ROOTS OF TOEPLITZ OPERATORS
ON THE BERGMAN SPACE

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A major goal in the theory of Toeplitz operators on the Bergman space over the unit disk \( \mathbb{D} \) in the complex plane \( \mathbb{C} \) is to completely describe the commutant of a given Toeplitz operator, that is, the set of all Toeplitz operators that commute with it. In [2007], the first author characterized the commutant of a Toeplitz operator \( T \) that has a quasihomogeneous symbol \( \phi(r)e^{ip\theta} \) with \( p > 0 \), in case it has a Toeplitz \( p \)-th root \( S \) with symbol \( \psi(r)e^{i\theta} \): The commutant of \( T \) is the closure of the linear space generated by powers \( S^n \) that are Toeplitz. But the existence of a \( p \)-th root was known until now only when \( \phi(r) = r^m \) with \( m \geq 0 \). Here we will show the existence of \( p \)-th roots for a much larger class of symbols, for example, those symbols for which

\[
\phi(r) = \sum_{i=1}^{k} r^{a_i} (\ln r)^{b_i}, \quad \text{where } 0 \leq a_i, b_i \text{ for all } 1 \leq i \leq k.
\]

1. Introduction

Let \( \mathbb{D} \) be the unit disk in the complex plane \( \mathbb{C} \), and let \( dA = rdrd\theta/\pi \) be the Lebesgue area measure normalized so that \( \mathbb{D} \) has unit measure. Let \( L^2_\alpha \) be the Bergman space, the Hilbert space of functions that are analytic on \( \mathbb{D} \) and square integrable with respect to \( dA \). We denote the inner product in \( L^2(\mathbb{D}, dA) \) by \( \langle \cdot, \cdot \rangle \).

It is well known that \( L^2_\alpha \) is a closed subspace of the Hilbert space \( L^2(\mathbb{D}, dA) \), and the set \( \{ \sqrt{n+1}z^n \mid n \geq 0 \} \) of functions is an orthonormal basis. Let \( P \) be the orthogonal projection from \( L^2(\mathbb{D}, dA) \) onto \( L^2_\alpha \). For a bounded function \( f \) on \( \mathbb{D} \), the Toeplitz operator \( T_f \) with symbol \( f \) is defined by

\[
T_f(h) = P(fh) \quad \text{for } h \in L^2_\alpha.
\]

A symbol \( f \) is said to be quasihomogeneous of order \( p \) an integer if it can be written as \( f(re^{i\theta}) = e^{ip\theta}\phi(r) \), where \( \phi \) is a radial function on \( \mathbb{D} \). In this case, the associated Toeplitz operator \( T_f \) is also called quasihomogeneous Toeplitz of order \( p \). Quasihomogeneous Toeplitz operators were first introduced in [Louhichi

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and Zakariasy 2005] while generalizing the results of [Čučković and Rao 1998]. We assume $p > 0$ from now on.

For a given a quasihomogeneous operator $T$ of degree $p$, we seek a quasihomogeneous operator $S$ of degree 1 such that $S^p = T$. Louhichi [2007] proved that if any such root exists, it is unique up to a multiplicative constant. Using the results in [Čučković and Rao 1998], Louchichi also proved the existence of $p$-th roots for the case $\phi(r) = r^m$ for any arbitrary $m \geq 0$ and $p > 0$. Here we plan to deal with more general $\phi(r)$.

2. The Mellin transform and two lemmas

For any two functions $f(r)$ and $g(r)$ defined on $I = [0, 1]$, we define the Mellin convolution by

$$(f *_M g)(r) = \int_r^1 f\left(\frac{t}{r}\right) g(t) \frac{dt}{t}.$$ 

Often we are interested in knowing when the Mellin convolution is a bounded function in the interval $I$. We say a function $f$ is of type $(a, b)$ with $a \geq 0$ and $b > 0$ if

$$|f(r)| \leq Cr^a(1-r)^{b-1} \quad \text{on} \ I,$$

where $C$ is a constant depending on $f$. Also we express the same thing as

$$f(r) \ll r^a(1-r)^{b-1},$$

omitting the constants and the absolute value signs.

**Lemma A.** Suppose $f(r)$ is of type $(a, b)$ and $g(r)$ is of type $(c, d)$. Then

$$\begin{cases} (f *_M g) \text{ is of type } (\min\{a, c\}, b + d) \quad \text{if } a \neq c, \text{ and} \\ (f *_M g)(r) \ll r^{\min\{a, c\}}(1-r)^{b+d-1} \ln(e/r) \quad \text{if } a = c. \end{cases}$$

This can be generalized to any finite product as follows: Suppose for $1 \leq i \leq n$, $f_i(r)$ is of type $(a_i, b_i)$. Then their Mellin convolution product $h(r)$ satisfies

$$h(r) \ll r^{\alpha}(1-r)^{\beta-1}\left(\ln\left(\frac{e}{r}\right)\right)^{n-1}$$

where $\alpha = \min\{a_i\}$ and $\beta = \sum b_i$. Further, if we know that the number of $a_i$ that are equal to $\min\{a_i\}$ is (say) $l$, the estimate (1) can be improved to

$$h(r) \ll r^{\alpha}(1-r)^{\beta-1}\left(\ln\left(\frac{e}{r}\right)\right)^{l-1}.$$  

Thus the log term will disappear if $l = 1$.

**Remark 2.1.** Most of the time our aim is to prove $h$ is bounded; the presence of log does not interfere with that aim since $\alpha > 0$, which bounds $h(r)$ near zero,
and if we assume further that $\beta \geq 1$, it would be bounded near 1 also. But \log cannot be avoided. Take for example $f_i(r) = r$ for every $i$ and compute the Mellin convolution product. It turn out to be $r(\ln r)^{n-1}/(n-1)!$, by a simple integration.

