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ROOTS OF TOEPLITZ OPERATORS ON THE BERGMAN SPACE

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A major goal in the theory of Toeplitz operators on the Bergman space over the unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$ is to competely describe the commutant of a given Toeplitz operator, that is, the set of all Toeplitz operators that commute with it. In [2007], the first author characterized the commutant of a Toeplitz operator T that has a quasihomogeneous symbol $\phi(r)e^{ip\theta}$ with $p > 0$, in case it has a Toeplitz p-th root S with symbol $\psi(r)e^{i\theta}$: The commutant of T is the closure of the linear space generated by powers S^n that are Toeplitz. But the existence of a p-th root was known until now only when $\phi(r) = r^m$ with $m \geq 0$. Here we will show the existence of p-th roots for a much larger class of symbols, for example, those symbols for which

$$\phi(r) = \sum_{i=1}^{k} r^{a_i} (\ln r)^{b_i}, \quad \text{where } 0 \leq a_i, b_i \text{ for all } 1 \leq i \leq k.$$

1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$, and let $dA = r dr d\theta/\pi$ be the Lebesgue area measure normalized so that $\mathbb{D}$ has unit measure. Let L_a^2 be the Bergman space, the Hilbert space of functions that are analytic on $\mathbb{D}$ and square integrable with respect to dA. We denote the inner product in $L^2(\mathbb{D}, dA)$ by $\langle \cdot, \cdot \rangle$. It is well known that L_a^2 is a closed subspace of the Hilbert space $L^2(\mathbb{D}, dA)$, and the set $\{\sqrt{n+1} z^n \mid n \geq 0\}$ of functions is an orthonormal basis. Let P be the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto L_a^2. For a bounded function f on $\mathbb{D}$, the Toeplitz operator T_f with symbol f is defined by

$$T_f(h) = P(fh) \quad \text{for } h \in L_a^2.$$

A symbol f is said to be *quasihomogeneous* of order p an integer if it can be written as $f(re^{i\theta}) = e^{ip\theta}\phi(r)$, where ϕ is a radial function on $\mathbb{D}$. In this case, the associated Toeplitz operator T_f is also called quasihomogeneous Toeplitz of order p. Quasihomogeneous Toeplitz operators were first introduced in [Louhichi

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and Zakariasy 2005] while generalizing the results of [Čučković and Rao 1998]. We assume $p > 0$ from now on.

For a given a quasihomogeneous operator T of degree p, we seek a quasihomogeneous operator S of degree 1 such that $S^p = T$. Louhichi [2007] proved that if any such root exists, it is unique up to a multiplicative constant. Using the results in [Čučković and Rao 1998], Louchichi also proved the existence of p-th roots for the case $\phi(r) = r^m$ for any arbitrary $m \geq 0$ and $p > 0$. Here we plan to deal with more general $\phi(r)$.

2. The Mellin transform and two lemmas

For any two functions $f(r)$ and $g(r)$ defined on $I = [0, 1]$, we define the Mellin convolution by

$$(f *_M g)(r) = \int_r^1 f\Big(\frac{r}{t}\Big) g(t) \frac{dt}{t}.$$

Often we are interested in knowing when the Mellin convolution is a bounded function in the interval I. We say a function f is of type (a, b) with $a \geq 0$ and $b > 0$ if

$$|f(r)| \leq Cr^a (1-r)^{b-1} \quad \text{on } I,$$

where C is a constant depending on f. Also we express the same thing as

$$f(r) \ll r^a (1-r)^{b-1},$$

omitting the constants and the absolute value signs.

Lemma A. *Suppose $f(r)$ is of type (a, b) and $g(r)$ is of type (c, d). Then*

$$\begin{cases} (f *_M g) \quad \text{is of type} \quad (\min\{a, c\}, b+d) & \text{if } a \neq c, \text{ and} \\ (f *_M g)(r) \ll r^{\min\{a,c\}} (1-r)^{b+d-1} \ln(e/r) & \text{if } a = c. \end{cases}$$

This can be generalized to any finite product as follows: Suppose for $1 \leq i \leq n$, $f_i(r)$ is of type (a_i, b_i). Then their Mellin convolution product $h(r)$ satisfies

$$h(r) \ll r^{\alpha} (1-r)^{\beta-1} \Big(\ln\Big(\frac{e}{r}\Big)\Big)^{n-1} \tag{1}$$

where $\alpha = \min\{a_i\}$ and $\beta = \sum b_i$. Further, if we know that the number of a_i that are equal to $\min\{a_i\}$ is (say) l, the estimate (1) can be improved to

$$h(r) \ll r^{\alpha} (1-r)^{\beta-1} \Big(\ln\Big(\frac{e}{r}\Big)\Big)^{l-1}. \tag{2}$$

Thus the log term will disappear if $l = 1$.

Remark 2.1. Most of the time our aim is to prove h is bounded; the presence of log does not interfere with that aim since $\alpha > 0$, which bounds $h(r)$ near zero, and if we assume further that $\beta \geq 1$, it would be bounded near 1 also. But log cannot be avoided. Take for example $f_i(r) = r$ for every i and compute the Mellin convolution product. It turn out to be $r(\ln r)^{n-1}/(n-1)!$, by a simple integration.

