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We construct generalized Weyman complexes for coherent sheaves on projective space and describe explicitly how the differentials depend on the differentials in the corresponding Tate resolution. We apply this to define the Weyman complex of a coherent sheaf on a projective variety and explain how certain Weyman complexes can be regarded as Fourier–Mukai transforms.

Introduction

In this paper we study the relation between the terms and the maps of two very important complexes in algebra and algebraic geometry: Tate resolutions and Weyman complexes. In the first section of the paper, we define the generalized Weyman complex of a coherent sheaf on projective space and construct an explicit functor that takes the Tate resolution of the sheaf to its generalized Weyman complex. The second section then applies this to coherent sheaves on projective varieties and Fourier–Mukai transforms.

We begin by recalling Tate resolutions and Weyman complexes.

Tate resolutions. Let K be a field of characteristic 0 and W a vector space over K of dimension $N + 1$ with dual W^* . Let $E = \bigwedge(W^*)$ be the exterior algebra of W^* and $S = \text{Sym}(W)$ the symmetric algebra of W . We grade E and S so that elements of W have degree 1 and elements of W^* have degree -1 . Thus $\text{Sym}^k(W)$ has the degree k and $\bigwedge^k(W^*)$ has the degree $-k$. It is well-known that coherent sheaves on the projective space $\mathbb{P}(W) = (W^* - \{0\})/\sim$ can be described in terms of graded S -modules. The Bernstein–Gel’fand–Gel’fand (BGG) correspondence described in [Bernšteĭn et al. 1978] (see also [Gel’fand and Manin 1988, Chapter 4, § 3; Kapranov 1983]) consists of a pair of adjoint functors \mathbf{R} and \mathbf{L} , which define an equivalence between the derived category of bounded complexes of finitely generated S -modules and the category of complexes of graded free E -modules.

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Eisenbud, Fløystad, and Schreyer [2003a] showed that the essential part of the BGG correspondence is given via the Tate resolution. Namely, a correspondence

$$\mathbf{R} : \{\text{sheaves on } \mathbb{P}(W)\} \rightarrow \{\text{complexes of } E\text{-modules}\}$$

is given by assigning to a coherent sheaf \mathcal{F} on $\mathbb{P}(W)$ a bi-infinite complex (the Tate resolution)

$$T^\bullet(\mathcal{F}) : \dots \rightarrow T^p(\mathcal{F}) \rightarrow T^{p+1}(\mathcal{F}) \rightarrow \dots, \quad p \in \mathbb{Z}$$

of free graded E -modules. By [Eisenbud et al. 2003a, Theorem 4.1] the terms of the Tate resolution $T^\bullet(\mathcal{F})$ are given by

$$T^p(\mathcal{F}) = \bigoplus_i \widehat{E}(i-p) \otimes_K H^i(\mathbb{P}(W), \mathcal{F}(p-i)),$$

where $\widehat{E} = \text{Hom}_K(E, K) = \omega_E$ is a dualizing module for E . Tate resolutions give constructive methods for computing Beilinson monads [Eisenbud et al. 2003a] and Chow forms [Eisenbud et al. 2003b], and are also related to Castelnuovo–Mumford regularity — see [Cox and Materov 2009] for an example.

The description of maps in Tate resolutions is a difficult and challenging problem. Knowledge of the maps leads to explicit formulas for sparse resultants, as in Khetan [2003; 2005]. Tate resolutions for Veronese embeddings were described in [Cox 2007], where it was shown that the maps can be given by the Bezoutian of homogeneous forms. For Segre embeddings, Cox and Materov [2008] give a description of the maps via hyperdeterminants.

Weyman complexes. We recall the Weyman complex for a resultant. Let X be an irreducible projective variety with a very ample line bundle $\mathcal{O}_X(1)$ and a vector bundle \mathcal{V} . Let $W = H^0(X, \mathcal{O}_X(1))$ and set $d = \dim(X)$. The incidence variety

$$Z = \{(f_0, \dots, f_d, x) \in W^{d+1} \times X \mid f_0(x) = \dots = f_d(x) = 0\}$$

gives the commutative diagram of projections

$$\begin{array}{ccccc} Z & & & & \\ & \searrow^{p_2} & & & \\ & & W^{d+1} \times X & \xrightarrow{\pi_2} & X \\ & \searrow^{p_1} & \downarrow \pi_1 & & \\ & & W^{d+1} & & \end{array}$$

Also let $A = \text{Sym}((W^{d+1})^*)$ be the coordinate ring of the affine space W^{d+1} . Then Weyman’s *Basic Theorem for Resultants* [2003, (9.1.2)] states that for the

free graded A -modules

$$(0-1) \quad F^p(\mathcal{V}) = \bigoplus_i \bigwedge^{i-p} K^{d+1} \otimes_K H^i(X, \mathcal{V}(p-i)) \otimes_K A(p-i)$$

there exist minimal graded differentials of degree 0,

$$F^p(\mathcal{V}) \longrightarrow F^{p+1}(\mathcal{V}),$$

such that the resulting complex $F^\bullet(\mathcal{V})$, when regarded as a complex of sheaves on W^{d+1} , satisfies

$$F^\bullet(\mathcal{V}) \simeq \mathbf{R}p_{1*}(p_2^*\mathcal{V}).$$

The complex $F^\bullet(\mathcal{V})$ has certain functorial properties [Weyman 2003, (9.1.2)] and can be used to compute resultants [Weyman 2003, (9.1.3)]. Other properties of $F^\bullet(\mathcal{V})$ are described in sections (5.1.3), (5.1.4), (5.1.6) of the same reference.

Weyman writes (0-1) as F_{-p} rather than F^p . We use F^p to emphasize the relation to the Tate resolution. The relation becomes clear when we write the Weyman complex using the projective embedding $i : X \hookrightarrow \mathbb{P}(W)$ induced by $\mathcal{O}_X(1)$. This gives the sheaf $\mathcal{F} = i_*\mathcal{V}$ on $\mathbb{P}(W)$, and (0-1) becomes

$$F^p(\mathcal{V}) = \bigoplus_i \bigwedge^{i-p} K^{d+1} \otimes_K H^i(\mathbb{P}(W), \mathcal{F}(p-i)) \otimes_K A(p-i).$$

In contrast, the Tate resolution of \mathcal{F} has

$$(0-2) \quad T^p(\mathcal{F}) = \bigoplus_i \widehat{E}(i-p) \otimes_K H^i(\mathbb{P}(W), \mathcal{F}(p-i)).$$

Our results. Our main results can be summarized as follows:

- First, we give an explicit formula for the differentials in the Weyman complex in terms of the differentials in the corresponding Tate resolution.
- Second, our construction works for *any* coherent sheaf \mathcal{F} on $\mathbb{P}(W)$ and allows us to replace W^{d+1} with W^ℓ for *any* ℓ satisfying $1 \leq \ell \leq \dim(W) - 1$.

The idea is that for $1 \leq \ell \leq \dim(W) - 1$, there is an additive functor \mathbf{W}_ℓ from the category of finitely generated graded free E -modules and homomorphisms of degree 0 such that $\mathbf{W}_\ell(\mathcal{F}) = (\mathbf{W}_\ell(T^\bullet(\mathcal{F})), \mathbf{W}_\ell(d^p))_{p \in \mathbb{Z}}$.

