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We study the existence of periodic solutions for a second order nonautonomous dynamical system. We make no assumptions on the gradient other than continuity. This allows both sublinear and superlinear problems. We also study the existence of nonconstant solutions.

1. Introduction

We consider the following problem. One wishes to solve

$$-\ddot{x}(t) = \nabla_x V(t, x(t)), \tag{1-1}$$

where

$$x(t) = (x_1(t), \ldots, x_n(t)) \tag{1-2}$$

is a map from $I = [0, T]$ to $\mathbb{R}^n$ such that each component $x_j(t)$ is a periodic function in H^1 with period T, and the function $V(t, x) = V(t, x_1, \ldots, x_n)$ is continuous from $\mathbb{R}^{n+1}$ to $\mathbb{R}$ with

$$\nabla_x V(t, x) = (\partial V/\partial x_1, \ldots, \partial V/\partial x_n) \in C(\mathbb{R}^{n+1}, \mathbb{R}^n). \tag{1-3}$$

For each $x \in \mathbb{R}^n$, the function $V(t, x)$ is periodic in t with period T.

We shall study this problem under several sets of assumptions. First, we make no assumption on $\nabla_x V(t, x)$ other than (1-3). This allows both sublinear and superlinear problems.

Theorem 1.1. *Assume*:

(1) *The function V satisfies*

$$0 \le \int_0^T V(t, x)\, dt \to \infty \quad \text{as } |x| \to \infty,\ x \in \mathbb{R}^n.$$

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(2) *There are positive constants* α, m *such that*

$$\int_0^T V(t,x)\,dt \le \alpha, \quad |x| \le m,\ x \in \mathbb{R}^n.$$

Then the system

$$-\ddot{x}(t) = \beta \nabla_x V(t, x(t)) \tag{1-4}$$

has a solution for almost all values of β *satisfying* $\beta \le 6m^2/\alpha T$. *If, in addition, there are a constant* $\gamma > 0$ *and a function* $W(t) \in L^1(I)$ *such that*

$$V(t,x) \ge \gamma |x|^2 - W(t),$$

then the system (1-4) *has a nonconstant solution for almost all* β *satisfying*

$$\frac{2\pi^2}{\gamma T^2} \le \beta \le \frac{6m^2}{\alpha T}.$$

Corollary 1.2. *Assume*:

(1) *The function* V *satisfies*

$$0 \le \int_0^T V(t,x)\,dt \to \infty \quad \text{as } |x| \to \infty,\ x \in \mathbb{R}^n.$$

(2) *There are positive constants* α, m *such that*

$$V(t,x) \le \alpha, \quad |x| \le m,\ x \in \mathbb{R}^n.$$

Then the system (1-4) *has a solution for almost all values of* β *satisfying* $0 \le \beta \le 6m^2/\alpha T^2$.

Theorem 1.3. *Assume*:

(1) *The function* V *satisfies*

$$0 \le \int_0^T V(t,x)\,dt \to \infty \quad \text{as } |x| \to \infty,\ x \in \mathbb{R}^n.$$

(2) *There is a constant* $q > 2$ *such that*

$$V(t,x) \le C(|x|^q + 1), \quad t \in I,\ x \in \mathbb{R}^n.$$

(3) *there are constants* $m > 0$, $\alpha > 0$ *such that*

$$V(t,x) \le \alpha |x|^2, \quad |x| \le m,\ t \in I,\ x \in \mathbb{R}^n.$$

Then the system (1-4) *has a solution for almost all* β *satisfying* $0 \le \beta \le 2\pi^2/\alpha T^2$.

Theorem 1.4. *Assume*:

(1) *The function V satisfies*

$$0 \le \int_0^T V(t,x)\,dt \to \infty \quad \text{as } |x| \to \infty,\ x \in \mathbb{R}^n.$$

(2) *There are a constant $\alpha > 0$ and a function $W(t) \in L^1(I)$ such that*

$$V(t,x) \le \alpha |x|^2 + W(t), \quad t \in I,\ x \in \mathbb{R}^n.$$

Then the system (1-4) has a solution for almost all $0 \le \beta \le 2\pi^2/\alpha T^2$. If we assume

$$B := \int_I W(t)\,dt < 0,$$

then (1-4) has a nonconstant solution for almost all such β.

Theorem 1.5. *The conclusions of Theorem 1.4 are valid if we replace condition (2) with*:

(2′) *There is a constant $\alpha > 0$ such that*

$$\sup_{|x|<m} \int_0^T V(t,x)\,dt \le \alpha m^2 + B \quad \text{for every } m > 0,$$

and require $0 \le \beta \le 6/\alpha T$.

The advantage of these theorems is that we obtain solutions under very weak hypotheses. In fact, we make no assumption on $\nabla_x V(t,x)$ other than (1-3). The disadvantage is that we do not obtain a solution for any particular value of β. If we wish to prove existence for every such β, we will have to make assumptions concerning $\nabla_x V(t,x)$ as well. We now present additional hypotheses which guarantee existence of solutions for all values of β in the given intervals. We do this for Theorems 1.1 and 1.3. The hypotheses are:

(1) $0 \le V(t,x)/|x|^2 \to \infty$ as $|x| \to \infty$.

(2) There are a constant C and a function $W(t) \in L^1(I)$ such that

$$H(t,\theta x) \le C(H(t,x) + W(t)), \quad 0 \le \theta \le 1,\ t \in I,\ x \in \mathbb{R}^n,$$

where

$$H(t,x) := \nabla_x V(t,x) \cdot x - 2V(t,x).$$

Theorem 1.6. *Assume*:

(1) $0 \le V(t,x)/|x|^2 \to \infty$ as $|x| \to \infty$.

(2) *There are positive constants α, m such that*

$$\int_0^T V(t,x)\,dt \le \alpha, \quad |x| \le m,\ x \in \mathbb{R}^n.$$

(3) *There are a constant C and a function $W(t) \in L^1(I)$ such that*

$$H(t, \theta x) \le C(H(t, x) + W(t)), \quad 0 \le \theta \le 1, \ t \in I, \ x \in \mathbb{R}^n.$$

Then the system (1-4) has a solution for all values of β satisfying $0 < \beta < 6m^2/\alpha T$.

Theorem 1.7. *Assume*:

(1) $0 \le V(t, x)/|x|^2 \to \infty$ as $|x| \to \infty$.