**Lemma B.** Let $f_i(r) = r^{a_i}(1 - r)^{b_i-1}$, where $a_i$ and $b_i$ are positive for $1 \leq i \leq n$. Let $\alpha$ and $\beta$ be as defined in Lemma A. Let $h$ be the Mellin convolution product of the $f_i$. For any integer $k \geq 0$, the $k$-th derivative of $h$ satisfies

$$h^{(k)}(r) \ll r^{\alpha-k}(1-r)^{\beta-k-1}\left(\ln\left(\frac{e}{r}\right)\right)^{n-1}.$$ 

Here the implied constant depends on $k$ and $h$.

3. **Applications of Lemmas A and B**

The Mellin transform $\hat{\phi}$ of a radial function $\phi$ in $L^1([0, 1], r\,dr)$ is defined by

$$\hat{\phi}(z) = \int_0^1 \phi(r) r^{z-1} \,dr = \mathcal{M}(\phi)(z).$$

It is well known that for these functions the Mellin transform is well-defined on the right half-plane $\{z : \text{Re } z \geq 2\}$ and analytic on $\{z : \text{Re } z > 2\}$. The Mellin transform $\hat{\phi}$ is uniquely determined by its values on any arithmetic sequence of integers. In fact we have the following classical theorem [Remmert 1998, page 102].

**Theorem 3.1.** Suppose $f$ is a bounded analytic function on $\{z : \text{Re } z > 0\}$ that vanishes at the pairwise distinct points $z_1, z_2, \ldots$, where

1. $\inf\{|z_n|\} > 0$ and
2. $\sum_{n \geq 1} \text{Re}(1/z_n) = \infty$.

Then $f$ vanishes identically on $\{z : \text{Re } z > 0\}$.

**Remark 3.2.** One can apply this theorem to prove that if $\phi \in L^1([0, 1], r\,dr)$ and if there exist $n_0, p \in \mathbb{N}$ such that

$$\hat{\phi}(pk + n_0) = 0 \quad \text{for all } k \in \mathbb{N},$$

then $\hat{\phi}(z) = 0$ for all $z \in \{z : \text{Re } z > 2\}$ and so $\phi = 0$.

It is easy to see that the Mellin transform converts the Mellin convolution product into a pointwise product, that is,

$$(\hat{\phi} *_{\mathcal{M}} \hat{\psi})(r) = \hat{\phi}(r) \hat{\psi}(r).$$

A direct calculation shows that a quasihomogeneous Toeplitz operator acts on the elements of the orthogonal basis of $L^2_a$ as a shift operator with a holomorphic
weight. In fact, for $p \geq 0$ and for all $k \geq 0$, we have

$$T_{e^{ip\theta}}(z^k) = P(e^{ip\theta} \phi z^k) = \sum_{n \geq 0} (n+1) \langle e^{ip\theta} \phi z^k, z^n \rangle z^n$$

$$= \sum_{n \geq 0} (n+1) \int_0^1 \int_0^{2\pi} \phi(r)r^{k+n+1} e^{i(k+p-n)\theta} \frac{d\theta}{\pi} dr z^n$$

$$= 2(k+p+1) \hat{\phi}(2k+p+2) z^{k+p}.$$ 

Now we are ready to start with a relatively easy example.

3.1. **Assuming $\phi(r) = r + r^2$, find the $p$-th roots of $T_{e^{ip\theta}}$.** If there exists a bounded radial function $\psi$ such that $(T_{e^{i\theta}} \psi)^p = T_{e^{ip\theta}}$, then

$$(T_{e^{i\theta}} \psi)^p(z^k) = T_{e^{ip\theta}}(z^k) \quad \text{for all } k \geq 0.$$ 

Since

$$(T_{e^{i\theta}} \psi)^p(z^k) = \left( \prod_{j=0}^{p-1} (2k+2j+4) \psi(2k+2j+3) \right) z^{k+p},$$

we obtain for all integers $k \geq 0$

$$(2k+2p+2) \hat{\phi}(2k+p+2) = \left( \prod_{j=0}^{p-1} (2k+2j+4) \psi(2k+2j+3) \right),$$

which is equivalent to

$$\frac{\hat{\phi}(2k+p+2)}{\prod_{j=0}^{p-2} (2k+2j+4)} = \prod_{j=0}^{p-1} \psi(2k+2j+3).$$

Note that $p$ is a positive integer and that our discussion is trivial for $p = 1$. So $p \geq 2$. By setting $z = 2k+3$, we notice that the function

$$f(z) = \frac{\hat{\phi}(z+p-1)}{\prod_{j=0}^{p-2} (z+2j+1)} - \prod_{j=0}^{p-1} \psi(z+2j)$$

is holomorphic and bounded in the right half-plane and vanishes for $z = 2k+3$, for $k$ any nonnegative integer. Now by Theorem 3.1, we get $f(z) \equiv 0$. Therefore

$$(2) \quad (z+2p-1) \hat{\phi}(z+p-1) = \left( \prod_{j=0}^{p-1} (z+2j+1) \psi(z+2j) \right).$$
If we divide the (2) by the equation obtained by replacing \( z \) by \( z + 2 \) in (2), we obtain after cancelation that in the right half-plane

\[
\frac{\hat{\psi}(z + 2p)}{\hat{\psi}(z)} = \frac{(z + 1)\hat{\phi}(z + p + 1)}{(z + 2p - 1)\hat{\phi}(z + p - 1)} \quad \text{for } \Re z > 0.
\]

Since

\[
\hat{\phi}(z) = \frac{1}{z + 1} + \frac{1}{z + 2} = \frac{2z + 3}{(z + 1)(z + 2)},
\]

it follows that, for \( \Re z > 0 \),

\[
\frac{\hat{\psi}(z + 2p)}{\hat{\psi}(z)} = \frac{(z + 1)}{(z + 2p - 1)} \frac{(2z + 2p + 5)}{(z + p + 2)(z + p + 3)} \frac{(z + p)(z + p + 1)}{(2z + 2p + 1)}.
\]

