,z)}$ a *cyclic quiver grassmannian* if z is a root of unity, and a *graded quiver grassmannian* otherwise.

Theorem 5.4. *Let V be a $Q_0 \times \mathbb{C}^*$-graded vector space. For a semisimple element $(g,z) \in \mathrm{GL}_w \times \mathbb{C}^*$, the graded/cyclic quiver grassmannian* $\mathrm{Gr}_{\mathcal{P}}(\operatorname{char} V, q^w)^{(g,z)}$ *is isomorphic to the lagrangian graded/cylic quiver variety $\mathfrak{L}^\bullet(V, s^w)$ defined in* [Nakajima 2004, §4], *where s^w is considered as a $Q_0 \times \mathbb{C}^*$-graded vector space as above.*

Proof. This follows immediately from Proposition 5.1 since $\mathfrak{L}^\bullet(V, W)$ is simply the set of points of $\mathfrak{L}(V, W)$ fixed by a semisimple element (g,z) of $\mathrm{GL}_w \times \mathbb{C}^*$. □

Remark 5.5. Nakajima [2004] assumes the quiver Q is of ADE type. However, the definitions in §4 of that article extend naturally to the more general case.

6. Geometric construction of representations of Kac–Moody algebras and compatibility with nested quiver grassmannians

Since certain quiver grassmannians are isomorphic to lagrangian Nakajima quiver varieties, one can translate Nakajima's geometric construction of representations of Kac–Moody algebras into the quiver grassmannian setting. Having done this, one sees that the quiver grassmannian construction is compatible with a natural nesting of these varieties — a property which seems to have no analog in the setting of quiver varieties. One benefit of this nesting compatibility is that it allows one to always work with quiver grassmannians in *finite-dimensional* modules, even though the injective objects q^w themselves may be infinite-dimensional (outside of finite type).

For the remainder of this section, we fix a Kac–Moody algebra $\mathfrak{g}$ with symmetric Cartan matrix and let $\mathcal{W}$ be its Weyl group. Let $Q = (Q_0, Q_1)$ be a quiver whose underlying graph is the Dynkin graph of $\mathfrak{g}$ and let $\mathcal{P} = \mathcal{P}(Q)$ denote the corresponding path algebra. We also fix a $\mathcal{P}_0$-module retract $\pi : q^w \to s^w$, allowing us to identify $\mathrm{Gr}_{\mathcal{P}}(v, q^w)$ with $\mathfrak{L}(v, w)$ as in Theorem 4.4.

6A. ***Constructible functions.*** Recall that for a topological space X, a *constructible set* is a subset of X that is obtained from open sets by a finite number of the usual set theoretic operations (complement, union and intersection). A *constructible function* on X is a function that is a finite linear combination of characteristic functions of constructible sets. For a complex variety X, let $M(X)$ denote the $\mathbb{C}$-vector space of constructible functions on X with values in $\mathbb{C}$. We define $M(\varnothing) = 0$. For a continuous map $p : X \to X'$, define

$$p^* : M(X') \to M(X), \quad (p^* f')(x) = f'(p(x)), \quad f' \in M(X')$$

and

$$p_! : M(X) \to M(X'), \quad (p_! f)(x) = \sum_{a \in \mathbb{Q}} a \chi(p^{-1}(x) \cap f^{-1}(a)), \quad f \in M(X),$$

where χ denotes the Euler characteristic of cohomology with compact support.

Lemma 6.1. *Suppose X is a constructible subset of a topological space Y and let $\iota : X \hookrightarrow Y$ be the inclusion map. Then*

(i) $\iota^*(f) = f|_X$ *for* $f \in M(Y)$, *and*

(ii) *for* $f \in M(X)$, $\iota_!(f)$ *is the extension of* f *by zero. That is,*

$$\iota_!(f)(x) = \begin{cases} f(x) & \text{if } x \in X, \\ 0 & \text{if } x \in Y \setminus X. \end{cases}$$

The proof is straightforward and will be omitted.

6B. ***Raising and lowering operators.*** Let V be a $\mathcal{P}$-module. For $u, u' \in \mathbb{N}Q_0$ with $u \le u'$ (i.e., $u = \sum u_i i$ and $u' = \sum u'_i i$ where $u_i \le u'_i$ for all $i \in Q_0$), define

$$\mathrm{Gr}_{\mathcal{P}}(u, u', V) = \{(U, U') \in \mathrm{Gr}_{\mathcal{P}}(u, V) \times \mathrm{Gr}_{\mathcal{P}}(u', V) \mid U \subseteq U'\}, \tag{6-1}$$

and let

$$\mathrm{Gr}_{\mathcal{P}}(u, V) \xleftarrow{\pi_1} \mathrm{Gr}_{\mathcal{P}}(u, u', V) \xrightarrow{\pi_2} \mathrm{Gr}_{\mathcal{P}}(u', V)$$

be the natural projections given by $\pi_1(U, U') = U$ and $\pi_2(U, U') = U'$. For each $i \in I$, define the operators

$$\begin{aligned} &\hat{E}_i : \quad M(\mathrm{Gr}_{\mathcal{P}}(u+i, V)) \to M(\mathrm{Gr}_{\mathcal{P}}(u, V)), \quad \hat{E}_i f = (\pi_1)_!(\pi_2^* f), \\ &\hat{F}_i : \quad M(\mathrm{Gr}_{\mathcal{P}}(u, V)) \to M(\mathrm{Gr}_{\mathcal{P}}(u+i, V)), \quad \hat{F}_i f = (\pi_2)_!(\pi_1^* f), \end{aligned} \tag{6-2}$$

where the maps π_1 and π_2 are as in (6-1) with $u' = u + i$.