(2) *There is a constant $q > 2$ such that*

$$V(t, x) \le C(|x|^q + 1), \quad t \in I, \ x \in \mathbb{R}^n.$$

(3) *There are constants $m > 0$, $\alpha > 0$ such that*

$$V(t, x) \le \alpha |x|^2, \quad |x| \le m, \ t \in I, \ x \in \mathbb{R}^n.$$

(4) *There are a constant C and a function $W(t) \in L^1(I)$ such that*

$$H(t, \theta x) \le C(H(t, x) + W(t)), \quad 0 \le \theta \le 1, \ t \in I, \ x \in \mathbb{R}^n.$$

Then the system (1-4) has a solution for all β satisfying $0 < \beta < 2\pi^2/\alpha T^2$.

The periodic nonautonomous problem

$$\ddot{x}(t) = \nabla_x V(t, x(t)) \tag{1-5}$$

has an extensive history in the case of singular systems (see, for example, [Ambrosetti and Coti Zelati 1993]). The first to consider it for potentials satisfying (1-3) were Berger and the author [1977]. We proved the existence of solutions to (1-4) under the condition that

$$V(t, x) \to \infty \quad \text{as } |x| \to \infty$$

uniformly for a.e. $t \in I$. Subsequently, Willem [1981], Mawhin [1987], Mawhin and Willem [1989], Tang [1995; 1998], Tang and Wu [1999; 2001; 2002] and others (see the references therein) proved existence under various conditions.

The periodic problem (1-1) was studied by Mawhin and Willem [1986; 1989], Long [1995], Tang and Wu [2003] and others. Tang and Wu [2003] proved existence of solutions of problem (1-1) under the following hypotheses:

(I) $V(t, x) \to \infty$ as $|x| \to \infty$ uniformly for a.e. $t \in I$.

(II) There exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1(0, T, \mathbb{R}^+)$ such that

$$|V(t, x)| + |\nabla V(t, x)| \le a(|x|)b(t) \quad \text{for all } x \in \mathbb{R}^n \text{ and a.e. } t \in [0, T].$$

and the superquadraticity condition:

(III) There exist $0 < \mu < 2$, $M > 0$ such that

$$V(t, x) > 0, \ H_\mu := \nabla V(t, x) \cdot x - \mu V(t, x) \le 0 \quad \text{for all } |x| \ge M \text{ and a.e. } t \in [0, T].$$

Rabinowitz [1980] proved existence under stronger hypotheses. In particular, in place of (I), he assumed:

(I′) There exist constants $a_1, a_2 > 0$, $\mu_0 > 1$ such that

$$V(t,x) \geq a_1|x|^{\mu_0} + a_2 \quad \text{for all } x \in \mathbb{R}^n \text{ and a.e. } t \in [0,T]$$

In place of (III), he assumed:

(III′) There exist $0 < \mu < 2$, $M > 0$ such that

$$0 < \nabla V(t,x)\cdot x \leq \mu V(t,x) \quad \text{for all } |x| \geq M \text{ and a.e. } t \in [0,T].$$

Mawhin and Willem [1986] proved existence for the case of convex potentials, while Long [1995] studied the problem for even potentials. They assumed that $V(t,x)$ is subquadratic in the sense that

there exist $a_3 < (2\pi/T)^2$ and a_4 such that
$|V(t,x)| \leq a_3|x|^2 + a_4$ for all $x \in \mathbb{R}^n$ and a.e. $t \in [0,T]$.

Mawhin and Willem [1989] also studied the problem for a bounded nonlinearity. Tang and Wu [2003] also proved existence of solutions if one replaces (I) with

$$\int_0^T V(t,x)\,dt \to \infty \quad \text{as } |x| \to \infty$$

and $V(t,x)$ is γ-subadditive with $\gamma > 0$ for a.e. $t \in [0,T]$. All of these authors studied only the existence of solutions.

All of the results mentioned above concerned the existence of solutions, which might be constants. Little was done concerning *nonconstant* solutions of problem (1-1). For the homogeneous case, Ben-Naoum, Troestler and Willem [Ben-Naoum et al. 1994] proved the existence of a nonconstant solution. For the case $T = 2\pi$, Theorem 1.7, with substantially stronger hypotheses, was proved by Nirenberg; see [Ekeland and Ghoussoub 2002]. Among other things, they assumed

$$V(t,x) \leq \frac{3}{2\pi^2}, \quad |x| \leq 1,\ t \in \mathbb{R},\ x \in \mathbb{R}^n,$$

and the superquadraticity condition

$$V(t,x) > 0,\ H_\mu(t,x) \leq 0, \quad |x| \geq C,\ t \in \mathbb{R},\ x \in \mathbb{R}^n,$$

for some $\mu > 2$, which implies our hypotheses, and

$$V(t,x) \geq C|x|^\mu - C', \quad x \in \mathbb{R}^n, C > 0,$$

among other things. These results were generalized in [Schechter 2006a; 2006b]. Further results, involving some of the hypotheses used in these last two papers, were obtained in [Wang et al. 2009].

We shall prove Theorems 1.1–1.5 in Section 5, and Theorems 1.6 and 1.7 in Section 7. We use linking and sandwich methods of critical point theory and then apply the monotonicity trick introduced by Struwe [1988; 1996] for minimization problems. (This trick was also used by others to solve Landesman–Lazer type problems, for bifurcation problems, for Hamiltonian systems and Schrödinger equations.)

Jeanjean [1999] shows that for a specific class of functionals having a mountain-pass (MP) geometry, almost every functional in this class has a bounded Palais–Smale sequence at the (MP) level. This theorem is used to obtain, for a given functional, a special Palais–Smale sequence possessing extra properties that help to ensure its convergence. Subsequently, these abstract results are applied to prove the existence of a positive solution for a problem of the form (P) $-\Delta u + Ku = f(x, u)$, $u \in H^1(R^N)$, $K > 0$. He assumed that the functional associated to (P) has an (MP) geometry. His results cover the case where the nonlinearity f satisfies (i) $f(x, s)s^{-1} \to a \in (0, \infty]$ as $s \to +\infty$ and (ii) $f(x, s)s^{-1}$ is nondecreasing as a function of $s \geq 0$, a.e. $x \in R^N$.

Here, we obtain a bounded Palais–Smale sequences for functionals that need not have (MP) geometry. We then apply the theory to situations in which the (MP) geometry is not present. In particular, we apply it to situations where there is linking without the (MP) geometry. We also apply it to situations in which there are sandwich pairs which do not link.

The theory of sandwich pairs began in [Silva 1991; Schechter 1992; 1993] and was developed in subsequent publications such as [Schechter 2008; 2009].