The statement and proof of our main result, Theorem 1.4, is the subject of Section 1. In Section 2 we define generalized Weyman complexes for irreducible projective varieties and, in the case of a vector bundle, interpret the Weyman complex as a Fourier–Mukai transform.

1. Generalized Weyman complexes

1.1. Notation. Fix an integer ℓ satisfying $1 \leq \ell \leq \dim(W) - 1$ and let

$$A_\ell = \text{Sym}((W^\ell)^*)$$

be the coordinate ring of the affine space W^ℓ . We give the polynomial ring A_ℓ the usual grading where every variable has degree 1. Note also that $W^\ell = K^\ell \otimes_K W$, so that $A_\ell = \text{Sym}(K^\ell \otimes_K W^*)$, provided we identify K^ℓ with its dual using the standard basis of K^ℓ .

1.2. The functor \mathbf{W}_ℓ . We define an additive functor \mathbf{W}_ℓ from finitely generated graded free E -modules and homomorphisms of degree 0 to finitely generated graded free A_ℓ -modules and homomorphisms of degree 0 as follows. For the free E -module $\widehat{E}(j)$, set

$$\mathbf{W}_\ell(\widehat{E}(j)) = \bigwedge^j K^\ell \otimes_K A_\ell(-j).$$

For morphisms, let H and H' be finite-dimensional K -vector spaces and let

$$\varphi : \widehat{E}(a) \otimes_K H \longrightarrow \widehat{E}(b) \otimes_K H'$$

be an E -module homomorphism of degree 0. By [Eisenbud et al. 2003a, Proposition 5.6], φ is uniquely determined by

$$\varphi_{-b} : \widehat{E}(a)_{-b} \otimes_K H = \bigwedge^{a-b} W \otimes_K H \longrightarrow \widehat{E}(b)_{-b} \otimes_K H' = H'.$$

We define

$$(1-1) \quad \mathbf{W}_\ell(\varphi) : \mathbf{W}_\ell(\widehat{E}(a)) \longrightarrow \mathbf{W}_\ell(\widehat{E}(b))$$

as follows. Recall the comultiplication map

$$\Phi : \bigwedge^a K^\ell \longrightarrow \bigwedge^b K^\ell \otimes_K \bigwedge^{a-b} K^\ell$$

and the injective map

$$\Psi : \bigwedge^{a-b} K^\ell \otimes_K \bigwedge^{a-b} W^* \longrightarrow \text{Sym}^{a-b}(K^\ell \otimes_K W^*) = (A_\ell)_{a-b}$$

coming from the Schur functor decomposition of $\text{Sym}^{a-b}(K^\ell \otimes_K W^*)$ (see [Weyman 2003, (2.3.3)]). Finally, φ_{-b} determines $\widetilde{\varphi}_{-b} : H \rightarrow \bigwedge^{a-b} W^* \otimes_K H'$. Putting these maps together, we get the composition

$$(1-2) \quad \begin{array}{ccc} \bigwedge^a K^\ell \otimes_K H & \xrightarrow{\Phi \otimes \widetilde{\varphi}_{-b}} & \bigwedge^b K^\ell \otimes_K \bigwedge^{a-b} K^\ell \otimes_K \bigwedge^{a-b} W^* \otimes_K H' \\ & & \downarrow 1 \otimes \Psi \otimes 1 \\ & & \bigwedge^b K^\ell \otimes_K (A_\ell)_{a-b} \otimes_K H', \end{array}$$

which in turn determines an A_ℓ -module homomorphism of degree 0,

$$\mathbf{W}_\ell(\varphi) : \bigwedge^a K^\ell \otimes_K H \otimes_K A_\ell(-a) \longrightarrow \bigwedge^b K^\ell \otimes_K H' \otimes_K A_\ell(-b).$$

This is the desired map (1-1). Note that Ψ is given explicitly by

$$v_1 \wedge \cdots \wedge v_{a-b} \otimes w_1 \wedge \cdots \wedge w_{a-b} \mapsto \sum_{\sigma \in \mathcal{S}_{a-b}} \text{sgn}(\sigma) v_1 \otimes w_{\sigma(1)} \cdots v_{a-b} \otimes w_{\sigma(a-b)},$$

and Φ is given by the formula

$$v_1 \wedge \cdots \wedge v_a \mapsto \sum \text{sgn}(I, I') v_I \otimes v_{I'},$$

where $v_I = v_{i_1} \wedge \cdots \wedge v_{i_p}$, $I = \{i_1, \dots, i_p\} \subset \{1, \dots, a\}$, $\text{sgn}(I, I')$ is the sign of the permutation whose bottom row is (I, I') , and $I' = \{1, \dots, a\} \setminus I$ is the complement to I . It follows that we have a completely explicit formula for $\mathbf{W}_\ell(\varphi)$.

1.3. Generalized Weyman complexes. We now define the generalized Weyman complex of a coherent sheaf on $\mathbb{P}(W)$.

Definition 1.1. Let $T^\bullet(\mathcal{F}) = (T^p, d^p)_{p \in \mathbb{Z}}$ be the Tate resolution of a coherent sheaf \mathcal{F} on $\mathbb{P}(W)$. Then the ℓ -th Weyman complex of \mathcal{F} is the complex of free graded A_ℓ -modules given by

$$\mathbf{W}_\ell^\bullet(\mathcal{F}) = (\mathbf{W}_\ell(T^p), \mathbf{W}_\ell(d^p))_{p \in \mathbb{Z}}.$$

Using (0-2), it follows that

$$\mathbf{W}_\ell^p(\mathcal{F}) = \bigoplus_i \bigwedge^{i-p} K^\ell \otimes_K H^i(\mathbb{P}(W), \mathcal{F}(p-i)) \otimes_K A_\ell(p-i).$$

Note also that $\mathbf{W}_\ell^\bullet(\mathcal{F})$ is a complex by functoriality and is minimal since the Tate resolution $T^\bullet(\mathcal{F})$ is minimal. In contrast to the bi-infinite Tate resolution, however, the Weyman complex is finite. More precisely, one sees easily that

$$\mathbf{W}_\ell^p(\mathcal{F}) = 0 \quad \text{whenever } p < -\ell \text{ or } p > \dim(\text{supp}(\mathcal{F})).$$

Remark 1.2. Our results about $\mathbf{W}_\ell^\bullet(\mathcal{F})$ are parallel (and were inspired by) the properties of the functor \mathbf{U}_ℓ^\bullet introduced by Eisenbud and Schreyer [2003b]. They define this functor on graded free E -modules by

$$\mathbf{U}_\ell(\widehat{E}(j)) = \bigwedge^j U_\ell,$$

where U_ℓ is the tautological subbundle on the Grassmannian \mathbb{G}_ℓ of codimension ℓ subspaces of $\mathbb{P}(W)$. When applied to the Tate resolution of a coherent sheaf \mathcal{F} on $\mathbb{P}(W)$, this gives the complex $\mathbf{U}_\ell^\bullet(\mathcal{F})$ of locally free sheaves on \mathbb{G}_ℓ defined in [Eisenbud et al. 2003b]. As we develop the properties of \mathbf{W}_ℓ^\bullet , we will explain how our functor relates to the corresponding properties of \mathbf{U}_ℓ^\bullet .