6C. ***Compatibility with nested quiver grassmannians.*** Suppose $V_1 \subseteq V_2$ are $\mathcal{P}$-modules. Then we have the commutative diagram

$$\begin{tikzcd} \mathrm{Gr}_{\mathcal{P}}(u, V_1) \arrow[d, hook, "\iota_u"] & \mathrm{Gr}_{\mathcal{P}}(u, u', V_1) \arrow[l, "\pi_1^1"'] \arrow[r, "\pi_2^1"] \arrow[d, hook, "\iota_{u,u'}"'] & \mathrm{Gr}_{\mathcal{P}}(u', V_1) \arrow[d, hook, "\iota_{u'}"'] \\ \mathrm{Gr}_{\mathcal{P}}(u, V_2) & \mathrm{Gr}_{\mathcal{P}}(u, u', V_2) \arrow[l, "\pi_1^2"'] \arrow[r, "\pi_2^2"] & \mathrm{Gr}_{\mathcal{P}}(u', V_2) \end{tikzcd}$$

where ι_u, $\iota_{u'}$ and $\iota_{u,u'}$ denote the canonical inclusions. Denote by $\hat{E}_i^j$ and $\hat{F}_i^j$, $j = 1, 2$, the operators defined in (6-2) for $V = V_j$.

Proposition 6.2. *We have*

(i) $\hat{E}_i^1 = \iota_u^* \circ \hat{E}_i^2 \circ (\iota_{u+i})_!$, *and*

(ii) $\hat{F}_i^1 = \iota_{u+i}^* \circ \hat{F}_i^2 \circ (\iota_u)_!$.

Proof. Let $u' = u + i$. By linearity, it suffices to prove the first statement for functions of the form 1_X where X is a constructible subset of $\mathrm{Gr}_{\mathcal{P}}(u', V_1)$. Then $(\iota_{u'})_! 1_X = 1_X$, where on the right-hand side, X is viewed as a subset of $\mathrm{Gr}_{\mathcal{P}}(u', V_2)$. We have

$$(\pi_2^2)^* \circ (\iota_{u'})_! 1_X = (\pi_2^2)^* 1_X = 1_{(\pi_2^2)^{-1}(X)}$$

and

$$(\iota_{u,u'})_! (\pi_2^1)^* 1_X = (\iota_{u,u'})_! 1_{(\pi_2^1)^{-1}(X)} = 1_{(\pi_2^1)^{-1}(X)}.$$

Since $X \subseteq \mathrm{Gr}_{\mathcal{P}}(u', V_1)$, we have $(\pi_2^2)^{-1}(X) = (\pi_2^1)^{-1}(X)$ and thus

$$(\pi_2^2)^* \circ (\iota_{u'})_! 1_X = (\iota_{u,u'})_! \circ (\pi_2^1)^* 1_X.$$

Therefore

$$\begin{aligned}
\iota_u^* \circ \hat{E}_i^2 \circ (\iota_{u'})_! 1_X &= \iota_u^* \circ (\pi_1^2)_! \circ (\pi_2^2)^* \circ (\iota_{u'})_! 1_X \\
&= \iota_u^* \circ (\pi_1^2)_! \circ (\iota_{u,u'})_! \circ (\pi_2^1)^* 1_X \\
&= \iota_u^* \circ (\pi_1^2 \circ \iota_{u,u'})_! \circ (\pi_2^1)^* 1_X \\
&= \iota_u^* \circ (\iota_u \circ \pi_1^1)_! \circ (\pi_2^1)^* 1_X \\
&= \iota_u^* \circ (\iota_u)_! \circ (\pi_1^1)_! \circ (\pi_2^1)^* 1_X \\
&= (\pi_1^1)_! \circ (\pi_2^1)^* 1_X \\
&= \hat{E}_i^1 1_X,
\end{aligned}$$

where the sixth equality holds since $\iota_u^* \circ (\iota_u)_!$ is the identity on $M(\mathrm{Gr}_{\mathcal{P}}(u, V_1))$.

We now prove the second statement. Again, it suffices to prove it for functions of the form 1_X where X is a constructible subset of $\mathrm{Gr}_{\mathcal{P}}(u, V_1)$. Now, for $U \in \mathrm{Gr}_{\mathcal{P}}(u', V_1)$, we have

$$\begin{aligned}
\iota_{u'}^* \circ \hat{F}_i^2 \circ (\iota_u)_! 1_X(U) &= \iota_{u'}^* \circ (\pi_2^2)_! \circ (\pi_1^2)^* \circ (\iota_u)_! 1_X(U) \\
&= \iota_{u'}^* \circ (\pi_2^2)_! \circ (\pi_1^2)^* 1_X(U) \\
&= \iota_{u'}^* \circ (\pi_2^2)_! \circ 1_{(\pi_1^2)^{-1}(X)}(U) \\
&= \chi\left((\pi_2^2)^{-1}(U) \cap (\pi_1^2)^{-1}(X)\right) \\
&= \chi\left((\pi_2^1)^{-1}(U) \cap (\pi_1^1)^{-1}(X)\right) \\
&= (\pi_2^1)_! 1_{(\pi_1^1)^{-1}(X)}(U) \\
&= (\pi_2^1)_! \circ (\pi_1^1)^* 1_X(U) \\
&= \hat{F}_i^1 1_X(U),
\end{aligned}$$

where the fifth equality holds since $U \in \mathrm{Gr}_{\mathcal{P}}(u', V_1)$. □

It follows from Proposition 4.12 that the Demazure quiver grassmannians stabilize in the following sense.