2. Flows

Let E be a Banach space, and let Σ be the set of all continuous maps $\sigma = \sigma(t)$ from $E \times [0, 1]$ to E such that

(1) $\sigma(0)$ is the identity map,

(2) for each $t \in [0, 1]$, $\sigma(t)$ is a homeomorphism of E onto E,

(3) $\sigma'(t)$ is piecewise continuous on [0,1] and satisfies

$$\|\sigma'(t)u\| \leq \text{constant}, \quad u \in E. \tag{2-1}$$

The mappings in Σ are called *flows*.

Remark 2.1. If $\sigma_1,\ \sigma_2$ are in Σ, define $\sigma_3 = \sigma_1 \circ \sigma_2$ by

$$\sigma_3(s) = \begin{cases} \sigma_1(2s) & \text{if } 0 \leq s \leq \frac{1}{2}, \\ \sigma_2(2s-1)\sigma_1(1) & \text{if } \frac{1}{2} < s \leq 1. \end{cases}$$

Then $\sigma_1 \circ \sigma_2 \in \Sigma$.

3. Sandwich systems

Let E be a Banach space. Define a nonempty collection $\mathcal{K}$ of nonempty subsets $K \subset E$ to be a *sandwich system* if $\mathcal{K}$ has the following property:

$$\sigma(1)K \in \mathcal{K}, \quad \sigma \in \Sigma, \ K \in \mathcal{K}.$$

Theorem 3.1. *Let $\mathcal{K}$ be a sandwich system, and let $G(u)$ be a C^1 functional on E. Define*

$$a := \inf_{K \in \mathcal{K}} \sup_{K} G, \tag{3-1}$$

and assume that a is finite. Assume, in addition, that there is a constant C_0 such that for each $\delta > 0$ there is a $K \in \mathcal{K}$ satisfying

$$\sup_K G \leq a + \delta, \tag{3-2}$$

such that the inequality

$$G(u) \geq a - \delta, \quad u \in K, \tag{3-3}$$

implies $\|u\| \leq C_0$. Then there is a bounded sequence $\{u_k\} \subset E$ such that

$$G(u_k) \to a, \quad \|G'(u_k)\| \to 0. \tag{3-4}$$

Theorem 3.2. *Let $\mathcal{K}$ be a sandwich system, and let $G(u)$ be a C^1 functional on E. Assume that there are subsets A, B of E such that*

$$a_0 := \sup_A G < \infty, \quad b_0 := \inf_B G > -\infty, \tag{3-5}$$

$A \in \mathcal{K}$ and

$$B \cap K \neq \varnothing, \quad K \in \mathcal{K}. \tag{3-6}$$

Assume, in addition, that there is a constant C_0 such that for each $\delta > 0$ there is a $K \in \mathcal{K}$ satisfying (3-2) such that the inequality (3-3) implies $\|u\| \leq C_0$. Then the value a given by (3-1) satisfies $b_0 \leq a \leq a_0$ and there is a bounded sequence $\{u_k\} \subset E$ such that

$$G(u_k) \to a, \quad \|G'(u_k)\| \to 0. \tag{3-7}$$

Definition 3.3. We shall say that sets A, B in E form a *sandwich pair* if A is a member of a sandwich system $\mathcal{K}$ and B satisfies (3-6).

Theorem 3.4. *Let N be a finite dimensional subspace of a Banach space E, and let p be any point of N. Let F be a continuous map of E onto N such that $F = I$ on N. Then $A = N$ and $B = F^{-1}(p)$ form a sandwich pair.*

Corollary 3.5. *Let N be a closed subspace of a Hilbert space E and let $M = N^\perp$. Assume that at least one of the subspaces M, N is finite dimensional. Then M, N form a sandwich pair.*

Corollary 3.6. *Let N be a finite dimensional subspace of a Hilbert space E with complement $M' = M \oplus \{v_0\}$, where v_0 is an element in E having unit norm, and let δ be any positive number. Let $\varphi(t) \in C^1(\mathbb{R})$ be such that*

$$0 \le \varphi(t) \le 1, \ \varphi(0) = 1 \quad \textit{and} \quad \varphi(t) = 0, \ |t| \ge 1.$$

Let

$$F(v + w + sv_0) = v + \big(s + \delta - \delta\varphi(\|w\|^2/\delta^2)\big)v_0, \quad v \in N, \ w \in M, \ s \in \mathbb{R}. \tag{3-8}$$

Then $A = N' = N \oplus \{v_0\}$ and $B = F^{-1}(\delta v_0)$ form a sandwich pair.

Proof. One checks that the mapping F given by (3-8) satisfies the hypotheses of Theorem 3.4 for N'. □

4. The parameter problem

Let E be a reflexive Banach space with norm $\|\cdot\|$, and let A, B be two closed subsets of E. Suppose that $G \in \mathscr{C}^1(E, \mathbb{R})$ is of the form $G(u) := I(u) - J(u)$, $u \in E$, where I, $J \in \mathscr{C}^1(E, \mathbb{R})$ map bounded sets to bounded sets. Define

$$G_\lambda(u) = \lambda I(u) - J(u), \quad \lambda \in \Lambda,$$

where Λ is an open interval contained in $(0, +\infty)$. Assume one of the following alternatives holds.

(H_1) $I(u) \ge 0$ for all $u \in E$ and $I(u) + |J(u)| \to \infty$ as $\|u\| \to \infty$.

(H_2) $I(u) \le 0$ for all $u \in E$ and $|I(u)| + |J(u)| \to \infty$ as $\|u\| \to \infty$.

Furthermore, we suppose that $\mathscr{K}$ is a sandwich system satisfying

(H_3) $a(\lambda) := \inf_{K \in \mathscr{K}} \sup_K G_\lambda$ is finite for each $\lambda \in \Lambda$.