Our first result, similar to [Eisenbud et al. 2003b, Proposition 1.3], shows that \mathcal{F} is uniquely determined by $\mathbf{W}_\ell^\bullet(\mathcal{F})$ provided ℓ is sufficiently large.

Proposition 1.3. *Let \mathcal{F} be a coherent sheaf on $\mathbb{P}(W)$. If $\ell > \dim(\text{supp}(\mathcal{F}))$, then \mathcal{F} is determined by the complex $\mathbf{W}_\ell^\bullet(\mathcal{F})$.*

Proof. A map $\varphi: \widehat{E}(a) \otimes_K H \rightarrow \widehat{E}(b) \otimes_K H'$ gives $\mathbf{W}_\ell(\varphi): \mathbf{W}_\ell(\widehat{E}(a)) \rightarrow \mathbf{W}_\ell(\widehat{E}(b))$. We claim that $\varphi \mapsto \mathbf{W}_\ell(\varphi)$ is injective when $0 \leq b \leq a \leq \ell$.

To prove this, assume that $\mathbf{W}_\ell(\varphi)$ is trivial. This implies that the map defined in (1-2) vanishes. Since Ψ is injective, it follows that

$$\Phi \otimes \tilde{\varphi}_{-b}: \bigwedge^a K^\ell \otimes_K H \rightarrow \bigwedge^b K^\ell \otimes_K \bigwedge^{a-b} K^\ell \otimes_K \bigwedge^{a-b} W^* \otimes_K H'$$

is trivial, which in turn implies that

$$\Phi \otimes \varphi_{-b}: \bigwedge^a K^\ell \otimes_K \bigwedge^{a-b} W \otimes_K H \rightarrow \bigwedge^b K^\ell \otimes_K \bigwedge^{a-b} K^\ell \otimes_K H'$$

is trivial. Then the commutative diagram

$$\begin{array}{ccc} \bigwedge^a K^\ell \otimes_K \bigwedge^{a-b} W \otimes_K H & \xrightarrow{\Phi \otimes 1 \otimes 1} & \bigwedge^b K^\ell \otimes_K \bigwedge^{a-b} K^\ell \otimes_K \bigwedge^{a-b} W \otimes_K H \\ \downarrow 1 \otimes \varphi_{-b} & & \downarrow 1 \otimes 1 \otimes \varphi_{-b} \\ \bigwedge^a K^\ell \otimes_K H' & \xrightarrow{\Phi \otimes 1} & \bigwedge^b K^\ell \otimes_K \bigwedge^{a-b} K^\ell \otimes_K H' \end{array}$$

and the injectivity of Φ (this uses $0 \leq b \leq a \leq \ell$) show that $1 \otimes \varphi_{-b} = 0$. It follows that φ_{-b} and hence φ are trivial, as claimed.

Now we argue as in the proof of [Eisenbud et al. 2003b, Proposition 1.3]. Set $d = \dim(\text{supp}(\mathcal{F}))$, so that $T^{-1}(\mathcal{F}) \rightarrow T^0(\mathcal{F})$ looks like

$$\begin{array}{ccc} \widehat{E}(d+1) \otimes_K H^d(\mathbb{P}(W), \mathcal{F}(-d-1)) & \longrightarrow & \widehat{E}(d) \otimes_K H^d(\mathbb{P}(W), \mathcal{F}(-d)) \\ \oplus & \searrow & \oplus \\ \vdots & \searrow & \vdots \\ \oplus & \searrow & \oplus \\ \widehat{E}(1) \otimes_K H^0(\mathbb{P}(W), \mathcal{F}(-1)) & \longrightarrow & \widehat{E}(0) \otimes_K H^0(\mathbb{P}(W), \mathcal{F}(0)). \end{array}$$

Since $\ell > d = \dim(\text{supp}(\mathcal{F}))$, all of the maps φ in this diagram satisfy $0 \leq a \leq b \leq \ell$, which means that these maps are determined uniquely by the map

$$\mathbf{W}_\ell^{-1}(\mathcal{F}) \rightarrow \mathbf{W}_\ell^0(\mathcal{F})$$

in the Weyman complex. Thus the Weyman complex of \mathcal{F} determines

$$T^{-1}(\mathcal{F}) \rightarrow T^0(\mathcal{F}),$$

which in turn determines \mathcal{F} by usual properties of the Tate resolution (see the proof of [Eisenbud et al. 2003b, Proposition 1.3] for the details). \square

1.4. Main theorem. Now consider the incidence variety

$$Z_\ell = \{(f_1, \dots, f_\ell, x) \in W^\ell \times \mathbb{P}(W) \mid f_1(x) = \dots = f_\ell(x) = 0\}$$

and the commutative diagram of projections

$$(1-3) \quad \begin{array}{ccc} Z_\ell & & \\ & \searrow^{p_2} & \\ & & \mathbb{P}(W) \\ & \searrow^{p_1} & \swarrow^{\pi_2} \\ & & W^\ell \times \mathbb{P}(W) & \xrightarrow{\pi_2} & \mathbb{P}(W) \\ & & \downarrow \pi_1 & & \\ & & W^\ell & & \end{array}$$

Here is the main theorem of this section.

Theorem 1.4. *Let \mathcal{F} be a coherent sheaf on $\mathbb{P}(W)$ and assume $1 \leq \ell \leq \dim(W) - 1$. Then $\mathbf{W}_\ell^\bullet(\mathcal{F})$, if regarded as a complex of sheaves on W^ℓ , represents $\mathbf{R}p_{1*}(p_2^*\mathcal{F})$ in the derived category of coherent sheaves on W^ℓ .*

Before we can prove this theorem, however, we need some preliminary work.

1.5. A special Weyman complex. First observe that since $W^\ell = \text{Spec}(A_\ell)$ is affine, $\mathbf{R}p_{1*}(p_2^*\mathcal{F})$ can be regarded as a complex of sheaves that computes the cohomology groups $H^q(Z_\ell, p_2^*\mathcal{F})$. The natural action of $K^\times = K \setminus \{0\}$ on W^ℓ (and corresponding trivial action on $\mathbb{P}(W)$) makes $H^q(Z_\ell, p_2^*\mathcal{F})$ into a graded A_ℓ -module. Here is a special case where this module is known explicitly.