Corollary 6.3. *For $u, w \in \mathbb{N}Q_0$, there exists $\sigma \in \mathcal{W}$, such that $\mathrm{Gr}_{\mathcal{P}}(v, q^{w,\sigma'})$ is isomorphic to $\mathfrak{L}(v, w)$ for all $\sigma' \succeq \sigma$.*

Proof. It follows from [Savage 2006d, Proposition 6.1] that there exists a $\sigma \in \mathcal{W}$ such that $\mathrm{Gr}_{\mathcal{P}}(v, q^{w,\sigma}) \cong \mathfrak{L}_\sigma(v, w) = \mathfrak{L}(v, w)$. It follows from the same proposition that for $\sigma' \succeq \sigma$, we have $\mathfrak{L}_{\sigma'}(v, w) = \mathfrak{L}(v, w)$. The result then follows from Proposition 4.12. □

Corollary 6.4. *For $v, w \in \mathbb{N}Q_0$, let $\sigma^{v,w} \in \mathcal{W}$ be minimal among the $\sigma \in \mathcal{W}$ such that $\mathrm{Gr}_{\mathcal{P}}(v, q^{w,\sigma})$ is isomorphic to $\mathfrak{L}(v, w)$. Then $\mathrm{Gr}_{\mathcal{P}}(v, q^{w,\sigma}) \cong \mathrm{Gr}_{\mathcal{P}}(v, q^w)$ for all $\sigma \succeq \sigma^{v,w}$. In particular, every submodule of the injective module q^w of graded dimension v is a submodule of $q^{w,\sigma}$ for $\sigma \succeq \sigma^{v,w}$.*

Remark 6.5. In the case when $\mathfrak{g}$ is of finite type, we can take $\sigma = \sigma_0$, where σ_0 is the longest element of the Weyl group. Then $\mathrm{Gr}_{\mathcal{P}}(v, q^w)$ is isomorphic to $\mathrm{Gr}_{\mathcal{P}}(v, q^{w,\sigma_0})$ for all $v \in \mathbb{N}Q_0$.

Lemma 6.6. *Suppose $w, v, v' \in \mathbb{N}Q_0$ with $v \le v'$ and $\sigma \in \mathcal{W}$. Then the diagram*

$$\begin{tikzcd}
\mathrm{Gr}_{\mathcal{P}}(v, q^{w,\sigma}) \arrow[d, hook] & \mathrm{Gr}_{\mathcal{P}}(v, v', q^{w,\sigma}) \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, hook] & \mathrm{Gr}_{\mathcal{P}}(v', q^{w,\sigma}) \arrow[d, hook] \\
\mathrm{Gr}_{\mathcal{P}}(v, q^{w}) & \mathrm{Gr}_{\mathcal{P}}(v, v', q^{w}) \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & \mathrm{Gr}_{\mathcal{P}}(v', q^{w})
\end{tikzcd}$$

commutes, where the vertical arrows are the natural inclusions. If $\sigma \succeq \sigma^{v,w}, \sigma^{v',w}$, then the vertical arrow are isomorphisms.

Proof. This follows immediately from Corollary 6.4. □

6D. *Quiver grassmannian realization of representations.* For each $i \in I$, define

(6-3) $$H_i : M(\mathrm{Gr}_{\mathcal{P}}(v, q^w)) \to M(\mathrm{Gr}_{\mathcal{P}}(v, q^w)), \quad H_i f = (w - Cv)_i f,$$

where C is the Cartan matrix of $\mathfrak{g}$. Also, in the special case when $V = q^w$ for some w, we denote the operators $\hat{E}_i$ and $\hat{F}_i$ by E_i and F_i respectively.

Proposition 6.7. *The operators E_i, F_i, H_i define an action of $\mathfrak{g}$ on*

$$\bigoplus_u M(\mathrm{Gr}_{\mathcal{P}}(u, q^w)).$$

Proof. Throughout this proof, for varieties X and Y, the notation $X \cong Y$ means that X and Y are homeomorphic. In [Nakajima 1994, §10], Nakajima defines the variety

$$\mathfrak{F}(v, w; i) \stackrel{\text{def}}{=} \tilde{\mathfrak{F}}(v, w; i)/\mathrm{GL}_V,$$

where

$$\tilde{\mathfrak{F}}(v, w; i) = \{(x, t, Z) \mid (x, t) \in \Lambda(V, W)^{\mathrm{st}},\ Z \subseteq V,\ x(Z) \subseteq Z,\ \dim Z = v - i\}.$$

Using the homeomorphism of Theorem 4.4, we have

$$\tilde{\mathfrak{F}}(v, w; i) \cong \{(\gamma, Z) \mid \gamma \in \widehat{\mathrm{Gr}}_{\mathcal{P}}(v, q^w),\ Z \subseteq V,\ \dim Z = v - i,\ \mathcal{P} \cdot \gamma(Z)) \subseteq \gamma(Z)\}.$$

The map from the set

$$\big\{(\gamma, Z) \mid \gamma \in \widehat{\mathrm{Gr}}_{\mathcal{P}}(v, q^w),\ Z \subseteq V,\ \dim Z = v - i,\ \mathcal{P} \cdot \gamma(Z) \subseteq \gamma(Z)\big\}$$

into $\mathrm{Gr}_{\mathcal{P}}(v - i, v, q^w)$ given by

$$(\gamma, Z) \mapsto (\gamma(Z), \gamma(V))$$

is a principal GL_V-bundle and thus

$$\begin{aligned}
&\mathfrak{F}(v, w; i) \\
&\quad = \tilde{\mathfrak{F}}(v, w; i)/\mathrm{GL}_V \\
&\quad \cong \big\{(\gamma, Z) \mid \gamma \in \widehat{\mathrm{Gr}}_{\mathcal{P}}(v, q^w),\ Z \subseteq V,\ \dim Z = v - i,\ \mathcal{P} \cdot \gamma(Z) \subseteq \gamma(Z)\big\}/\mathrm{GL}_V \\
&\quad = \mathrm{Gr}_{\mathcal{P}}(u - i, u, q^w).
\end{aligned}$$