Theorem 4.1. *Assume that (H_1) (or (H_2)) and (H_3) hold.*

(1) *For almost all $\lambda \in \Lambda$ there exists a constant $k_0(\lambda) := k_0$ (depending only on λ) such that for each $\delta > 0$ there exists a $K \in \mathscr{K}$ such that*

$$\sup_K G_\lambda \le a(\lambda) + \delta,$$

$$\|u\| \le k_0 \quad \text{whenever} \ u \in K \quad \text{and} \quad G_\lambda(u) \ge a(\lambda) - \delta. \tag{4-1}$$

(2) *For almost all $\lambda \in \Lambda$ there exists a bounded sequence $u_k(\lambda) \in E$ such that*

$$\|G'_\lambda(u_k)\| \to 0, \quad G_\lambda(u_k) \to a(\lambda) := \inf_{K \in \mathscr{K}} \sup_K G_\lambda \quad \text{as } k \to \infty.$$

Corollary 4.2. *The conclusions of Theorem 4.1 hold if we replace Hypothesis* (H_3) *with*:

(H_3') *There is a sandwich pair* A, B *such that for each* $\lambda \in \Lambda$,

$$a_0 := \sup_A G_\lambda < \infty, \quad b_0 := \inf_B G_\lambda > -\infty. \tag{4-2}$$

Corollary 4.3. *The conclusions of Theorem 4.1 hold if we replace Hypothesis* (H_3) *with*:

(H_3'') *There are sets* A, B *such that* A *links* B *and for each* $\lambda \in \Lambda$,

$$a_0 := \sup_A G_\lambda \le b_0 := \inf_B G_\lambda. \tag{4-3}$$

5. Proofs of the theorems

We now give the proof of Theorem 1.4.

Proof. Let X be the set of vector functions $x(t)$ described above. It is a Hilbert space with norm satisfying

$$\|x\|_X^2 = \sum_{j=1}^{n} \|x_j\|_{H^1}^2.$$

We also write

$$\|x\|^2 = \sum_{j=1}^{n} \|x_j\|^2,$$

where $\|\cdot\|$ is the $L^2(I)$ norm.

Let

$$N = \{x(t) \in X : x_j(t) \equiv \text{constant for } 1 \le j \le n\},$$

and set $M = N^\perp$. The dimension of N is n, and $X = M \oplus N$. See, for example, [Mawhin and Willem 1989, Proposition 1.3] for details on the following lemma.

Lemma 5.1. *If* $x \in M$, *then*

$$\|x\|_\infty^2 \le \frac{T}{12}\|\dot{x}\|^2 \quad \text{and} \quad \|x\| \le \frac{T}{2\pi}\|\dot{x}\|.$$

Define

$$G(x) = \|\dot{x}\|^2 - 2\int_I V(t, x(t))\, dt, \qquad x \in X. \tag{5-1}$$

For each $x \in X$ write $x = v + w$, where $v \in N$, $w \in M$. For convenience, we shall follow [Mawhin and Willem 1989] and use the equivalent norm for X:

$$\|x\|_X^2 = \|\dot{w}\|^2 + \|v\|^2.$$

Let

$$I(x)=\|\dot{x}\|^2, \quad J(x)=2\int_I V(t,x(t))\,dt.$$

By Hypothesis (1),

$$J(v)\to\infty \quad \text{as } \|v\|\to\infty,\ v\in N.$$

Hence,

$$I(x)+|J(x)|\to\infty \quad \text{as } \|x\|_X\to\infty.$$

Let

$$G_\lambda(x)=\lambda\|\dot{x}\|^2-2\int_I V(t,x(t))\,dt=\lambda I(x)-J(x), \quad x\in X. \tag{5-2}$$

Hypothesis (1) implies

$$\sup\nolimits_N G_\lambda(v)=-\inf\nolimits_N J(v)<\infty. \tag{5-3}$$

If $x\in M$, we have by Hypothesis (2) and Lemma 5.1 that

$$\begin{aligned} G_\lambda(x)&\geq\lambda\|\dot{x}\|^2-2\int\alpha|x(t)|^2\,dt-B \\ &\geq\Big(\frac{4\pi^2\lambda}{T^2}-2\alpha\Big)\|x\|^2-B\geq -B, \end{aligned} \tag{5-4}$$

provided

$$\lambda\geq\alpha T^2/2\pi^2. \tag{5-5}$$

By Corollary 3.5, M and N form a sandwich pair. Then by Corollary 4.2, for almost every λ satisfying (5-5) there is a bounded sequence $\{x^{(k)}\}\subset X$ such that

$$G_\lambda(x^{(k)})=\lambda\|\dot{x}^{(k)}\|^2-2\int_I V(t,x^{(k)}(t))\,dt\to c\geq -B, \tag{5-6}$$

$$(G'_\lambda(x^{(k)}),z)/2=\lambda(\dot{x}^{(k)},\dot{z})-\int_I\nabla_x V(t,x^{(k)})\cdot z(t)\,dt\to 0, \quad z\in X, \tag{5-7}$$

$$(G'_\lambda(x^{(k)}),x^{(k)})/2=\lambda\|\dot{x}^{(k)}\|^2-\int_I\nabla_x V(t,x^{(k)})\cdot x^{(k)}\,dt\to 0. \tag{5-8}$$

Since

$$\rho_k=\|x^{(k)}\|_X\leq C,$$

there is a renamed subsequence such that $x^{(k)}$ converges to a limit $x\in X$ weakly in X and uniformly on I. From (5-7) we see that

$$(G'_\lambda(x),z)/2=\lambda(\dot{x},\dot{z})-\int_I\nabla_x V(t,x(t))\cdot z(t)\,dt=0, \quad z\in X,$$

from which we conclude easily that x is a solution of (1-4) with $\beta = 1/\lambda$, proving the first statement of the theorem. To prove the second, note that (5-4) implies

$$G_\lambda(x) \geq -B, \quad x \in M.$$

Consequently, if $B < 0$, we see that

$$b_0 = \inf_M G_\lambda(x) > 0.$$

Thus, the solution x satisfies $G_\lambda(x) \geq b_0 > 0$. If x were a constant, we would have $G_\lambda(x) = -J(x) \leq 0$, a contradiction. This gives the result. □

The proof of Theorem 1.5 is similar to that of Theorem 1.4 with the exception of the inequality (5-4) resulting from Hypothesis (2). In its place we reason as follows: If $x \in M$ and $\|\dot{x}\|^2 = 12m^2/T$, then $|x| \leq m$ by Lemma 5.1. Thus, we have by Hypothesis (2′),

$$\begin{aligned} G_\lambda(x) &\geq \lambda\|\dot{x}\|^2 - 2\alpha m^2 - B \\ &\geq (12\lambda - 2\alpha T)m^2/T - B \geq -B, \end{aligned}$$

provided $\lambda \geq \alpha T/6$. The remainder of the proof is essentially the same.

In proving Theorem 1.1, we follow the proof of Theorem 1.4. Hypothesis (1) implies

$$G_\lambda(v) \leq 0, \quad v \in N. \tag{5-9}$$

If $x \in M$ and

$$\|\dot{x}\|^2 = \rho^2 = \frac{12}{T}m^2,$$

then Lemma 5.1 implies that $\|x\|_\infty \leq m$, and we have by Hypothesis (2) that $\int_0^T V(t,x)\,dt \leq \alpha$. Hence,

$$\begin{aligned} G_\lambda(x) &\geq \lambda\|\dot{x}\|^2 - 2\int_0^T V(t,x)\,dt \\ &\geq \lambda\rho^2 - 2\alpha \geq 0, \end{aligned} \tag{5-10}$$

provided $\lambda \geq \alpha T/6m^2$.