Proposition 1.5. *If ℓ satisfies $1 \leq \ell \leq \dim(W) - 1$, then the graded A_ℓ -module $H^q(Z_\ell, p_2^*\Omega_{\mathbb{P}(W)}^a)$ is given by*

$$H^q(Z_\ell, p_2^*\Omega_{\mathbb{P}(W)}^a) = \begin{cases} \bigwedge^a K^\ell \otimes_K A_\ell(-a) = \mathbf{W}_\ell(\widehat{E}(a)) & q = 0 \\ 0 & q > 0. \end{cases}$$

Proof. The proof of [Weyman 2003, (5.1.2)] is easily adapted to show that

$$H^q(Z_\ell, p_2^*\Omega_{\mathbb{P}(W)}^a)$$

is the q -th cohomology of the complex

$$(1-4) \quad \begin{aligned} & \dots \rightarrow \bigoplus_{i \geq 0} \bigwedge^{i-q} K^\ell \otimes_K H^i(\mathbb{P}(W), \Omega_{\mathbb{P}(W)}^a(a+q-i)) \otimes_K A_\ell(q-i) \rightarrow \\ & \bigoplus_{i \geq 0} \bigwedge^{i-q-1} K^\ell \otimes_K H^i(\mathbb{P}(W), \Omega_{\mathbb{P}(W)}^a(a+q+1-i)) \otimes_K A_\ell(q+1-i) \rightarrow \dots \end{aligned}$$

Note $\dim(\mathbb{P}(W)) = N$ since $\dim(W) = N + 1$. The Bott vanishing theorem (proved in many papers, including [Huang 2001]) states that

$$H^i(\mathbb{P}(W), \Omega_{\mathbb{P}(W)}^a(k)) = 0$$

whenever

$$i \notin \{0, a, N\} \text{ or } i = 0, k \leq a \text{ or } i = a, k \neq 0 \text{ or } i = N, k \geq a - N.$$

Then the general q -th term of the complex above for $i \in \{0, a, N\}$ reduces to

$$\begin{aligned} & \bigwedge^{-q} K^\ell \otimes_K H^0(\mathbb{P}(W), \Omega_{\mathbb{P}(W)}^a(a+q)) \otimes_K A_\ell(q) \\ & \bigoplus \bigwedge^{a-q} K^\ell \otimes_K H^a(\mathbb{P}(W), \Omega_{\mathbb{P}(W)}^a(q)) \otimes_K A_\ell(q-a) \\ & \bigoplus \bigwedge^{N-q} K^\ell \otimes_K H^N(\mathbb{P}(W), \Omega_{\mathbb{P}(W)}^a(a+q-N)) \otimes_K A_\ell(q-N). \end{aligned}$$

We have the following cases with respect to the different i :

- If $i = 0$, then either $\bigwedge^{-q} K^\ell = 0$ for $q > 0$, or $H^0(\mathbb{P}(W), \Omega_{\mathbb{P}(W)}^a(a+q)) = 0$ for $q \leq 0$ from the Bott vanishing theorem. Therefore, the first summand is zero for all q .
- If $i = a$, then the Bott vanishing theorem implies

$$\begin{aligned} & \bigwedge^{a-q} K^\ell \otimes_K H^a(\mathbb{P}(W), \Omega_{\mathbb{P}(W)}^a(q)) \otimes_K A_\ell(q-a) \\ & = \begin{cases} \bigwedge^a K^\ell \otimes_K H^a(\mathbb{P}(W), \Omega_{\mathbb{P}(W)}^a(q)) \otimes_K A_\ell(-a) & \text{if } q = 0, \\ 0 & \text{if } q \neq 0. \end{cases} \end{aligned}$$

- If $i = N$, then $\bigwedge^{N-q} K^\ell = 0$ for $q < 0$ because we assume that

$$\ell \leq \dim(W) - 1 = N.$$

If $q \geq 0$, then the Bott vanishing theorem says that

$$H^N(\mathbb{P}(W), \Omega_{\mathbb{P}(W)}^a(a+q-N)) = 0.$$

Our conclusion is that the complex reduces to

$$\cdots \longrightarrow \bigwedge^a K^\ell \otimes_K H^a(\mathbb{P}(W), \Omega_{\mathbb{P}(W)}^a(q)) \otimes_K A_\ell(q-a) \longrightarrow \cdots,$$

where the nonzero module occurs where $q = 0$. Since $H^a(\mathbb{P}(W), \Omega_{\mathbb{P}(W)}^a(q)) = K$, it follows that

$$H^q(Z_\ell, p_2^* \Omega_{\mathbb{P}(W)}^a(a)) = \begin{cases} \bigwedge^a K^\ell \otimes_K A_\ell(-a) = \mathbf{W}_\ell(\widehat{E}(a)) & \text{if } q = 0, \\ 0 & \text{if } q > 0. \end{cases} \quad \square$$

Corollary 1.6. *The derived functor $\mathbf{R}p_{1*}(p_2^*\Omega_{\mathbb{P}(W)}^a(a))$ is canonically represented by the complex of graded A_ℓ -modules consisting of $\mathbf{W}_\ell(\widehat{E}(a))$ concentrated in degree 0. In other words,*

$$\mathbf{R}p_{1*}(p_2^*\Omega_{\mathbb{P}(W)}^a(a)) = p_{1*}(p_2^*\Omega_{\mathbb{P}(W)}^a(a)) = \mathbf{W}_\ell(\widehat{E}(a)).$$

1.6. Homomorphisms. Proposition 1.5 gives isomorphisms

$$\gamma^a : \mathbf{W}_\ell(\widehat{E}(a)) \simeq H^0(Z_\ell, p_2^*\Omega_{\mathbb{P}(W)}^a(a)), \quad a \geq 0.$$

We next explore how these isomorphisms interact with the homomorphisms $\mathbf{W}_\ell(\varphi)$ from (1-1). The simplest case begins with an element

$$\psi \in \wedge^{a-b} W^*,$$

which gives two maps as follows. To construct the first map, note that ψ induces a map $\varphi_{-b} : \wedge^{a-b} W \rightarrow K$, which gives an E -module homomorphism of degree 0,

$$\varphi : \widehat{E}(a) \longrightarrow \widehat{E}(b).$$

Applying (1-1), we obtain a graded A_ℓ -module homomorphism

$$\mathbf{W}_\ell(\varphi) : \mathbf{W}_\ell(\widehat{E}(a)) \longrightarrow \mathbf{W}_\ell(\widehat{E}(b)).$$

This is the first map. For the second map, we recall the standard exact sequence

$$(1-5) \quad 0 \longrightarrow \Omega_{\mathbb{P}(W)}^1(1) \longrightarrow W \otimes \mathcal{O}_{\mathbb{P}(W)} \longrightarrow \mathcal{O}_{\mathbb{P}(W)}(1) \longrightarrow 0.$$

From contraction with ψ , we get a sheaf morphism

$$\widetilde{\psi} : \Omega_{\mathbb{P}(W)}^a(a) \longrightarrow \Omega_{\mathbb{P}(W)}^b(b).$$

This gives a graded A_ℓ -module homomorphism

$$\widehat{\psi} : H^0(Z_\ell, p_2^*\Omega_{\mathbb{P}(W)}^a(a)) \longrightarrow H^0(Z_\ell, p_2^*\Omega_{\mathbb{P}(W)}^b(b)),$$

which is the second map. These maps are related as follows.

Proposition 1.7. *The maps above fit into a commutative diagram:*

$$\begin{array}{ccc} H^0(Z_\ell, p_2^*\Omega_{\mathbb{P}(W)}^a(a)) & \xrightarrow{\widehat{\psi}} & H^0(Z_\ell, p_2^*\Omega_{\mathbb{P}(W)}^b(b)) \\ \gamma^a \uparrow & & \uparrow \gamma^b \\ \mathbf{W}_\ell(\widehat{E}(a)) & \xrightarrow{\mathbf{W}_\ell(\varphi)} & \mathbf{W}_\ell(\widehat{E}(b)). \end{array}$$

Proving this will require an intrinsic description of the graded structure of

$$H^0(Z_\ell, p_2^*\Omega_{\mathbb{P}(W)}^a(a))$$

in terms of Schur functors.