Therefore, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Gr}_{\mathcal{P}}(v-i, q^w) \arrow[d, "\cong"] & \mathrm{Gr}_{\mathcal{P}}(v-i, v, q^w) \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\cong"] & \mathrm{Gr}_{\mathcal{P}}(v, q^w) \arrow[d, "\cong"] \\
\mathfrak{L}(v-i, w) & \mathfrak{F}(v, w; i) \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & \mathfrak{L}(v, w)
\end{tikzcd} \tag{6-4}$$

where the maps π_1 and π_2 appearing on the bottom row are described in §10 of [Nakajima 1994]. The result then follows immediately from Proposition 10.12 of the same reference. □

Let $U(\mathfrak{g})^-$ be the lower half of the enveloping algebra of $\mathfrak{g}$. Let α be the constant function on $\mathrm{Gr}_{\mathcal{P}}(0, q^w)$ with value 1 and let

$$L_w \overset{\text{def}}{=} U(\mathfrak{g})^- \cdot \alpha \subseteq \bigoplus_v M(\mathrm{Gr}_{\mathcal{P}}(v, q^w)), \tag{6-5}$$

$$L_w(v) \overset{\text{def}}{=} M(\mathrm{Gr}_{\mathcal{P}}(v, q^w)) \cap L_w. \tag{6-6}$$

Theorem 6.8. *The operators E_i, F_i, H_i preserve L_w and L_w is isomorphic to the irreducible highest-weight integrable representation of $\mathfrak{g}$ with highest weight ω_w. The summand $L_w(v)$ in the decomposition $L_w = \bigoplus_v L_w(v)$ is a weight space with weight $\omega_w - \alpha_v$.*

Proof. In light of the commutative diagram (6-4), the result follows immediately from [Nakajima 1994, Theorem 10.14]. □

Remark 6.9. It follows from Proposition 6.2 and Lemma 6.6 that we can always work with $\mathrm{Gr}_{\mathcal{P}}(v, q^{w,\sigma})$ for large enough σ. Therefore, we can avoid quiver grassmannians in infinite-dimensional injectives if desired.

From the realization of irreducible highest-weight representations given in Theorem 6.8, we obtain some natural automorphisms of these representations. Recall from Definition 2.16 the natural action of $\mathrm{Aut}_{\mathcal{P}}\, q^w$ on $\mathrm{Gr}_{\mathcal{P}}(v, q^w)$ for any v given by $(g, V) \mapsto g(V)$. This induces an action on $\bigoplus_v M(\mathrm{Gr}_{\mathcal{P}}(v, q^w))$ given by

$$(g, f) \mapsto f \circ g^{-1}, \quad f \in \bigoplus_v M(\mathrm{Gr}_{\mathcal{P}}(v, q^w)), \quad g \in \mathrm{Aut}_{\mathcal{P}}\, q_w.$$

This action clearly commutes with the operators E_i and F_i and thus induces an action on L_w. Such actions do not seem to be clear in the original quiver variety picture. Similar actions were considered in [Lusztig 2000, §1.22] in the case when Q is of finite type.

7. Relation to Lusztig's grassmannian realization

Lusztig [1998; 2000] gave a grassmannian type realization of the lagrangian Nakajima quiver varieties inside the projective modules p^w. In the case when Q is a quiver of finite type, the injective hulls of the simple objects are also projective covers (of different simple objects). Thus, Lusztig's and our construction are closely related. In this section, we extend Lusztig's construction to give a realization of the Demazure quiver varieties. We then give a precise relationship between his construction and ours in the finite type case. We will see that the natural identification of the two constructions corresponds to the Chevalley involution on the level of representations of the Lie algebra $\mathfrak{g}$ associated to our quiver.

7A. *Lusztig's construction and Demazure quiver varieties.*

Definition 7.1. For $V \in \mathcal{P}$-Mod, define

$$\tilde{\mathrm{Gr}}_{\mathcal{P}}(V) = \{U \in \mathrm{Gr}_{\mathcal{P}}(V) \mid \mathcal{P}_n \cdot V \subseteq U \text{ for some } n \in \mathbb{N}\}.$$

In other words, $\tilde{\mathrm{Gr}}_{\mathcal{P}}(V)$ consists of all $\mathcal{P}$-submodules of V such that the quotient V/U is nilpotent. For $u \in \mathbb{N}Q_0$, we define

$$\tilde{\mathrm{Gr}}_{\mathcal{P}}(u, V) = \{U \in \tilde{\mathrm{Gr}}_{\mathcal{P}}(V) \mid \dim_{Q_0}(V/U) = u\}.$$

Proposition 7.2. *Fix $v, w \in \mathbb{N}Q_0$. Then $\mathfrak{L}(v, w)$ is isomorphic to $\tilde{\mathrm{Gr}}_{\mathcal{P}}(v, p^w)$ as an algebraic variety.*

Proof. This is proven in Corollary 3.2 of [Shipman 2010]. Note that, in that article, a different stability condition is used in the definition of $\mathfrak{L}(v, w)$. However, it is well-known that the different stability conditions give rise to isomorphic varieties. We refer the reader to [Nakajima 1996] for a discussion of various stability conditions. □

Proposition 7.3. *For $v \in \mathbb{N}Q_0$, the following statements are equivalent:*

(i) *v is w-extremal.*

(ii) *$\mathfrak{L}(v, w)$ consists of a single point.*

(iii) *$\tilde{\mathrm{Gr}}_{\mathcal{P}}(v, p^w)$ consists of a single point.*

(iv) *There is a unique $\mathcal{P}$-submodule V of p^w of codimension v such that p^w/V is nilpotent.*

Proof. The equivalence of (i) and (ii) is given in [Savage 2006d, Proposition 5.1]. The equivalence of (ii) and (iii) follows from Proposition 7.2. Finally, the equivalence of (iii) and (iv) follows directly from Definition 7.1 □

Definition 7.4. For $\sigma \in \mathcal{W}$, we let $p^{w,\sigma}$ denote the unique submodule of p^w of graded codimension $\sigma \cdot_w 0$ and define

$$\tilde{\mathrm{Gr}}_{Q,\sigma}(v, p^w) = \{V \in \tilde{\mathrm{Gr}}_{\mathcal{P}}(v, p^w) \mid p^{w,\sigma} \subseteq V\}.$$