If we take

$$A = M \cap B_\rho, \quad B = N,$$

then A links B by [Schechter 1999, Corollary 13.5]. Thus, we see that Hypothesis (H_3'') of Corollary 4.3 holds with G_λ replaced with $-G_\lambda$. By that corollary, there is a bounded sequence satisfying (5-6)–(5-8). The first result now follows as before. To prove the second, let

$$y(t) = v + sw_0,$$

where $v \in N$, $s \geq 0$, and

$$w_0 = (\sin(2\pi t/T), 0, \dots, 0).$$

Then $w_0 \in M$, and

$$\|w_0\|^2 = T/2, \quad \|\dot{w}_0\|^2 = 2\pi^2/T.$$

Note that

$$\|y\|^2 = \|v\|^2 + s^2T/2 = T|v|^2 + Ts^2/2.$$

Consequently,

$$\begin{aligned} G_\lambda(y) = \lambda s^2\|\dot{w}_0\|^2 - 2\int_I V(t, y(t))\,dt &\leq 2\lambda\pi^2 s^2/T - 2\gamma\int_I |y(t)|^2\,dt + B \\ &\leq 2\lambda\pi^2 s^2/T - 2\gamma(\|v\|^2 + Ts^2/2) + B \\ &\leq (2\lambda\pi^2 - \gamma T^2)s^2/T - 2T\gamma|v|^2 + B \to -\infty \text{ as } s^2 + |v|^2 \to \infty. \end{aligned}$$

Take

$$\begin{aligned} &A = \{v \in N : |v\| \leq R\} \cup \{sw_0 + v : v \in N, s \geq 0, \|sw_0 + v\| = R\}, \\ &B = \partial B_\rho \cap M,\ 0 < \rho < R, \end{aligned}$$

where

$$B_\sigma = \{x \in X : \|x\|_X < \sigma\}.$$

By [Schechter 1999, Example 3, page 38], A links B. Moreover, if R is sufficiently large,

$$\sup\nolimits_A G_\lambda \leq 0 \leq \inf\nolimits_B G_\lambda. \tag{5-11}$$

Hence, we may apply [Schechter 1999, Corollary 2.8.2] and Corollary 4.3 to conclude that there is a sequence $\{x^{(k)}\} \subset X$ such that

$$G_\lambda(x^{(k)}) = \lambda\|\dot{x}^{(k)}\|^2 - 2\int_I V(t, x^{(k)}(t))\,dt \to c \geq 0, \tag{5-12}$$

$$(G'_\lambda(x^{(k)}), z)/2 = \lambda(\dot{x}^{(k)}, \dot{z}) - \int_I \nabla_x V(t, x^{(k)}) \cdot z(t)\,dt \to 0, \quad z \in X, \tag{5-13}$$

$$(G'_\lambda(x^{(k)}), x^{(k)})/2 = \lambda\|\dot{x}^{(k)}\|^2 - \int_I \nabla_x V(t, x^{(k)}) \cdot x^{(k)}\,dt \to 0. \tag{5-14}$$

Since

$$\rho_k = \|x^{(k)}\|_X \leq C,$$

there is a renamed subsequence such that $x^{(k)}$ converges to a limit $x \in X$ weakly in X and uniformly on I. From (5-13) we see that

$$(G'_\lambda(x), z)/2 = \lambda(\dot{x}, \dot{z}) - \int_I \nabla_x V(t, x(t)) \cdot z(t)\,dt = 0, \quad z \in X,$$

from which we conclude easily that x is a solution of (1-1). By (5-12) we see that

$$G_\lambda(x) \geq c \geq 0,$$

showing that $x(t)$ is not a constant. For if $c > 0$ and $x \in N$, then

$$G_\lambda(x) = -2 \int_I V(t, x(t))\, dt \leq 0.$$

If $c = 0$, we know that $d(x^{(k)}, B) \to 0$ by [Schechter 1999, Theorem 2.1.1]. Hence, there is a sequence $\{y^{(k)}\} \subset B$ such that $x^{(k)} - y^{(k)} \to 0$ in X. If $v \in N$, then

$$(x, v) = (x - x^{(k)}, v) + (x^{(k)} - y^{(k)}, v) \to 0,$$

since $y^{(k)} \in M$. Thus $x \in M$. This completes the proof.

To prove Theorem 1.3, note that Hypothesis (1) implies

$$G_\lambda(v) \leq 0, \quad v \in N. \tag{5-15}$$

If $x \in M$, we have by Hypothesis (2)

$$\begin{aligned}
G_\lambda(x) &\geq \lambda \|\dot{x}\|^2 - 2 \int_{|x|<m} \alpha |x(t)|^2\, dt - C \int_{|x|>m} (|x|^q + 1)\, dt \\
&\geq \lambda \|\dot{x}\|^2 - 2\alpha \|x\|^2 - C(1 + m^{2-q} + m^{-q}) \int_{|x|>m} |x|^q\, dt \\
&\geq \|\dot{x}\|^2 \big(\lambda - (2\alpha T^2/4\pi^2)\big) - C' \int_{|x|>m} |x|^q\, dt \\
&\geq \big(\lambda - (\alpha T^2/2\pi^2)\big) \|x\|_X^2 - C'' \int_I \|x\|_X^q\, dt \\
&\geq \big(\lambda - (\alpha T^2/2\pi^2)\big) \|x\|_X^2 - C''' \|x\|_X^q = \big(\lambda - (\alpha T^2/2\pi^2) - C''' \|x\|_X^{q-2}\big) \|x\|_X^2.
\end{aligned}$$

Hence,

$$G_\lambda(x) \geq \varepsilon \|x\|_X^2, \quad \|x\|_X \leq \rho,\ x \in M \tag{5-16}$$

for $\rho > 0$ sufficiently small, where $\varepsilon < \lambda - (\alpha T^2/2\pi^2)$ is positive. If we take

$$A = M \cap B_\rho, \quad B = N,$$

then A links B by [Schechter 1999, Corollary 13.5]. Thus, Hypothesis (H_3'') of Corollary 4.3 holds with G_λ replaced with $-G_\lambda$. By that corollary, there is a bounded sequence satisfying (5-6)–(5-8). The result now follows as before.