1.7. Graded structures and Schur functors. For simplicity, we write the exact sequence (1-5) as

$$0 \longrightarrow \mathcal{R} \longrightarrow W \otimes \mathbb{C}_{\mathbb{P}(W)} \longrightarrow \mathcal{Q} \longrightarrow 0.$$

For simplicity, we will use \otimes to denote both $\otimes_{\mathbb{C}_{\mathbb{P}(W)}}$ (for sheaves) and \otimes_K (for vector spaces) in the remainder of the paper. The meaning of \otimes should be clear from the context.

One easily sees that $Z_\ell \subseteq W^\ell \times \mathbb{P}(W)$ is the total space of the vector bundle $\mathcal{R}^\ell = K^\ell \otimes \mathcal{R}$, and the direct image $p_{2*}(\mathbb{C}_{Z_\ell})$ can be identified with the sheaves of algebras $\text{Sym}((\mathcal{R}^\ell)^*) \cong \text{Sym}(K^\ell \otimes \mathcal{R}^*)$ (see [Weyman 2003, Proposition 5.1.1(b)]). Then the projection formula yields

$$\begin{aligned} H^0(Z_\ell, p_2^* \Omega_{\mathbb{P}(W)}^a(a)) &\simeq H^0(\mathbb{P}(W), p_{2*}(p_2^* \Omega_{\mathbb{P}(W)}^a(a))) \\ &= H^0(\mathbb{P}(W), \text{Sym}(K^\ell \otimes \mathcal{R}^*) \otimes \Omega_{\mathbb{P}(W)}^a(a)) \\ &= \bigoplus_{k \geq 0} H^0(\mathbb{P}(W), \text{Sym}^k(K^\ell \otimes \mathcal{R}^*) \otimes \Omega_{\mathbb{P}(W)}^a(a)). \end{aligned}$$

We identify K^ℓ with its dual using the standard basis. Hence $H^0(Z_\ell, p_2^* \Omega_{\mathbb{P}(W)}^a(a))$ is the graded A_ℓ -module whose graded piece in degree k is

$$(1-6) \quad H^0(Z_\ell, p_2^* \Omega_{\mathbb{P}(W)}^a(a))_k = H^0(\mathbb{P}(W), \text{Sym}^k(K^\ell \otimes \mathcal{R}^*) \otimes \Omega_{\mathbb{P}(W)}^a(a)).$$

Note that $\mathcal{R} = \Omega_{\mathbb{P}(W)}^1(1)$ has rank N since $\dim(W) = N + 1$. Combining this with $\mathcal{Q} = \mathbb{C}_{\mathbb{P}(W)}(1)$, we obtain

$$\Omega_{\mathbb{P}(W)}^a(a) = \bigwedge^a \mathcal{R} \simeq \bigwedge^{N-a} \mathcal{R}^* \otimes \bigwedge^N \mathcal{R} \simeq \bigwedge^{N-a} \mathcal{R}^* \otimes \mathcal{Q}^*,$$

where the second equality is standard duality and the third follows from

$$\bigwedge^N \mathcal{R} = \Omega_{\mathbb{P}(W)}^N(N) = \mathbb{C}_{\mathbb{P}(W)}(-N - 1 + N) = \mathbb{C}_{\mathbb{P}(W)}(-1) = \mathcal{Q}^*.$$

Hence the graded piece (1-6) is

$$(1-7) \quad H^0(Z_\ell, p_2^* \Omega_{\mathbb{P}(W)}^a(a))_k = H^0(\mathbb{P}(W), \text{Sym}^k(K^\ell \otimes \mathcal{R}^*) \otimes \bigwedge^{N-a} \mathcal{R}^* \otimes \mathcal{Q}^*).$$

We use Schur functors to decompose (1-7), based on [Weyman 2003, Chapter 2]. In Weyman's [2003] notation, the Schur functor of a partition λ is denoted L_λ . This is not consistent with [Fulton and Harris 1991], which parametrizes Schur functors using conjugate partitions. Weyman's treatment also uses Weyl functors, which he denotes K_λ . Fortunately, since we are in characteristic 0, there is a canonical isomorphism $K_\lambda \simeq L_{\lambda'}$, where λ' is the conjugate partition of λ .

We will also need this consequence of Weyman's version [2003, Chapter 4] of the Bott theorem:

Lemma 1.8. *Given a partition π , we have*

$$H^q(\mathbb{P}(W), L_\pi(\mathcal{R}^*) \otimes \mathcal{Q}^*) = 0, \quad q > 0$$

and

$$H^0(\mathbb{P}(W), L_\pi(\mathcal{R}^*) \otimes \mathcal{Q}^*) = \begin{cases} L_\mu(W^*) & \text{if } \pi = (N, \mu), \mu = (a_2, \dots, a_s), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Write $\pi = (a_1, \dots, a_s)$. If $a_1 > N = \text{rank}(\mathcal{R}^*)$, then $L_\pi(\mathcal{R}^*) = 0$ and we are done. Hence we may assume $a_1 \leq N$. This allows us to write the conjugate partition as

$$\pi' = (b_1, \dots, b_N), \quad b_1 \geq \dots \geq b_N \geq 0.$$

If we set $\alpha = (b_1, \dots, b_N, 1)$, then Weyman [2003, p. 115] defines the vector bundle

$$\mathcal{V}(\alpha) = K_{\pi'}(\mathcal{R}^*) \otimes \mathcal{Q}^*.$$

Using the relation between Weyl and Schur functors noted above, we write this as

$$\mathcal{V}(\alpha) = L_\pi(\mathcal{R}^*) \otimes \mathcal{Q}^*.$$

There are now two cases to consider.

Case 1: If $a_1 < N$, then $b_N = 0$, so that $\alpha = (b_1, \dots, b_{N-1}, 0, 1)$. Then part (1) of [Weyman 2003, (4.1.8)] implies that $H^q(\mathbb{P}(W), L_\pi(\mathcal{R}^*) \otimes \mathcal{Q}^*) = 0$ for all $q \geq 0$. See also the discussion of item (1) of [Weyman 2003, (4.1.5)].