Letting \( \lambda(\zeta) = \hat{\psi}(2p\zeta) \), this equation becomes, for \( \Re \zeta > 0 \),

\[
\frac{\lambda(\zeta + 1)}{\lambda(\zeta)} = \frac{(2p\zeta + 1)(4p\zeta + 2p + 5)(2p\zeta + p)(2p\zeta + p + 1)}{(2p\zeta + 2p - 1)(2p\zeta + p + 2)(2p\zeta + p + 3)(4p\zeta + 2p + 1)}.
\]

Using the well-known identity \( \Gamma(z + 1) = z\Gamma(z) \), where \( \Gamma \) is the Gamma function, we can write

\[
\frac{\lambda(\zeta + 1)}{\lambda(\zeta)} = \frac{F(\zeta + 1)}{F(\zeta)} \quad \text{for } \Re \zeta > 0,
\]

where

\[
F(\zeta) = \frac{\Gamma(\zeta + a_1)\Gamma(\zeta + a_2)\Gamma(\zeta + a_3)\Gamma(\zeta + a_4)}{\Gamma(\zeta + a'_1)\Gamma(\zeta + a'_2)\Gamma(\zeta + a'_3)\Gamma(\zeta + a'_4)},
\]

and the \( a_i \) are in increasing order

\[
\frac{2}{4p}, \quad \frac{2p}{4p}, \quad \frac{2p + 2}{4p}, \quad \frac{2p + 5}{4p}
\]

respectively and the \( a'_i \) are in almost increasing order

\[
\frac{2p + 1}{4p}, \quad \frac{2p + 4}{4p}, \quad \frac{4p - 2}{4p}, \quad \frac{2p + 6}{4p}
\]

respectively for \( i = 1, \ldots, 4 \). We shall show in a moment that \( F(\zeta) \) is a bounded holomorphic function in the right half-plane. Granting that, Equation (4) combined with [Louhichi 2007, Lemma 6, page 1468] implies exists a constant \( C \) such that

\[
\lambda(\zeta) = CF(\zeta) \quad \text{for } \Re \zeta > 0.
\]

A basic observation is that the quotient of two Gamma functions

\[
\frac{\Gamma(\zeta + a_i)}{\Gamma(\zeta + a'_i)}, \quad \text{where } 0 < a_i < a'_i,
\]
is a constant times the Beta function

\[ B(\zeta + a_i, a'_i - a_i) = \int_0^1 x^{\zeta + a_i - 1} (1 - x)^{a'_i - a_i - 1} \, dx. \]

According to our definition of the Mellin transform, \( B(\zeta + a_i, a'_i - a_i) \) is the Mellin transform of \( x^{a_i} (1 - x)^{a'_i - a_i - 1} \), which is of type \( (a_i, a'_i - a_i) \). Since \( a_i < a'_i \) for \( i = 1, \ldots, 4 \) (in fact, \( a'_3 \geq a_3 \) if and only if \( 2p \geq 4 \), which is always true), each of the Beta functions is a bounded holomorphic function in the right half-plane and \( F(\zeta) \), which is a constant times the product of these four Beta functions, is a bounded holomorphic function in the right half-plane. Equation (5) implies that

\[ \lambda(\zeta) = C \sum_{i=1}^4 B(\zeta + a_i, a'_i - a_i), \]

where \( C \) is a constant. Since the product of Mellin transforms equals the Mellin transform of the Mellin convolution product, we have

\[ \lambda(\zeta) = Ch(\zeta), \]

where \( h \) is the convolution product of four functions of type \( (a_i, a'_i - a_i) \) for \( i = 1, \ldots, 4 \). Now Lemma A tells us that

\[ h(r) \ll r^{\min\{a_i\}} (1 - r)^{\sum_i (a'_i - a_i) - 1} \ln(e/r). \]

Because \( \sum_i a'_i - a_i = 1 \), we have

\[ h(r) \ll r^{\min\{a_i\}} \ln(e/r), \]

and hence \( h \) is a bounded function. Therefore the function \( \psi \), if it exists, satisfies the equation

\[ \hat{\psi}(2p\zeta) = C\hat{h}(\zeta) \]

for some constant \( C \), which is equivalent to

\[ \int_0^1 \psi(r)r^{2p\zeta - 1}dr = C \int_0^1 h(t)t^{\zeta - 1}dt. \]

Now, by a change of variables \( t = r^{2p} \), we obtain

\[ \int_0^1 \psi(r)r^{2p\zeta - 1}dr = \int_0^1 h(r^{2p})r^{2p\zeta - 1}2pdr. \]

Thus \( \psi(r) = 2ph(r^{2p}) \), and so \( \psi \) is bounded. Hence the operator \( T_{e^{i\theta} \psi} \) is a genuine Toeplitz operator and a \( p \)-th root of \( T_{e^{i\theta} \phi} \).
3.2. $p$-th roots of $T_{ei\theta \phi}$, where $\hat{\phi}(z)$ is a proper rational fraction. Such functions are plenty. For example, take $\Phi(r) = r^a \ln(r)^b$, where $a > 0$ and $b$ is a nonnegative integer. By integration by parts we see that $\hat{\phi}(z) = (-1)^b b!/((a + z)^{b+1}$.