6. Finding the sequences

Proof of Theorem 3.1. Let $M = C_0 + 1$. Then

$$\|\sigma(1)v\| \le M$$

whenever $\sigma \in \Sigma$ satisfies $\|\sigma'(t)\| \le 1$ and $v \in E$ satisfies $\|v\| \le C_0$. If the theorem were false, then there would be a $\delta > 0$ such that

$$\|G'(u)\| \ge 3\delta \tag{6-1}$$

when

$$u \in \{u \in E : \|u\| \le M+1, |G(u) - a| \le 3\delta\}. \tag{6-2}$$

Take $\delta < 1/3$. Since $G \in C^1(E, \mathbb{R})$, for each $\theta < 1$ there is a locally Lipschitz continuous mapping $Y(u)$ of $\hat{E} = \{u \in E : G'(u) \ne 0\}$ into E such that

$$\|Y(u)\| \le 1, \quad \theta\|G'(u)\| \le (G'(u), Y(u)), \quad u \in \hat{E} \tag{6-3}$$

(see, for example, [Schechter 1999]). Take $\theta > 2/3$. Let

$$\begin{aligned} Q_0 &= \{u \in E : \|u\| \le M+1, |G(u) - a| \le 2\delta\}, \\ Q_1 &= \{u \in E : \|u\| \le M, |G(u) - a| \le \delta\}, \\ Q_2 &= E \setminus Q_0, \\ \eta(u) &= d(u, Q_2)/\big(d(u, Q_1) + d(u, Q_2)\big). \end{aligned}$$

It is easily checked that $\eta(u)$ is locally Lipschitz continuous on E and satisfies

$$\begin{cases} \eta(u) = 1 & \text{if } u \in Q_1, \\ \eta(u) = 0 & \text{if } u \in \bar{Q}_2, \\ \eta(u) \in (0, 1) & \text{otherwise.} \end{cases} \tag{6-4}$$

Let

$$W(u) = -\eta(u)Y(u).$$

Then

$$\|W(u)\| \le 1, \quad u \in E.$$

By [Schechter 2009, Theorem 4.5], for each $v \in E$ there is a unique solution $\sigma(t)v$ of the system

$$\sigma'(t) = W(\sigma(t)), \; t \in \mathbb{R}^+, \quad \sigma(0) = v. \tag{6-5}$$

We have

$$\begin{aligned} dG(\sigma(t)v)/dt &= -\eta(\sigma(t)v)(G'(\sigma(t)v), Y(\sigma(t)v)) \\ &\le -\theta\eta(\sigma)\|G'(\sigma)\| \le -3\theta\delta\eta(\sigma). \end{aligned} \tag{6-6}$$

Let $K \in \mathcal{K}$ satisfy the hypotheses of the theorem. Let v be any element of $K \cap Q_1$. Then $\|v\| \le C_0$. If there is a $t_1 \le 1$ such that $\sigma(t_1)v \notin Q_1$, then

$$G(\sigma(1)v) < a - \delta, \tag{6-7}$$

since $\|\sigma(1)v\| \le M$,

$$G(\sigma(1)v) \le G(\sigma(t_1)v)$$

and the right hand side cannot be greater than $a + \delta$ by (6-6). On the other hand, if $\sigma(t)v \in Q_1$ for all $t \in [0, 1]$, then we have by (6-6)

$$G(\sigma(1)v) \le a + \delta - 3\delta\theta < a - \delta.$$

If $v \in K \backslash Q_1$, then we must have

$$G(\sigma(1)v) \le G(v) < a - \delta,$$

since $G(v) \ge a - \delta$ would put v into Q_1. Hence

$$G(\sigma(1)v) < a - \delta, \quad v \in K. \tag{6-8}$$

By hypothesis, $\widetilde{K} = \sigma(1)K \in \mathcal{K}$. This means that

$$G(w) < a - \delta, \quad w \in \widetilde{K}. \tag{6-9}$$

But this contradicts the definition (3-1) of a. Hence (6-1) cannot hold for u satisfying (6-2). This proves the theorem. □

Proof of Theorem 3.2. Since $A \in \mathcal{K}$, clearly $a \le a_0$. Moreover, for any $K \in \mathcal{K}$, we have

$$b_0 = \inf_B G_\lambda \le \inf_{B \cap K} G_\lambda \le \sup_{B \cap K} G_\lambda \le \sup_K G_\lambda.$$

Hence, $b_0 \le a$. Apply Theorem 3.1. □

Proof of Theorem 3.4. Define

$$\mathcal{K} = \{\sigma(1)A : \sigma \in \Sigma\}.$$

Then $\mathcal{K}$ is a sandwich system. To see this, let $K = \widetilde{\sigma}(1)A$ be a set in $\mathcal{K}$. If $\sigma \in \Sigma$, then $\sigma \circ \widetilde{\sigma}$ is also in Σ. Thus, $\mathcal{K}$ is a sandwich system. Let $B = F^{-1}(p)$. If we can show that B satisfies (3-6), then the result will follow from Theorem 3.2. Now (3-6) is equivalent to

$$F^{-1}(p) \cap \sigma(1)N \ne \varnothing, \quad \sigma \in \Sigma.$$

Let $\Omega_R(p)$ be a ball in N with radius R and center p, and let $\sigma(t)$ be any flow in Σ. Since

$$\sigma(t)u - u = \int_0^t \sigma'(\tau)u \, d\tau, \tag{6-10}$$

we have

$$\|\sigma(t)u - \sigma(s)u\| \le C|t-s|.$$

If $u \in A_R = \partial\Omega_R(p)$, and $v \in B$, we have

$$h(s) := d(\sigma(s)u, B) \le \|\sigma(s)u - v\| \le \|\sigma(t)u - v\| + C|t-s|.$$

This implies

$$h(s) \le h(t) + C|t-s|. \tag{6-11}$$

Moreover, by [Schechter 2009, Lemmas 4.3 and 4.8], $h(s)$ satisfies

$$h(s) \ge m(R) \to \infty \quad \text{as } R \to \infty, \quad 0 \le s \le 1, \ u \in \partial\Omega_R(p).$$

Thus,

$$\|\sigma(s)u - F^{-1}(p)\| \ge h(s) \ge m(R) \to \infty, \quad u \in A_R.$$

Consequently,

$$F^{-1}(p) \cap \sigma(1)A_R = \varnothing, \quad \sigma \in \Sigma, \tag{6-12}$$

for R sufficiently large. Now A_R links B; see, for example, [Schechter 1999]. For $\Gamma \in \Phi$, define

$$\Gamma_1(s) = \begin{cases} \sigma(2s) & \text{if } 0 \le s \le \frac{1}{2}, \\ \sigma(1)\Gamma(2s-1) & \text{if } \frac{1}{2} < s \le 1. \end{cases}$$