Proposition 7.5. *Fix $\sigma \in \mathcal{W}$ and $v, w \in \mathbb{N}Q_0$. Then $\tilde{\mathrm{Gr}}_{Q,\sigma}(v, p^w)$ is isomorphic to the Demazure quiver variety $\mathfrak{L}_\sigma(v, w)$.*

Proof. This follows directly from Definitions 3.5 and 7.4 and Proposition 7.2. □

7B. *Relation between the projective and injective constructions.* We now suppose Q is of finite type and let $\mathfrak{g}$ be the Kac–Moody algebra whose Dynkin diagram is the underlying graph of Q. Let σ_0 be the longest element of the Weyl group of $\mathfrak{g}$. There is a unique Dynkin diagram automorphism θ such that $-w_0(\alpha_i) = \alpha_{\theta(i)}$. Extend θ to an automorphism of the root lattice $\bigoplus_{i \in Q_0} \mathbb{Z}\alpha_i$ by linearly extending the map $\alpha_i \mapsto \alpha_{\theta(i)}$. We also have an involution of $\mathbb{N}Q_0$ given by $w \mapsto \theta(w)$ where $\theta(w)_i = w_{\theta(i)}$.

Definition 7.6 (Chevalley involution). The *Chevalley involution* ζ of $\mathfrak{g}$ is given by

$$\zeta(E_i) = F_i, \quad \zeta(F_i) = E_i, \quad \zeta(H_i) = -H_i.$$

For any representation V of $\mathfrak{g}$, let ${}^\zeta V$ be the representation with the same underlying vector space as V, but with the action of $\mathfrak{g}$ twisted by ζ. More precisely, the $\mathfrak{g}$-action on ${}^\zeta V$ is given by $(a, v) \mapsto \zeta(a) \cdot v$.

For a dominant weight λ of $\mathfrak{g}$, let L_λ denote the corresponding irreducible highest-weight representation and let v_λ be a highest weight vector. Recall that an isomorphism of irreducible representations is uniquely determined by the image of v_λ. The following lemma is well known.

Lemma 7.7. *The lowest weight of L_λ is $\sigma_0(\lambda) = -\theta(\lambda)$. If $v_{-\theta(\lambda)}$ denotes a lowest weight vector, then the map $v_\lambda \mapsto v_{-\theta(\lambda)}$ induces an isomorphism ${}^\zeta L_\lambda \cong L_{\theta(\lambda)}$.*

Lemma 7.8. *We have* $\dim_{Q_0} p^w = \dim_{Q_0} q^w = \sigma_0 \cdot_w 0$.

Proof. Since the lowest weight of the representation $L(w)$ is $\sigma_0(w)$, the result follows immediately from Theorem 4.4 and Proposition 7.2. □

Lemma 7.9. *For $w \in \mathbb{N}Q_0$, we have $\sigma_0 \cdot_w 0 = \sigma_0 \cdot_{\theta(w)} 0$. Furthermore, $\theta(\sigma_0 \cdot_w 0) = \sigma_0 \cdot_w 0$.*

Proof. Let $v = \sigma_0 \cdot_w 0$. Then $\alpha_v = \omega_w - \sigma_0(\omega_w) = \omega_w + \theta(\omega_w)$ and the results follow easily from the fact that $\theta^2 = \mathrm{Id}$. □

Proposition 7.10. *If Q is a quiver of finite type and $w \in \mathbb{N}Q_0$, then $p^w \cong q^{\theta(w)}$.*

Proof. Since $p^w = \bigoplus_{i \in Q_0} (p^i)^{\oplus w_i}$ and $q^w = \bigoplus_{i \in Q_0} (q^i)^{\oplus w_i}$, it suffices to prove the result for w equal to i for arbitrary $i \in Q_0$.

Let $v = \sigma_0 \cdot_w 0 = \dim_{Q_0} p^i$. In the geometric realization of crystals via quiver varieties [Saito 2002], the point $\tilde{\mathrm{Gr}}_{\mathcal{P}}(v, p^w) \cong \mathfrak{L}(v, w)$ corresponds to the lowest weight element of the crystal B_{ω_i}. The lowest weight of the representation L_{ω_i} is $\sigma_0(\omega_i) = -\omega_{\theta(i)}$. Therefore, it follows from the geometric description of the crystals that $\dim_{Q_0} \operatorname{socle} p^i = \theta(i)$. By Lemmas 7.8 and 7.9, we have

$$\dim_{Q_0} p^i = \sigma_0 \cdot_w 0 = \sigma_0 \cdot_{\theta(w)} 0 = \dim_{Q_0} q^{\theta(i)}.$$

Thus, by Proposition 4.9, we have $p^i \cong q^{\theta(i)}$. □

Corollary 7.11. *Suppose Q is a quiver of finite type, $w \in \mathbb{N}Q_0$, and $\sigma \in \mathcal{W}$. Then $q^{w,\sigma} \cong p^{\theta(w),\sigma\sigma_0}$.*

Proof. Let $\tau = \sigma\sigma_0$ (and so $\sigma = \tau\sigma_0$). In light of Propositions 4.9, 7.3 and 7.10 and Definitions 4.10 and 7.4, it suffices to prove that the codimension of $q^{w,\sigma}$ in q^w is $\tau \cdot_{\theta(w)} 0$.