Assume we are given a radial function $\phi(r)$ such that $\hat{\phi}(r)$ is a proper rational function. Recall that if there is a radial function $\psi$ such that $(T_{ei\theta \psi})^p = T_{ei\theta \phi}$, then we have Equation (3), which is

$$\hat{\psi}(z + 2p) = \hat{\psi}(z) \frac{(z + 1)\hat{\phi}(z + p + 1)}{(z + 2p - 1)\hat{\phi}(z + p - 1)} \text{ for } \text{Re } z > 0.$$ 

Here we are assuming $\hat{\phi}(z) = P(z)/Q(z)$, where

$$P(z) = \prod_{j=1}^{m}(z + a_j) \quad \text{and} \quad Q(z) = \prod_{k=1}^{n}(z + b_k)$$

with $1 \leq m < n$. So,

$$\hat{\psi}(z + 2p) = \hat{\psi}(z) \frac{(z + 1)P(z + p + 1)Q(z + p - 1)}{(z + 2p - 1)P(z + p - 1)Q(z + p + 1)}$$

$$= \frac{(z + 1)}{(z + 2p + 1)} \prod_{j=1}^{m} \frac{z + a_j + p + 1}{z + a_j + p - 1} \prod_{k=1}^{n} \frac{z + b_k + p + 1}{z + b_k + p - 1}$$

Let $\lambda(\xi) = \hat{\psi}(2p\xi)$. Then the equality above becomes

$$\frac{\lambda(\xi + 1)}{\lambda(\xi)} = \frac{(2p\xi + 1)}{(2p\xi + 2p - 1)} \prod_{j=1}^{m} \frac{2p\xi + a_j + p + 1}{2p\xi + a_j + p - 1} \prod_{k=1}^{n} \frac{2p\xi + b_k + p + 1}{2p\xi + b_k + p - 1}$$

$$= \frac{F(\xi + 1)G(\xi)}{F(\xi)G(\xi + 1)},$$

where

$$F(\xi) = \frac{\Gamma(\xi + A_0)}{\Gamma(\xi + A'_0)} \prod_{k=1}^{n} \frac{\Gamma(\xi + B_k)}{\Gamma(\xi + B'_k)} \quad \text{and} \quad G(\xi) = \prod_{j=1}^{m} \frac{\Gamma(\xi + A'_j)}{\Gamma(\xi + A_j)},$$

$$A_0 = \frac{1}{2p}, \quad A'_0 = \frac{2p - 1}{2p},$$

$$A_j = \frac{a_j + p + 1}{2p}, \quad A'_j = \frac{a_j + p - 1}{2p},$$

$$B_k = \frac{b_k + p - 1}{2p}, \quad B'_k = \frac{b_k + p + 1}{2p} \quad \text{for } 1 \leq j \leq m \text{ and } 1 \leq k \leq n.$$
Note that any quotient of two Gamma functions, say,
\[
\frac{\Gamma(\zeta + \alpha)}{\Gamma(\zeta + \gamma)} = \beta(\zeta + \alpha, \gamma - \alpha)\Gamma(\gamma - \alpha)
\]
is a bounded holomorphic function in the right half-plane if \(\alpha\) and \(\gamma - \alpha\) are positive. Hence both \(F(\zeta)\) and \(G(\zeta)\) are bounded holomorphic functions in the right half-plane if we assume all \(A_j, A'_j, B_k, B'_k\) are positive. We will assume that.

Therefore, by [Louhichi 2007, Lemma 6, page 1468], \(\lambda\) is a constant times the quotient of \(m + n + 1\) Gamma functions in the numerator and about the same in the denominator, as follows:

\[
\lambda(\zeta) = C \frac{\Gamma(\zeta + A_0)}{\Gamma(\zeta + A'_0)} \prod_{j=1}^{m} \frac{\Gamma(\zeta + A_j)}{\Gamma(\zeta + A'_j)} \prod_{k=1}^{n} \frac{\Gamma(\zeta + B_k)}{\Gamma(\zeta + B'_k)}.
\]  

(6)

Based on the argument of the previous subsection, we would like to write each quotient of two Gamma functions as a constant times a Beta function. In order to do that, we must assume that all \(A_j\) and \(B_k\) are positive for every \(0 \leq j \leq m\) and \(1 \leq k \leq n\). Moreover, we observe that

\[
A'_0 - A_0 = \frac{p - 1}{p}, \quad A'_j - A_j = -\frac{1}{p}, \quad B'_k - B_k = \frac{1}{p}.
\]

So each quotient of two Gamma functions in Equation (6) can be written as a constant times a Beta function except those involving \(A_j\) for \(1 \leq j \leq m\). We fix this matter by noting that \(\Gamma(\zeta + A'_j + 1) = (\zeta + A'_j)\Gamma(\zeta + A'_j)\), and so here \(A'_j + 1 - A_j = (p - 1)/p\). Hence, Equation (6) becomes

\[
\frac{\lambda(\zeta)}{\prod_{j=1}^{m} (\zeta + A'_j)} = C \frac{\Gamma(\zeta + A_0)}{\Gamma(\zeta + A'_0)} \prod_{j=1}^{m} \frac{\Gamma(\zeta + A_j)}{\Gamma(\zeta + A'_j + 1)} \prod_{j=1}^{n} \frac{\Gamma(\zeta + B_j)}{\Gamma(\zeta + B'_j)}.
\]

As in the previous subsection, this quotient of \(m + n + 1\) Gamma functions on the numerator and the same in the denominator, respectively would be the Mellin transform of the convolution product of \(m + n + 1\) functions of type \((a_i, b_i)\). Let us call it \(h\). By Lemma A, we have

\[
h(r) \ll r^A (1 - r)^B \left(\ln \left(\frac{e}{r}\right)\right)^{m+n},
\]

where \(A = \min\{A_j\}\), which is definitely positive, and \(B\) is given by

\[
A'_0 - A_0 + \sum_{j=1}^{m} A'_j + 1 - A_j + \sum_{k=1}^{n} B'_k - B_k = (m+1)\frac{p-1}{p} + n = m+1 + \frac{n-m-1}{p}.
\]
Therefore we obtain
\[ h(r) \ll r^A(1 - r)^{m + (n - m - 1)/p} \left( \ln \left( \frac{e}{r} \right) \right)^{m+n} = r^A(1 - r)^{m+\nu} \left( \ln \left( \frac{e}{r} \right) \right)^{m+n}, \]
where \( \nu = (n - m - 1)/p \) is a nonnegative number. Using Lemma B, we see that \( h \) has all derivatives of order not exceeding \( m \) and they satisfy the inequality
\[ r^j h^{(j)}(r) \ll r^A(1 - r)^{m-j+\nu} \left( \ln \left( \frac{e}{r} \right) \right)^{m+n}. \]