Case 2: If $a_1 = N$, then $b_N \geq 1$, so that α is nonincreasing. Here, part (2) of [Weyman 2003, (4.1.8) and (4.1.4)] imply that $H^q(\mathbb{P}(W), L_\pi(\mathcal{R}^*) \otimes \mathcal{Q}^*) = 0$ for $q > 0$ and that

$$H^0(\mathbb{P}(W), L_\pi(\mathcal{R}^*) \otimes \mathcal{Q}^*) = K_{(b_1-1, \dots, b_N-1)}(W^*) \otimes (\bigwedge^{N+1} W)^{\otimes (-1)}.$$

One easily sees that $(b_1 - 1, \dots, b_N - 1)' = (a_2, \dots, a_s) = \mu$. Hence

$$K_{(b_1-1, \dots, b_N-1)} = L_{(b_1-1, \dots, b_N-1)'} = L_\mu,$$

and the lemma follows since $\bigwedge^{N+1} W \simeq K$. \square

We now decompose (1-7) by writing $\text{Sym}^k(K^\ell \otimes \mathcal{R}^*) \otimes \bigwedge^{N-a} \mathcal{R}^*$ in terms of Schur functors and then apply Lemma 1.8. We begin with the Cauchy formula

$$\text{Sym}^k(K^\ell \otimes \mathcal{R}^*) = \bigoplus_{|\lambda|=k} L_\lambda(K^\ell) \otimes L_\lambda(\mathcal{R}^*),$$

where the direct sum is over all partitions λ of k [Weyman 2003, (2.3.3)]. Then the Pieri formula gives

$$L_\lambda(\mathcal{R}^*) \otimes \bigwedge^{N-a} \mathcal{R}^* = \bigoplus_{\pi \in Y(\lambda, N-a)} L_\pi(\mathcal{R}^*),$$

where the direct sum is over all partitions $\pi \in Y(\lambda, N - a)$ whose Young diagram is obtained by adding $N - a$ boxes to the Young diagram of λ , with no two boxes in the same column [Weyman 2003, (2.3.5)]. Combining these, we have

$$(1-8) \quad \text{Sym}^k(K^\ell \otimes \mathcal{R}^*) \otimes \bigwedge^{N-a} \mathcal{R}^* \otimes \mathcal{Q}^* \\ = \bigoplus_{|\lambda|=k} \left(\bigoplus_{\pi \in Y(\lambda, N-a)} L_\lambda(K^\ell) \otimes L_\pi(\mathcal{R}^*) \otimes \mathcal{Q}^* \right).$$

Taking global sections and applying Lemma 1.8, we get a Schur functor decomposition of the graded piece (1-7).

When we do this, Lemma 1.8 tells us that we need only consider partitions λ of k such that adding $N - a$ boxes to a partition λ gives a partition π of the form $\pi = (N, a_2, \dots, a_s)$. This has some nice consequences for the graded pieces (1-7):

- If $k < a$, then adding $N - a$ boxes to λ gives a partition π of $k + N - a < N$, so that π is never of the form $\pi = (N, a_2, \dots, a_s)$. Then (1-8) and Lemma 1.8 easily imply that the graded piece (1-7) in degree $k < a$ vanishes, that is,

$$H^0(Z_\ell, p_2^* \Omega_{\mathbb{P}(W)}^a(a))_k = 0.$$

- If $k = a$, then adding $N - a$ boxes to λ gives a partition π of $a + N - a = N$, so that π is of the form $\pi = (N, a_2, \dots, a_s)$ if and only if $\lambda = (a)$ and $\pi = (N)$. Then (1-8) and Lemma 1.8 imply that the graded piece in degree a is

$$\begin{aligned} H^0(Z_\ell, p_2^* \Omega_{\mathbb{P}(W)}^a(a))_a &= H^0(\mathbb{P}(W), L_{(a)}(K^\ell) \otimes L_{(N)}(\mathcal{R}^*) \otimes \mathcal{Q}^*) \\ &= L_{(a)}(K^\ell) \otimes H^0(\mathbb{P}(W), L_{(N)}(\mathcal{R}^*) \otimes \mathcal{Q}^*) \\ &= \bigwedge^a K^\ell \otimes L_{(0)}(W^*) = \bigwedge^a K^\ell, \end{aligned}$$

$$\text{since } L_{(a)}(K^\ell) = \bigwedge^a K^\ell \text{ and } L_{(0)}(W^*) = \bigwedge^0 W^* = K.$$

These bullets explain the shift by a in the isomorphism

$$\begin{aligned} H^0(Z_\ell, p_2^* \Omega_{\mathbb{P}(W)}^a(a)) &\simeq \bigoplus_{k \geq 0} H^0(\mathbb{P}(W), \text{Sym}^k(K^\ell \otimes \mathcal{R}^*) \otimes \bigwedge^{N-a} \mathcal{R}^* \otimes \mathcal{Q}^*) \\ &\simeq \bigwedge^a K^\ell \otimes A_\ell(-a) = \mathbf{W}_\ell(\widehat{E}(a)) \end{aligned}$$

from Proposition 1.5.

1.8. We turn to the proof of Proposition 1.7 For

$$\gamma^a : \mathbf{W}_\ell(\widehat{E}(a)) \simeq H^0(Z_\ell, p_2^* \Omega_{\mathbb{P}(W)}^a(a)),$$

the analysis above shows that the degree k component of γ^a is an isomorphism

$$\gamma_k^a : \bigwedge^a K^\ell \otimes (A_\ell)_{k-a} \simeq H^0(\mathbb{P}(W), \text{Sym}^k(K^\ell \otimes \mathcal{R}^*) \otimes \bigwedge^{N-a} \mathcal{R}^* \otimes \mathcal{Q}^*).$$

Furthermore, Lemma 1.8 and (1-8) give a Schur functor decomposition of the right-hand side, and the Cauchy and Pieri formulas also give a Schur functor decomposition of the left-hand side, since $(A_\ell)_{k-a} = \text{Sym}^{k-a}(K^\ell \otimes W^*)$. The isomorphisms γ_k^a are clearly compatible with these decompositions.

Proof of Proposition 1.7. Since $\mathbf{W}_\ell(\widehat{E}(a))$ is generated in degree a , it suffices to show that the diagram commutes in degree a . Using (1-7), the map

$$\tilde{\varphi}_{-b} : K \rightarrow \bigwedge^{a-b} W^*$$

induced by ψ , and the maps Φ and Ψ from diagram (1-2), we can write this as

$$\begin{array}{ccc} H^0(\text{Sym}^a(K^\ell \otimes \mathcal{R}^*) \otimes \bigwedge^{N-a} \mathcal{R}^* \otimes \mathcal{Q}^*) & \xrightarrow{\widehat{\psi}} & H^0(\text{Sym}^a(K^\ell \otimes \mathcal{R}^*) \otimes \bigwedge^{N-b} \mathcal{R}^* \otimes \mathcal{Q}^*) \\ \uparrow \gamma_a^a & \nearrow \mathbf{W}_\ell(\varphi) & \uparrow \gamma_a^b \\ \bigwedge^a K^\ell & \xrightarrow{\Phi \otimes \tilde{\varphi}_{-b}} & \bigwedge^b K^\ell \otimes \bigwedge^{a-b} K^\ell \otimes \bigwedge^{a-b} W^* \\ & & \uparrow 1 \otimes \Psi \\ & & \bigwedge^b K^\ell \otimes \text{Sym}^{a-b}(K^\ell \otimes W^*) \end{array}$$

We have simplified notation by omitting $\mathbb{P}(W)$ when we write sheaf cohomology.