Clearly, $\Gamma_1 \in \Phi$. Consequently, there is a $t_0 \in [0, 1]$ such that

$$\Gamma_1(t_0)A_R \cap B \neq \varnothing.$$

If $t_0 \le \frac{1}{2}$, then

$$\sigma(2t_0)A_R \cap B \neq \varnothing,$$

contradicting (6-12). If $t_0 > \frac{1}{2}$, then

$$\sigma(1)\Gamma(2t_0 - 1)A_R \cap B \neq \varnothing.$$

Take $\Gamma(s)u = (1-s)u$. Then $\Gamma \in \Phi$ and $\Gamma(2t_0 - 1)A_R \subset N$. Hence,

$$\sigma(1)N \cap B \neq \varnothing.$$

Thus (3-6) holds, and the theorem is proved. □

7. The monotonicity trick

Proof of Theorem 4.1. We prove conclusion (1) assuming the first of the alternative hypotheses, (H_1).

By (H_1), the map $\lambda \mapsto a(\lambda)$ is nondecreasing. Hence, $a'(\lambda) := da(\lambda)/d\lambda$ exists for almost every $\lambda \in \Lambda$. From this point on, we consider those λ where $a'(\lambda)$ exists. For fixed $\lambda \in \Lambda$, let $\lambda_n \in (\lambda, 2\lambda) \cap \Lambda$, $\lambda_n \to \lambda$ as $n \to \infty$. Then there exists $\bar{n}(\lambda)$ such that

$$a'(\lambda) - 1 \le \frac{a(\lambda_n) - a(\lambda)}{\lambda_n - \lambda} \le a'(\lambda) + 1 \quad \text{for } n \ge \bar{n}(\lambda). \tag{7-1}$$

Next, there exist $K_n \in \mathcal{K}_Q$, $k_0 := k_0(\lambda) > 0$ such that

$$\|u\| \le k_0 \quad \text{whenever } \; G_\lambda(u) \ge a(\lambda) - (\lambda_n - \lambda). \tag{7-2}$$

In fact, by the definition of $a(\lambda_n)$, there exists K_n such that

$$\sup_{K_n} G_\lambda(u) \le \sup_{K_n} G_{\lambda_n}(u) \le a(\lambda_n) + (\lambda_n - \lambda). \tag{7-3}$$

If $G_\lambda(u) \ge a(\lambda) - (\lambda_n - \lambda)$ for some $u \in K_n$, then, by (7-1) and (7-3), we have that

$$\begin{aligned} I(u) &= \frac{G_{\lambda_n}(u) - G_\lambda(u)}{\lambda_n - \lambda} \\ &\le \frac{a(\lambda_n) + (\lambda_n - \lambda) - a(\lambda) + (\lambda_n - \lambda)}{\lambda_n - \lambda} \\ &\le a'(\lambda) + 3, \end{aligned} \tag{7-4}$$

and it follows that

$$\begin{aligned} J(u) &= \lambda_n I(u) - G_{\lambda_n}(u) \\ &\le \lambda_n(a'(\lambda) + 3) - G_\lambda(u) \\ &\le \lambda_n(a'(\lambda) + 3) - a(\lambda) + (\lambda_n - \lambda) \\ &\le 2\lambda(a'(\lambda) + 3) - a(\lambda) + \lambda. \end{aligned} \tag{7-5}$$

On the other hand, by (H_1), (7-1), and (7-3),

$$\begin{aligned} J(u) &= \lambda_n I(u) - G_{\lambda_n}(u) \\ &\ge -G_{\lambda_n}(u) \\ &\ge -(a(\lambda_n) + (\lambda_n - \lambda)) \\ &\ge -(a(\lambda) + (\lambda_n - \lambda)(a'(\lambda) + 2)) \\ &\ge -a(\lambda) - \lambda|a'(\lambda) + 2|. \end{aligned} \tag{7-6}$$

Combining (7-4)–(7-7) and (H_1), we see that there exists $k_0(\lambda) := k_0$ (depending only on λ) such that (7-2) holds.

By the choice of K_n and (7-1), we see that

$$\begin{aligned} G_\lambda(u) &\le G_{\lambda_n}(u) \le \sup_{K_n} G_{\lambda_n}(u) \\ &\le a(\lambda_n) + (\lambda_n - \lambda) \\ &\le (a'(\lambda)+1)(\lambda_n - \lambda) + a(\lambda) + (\lambda_n - \lambda) \\ &\le a(\lambda) + (a'(\lambda)+2)(\lambda_n - \lambda) \end{aligned}$$

for all $u \in K_n$. Take n sufficiently large to ensure that $|a'(\lambda)+2|(\lambda_n - \lambda) < \delta$. This proves conclusion (1). Conclusion (2) now follows from Theorem 3.1. The proof under Hypothesis (H_2) is similar, and is omitted. □

In proving Corollary 4.3, we shall make use of the following results of linking. Let E be a Banach space. The set Φ of mappings $\Gamma(t) \in C(E \times [0, 1], E)$ is to have following properties:

(a) For each $t \in [0, 1)$, $\Gamma(t)$ is a homeomorphism of E onto itself and $\Gamma(t)^{-1}$ is continuous on $E \times [0, 1)$.

(b) $\Gamma(0) = I$.

(c) For each $\Gamma(t) \in \Phi$ there is a $u_0 \in E$ such that $\Gamma(1)u = u_0$ for all $u \in E$ and $\Gamma(t)u \to u_0$ as $t \to 1$ uniformly on bounded subsets of E.

(d) For each $t_0 \in [0, 1)$ and each bounded set $A \subset E$ we have

$$\sup_{\substack{0 \le t \le t_0 \\ u \in A}} \{\|\Gamma(t)u\| + \|\Gamma^{-1}(t)u\|\} < \infty.$$

A subset A of E links a subset B of E if $A \cap B = \varnothing$ and, for each $\Gamma(t) \in \Phi$, there is a $t \in (0, 1]$ such that $\Gamma(t)A \cap B \neq \varnothing$.