Further the function \( \psi \), were it to exist, would satisfy the equation
\[ (7) \quad \hat{\psi}(2p\xi) = C \left( \prod_{j=1}^{m} (\xi + A'_j) \right) \hat{h}(\xi). \]

Now it is easy to check by integration by parts the identity
\[ \xi \hat{h}(\xi) = -\mathcal{M} \left( r \frac{dh}{dr} \right)(\xi) \]
powered h vanishes at 1 and \( rh' \) is bounded in \((0, 1)\). Thus in the current case, letting \( h' = Dh \), where \( D = d/dr \), we can see
\[ (\xi + A'_j)\hat{h}(\xi) = \mathcal{M}((A'_j - rD)h)(\xi) \]
and
\[ \left( \prod_{j=1}^{m} (\xi + A'_j) \right) \hat{h}(\xi) = \mathcal{M} \left( \prod_{j=1}^{m} (A'_j - rD)h \right)(\xi). \]

Let us set
\[ H(r) = \left( \prod_{j=1}^{m} (A'_j - rD)h \right)(r), \]
which allows us to rewrite Equation (7) as
\[ \int_0^1 \psi(r)r^{2p\xi-1}dr = C \int_0^1 H(t)t^{\xi-1}dt. \]

Now, by a change of variables \( t = r^{2p} \), we obtain
\[ \int_0^1 \psi(r)r^{2p\xi-1}dr = C \int_0^1 H(r^{2p})r^{2p\xi-1}2pdr. \]

Thus \( \psi(r) = 2pC H(r^{2p}) \), and hence is bounded and the operator \( T_{e^{i\theta} \phi} \) is a genuine Toeplitz operator and a \( p \)-th root of \( T_{e^{i\theta} \phi} \).
4. **Proof of Lemma A for two functions**

We start by proving Lemma A for functions \( f \) and \( g \) of type \((a, b)\) and \((c, d)\) respectively, where \( a, b, c \) and \( d \) are all positive. A similar thing was discussed in [Čučković and Rao 1998, pages 210-212] but with less generality since the goal was different.

Let \( h(r) = (f * M g)(r) \). By definition of the Mellin convolution, it is easy to see that

\[
h(r) \ll \int_{r}^{1} \left( \frac{r}{t} \right)^{a} \left( 1 - \frac{r}{t} \right)^{b-1} t^{c(1-t)^{d-1}} \frac{dt}{t},
\]

which after changing variables as \( \frac{t-r}{1-r} = u \) and using the consequent identities

\[
t = r + u - ru, \quad t - r = u(1-r), \quad 1-t = (1-u)(1-r), \quad dt = (1-r)du
\]

while keeping \( r \) fixed, leads to

\[
h(r) \ll \int_{r}^{1} \left( \frac{r}{t} \right)^{a} \left( 1 - \frac{r}{t} \right)^{b-1} t^{c(1-t)^{d-1}} \frac{dt}{t} = \int_{r}^{1} \left( \frac{r}{t} \right)^{a} \left( \frac{t-r}{t} \right)^{b-1} t^{c(1-t)^{d-1}} \frac{dt}{t} = \int_{0}^{1} r^{a} t^{-a} u^{b-1} (1-r)^{b-1} t^{-b+1} t^{c(1-u)^{d-1}} (1-r)^{d-1} \frac{1}{t} \frac{du}{t}
\]

\[
= r^{a} (1-r)^{b+d-1} \int_{0}^{1} t^{c-a-b} u^{b-1} (1-u)^{d-1} du.
\]

We have the following cases.

- \( c - a - b \geq 0 \). Since \( 0 \leq t \leq 1 \), we have

\[
h(r) \ll r^{a} (1-r)^{b+d-1},
\]

and hence \( h \) is of type \((a, b+d)\).

- \( c - a - b < 0 \). Assuming \( c - a > 0 \) and noting that \( t \geq u \), we obtain

\[
h(r) \ll r^{a} (1-r)^{b+d-1} \int_{0}^{1} u^{c-a-b} u^{b-1} (1-u)^{d-1} du \leq r^{a} (1-r)^{b+d-1} \int_{0}^{1} u^{c-a-1} (1-u)^{d-1} du = r^{a} (1-r)^{b+d-1} B(c-a, d),
\]

and therefore \( h \) is of type \((a, b+d)\).
In this context we can assume that the function $f$, of type $(a_i, b_i)$, is given by
\[ f_i(x) = x^{a_i}(1 - x)^{b_i - 1} \quad \text{for } 1 \leq i \leq n. \]
The Mellin convolution product of these $n$ functions is defined by a repeated integral

\begin{equation}
(9) \quad h(r) = \int_r^1 \int_{r/x_1}^1 \int_{r/x_1x_2}^1 \ldots \int_{r/x_1x_2 \ldots x_{n-2}}^1 f_1(x_1)f_2(x_2)\ldots f_{n-1}(x_{n-1})f_n\left(\frac{r}{x_1 \ldots x_{n-1}}\right)\frac{dx_{n-1}}{x_1} \ldots \frac{dx_3}{x_2} \frac{dx_1}{x_1}
\end{equation}

As in the case of two functions where we changed variables as $u = (t - r)/(1 - r)$, we change variables so that the new integral is over the unit cube $I^n$, where limits of integration do not depend on other variables. Let $y_0 = 1$ and inductively define $y_i = \prod_{j=1}^i x_i$ for $i \geq 1$. Now we change variables as

$$x_i = \frac{r}{y_{i-1}} + \left(1 - \frac{r}{y_{i-1}}\right)\xi_i \quad \text{for} \quad i \geq 1,$$

so that the limits for each $\xi_i$ are 0 and 1. Further we note

$$y_i - r = x_i y_{i-1} - r = (y_{i-1} - r)\xi_i \quad \text{for} \quad i \geq 0.$$