Let $y = \tau \cdot_{\theta(w)} 0$, so that $\tau(\theta(w)) = \theta(w) - \alpha_y$, that is,

$$\alpha_y = \theta(w) - \tau(\theta(w)).$$

Next, let

$$v = \dim_{Q_0} q^w = \sigma_0 \cdot_w 0 \quad \text{and} \quad u = \dim_{Q_0} q^{w,\sigma} = \sigma \cdot_w 0,$$

which implies $\sigma_0(w) = w - \alpha_v$ and $\sigma(w) = w - \alpha_u$. Then

$$\sum_{i \in Q_0} (v_i - u_i)\alpha_i = -\sigma_0(w) + \sigma(w) = \theta(w) + \tau\sigma_0(w) = \theta(w) - \tau(\theta(w)),$$

and so $y = v - u$ as desired. □

Proposition 7.12. *If Q is a quiver of finite type, then*

$$\mathrm{Gr}_{\mathcal{P}}(u, q^w) \cong \tilde{\mathrm{Gr}}_{\mathcal{P}}((\sigma_0 \cdot_w 0) - u, p^{\theta(w)}).$$

Proof. Let (x, V) be the quiver representation corresponding to the $\mathcal{P}$-module q^w and let $v = \dim_{Q_0} V = \sigma_0 \cdot_w 0$. By Proposition 7.10, (x, V) also corresponds to the $\mathcal{P}$-module $p^{\theta(w)}$. By Remark 2.10, $\mathcal{P}_n \cdot p^w = 0$ for sufficiently large n. Therefore

$$\begin{aligned} \mathrm{Gr}_{\mathcal{P}}(u, q^w) &= \{U \subseteq V \mid x(U) \subseteq U,\ \dim U = u\} \\ &= \{U \subseteq V \mid x(U) \subseteq U,\ \dim_{Q_0} V/U = v - u\} \\ &\cong \tilde{\mathrm{Gr}}_{\mathcal{P}}(v - u, p^{\theta(w)}). \end{aligned}$$

□

By Proposition 7.12, we have

$$\mathfrak{L}(u, w) \xleftarrow[\cong]{\phi_w(u)} \mathrm{Gr}_{\mathcal{P}}(u, q^w) \cong \tilde{\mathrm{Gr}}_{\mathcal{P}}((\sigma_0 \cdot_w 0) - u, p^{\theta(w)}) \xrightarrow[\cong]{\psi_{\theta(w)}((\sigma_0 \cdot_w 0)-u)} \mathfrak{L}((\sigma_0 \cdot_w 0) - u, \theta(w)), \tag{7-1}$$

where $\phi_w(u)$ is the isomorphism of Theorem 4.4 (see Corollary A.6), and $\psi_{\theta(w)}(u)$ is the isomorphism of Proposition 7.2. Define

$$\phi_w = (\phi_w(u))_u : \mathrm{Gr}_{\mathcal{P}}(q^w) \to \bigsqcup_u \mathfrak{L}(u, w),$$
$$\psi_w = (\psi_w(u))_u : \tilde{\mathrm{Gr}}_{\mathcal{P}}(p^w) \to \bigsqcup_u \mathfrak{L}(u, w).$$

Theorem 7.13. *The isomorphism $\psi_{\theta(w)} \circ \phi_w^{-1}$ induces the involution ζ. More precisely, we have $a \circ (\psi_{\theta(w)} \circ \phi_w^{-1})^* = (\psi_{\theta(w)} \circ \phi_w^{-1})^* \circ \zeta(a)$, $a \in \mathfrak{g}$, as operators on L_w, where $(\psi_{\theta(w)} \circ \phi_w^{-1})^*$ denotes the pullback of functions along $\psi_{\theta(w)} \circ \phi_w^{-1}$.*

Proof. For $u, u' \in \mathbb{N}Q_0$, define

$$\tilde{\mathrm{Gr}}_{\mathcal{P}}(u, u', p^{\theta(w)}) = \{(U, U') \in \tilde{\mathrm{Gr}}_{\mathcal{P}}(u, p^{\theta(w)}) \times \tilde{\mathrm{Gr}}_{\mathcal{P}}(u', p^{\theta(w)}) \mid U' \subseteq U\}.$$

The map $\psi_{\theta(w)}$ induces a isomorphism

$$\tilde{\mathrm{Gr}}_{\mathcal{P}}(u, u', p^{\theta(w)}) \xrightarrow{\cong} \mathfrak{F}(u, \theta(w); u - u')$$

for all $u, u' \in \mathbb{N}Q_0$ and we will also denote this collection of isomorphisms by $\psi_{\theta(w)}$. Then we have the commutative diagram

$$\begin{tikzcd}
\mathfrak{L}(u-i, w) & \mathfrak{F}(u, w; i) \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & \mathfrak{L}(u, w) \\
\mathrm{Gr}_{\mathcal{P}}(u-i, q^w) \arrow[u, "\phi_w", "\cong"'] \arrow[d, "\cong"] & \mathrm{Gr}_{\mathcal{P}}(u-i, u, q^w) \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[u, "\phi_w", "\cong"'] \arrow[d, "\cong"] & \mathrm{Gr}_{\mathcal{P}}(u, q^w) \arrow[u, "\phi_w", "\cong"'] \arrow[d, "\cong"] \\
\tilde{\mathrm{Gr}}_{\mathcal{P}}((\sigma_0 \cdot_w 0)-(u-i), p^{\theta(w)}) \arrow[d, "\psi_{\theta(w)}"', "\cong"] & \Xi \arrow[l, "\pi_2"'] \arrow[r, "\pi_1"] \arrow[d, "\psi_{\theta(w)}"', "\cong"] & \tilde{\mathrm{Gr}}_{\mathcal{P}}((\sigma_0 \cdot_w 0)-u, p^{\theta(w)}) \arrow[d, "\psi_{\theta(w)}"', "\cong"] \\
\mathfrak{L}((\sigma_0 \cdot_w 0)-(u-i), \theta(w)) & \mathfrak{F}((\sigma_0 \cdot_w 0)-u, \theta(w); i) \arrow[l, "\pi_2"'] \arrow[r, "\pi_1"] & \mathfrak{L}((\sigma_0 \cdot_w 0)-u, \theta(w))
\end{tikzcd}$$

where $\Xi = \tilde{\mathrm{Gr}}_{\mathcal{P}}((\sigma_0 \cdot_w 0) - u, (\sigma_0 \cdot_w 0) - (u - i), p^{\theta(w)})$. It follows that, for f in $\bigoplus_u M(\mathfrak{L}(u, w))$, we have

$$E_i \circ (\psi_{\theta(w)} \circ \phi_w^{-1})^*(f) = (\psi_{\theta(w)} \circ \phi_w^{-1})^* \circ F_i(f),$$
$$F_i \circ (\psi_{\theta(w)} \circ \phi_w^{-1})^*(f) = (\psi_{\theta(w)} \circ \phi_w^{-1})^* \circ E_i(f).$$