To prove commutativity, first recall that in the Schur functor decomposition of the cohomology group on the top left, the only partition λ that appears is $\lambda = (a)$ and that the top left cohomology group can be replaced with

$$H^0(\text{Sym}^a(K^\ell \otimes \mathcal{R}^*) \otimes \bigwedge^{N-a} \mathcal{R}^* \otimes \mathcal{Q}^*) = \bigwedge^a K^\ell.$$

This maps to the $\lambda = (a)$ part of the cohomology group on the top right, where the only element $\pi \in Y(\lambda, N-b)$ of the form $\pi = (N, a_2, \dots, a_s)$ is $\pi = (N, a-b)$. Hence the top right cohomology can be replaced with

$$H^0(\bigwedge^a K^\ell \otimes L_{(N, a-b)}(\mathcal{R}^*) \otimes \mathcal{Q}^*) = \bigwedge^a K^\ell \otimes L_{(a-b)}(W^*) = \bigwedge^a K^\ell \otimes \bigwedge^{a-b} W^*.$$

The map between these cohomology groups is clearly $1 \otimes \tilde{\varphi}_{-b}$. Furthermore, one sees that the only way the partition $\lambda = (a)$ occurs in the Schur decomposition of

$$\bigwedge^b K^\ell \otimes \text{Sym}^{a-b}(K^\ell \otimes W^*)$$

is via the map

$$\begin{array}{ccc} \bigwedge^b K^\ell \otimes \text{Sym}^{a-b}(K^\ell \otimes W^*) & & \\ \uparrow 1 \otimes \Psi & & \\ \bigwedge^a K^\ell \otimes \bigwedge^{a-b} W^* & \xrightarrow{\Phi \otimes 1} & \bigwedge^b K^\ell \otimes \bigwedge^{a-b} K^\ell \otimes \bigwedge^{a-b} W^* \end{array}$$

From here, the desired commutativity follows easily. \square

1.9. Proof of the main theorem. We now have all of the tools needed for our main result.

Proof of Theorem 1.4. The key idea of the proof (taken from [Eisenbud et al. 2003b, Theorem 1.2]) is to replace a coherent sheaf \mathcal{F} on $\mathbb{P}(W)$ with its Beilinson monad, which is the complex of sheaves $B^\bullet(\mathcal{F})$ on $\mathbb{P}(W)$ with

$$B^p(\mathcal{F}) = \bigoplus_i H^i(\mathcal{F}(p-i)) \otimes \Omega_{\mathbb{P}(W)}^{i-p}(i-p),$$

and differentials coming from the corresponding differentials in the Tate resolution via the correspondence

$$\varphi : \widehat{E}(a) \otimes H \rightarrow \widehat{E}(b) \otimes H' \longmapsto \widetilde{\psi} : \Omega_{\mathbb{P}(W)}^a(a) \otimes H \rightarrow \Omega_{\mathbb{P}(W)}^b(b) \otimes H'$$

defined as follows: $\varphi : \widehat{E}(a) \otimes H \rightarrow \widehat{E}(b) \otimes H'$ induces $\psi : H \rightarrow \bigwedge^{a-b} W^* \otimes H'$, and contraction with ψ gives $\widetilde{\psi} : \Omega_{\mathbb{P}(W)}^a(a) \otimes H \rightarrow \Omega_{\mathbb{P}(W)}^b(b) \otimes H'$. Theorem 6.1 of [Eisenbud et al. 2003a] is Beilinson's result that $B^\bullet(\mathcal{F})$ is exact except at $p=0$, where the homology is \mathcal{F} . It follows that

$$\mathbf{R}p_{1*}(p_2^*\mathcal{F}) = \mathbf{R}p_{1*}(p_2^*B^\bullet(\mathcal{F}))$$

in the derived category. Since $B^\bullet(\mathcal{F})$ consists of direct sums of sheaves $\Omega_{\mathbb{P}(W)}^a(a)$, Corollary 1.6 implies that

$$\mathbf{R}p_{1*}(p_2^*B^\bullet(\mathcal{F})) = p_{1*}(p_2^*B^\bullet(\mathcal{F})),$$

where $p_{1*}(p_2^*B^p(\mathcal{F}))$ is the sheaf associated to the graded A_ℓ -module

$$\bigoplus_i H^i(\mathcal{F}(p-i)) \otimes \mathbf{W}_\ell(\widehat{E}(i-p)) = \mathbf{W}_\ell(T^p(\mathcal{F})).$$

Furthermore, Proposition 1.7 easily implies that the differentials in this complex come from the Tate resolution via the \mathbf{W}_ℓ functor. This completes the proof of the theorem. \square

This theorem implies the following uniqueness result for our generalized Weyman complexes.

Corollary 1.9. *The complex $\mathbf{W}_\ell^\bullet(\mathcal{F})$ is the unique minimal free complex quasi-isomorphic to $\mathbf{R}p_{1*}(p_2^*B^\bullet(\mathcal{F}))$.*

Proof. The proof is similar to the proof of [Weyman 2003, (5.2.5)]. \square

2. Properties of generalized Weyman complexes

In Section 1, we worked with coherent sheaves on projective space, yet Weyman complexes were originally defined for vector bundles on projective varieties. Fortunately, the theory of Section 1 adapts easily to projective varieties.

2.1. Generalized Weyman complexes for irreducible projective varieties. Let X be an irreducible projective variety with a very ample line bundle $\mathcal{O}_X(1)$. Let $W = H^0(X, \mathcal{O}_X(1))$ and fix an integer $1 \leq \ell \leq \dim(W) - 1$. The incidence variety

$$Z_\ell = \{(f_1, \dots, f_\ell, x) \in W^\ell \times X \mid f_1(x) = \dots = f_\ell(x) = 0\}$$

gives the commutative diagram of projections

$$\begin{array}{ccc} Z_\ell & & \\ & \searrow^{p_2} & \\ & & X \\ & \searrow^{p_1} & \swarrow^{\pi_2} \\ & & W^\ell \times X \\ & & \downarrow^{\pi_1} \\ & & W^\ell \end{array}$$

Also let $A_\ell = \text{Sym}((W^\ell)^*)$ be the coordinate ring of the affine space W^ℓ .

Theorem 2.1. *For every coherent sheaf \mathcal{G} on X , there is a complex $\mathbf{W}_\ell^\bullet(\mathcal{G})$ with the following properties:*

- (1) $\mathbf{W}_\ell^\bullet(\mathcal{G})$ is functorial in \mathcal{G} .
- (2) $\mathbf{W}_\ell^\bullet(\mathcal{G})$ is a minimal complex of free graded A_ℓ -modules.
- (3) $\mathbf{W}_\ell^p(\mathcal{G}) = \bigoplus_i \bigwedge^{i-p} K^\ell \otimes H^i(X, \mathcal{G}(p-i)) \otimes A_\ell(p-i)$.
- (4) When we regard $\mathbf{W}_\ell^\bullet(\mathcal{G})$ as a complex of sheaves on W^ℓ , we have

$$\mathbf{W}_\ell^\bullet(\mathcal{G}) \simeq \mathbf{R}p_{1*}(p_2^*\mathcal{G}).$$

- (5) $\mathbf{W}^\bullet(\mathcal{G})$ is the unique minimal free graded complex of A_ℓ -modules quasi-isomorphic to $\mathbf{R}p_{1*}(p_2^*\mathbf{B}^\bullet(\mathcal{G}))$.
- (6) If $\ell > \dim(\text{supp}(\mathcal{G}))$, then \mathcal{G} is determined up to isomorphism by $\mathbf{W}_\ell^\bullet(\mathcal{G})$.
- (7) Let $i : X \hookrightarrow \mathbb{P}(W)$ be the projective embedding induced by $\mathcal{O}_X(1)$, and write the Tate resolution of $i_*\mathcal{G}$ on $\mathbb{P}(W)$ as

$$T^p(i_*\mathcal{G}) = \bigoplus_i \widehat{E}(i-p) \otimes H^i(X, \mathcal{G}(p-i)).$$

Then $\mathbf{W}_\ell^\bullet(\mathcal{G}) = \mathbf{W}_\ell(T^\bullet(i_*\mathcal{G}))$. In particular, the differentials in $\mathbf{W}_\ell^\bullet(\mathcal{G})$ are completely determined by the corresponding differentials in the Tate resolution.