Set $\eta_0 = 1$ and $\eta_i = \prod_{j=1}^i \xi_i$ for $i \geq 1$. It is easy to show, by induction on $i$, that

$$y_i - r = (1 - r)\eta_i \quad \text{for all} \quad i \geq 1.$$

Further

\begin{equation}
(10) \quad (1 - x_i) = (1 - \xi_i)\left(1 - \frac{r}{y_{i-1}}\right) = \frac{(1 - \xi_i)(1 - r)\eta_{i-1}}{y_{i-1}} \quad \text{for all} \quad i \geq 1.
\end{equation}

Thus

$$f_i(x_i) = x_i^{a_i}(1 - x_i)^{b_i-1} = \left(\frac{y_i}{y_{i-1}}\right)^{a_i} \left(\frac{(1 - \xi_i)(1 - r)\eta_{i-1}}{y_{i-1}}\right)^{b_i-1} \quad \text{for} \quad 1 \leq i \leq n - 1.$$

But for $i = n$, we have

$$f_n(x_n) = \left(\frac{r}{y_{n-1}}\right)^{a_n} \left(1 - \frac{r}{y_{n-1}}\right)^{b_{n-1}} = \left(\frac{r}{y_{n-1}}\right)^{a_n} \left(\frac{y_{n-1} - r}{y_{n-1}}\right)^{b_{n-1}} = \left(\frac{r}{y_{n-1}}\right)^{a_n} \left(\frac{(1 - r)\eta_{n-1}}{y_{n-1}}\right)^{b_{n-1}}.$$

Writing the product of functions in (9) in terms of $\xi_i$, $\eta_i$, $r$ and $y_i$ for $1 \leq i \leq n - 1$ yields

\begin{equation}
(11) \quad r^{a_n} \prod_{i=1}^{n-1} \eta_i^{b_{i+1}-1} \prod_{i=1}^{n-1} y_i^{a_i-a_{i+1}+b_{i+1}} \prod_{i=1}^{n-1} (1 - \xi_i)^{b_i-1} \prod_{i=1}^{n} (1 - r)^{b_i-1}.
\end{equation}
Using equalities of (10), we calculate the differential form

\[
\left(1 - r\right)^{n-1} \prod_{i=1}^{n-1} \frac{x_i}{x_i y_i} = \left(1 - r\right)^{n-1} \prod_{i=1}^{n-1} \frac{\eta_i}{y_i} \prod_{i=1}^{n-1} dx_i.
\]

From (9), (10), and (11) we derive

\[
h(r) = r^{a_n} (1 - r)^{b_1 + \cdots + b_n - 1} \int_{f_n-1} \eta_n b_n \prod_{i=1}^{n-1} \eta_i b_i+1 \prod_{i=1}^{n-1} y_n a_i - a_i + 1 \prod_{i=1}^{n-1} (1 - \xi_i) b_i-1 \prod_{i=1}^{n-1} d\xi_i.
\]

Let us assume that the \(a_i\) are arranged in decreasing order. Then

\[
\eta_i b_i+1 \prod_{i=1}^{n-1} y_n a_i - a_i + 1 \prod_{i=1}^{n-1} (1 - \xi_i) b_i-1 \leq 1
\]

since \(\eta_i \leq y_i \leq 1\). Therefore

\[
h(r) \leq r^{a_n} (1 - r)^{b_1 + \cdots + b_n - 1} \int_{f_n-1} \eta_n b_n \prod_{i=1}^{n-1} (1 - \xi_i) b_i-1 \prod_{i=1}^{n-1} d\xi_i
\]

Here four cases have to be discussed.

**Case 1:** \(a_{n-1} - a_n - b_n \geq 0\). In this case

\[
h(r) \leq r^{a_n} (1 - r)^{b_1 + \cdots + b_n - 1} \int_{f_n-1} \eta_n b_n \prod_{i=1}^{n-1} (1 - \xi_i) b_i-1 \prod_{i=1}^{n-1} \frac{\Gamma(b_n) \Gamma(b_i)}{\Gamma(b_n + b_i)}
\]

**Case 2:** \(a_{n-1} - a_n - b_n < 0\) and \(a_{n-1} \neq a_n\). Then

\[
h(r) \leq r^{a_n} (1 - r)^{b_1 + \cdots + b_n - 1} \int_{f_n-1} \eta_n b_n \prod_{i=1}^{n-1} (1 - \xi_i) b_i-1 \prod_{i=1}^{n-1} \frac{\Gamma(a_n - a_n) \Gamma(b_i)}{\Gamma(a_n - a_n + b_i)}
\]
Case 3: $a_{n-1} = a_n$. Choose an arbitrary $0 < \epsilon \leq b_n$. Note that $y_{n-1} \geq r$ and $\eta_{n-1} > 0$. Then

$$h(r) \leq r^{a_n}(1 - r)^{b_1 + \cdots + b_{n-1}} \int_{I_{n-1}} y_n^{-b_n} y_{n-1}^{\eta_{n-1}} \prod_{i=1}^{n-1} (1 - \xi_i)^{b_i} \prod_{i=1}^{n} d\xi_i$$

$$\leq r^{a_n}(1 - r)^{b_1 + \cdots + b_{n-1}} \int_{I_{n-1}} r^{-\epsilon} y_n^{\epsilon} \eta_{n-1}^{b_{n-1}} \prod_{i=1}^{n-1} (1 - \xi_i)^{b_i} \prod_{i=1}^{n} d\xi_i$$

$$\leq r^{a_n}(1 - r)^{b_1 + \cdots + b_{n-1}} \int_{I_{n-1}} r^{-\epsilon} \eta_{n-1} \prod_{i=1}^{n-1} (1 - \xi_i)^{b_i} \prod_{i=1}^{n} d\xi_i$$

$$= r^{a_n}(1 - r)^{b_1 + \cdots + b_{n-1}} - r^{-\epsilon} \prod_{i=1}^{n-1} \frac{\Gamma(\epsilon) \Gamma(b_i)}{\Gamma(\epsilon + b_i)}.$$