Furthermore, $(\psi_{\theta(w)} \circ \phi_w^{-1})^*$ maps the constant function on $\mathfrak{L}(0, w)$ with value one to the constant function on $\mathfrak{L}(\sigma_0 \cdot_w 0, \theta(w))$ with value one. The result follows. $\square$

Remark 7.14. Note that the middle isomorphism in (7-1) depends on our identification of q^w and $p^{\theta(w)}$. The isomorphism $\phi_w(u)$ also depends on our fixed retract $\pi : q^w \to s^w$. By Proposition 4.1, all such choices are related by the natural action of $\mathrm{Aut}_{\mathcal{P}}\, q^w$; see Definition 2.16. A similar group action appears in the identification of $\tilde{\mathrm{Gr}}_{\mathcal{P}}((\sigma_0 \cdot_w 0) - u, p^{\theta(w)})$ with $\mathfrak{L}((\sigma_0 \cdot_w 0) - u, \theta(w))$; see [Lusztig 2000]. Via the isomorphisms $\phi_w(u)$, the group $\mathrm{Aut}_{\mathcal{P}}\, q^w$ acts on the space of constructible functions on $\bigsqcup_v \mathfrak{L}(v, w)$ and L_w is a subspace of the space of invariant functions. The pullback $(\psi_{\theta(w)} \circ \phi_w^{-1})^*$ acting on the space of invariant functions is independent of the choice of π and the chosen identification of q^w with $p^{\theta(w)}$.

Appendix: Isomorphisms of varieties

After an earlier version of the current paper was released [Savage and Tingley 2009], Shipman proved [2010] that the grassmannian type varieties $\tilde{\mathrm{Gr}}_{\mathcal{P}}(v, p^w)$ defined by Lusztig are indeed isomorphic as algebraic varieties to the lagrangian Nakajima quiver varieties $\mathfrak{L}(v, w)$. A simple "duality" map gives an isomorphism of varieties between the quiver grassmannian $\mathrm{Gr}_{\mathcal{P}}(v, q^w)$ and $\tilde{\mathrm{Gr}}_{\mathcal{P}}(v, p^w)$. The purpose of this appendix is to describe this map precisely, and from there to conclude that the map from $\mathrm{Gr}_{\mathcal{P}}(v, q^w)$ to $\mathfrak{L}(v, w)$ constructed in Theorem 4.4 is in fact an isomorphism of algebraic varieties. An alternative approach (not pursued here) would be an injective version of the argument of [Shipman 2010] that would directly show that $\mathrm{Gr}_{\mathcal{P}}(v, q^w)$ is isomorphic to $\mathfrak{L}(v, w)$.

Let $i \in Q_0$ and fix a nondegenerate bilinear pairing

$$\langle \cdot, \cdot \rangle_{s^i} : s^i \times s^i \to \mathbb{C},$$

and a retract $\pi : q^i \to s^i$ of $\mathcal{P}_0$-modules. For a path $\beta = a_1 \cdots a_n$ in the double quiver $\tilde{Q}$, let

$$\beta^\vee = \bar{a}_n \cdots \bar{a}_1 \tag{A-1}$$

be the reverse path. Extending by linearity, this defines an algebra anti-involution of $\mathbb{C}\tilde{Q}$ that induces an algebra anti-involution of $\mathcal{P}$. Then define a bilinear pairing

$$\langle \cdot, \cdot \rangle : \tilde{q}^i \times p^i \to \mathbb{C}, \quad \langle v, \beta e_i \rangle = \langle \pi(\beta^\vee v), e_i \rangle_{s^i}. \tag{A-2}$$

For $n \geq 0$, let

$$p_n^i = \mathcal{P}_{\geq n} e_i \subseteq p^i,$$
$$q_n^i = \{v \in q^i \mid \mathcal{P}_n \cdot v = 0\} = \{v \in \tilde{q}^i \mid \mathcal{P}_n \cdot v = 0\},$$

where the last equality holds since $\tilde{q}^i$ contains all nilpotent elements of q^i by Lemma 4.14. Note that each q^i_n is finite-dimensional. We have the obvious inclusions

$$q^i_0 \subseteq q^i_1 \subseteq q^i_2 \subseteq \cdots ,$$

and it follows from Lemma 4.14 and Theorem 4.15 that $\tilde{q}^i = \bigcup_{n=0}^{\infty} q^i_n$. It is clear from the definitions that

$$\langle q^i_n, p^i_{n+1} \rangle = 0, \quad \text{for all } n \geq 0.$$

Thus we have the induced bilinear pairing on $q^i_n \times (p^i / p^i_{n+1})$.

Lemma A.1. *The pairing*

$$\langle \cdot , \cdot \rangle : q^i_n \times (p^i / p^i_{n+1}) \to \mathbb{C}$$

is nondegenerate.