Lemma B. *Let $f_i(r) = r^{a_i}(1-r)^{b_i-1}$, where a_i and b_i are positive for $1 \leq i \leq n$. Let α and β be as defined in Lemma A. Let h be the Mellin covolution product of the f_i. For any integer $k \geq 0$, the k-th derivative of h satisfies*

$$h^{(k)}(r) \ll r^{\alpha-k}(1-r)^{\beta-k-1}\Big(\ln\Big(\frac{e}{r}\Big)\Big)^{n-1}.$$

Here the implied constant depends on k and h.

3. Applications of Lemmas A and B

The Mellin transform $\hat{\phi}$ of a radial function ϕ in $L^1([0, 1], r dr)$ is defined by

$$\hat{\phi}(z) = \int_0^1 \phi(r) r^{z-1}\, dr = \mathcal{M}(\phi)(z).$$

It is well known that for these functions the Mellin transform is well-defined on the right half-plane $\{z : \operatorname{Re} z \geq 2\}$ and analytic on $\{z : \operatorname{Re} z > 2\}$. The Mellin transform $\hat{\phi}$ is uniquely determined by its values on any arithmetic sequence of integers. In fact we have the following classical theorem [Remmert 1998, page 102].

Theorem 3.1. *Suppose f is a bounded analytic function on $\{z : \operatorname{Re} z > 0\}$ that vanishes at the pairwise distinct points $z_1, z_2, \ldots,$ where*

(1) $\inf\{|z_n|\} > 0$ *and*

(2) $\sum_{n\geq 1} \operatorname{Re}(1/z_n) = \infty$.

Then f vanishes identically on $\{z : \operatorname{Re} z > 0\}$.

Remark 3.2. One can apply this theorem to prove that if $\phi \in L^1([0, 1], r dr)$ and if there exist $n_0, p \in \mathbb{N}$ such that

$$\hat{\phi}(pk + n_0) = 0 \quad \text{for all } k \in \mathbb{N},$$

then $\hat{\phi}(z) = 0$ for all $z \in \{z : \operatorname{Re} z > 2\}$ and so $\phi = 0$.

It is easy to see that the Mellin transform converts the Mellin convolution product into a pointwise product, that is,

$$(\widehat{\phi *_M \psi})(r) = \hat{\phi}(r)\hat{\psi}(r).$$

A direct calculation shows that a quasihomogeneous Toeplitz operator acts on the elements of the orthogonal basis of L^2_a as a shift operator with a holomorphic weight. In fact, for $p \geq 0$ and for all $k \geq 0$, we have

$$\begin{aligned} T_{e^{ip\theta}\phi}(z^k) = P(e^{ip\theta}\phi z^k) &= \sum_{n\geq 0}(n+1)\langle e^{ip\theta}\phi z^k, z^n\rangle z^n \\ &= \sum_{n\geq 0}(n+1)\int_0^1\int_0^{2\pi}\phi(r)r^{k+n+1}e^{i(k+p-n)\theta}\frac{d\theta}{\pi}dr z^n \\ &= 2(k+p+1)\hat{\phi}(2k+p+2)z^{k+p}. \end{aligned}$$

Now we are ready to start with a relatively easy example.

3.1. ***Assuming $\phi(r) = r + r^2$, find the p-th roots of $T_{e^{ip\theta}\phi}$.*** If there exists a bounded radial function ψ such that $(T_{e^{i\theta}\psi})^p = T_{e^{ip\theta}\phi}$, then

$$(T_{e^{i\theta}\psi})^p(z^k) = T_{e^{ip\theta}\phi}(z^k) \quad \text{for all } k \geq 0.$$

Since

$$(T_{e^{i\theta}\psi})^p(z^k) = \left(\prod_{j=0}^{p-1}(2k+2j+4)\hat{\psi}(2k+2j+3)\right)z^{k+p},$$

we obtain for all integers $k \geq 0$

$$(2k+2p+2)\hat{\phi}(2k+p+2) = \left(\prod_{j=0}^{p-1}(2k+2j+4)\hat{\psi}(2k+2j+3)\right),$$

which is equivalent to

$$\frac{\hat{\phi}(2k+p+2)}{\prod_{j=0}^{p-2}(2k+2j+4)} = \prod_{j=0}^{p-1}\hat{\psi}(2k+2j+3).$$

Note that p is a positive integer and that our discussion is trivial for $p = 1$. So $p \geq 2$. By setting $z = 2k+3$, we notice that the function

$$f(z) = \frac{\hat{\phi}(z+p-1)}{\prod_{j=0}^{p-2}(z+2j+1)} - \prod_{j=0}^{p-1}\hat{\psi}(z+2j)$$

is holomorphic and bounded in the right half-plane and vanishes for $z = 2k+3$, for k any nonnegative integer. Now by Theorem 3.1, we get $f(z) \equiv 0$. Therefore

$$(z+2p-1)\hat{\phi}(z+p-1) = \left(\prod_{j=0}^{p-1}(z+2j+1)\hat{\psi}(z+2j)\right). \tag{2}$$