Similarly to Section 2, this product of quotients of Gamma functions is meromorphic on the interval $[0, b_n]$ except at zero, where it has a pole of order $n - 1$, and so there exists a constant $C$ such that

$$\prod_{i=1}^{n-1} \frac{\Gamma(\epsilon) \Gamma(b_i)}{\Gamma(\epsilon + b_i)} \leq C \epsilon^{1-n}.$$

Hence

$$h(r) \ll r^{a_n}(1 - r)^{b_1 + \cdots + b_{n-1}} r^{-\epsilon} \epsilon^{1-n}.$$

Now there are two subcases: If $r \leq e^{-1/b_n}$, then $(\ln(1/r))^{-1} \leq b_n$. In this case, we choose $\epsilon = (\ln(1/r))^{-1}$ and obtain

$$h(r) \ll r^{a_n}(1 - r)^{b_1 + \cdots + b_{n-1}} e(\ln(1/r))^{n-1}. \quad (14)$$

On the other hand, if $r \geq e^{-1/b_n}$, we choose $\epsilon = b_n$ and obtain

$$h(r) \ll r^{a_n}(1 - r)^{b_1 + \cdots + b_{n-1}} r^{-b_n b_{n-1}^{1-n}} \leq r^{a_n}(1 - r)^{b_1 + \cdots + b_{n-1}} e b_{n-1}^{1-n}. \quad (15)$$

Combining (14) and (15) yields

$$h(r) \ll r^{a_n}(1 - r)^{b_1 + \cdots + b_{n-1}} e(\ln(1/r))^{n-1} + e b_{n-1}^{1-n}$$

$$\ll r^{a_n}(1 - r)^{b_1 + \cdots + b_{n-1}} (\ln(1/r))^{n-1} \text{ for all } 0 < r < 1.$$

This is enough for our purposes but we can get a more refined estimate as mentioned in the second case of Lemma A. Thus we reach the final case:
Case 4: there exists \( k \) such that \( a_k > a_{k+1} = \cdots = a_n = a \). Let \( F(r) \) be the Mellin convolution product of \( f_1, f_2, \ldots, f_k+1 \) and \( G(r) \) be the Mellin convolution product of the rest, namely \( f_{k+2}, \ldots, f_n \). From the previous discussion it is clear that

\[
F(r) \ll r^a (1-r)^{b_1+\cdots+b_{k+1}-1}
\]

and

\[
G(r) \ll r^a (1-r)^{b_{k+2}+\cdots+b_n-1}(\ln(e/r))^{n-k-2}.
\]

Let \( b = b_1 + \cdots + b_{k+1} \), \( d = b_{k+2} + \cdots + b_n \) and \( n - k - 1 = l \). The case \( l = 1 \) has been treated previously. So assume \( l > 1 \). We see that

\[
h(r) = (F \ast_M G)(r)
\]

\[
\leq \int_r^1 (r/t)^a (1-r/t)^{b-1} t^{a-1} (\ln(t/e))^l dt
\]

\[
\leq r^a \int_r^1 (t-r)^{b-1} t^{-b} (1-t)^{d-1} (\ln(t/e))^l dt.
\]

Now the change of variables \( t = u + r - ur \) leads to \( t - r = u(1-r), \ 1-t = (1-u)(1-r), \ dt = (1-r)du \) and

\[
h(r) \ll r^a \int_0^1 (1-r)^{b-1} u^{b-1} t^{-b} (1-r)^{d-1} (1-u)^{d-1} (\ln(t/e))^l (1-r) du
\]

\[
\leq r^a (1-r)^{b+d-1} \int_0^1 u^{b-1} t^{-b} (1-u)^{d-1} (\ln(t/e))^l du.
\]

Noting that \( t \geq u \) and \( r > 0 \), and choosing an arbitrary \( 0 < \epsilon \leq b \) implies

\[
h(r) \ll r^a (1-r)^{b+d-1} \int_0^1 u^{b-1} t^{-\epsilon} t^{\epsilon-b} (1-u)^{d-1} (\ln(t/e))^l du
\]

\[
\leq r^a (1-r)^{b+d-1} \int_0^1 u^{b-1} r^{-\epsilon} u^{\epsilon-b} (1-u)^{d-1} (\ln(t/e))^l du
\]

\[
\leq r^a (1-r)^{b+d-1} r^{-\epsilon} \int_0^1 u^{b-1} u^{\epsilon-b} (1-u)^{d-1} (\ln(t/e))^l du
\]

\[
\leq r^a (1-r)^{b+d-1} r^{-\epsilon} \int_0^1 u^{\epsilon-1} (1-u)^{d-1} (\ln(t/e))^l du.
\]

Let \( H_j(\epsilon) = \int_0^1 u^{\epsilon-1} (1-u)^{d-1} (\ln(u))^j du \). This is the \( j \)-th derivative of the beta function \( B(\epsilon, d) \) as a function of \( \epsilon \), and \( B(\epsilon, d) \) is holomorphic on \((-1, \infty)\) except at zero where it has a simple pole with residue 1. This is easy to verify. So \( \epsilon^{j+1} H_j(\epsilon) \) will be holomorphic on the interval \((-1, \infty)\). Observing that

\[
\int_0^1 u^{\epsilon-1} (1-u)^{d-1} (\ln(t/e))^l du
\]
is a linear sum of the derivatives of order less than or equal to \( l - 1 \) of the Beta function, we find

\[
\epsilon^l \int_0^1 u^{e-1} (1-u)^{d-1} \left( \ln \left( \frac{e}{u} \right) \right)^{l-1} du
\]

is bounded by a constant \( C \) in the interval \([0, b]\). Thus

\[
h(r) \ll r^a (1-r)^{b+d-1} e^{-\epsilon} e^{-l}.
\]