- (8) When \mathcal{G} is a vector bundle, $\mathbf{W}_\ell^\bullet(\mathcal{G})$ is isomorphic to the complex constructed by Weyman [1989; 1994; 2003].

Proof. This follows from Proposition 1.3, Theorem 1.4, and Corollary 1.9. \square

Definition 2.2. $\mathbf{W}_\ell^*(\mathcal{G})$ is the ℓ -th generalized Weyman complex of \mathcal{G} .

2.2. The excluded case. All of our results assume $\ell \leq \dim(W) - 1$. Here we discuss the complications that arise when $\ell = \dim(W) = N + 1$.

First observe that Proposition 1.5 fails when $\ell = \dim(W)$. To see why, set $\ell = \dim(W) = N + 1$ in (1-4) to obtain

$$\begin{aligned} \cdots \longrightarrow 0 \longrightarrow \bigwedge^{N+1} K^{N+1} \otimes H^N(\mathbb{P}(W), \Omega_{\mathbb{P}(W)}^a(a - N - 1)) \otimes A_{N+1}(-N - 1) \\ \longrightarrow \bigwedge^a K^{N+1} \otimes H^a(\mathbb{P}(W), \Omega_{\mathbb{P}(W)}^a(-a)) \longrightarrow 0 \longrightarrow \cdots, \end{aligned}$$

where the displayed terms are in degrees $q = -1, 0$. Note also that

$$H^N(\mathbb{P}(W), \Omega_{\mathbb{P}(W)}^a(a - N - 1)) \simeq \bigwedge^a W \quad \text{and} \quad H^a(\mathbb{P}(W), \Omega_{\mathbb{P}(W)}^a) \simeq K.$$

Since $H^q(Z_{N+1}, p_2^* \Omega_{\mathbb{P}(W)}^a(a))$ is the q -th cohomology of this complex, it follows that the isomorphism of Proposition 1.5 is replaced with an exact sequence

$$\begin{aligned} 0 \rightarrow \bigwedge^{N+1} K^{N+1} \otimes \bigwedge^a W \otimes A_{N+1}(-N - 1) \rightarrow \\ \mathbf{W}_{N+1}(\widehat{E}(a)) \rightarrow H^0(Z_{N+1}, p_2^* \Omega_{\mathbb{P}(W)}^a(a)) \rightarrow 0. \end{aligned}$$

It follows that Corollary 1.6 needs to be replaced with a similar exact sequence. We suspect that the proof of Theorem 1.4 could be adapted to this situation, though the proof would be considerably more complicated.

From the point of view of resultants, however, the assumption $\ell \leq \dim(W) - 1$ is not overly restrictive. When one has a vector bundle on $\mathbb{P}(W)$ and $\ell = \dim(W)$, our theory does not apply, but one still has the classical Weyman complex [1989; 1994; 2003]. Since the resultant is just a power of the determinant in this case, we do not lose much by excluding $\ell = \dim(W)$.

2.3. Weyman complexes and the Fourier–Mukai transform. Here we observe that the Weyman complex $\mathbf{W}_\ell^*(\mathcal{V})$ can be regarded as a Fourier–Mukai transform when \mathcal{V} is a vector bundle on an irreducible projective variety X .

We begin with the definition of a Fourier–Mukai transform [Huybrechts 2006, Section 5]. Denote by $\mathbf{D}^b(A)$ a bounded derived category of coherent sheaves of a scheme A .

Definition 2.3. Let M and N be projective varieties. Given the two projections

$$p : M \times N \rightarrow M, \quad q : M \times N \rightarrow N,$$

and an object $\mathcal{P} \in \mathbf{D}^b(M \times N)$, the *Fourier–Mukai transform* is the functor

$$\Phi_{\mathcal{P}} : \mathbf{D}^b(M) \rightarrow \mathbf{D}^b(N), \quad \mathcal{E}^\bullet \mapsto \mathbf{R}q_*(p^* \mathcal{E}^\bullet \otimes^{\mathbf{L}} \mathcal{P}).$$

The object \mathcal{P} is called a *Fourier–Mukai kernel* of the Fourier–Mukai transform.

Notice that if \mathcal{P} is a complex of locally free sheaves, then $\otimes^{\mathbf{L}}$ is the usual tensor product. Also, p^* is the usual pull-back as the projection map p is flat.

Denote by W the set of global sections of the very ample line bundle $\mathcal{O}_X(1)$ on X and recall the basic diagram of projections

$$\begin{array}{ccc}
 Z_\ell & & \\
 \searrow^{p_2} & & \\
 & W^\ell \times X & \xrightarrow{\pi_2} X \\
 \swarrow_{p_1} & \downarrow \pi_1 & \\
 & W^\ell &
 \end{array}$$

for the incidence variety

$$Z_\ell = \{(f_1, \dots, f_\ell, x) \in W^\ell \times X \mid f_1(x) = \dots = f_\ell(x) = 0\}.$$

Theorem 2.4. *Let X , W and Z_ℓ be as above and assume $1 \leq \ell \leq \dim(W) - 1$. If \mathcal{V} is a vector bundle on X , then the complex $\mathbf{W}_\ell^\bullet(\mathcal{V})$ represents the Fourier–Mukai transform*

$$\Phi_{\mathcal{O}_{Z_\ell}}(\mathcal{V}) = \mathbf{R}\pi_{1*}(\pi_2^*\mathcal{V} \otimes^{\mathbf{L}} \mathcal{O}_{Z_\ell})$$

with respect to the kernel \mathcal{O}_{Z_ℓ} .

Proof. We have seen that $\mathbf{W}_\ell^\bullet(\mathcal{V})$, when regarded as a complex in $\mathbf{D}^b(W^\ell)$, represents

$$\mathbf{R}p_{1*}(p_2^*\mathcal{V}) \simeq \mathbf{R}\pi_{1*}(\pi_2^*\mathcal{V} \otimes \mathcal{O}_{Z_\ell}).$$

However, \mathcal{V} is locally free, which implies that the same is true for $\pi_2^*\mathcal{V}$. Hence $\mathcal{F} \mapsto \pi_2^*\mathcal{V} \otimes \mathcal{F}$ is an exact functor, so that $\pi_2^*\mathcal{V} \otimes - = \pi_2^*\mathcal{V} \otimes^{\mathbf{L}} -$. The theorem follows immediately. \square

This theorem implies that we can compute the Fourier–Mukai transform of \mathcal{V} for the kernel \mathcal{O}_{Z_ℓ} using the Tate resolution of $i_*\mathcal{V}$ on $\mathbb{P}(W)$.

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