Proof. Since q^i_n is nilpotent of degree n and has socle s^i, for all nonzero $v \in q^i_n$, there exists $\beta \in \mathcal{P}_{\leq n}$ such that $0 \neq \beta \cdot v \in s^i$. Then $\langle v, \beta^\vee e_i \rangle \neq 0$. Thus, it suffices to show that $\dim(p^i / p^i_{n+1}) \leq \dim q^i_n$. Now, $(p^i / p^i_{n+1})^*$ is naturally a right $\mathcal{P}$-module. Via the anti-involution (A-1), this becomes a nilpotent left $\mathcal{P}$-module with socle s^i. Therefore, by Proposition 4.1, $(p^i / p^i_{n+1})^*$ injects into $\tilde{q}^i$. It is clear that the image of this injection is contained in q^i_n and thus the result follows since q^i_n is finite-dimensional. □

We then have the following corollary, whose proof is immediate.

Corollary A.2. *The pairing* (A-2) *is nondegenerate. Furthermore*,

$$\tilde{q}^i \cong \{ f \in \operatorname{Hom}_{\mathbb{C}}(p^i, \mathbb{C}) \mid f|_{p^i_n} = 0 \textit{ for } n \gg 0 \}$$

as $\mathcal{P}$-modules, where the $\mathcal{P}$-module structure on the right-hand side is given by

$$(\beta \cdot f')(v) = f'(\beta^\vee \cdot v),$$

for $\beta \in \mathcal{P}$, $v \in p^i$, and $f' \in \{ f \in \operatorname{Hom}_{\mathbb{C}}(p^i, \mathbb{C}) \mid f|_{p^i_n} = 0$ for $n \gg 0 \}$.

Remark A.3. One should compare this result to Definition 2.7 and Lemma 2.8 in finite type.

Recall that, for $w = \sum_i w_i i \in \mathbb{N} Q_0$, we have

$$s^w = \bigoplus_i (s^i)^{\oplus w_i}, \quad p^w = \bigoplus_i (p^i)^{\oplus w_i}, \quad \tilde{q}^w = \bigoplus_i (\tilde{q}^i)^{\oplus w_i}.$$

By declaring distinct summands to be orthogonal, we have a nondegenerate bilinear pairing

$$\langle \cdot , \cdot \rangle : \tilde{q}^w \times p^w \to \mathbb{C}. \tag{A-3}$$

For a subspace U of $\tilde{q}^w$, define the subspace

$$U^\perp = \{v \in p^w \mid \langle v', v\rangle = 0 \text{ for all } v' \in U\}$$

of p^w. Similarly, for a subspace U of p^w, define the subspace $U^\perp$ of $\tilde{q}^w$.

Proposition A.4. *For $U \in \mathrm{Gr}_{\mathcal{P}}(v, \tilde{q}^w)$, we have $U^\perp \in \tilde{\mathrm{Gr}}_{\mathcal{P}}(v, p^w)$, and the map*

$$\mathrm{Gr}_{\mathcal{P}}(v, \tilde{q}^w) \to \tilde{\mathrm{Gr}}_{\mathcal{P}}(v, p^w), \quad U \mapsto U^\perp,$$

is an isomorphism of algebraic varieties.

Proof. It follows from the definition of the pairing (A-3) that U is a submodule of $\tilde{q}^w$ if and only if $U^\perp$ is a submodule of p^w. Also, note that $U \subseteq \tilde{q}^w$ is finite-dimensional if and only if $U \subseteq q_n^w$ for some n. Therefore, it follows from Lemma A.1 that the maps $U \mapsto U^\perp$ (in either direction) are mutually inverse bijections between $\mathrm{Gr}_{\mathcal{P}}(v, \tilde{q}^w)$ and $\tilde{\mathrm{Gr}}_{\mathcal{P}}(v, p^w)$. Since these maps are clearly algebraic, the result follows. □

Theorem A.5. *The quiver grassmannian $\mathrm{Gr}_{\mathcal{P}}(v, q^w)$ is isomorphic to the lagrangian Nakajima quiver variety $\mathfrak{L}(v, w)$ as an algebraic variety.*

Proof. This follows from the isomorphisms of algebraic varieties

$$\mathrm{Gr}_{\mathcal{P}}(v, q^w) = \mathrm{Gr}_{\mathcal{P}}(v, \tilde{q}^w) \cong \tilde{\mathrm{Gr}}_{\mathcal{P}}(v, p^w) \cong \mathfrak{L}(v, w).$$

Recall that all finite-dimensional submodules of q^w are submodules of $\tilde{q}^w$. This gives the first equality. The first isomorphism is Proposition A.4 and the second is Proposition 7.2. □

Corollary A.6. *The map $\bar{\iota} : \mathrm{Gr}_{\mathcal{P}}(v, q^w) \to \mathfrak{L}(v, w)$ of Theorem 4.4 is an isomorphism of algebraic varieties.*

Proof. By Theorem A.5, we know that $\mathrm{Gr}_{\mathcal{P}}(v, q^w)$ and $\mathfrak{L}(v, w)$ are isomorphic as algebraic varieties. Since $\bar{\iota}$ is a bijective algebraic map by Theorem 4.4, the result follows by [Kaliman 2005, Lemma 1] (while the result there is stated for irreducible varieties, the proof applies to reducible ones — the only difference is that the normalization is now a disjoint union of components). □

Acknowledgements

The authors would like to thank B. Leclerc who, after hearing some of the preliminary results of the current paper, suggested extending these results to graded/cyclic versions. They are also grateful to W. Crawley-Boevey for many helpful discussions and for suggesting the proof of Proposition 2.11. Furthermore, they would like to thank P. Etingof, A. Hubery, H. Nakajima, M. Roth, O. Schiffmann, and I. Shipman for useful conversations and S.-J. Kang, Y.-T. Oh, and the Korean

Mathematical Society for the invitation to participate in the 2008 Global KMS International Conference in Jeju, Korea, where the ideas in this paper were originally developed.

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Received June 23, 2010. Revised January 21, 2011.

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Founded in 1951 by
E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

PACIFIC JOURNAL OF MATHEMATICS

Volume 251 No. 2 June 2011

0030-8730(201106)251:2;1-C