Now arguing as in Case 3, if \( r \leq e^{-1/b} \), we choose \( \epsilon = 1/\ln(1/r) \) and get

\[
h(r) \ll r^a (1-r)^{b+d-1} e(\ln(1/r))^l
\]

and if \( r > e^{-1/b} \), we let \( \epsilon = b \), and have

\[
h(r) \ll r^a (1-r)^{b+d-1} e/b^l.
\]

Combining these two cases, we obtain

\[
h(r) \ll r^a (1-r)^{b+d-1}(\ln(e/r))^l.
\]

This fully proves Lemma A.

\[\square\]

6. Proof of Lemma B

We recall (13):

\[
h(r) = r^a n (1-r)^{b_1+\cdots+b_n-1} \int_{l_{n-1}}^n \eta_{n-1}^{b_{n-1}} \prod_{i=1}^{n-2} \eta_i^{b_i+1} \times \prod_{i=1}^{n-1} y_i^{a_i-a_{i+1}-b_{i+1}} \prod_{i=1}^{n-1} (1-\xi_i)^{b_i-1} \wedge d\xi_i.
\]

To make the differentiation easier, we introduce some notation. Let

\[
A = a_n, \quad B = b_1 + \cdots + b_n,
\]

\[
\eta = (\eta_1, \ldots, \eta_{n-1}), \quad \xi = (\xi_1, \ldots, \xi_{n-1}), \quad y = (y_1, \ldots, y_{n-1}),
\]

\[
\alpha_i = a_i - a_{i+1} - b_{i+1} \quad \text{for } 1 \leq i \leq n-1,
\]

\[
\beta_i = b_{i+1} \quad \text{for } 1 \leq i \leq n-2, \quad \beta_{n-1} = b_{n-1},
\]

\[
\beta = (\beta_1, \ldots, \beta_{n-1}), \quad G(\xi) = \prod_{i=1}^{n-1} (1-\xi_i)^{b_i-1}, \quad d\xi = \wedge d\xi_i, \quad J = I^{n-1}.
\]
With this notation and the multiindex notation as for example $y^\alpha = y_1^{\alpha_1} \cdots y_n^{\alpha_n}$, (13) can be written as

$$h(r) = r^A (1 - r)^{B-1} \int J y^\alpha \eta^\beta G(\xi) \, d\xi. \tag{16}$$

Clearly the function $\eta^\beta G(\xi_i)$ is summable $d\xi$, and each $y_i = \eta_i + r(1 - \eta_i)$ satisfies $0 < r \leq y_i < 1$ for $0 < r < 1$. So one can differentiate under the integral sign with respect to $r$. But before we do that let us introduce the notation

$$g_1(r) = r^A, \quad g_2(r) = (1 - r)^{B-1}, \quad u_i = \eta_i^{\alpha_i} \quad \text{for } 1 \leq i \leq n - 1.$$

Rewrite (16) as

$$h(r) = \int J g_1 g_2 u_1 \cdots u_{n-1} \eta^\beta G(\xi) \, d\xi. \tag{17}$$

Now differentiating under the integral sign, we obtain

$$h^{(k)}(r) = \sum \int J g_1^{(l_1)} g_2^{(l_2)} u_1^{(j_1)} \cdots u_{n-1}^{(j_{n-1})} \eta^\beta G(\xi) \, d\xi, \tag{18}$$

where the sum is over all $(n+1)$-tuples of nonnegative integers $(l_1, l_2, j_1, \ldots, j_{n-1})$ such that $k = l_1 + l_2 + j_1 + \cdots + j_{n-1}$. Further it easy to check that

$$u_i^{(j_i)}(r) = \alpha_i(\alpha_i - 1) \cdots (\alpha_i - j_i + 1) \eta_i^{\alpha_i - j_i} (1 - \eta_i)^{j_i},$$

$$g_1^{(l_1)}(r) = A(A - 1) \cdots (A - l_1 + 1)r^{A-l_1},$$

$$g_2^{(l_2)}(r) = (B - 1)(B - 2) \cdots (B - l_2)(-1)^{l_2}(1 - r)^{B-l_2-1}.$$

Since $y_i \geq r$ and $0 \leq \eta_i \leq 1$, the equalities above imply

$$u_i^{(j_i)}(r) \ll y_i^{\alpha_i} r^{-j_i},$$

$$g_1^{(l_1)}(r) \ll g_1(r)r^{-l_1},$$

$$g_2^{(l_2)}(r) \ll (1 - r)^{B-k-1} = g_2(r)(1 - r)^{-k},$$

where the last inequality is obtained because $0 \leq l_2 \leq k$. From these three, we deduce that

$$g_2^{(l_2)}(r) g_1^{(l_1)}(r) u_1^{(j_1)}(r) \cdots u_{n-1}^{(j_{n-1})}(r) \ll g_2(r)(1 - r)^{-k} g_1(r) u_1(r) \cdots$$

$$\cdots u_{n-1}(r)r^{-l_1-j_1-\cdots-j_{n-1}}$$

$$\ll r^{-k}(1 - r)^{-k} g_2(r) g_1(r) u_1(r) \cdots u_{n-1}(r).$$

Multiplying both sides by $\eta^\beta G(\xi) \, d\xi$ and integrating over $J$ yield

$$h^{(k)}(r) \ll r^{-k}(1 - r)^{-k} h(r)$$
and by Lemma A,

\[ h(r) \ll r^{A} (1 - r)^{B-1} (\ln(e/r))^{n-1}. \]

Hence we have

\[ h^{(k)}(r) \ll r^{A-k} (1 - r)^{B-k-1} (\ln(e/r))^{n-1}. \]

This proves Lemma B. \qed

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