Theorem [Schechter 1999, Theorem 2.1.1]. *Let G be a C^1-functional on E, and let A, B be subsets of E such that A links B and*

$$a_0 := \sup_A G \le b_0 := \inf_B G.$$

Assume that

$$a := \inf_{\Gamma \in \Phi} \sup_{\substack{0 \le s \le 1 \\ u \in A}} G(\Gamma(s)u)$$

is finite. Then there is a sequence $\{u_k\} \subset E$ such that

$$G(u_k) \to a, \quad G'(u_k) \to 0.$$

If $a = b_0$, then we can also require that

$$d(u_k, B) \to 0.$$

Proof of Corollary 4.3. Let

$$\mathcal{K} = \{\Gamma(s)A : \Gamma \in \Phi,\ s \in I\}.$$

Then $\mathcal{K}$ is a sandwich system. In fact, if $\sigma \in \Sigma$ and $\Gamma \in \Phi$, define

$$\Gamma_1(s) = \begin{cases} \sigma(2s) & \text{if } 0 \le s \le \frac{1}{2}, \\ \sigma(1)\Gamma(2s-1) & \text{if } \frac{1}{2} < s \le 1. \end{cases}$$

Then $\Gamma_1 \in \Phi$. Thus,

$$\sigma(1)K \in \mathcal{K}, \quad \sigma \in \Sigma,\ K \in \mathcal{K}.$$

Since A links B, we have for each $\Gamma(t) \in \Phi$, there is a $t \in (0, 1]$ such that $\Gamma(t)A \cap B \neq \varnothing$. Consequently,

$$B \cap K \neq \varnothing, \quad K \in \mathcal{K}. \tag{7-7}$$

Thus, A, B form a sandwich pair. Let

$$a(\lambda) := \inf_{\Gamma \in \Phi} \sup_{\substack{0 \le s \le 1 \\ u \in A}} G_\lambda(\Gamma(s)u).$$

Then $a(\lambda) := \inf_{K \in \mathcal{K}} \sup_K G_\lambda$ is finite for any $\lambda \in \Lambda$. This shows that Hypothesis (H_3'') implies Hypothesis (H_3). We can now apply Theorem 4.1. □

Proof of Theorem 1.6. Take $\lambda = 1/\beta$. Let $\lambda_0 = \alpha T/6m^2$, and let $\nu < \infty$. By Theorem 1.1, for a.e. $\lambda \in (\lambda_0, \nu)$, there exists u_λ such that $G'_\lambda(u_\lambda) = 0$, $G_\lambda(u_\lambda) = a(\lambda) \ge a(\lambda_0)$. Let λ satisfy $\lambda_0 < \lambda < \nu$. Choose $\lambda_n \to \lambda$, $\lambda_n > \lambda$. Then there exists x_n such that

$$G'_{\lambda_n}(x_n) = 0, \quad G_{\lambda_n}(x_n) = a(\lambda_n) \ge a(\lambda_0).$$

Therefore,

$$\int_\Omega \frac{2V(t, x_n)}{\|x_n\|_X^2}\, dt \le C.$$

Now we prove that $\{x_n\}$ is bounded. If $\|x_n\|_X \to \infty$, let $w_n = x_n/\|x_n\|_X$. Then there is a renamed subsequence such that $w_n \to w$ weakly in X, strongly in $L^\infty(\Omega)$ and a.e. in Ω.

Let Ω_0 be the set where $w \neq 0$. Then $|x_n(t)| \to \infty$ for $t \in \Omega_0$. If Ω_0 had positive measure, then we would have

$$C \ge \int_\Omega \frac{2V(t, x_n)}{\|x_n\|_X^2}\, dt = \int_\Omega \frac{2V(t, x_n)}{x_n^2} |w_n|^2\, dt \ge \int_{w \neq 0} \frac{2V(t, x_n)}{x_n^2} |w_n|^2\, dt \to \infty,$$

showing that $w = 0$ a.e. in Ω. Hence, $w_n \to 0$. Since

$$\|\dot{w}_n\|^2 + \|w_n\|^2 = 1,$$

we have $\|\dot{w}_n\| \to 1$. Define $\theta_n \in [0, 1]$ by

$$G_{\lambda_n}(\theta_n x_n) = \max_{\theta \in [0,1]} G_{\lambda_n}(\theta x_n).$$

For any $c > 0$ and $\overline{w}_n = c w_n$, we have

$$\int_\Omega V(t, \overline{w}_n)\, dt \to 0$$

(see, for example, [Schechter 2008, page 64]). Thus,

$$G_{\lambda_n}(\theta_n x_n) \ge G_{\lambda_n}(c w_n) = c^2 \lambda_n \|\dot{w}_n\|^2 - 2\int_\Omega V(t, \overline{w}_n)\, dt \to \lambda c^2, \quad n \to \infty.$$

Hence, $G_{\lambda_n}(\theta_n x_n) \ge \lambda c^2/2$ for n sufficiently large. That is, $\lim_{n\to\infty} G_{\lambda_n}(\theta_n x_n) = \infty$. If there is a renamed subsequence such that $\theta_n = 1$, then

$$G_{\lambda_n}(x_n) \to \infty. \tag{7-8}$$

If $0 \le \theta_n < 1$ for all n, then we have $(G'_{\lambda_n}(\theta_n x_n), x_n) \le 0$. Therefore,

$$\begin{aligned}\int_\Omega H(t, \theta_n x_n)\, dt &= \int_\Omega \big(\nabla_x V(t, \theta_n x_n)\theta_n x_n - 2V(t, \theta_n x_n)\big)\, dt \\ &= G_{\lambda_n}(\theta_n x_n) - (G'_{\lambda_n}(\theta_n x_n), \theta_n x_n) \\ &\ge G_{\lambda_n}(\theta_n x_n) \to \infty.\end{aligned}$$

By hypothesis,

$$G_{\lambda_n}(x_n) = \int_\Omega H(t, x_n)\, dx \ge \int_\Omega H(t, \theta_n x_n)\, dt/C - \int_\Omega W(t)\, dt \to \infty.$$

Thus, (7-8) holds in any case. But

$$G_{\lambda_n}(x_n) = a(\lambda_n) \le a(\nu) < \infty,$$

Thus, $\|x_n\|_X \le C$. It now follows that for a renamed subsequence,

$$G'_\lambda(x_n) \to 0, \quad G_\lambda(x_n) \to a(\lambda) \ge a(\lambda_0).$$

Applying [Schechter 1999, Theorem 3.4.1, page 64] gives the desired solution. □

Proof of Theorem 1.7. This time we take $\lambda_0 = \alpha T^2/2\pi^2$, apply Theorem 1.3 and follow the proof of Theorem 1.6. □

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