If we divide the (2) by the equation obtained by replacing z by $z+2$ in (2), we obtain after cancelation that in the right half-plane

$$\frac{\hat{\psi}(z+2p)}{\hat{\psi}(z)} = \frac{(z+1)\hat{\phi}(z+p+1)}{(z+2p-1)\hat{\phi}(z+p-1)} \quad \text{for } \operatorname{Re} z > 0. \tag{3}$$

Since

$$\hat{\phi}(z) = \frac{1}{z+1} + \frac{1}{z+2} = \frac{2z+3}{(z+1)(z+2)},$$

it follows that, for $\operatorname{Re} z > 0$,

$$\frac{\hat{\psi}(z+2p)}{\hat{\psi}(z)} = \frac{(z+1)}{(z+2p-1)} \frac{(2z+2p+5)}{(z+p+2)(z+p+3)} \frac{(z+p)(z+p+1)}{(2z+2p+1)}.$$

Letting $\lambda(\zeta) = \hat{\psi}(2p\zeta)$, this equation becomes, for $\operatorname{Re} \zeta > 0$,

$$\frac{\lambda(\zeta+1)}{\lambda(\zeta)} = \frac{(2p\zeta+1)(4p\zeta+2p+5)(2p\zeta+p)(2p\zeta+p+1)}{(2p\zeta+2p-1)(2p\zeta+p+2)(2p\zeta+p+3)(4p\zeta+2p+1)}.$$

Using the well-known identity $\Gamma(z+1) = z\Gamma(z)$, where Γ is the Gamma function, we can write

$$\frac{\lambda(\zeta+1)}{\lambda(\zeta)} = \frac{F(\zeta+1)}{F(\zeta)} \quad \text{for } \operatorname{Re} \zeta > 0, \tag{4}$$

where

$$F(\zeta) = \frac{\Gamma(\zeta+a_1)\Gamma(\zeta+a_2)\Gamma(\zeta+a_3)\Gamma(\zeta+a_4)}{\Gamma(\zeta+a_1')\Gamma(\zeta+a_2')\Gamma(\zeta+a_3')\Gamma(\zeta+a_4')},$$

and the a_i are in increasing order

$$\frac{2}{4p}, \quad \frac{2p}{4p}, \quad \frac{2p+2}{4p}, \quad \frac{2p+5}{4p}$$

respectively and the a_i' are in almost increasing order

$$\frac{2p+1}{4p}, \quad \frac{2p+4}{4p}, \quad \frac{4p-2}{4p}, \quad \frac{2p+6}{4p}$$

respectively for $i = 1, \dots, 4$. We shall show in a moment that $F(\zeta)$ is a bounded holomorphic function in the right half-plane. Granting that, Equation (4) combined with [Louhichi 2007, Lemma 6, page 1468] implies exists a constant C such that

$$\lambda(\zeta) = CF(\zeta) \quad \text{for } \operatorname{Re} \zeta > 0. \tag{5}$$

A basic observation is that the quotient of two Gamma functions

$$\frac{\Gamma(\zeta+a_i)}{\Gamma(\zeta+a_i')}, \quad \text{where } 0 < a_i < a_i',$$

is a constant times the Beta function

$$B(\zeta+a_i, a_i'-a_i)=\int_0^1 x^{\zeta+a_i-1}(1-x)^{a_i'-a_i-1}\,dx.$$

According to our definition of the Mellin transform, $B(\zeta+a_i, a_i'-a_i)$ is the Mellin transform of $x^{a_i}(1-x)^{a_i'-a_i-1}$, which is of type $(a_i, a_i'-a_i)$. Since $a_i < a_i'$ for $i=1,\dots,4$ (in fact, $a_3' \geq a_3$ if and only if $2p \geq 4$, which is always true), each of the Beta functions is a bounded holomorphic function in the right half-plane and $F(\zeta)$, which is a constant times the product of these four Beta functions, is a bounded holomorphic function in the right half-plane. Equation (5) implies that

$$\lambda(\zeta)=C\sum_{i=1}^{4} B(\zeta+a_i, a_i'-a_i),$$

where C is a constant. Since the product of Mellin transforms equals the Mellin transform of the Mellin convolution product, we have

$$\lambda(\zeta)=Ch(\zeta),$$

where h is the convolution product of four functions of type $(a_i, a_i'-a_i)$ for $i=1,\dots,4$. Now Lemma A tells us that

$$h(r)\ll r^{\min\{a_i\}}(1-r)^{\sum_i(a_i'-a_i)-1}\ln(e/r).$$

Because $\sum_i a_i'-a_i=1$, we have

$$h(r)\ll r^{\min\{a_i\}}\ln(e/r),$$

and hence h is a bounded function. Therefore the function ψ, if it exists, satisfies the equation

$$\hat{\psi}(2p\zeta)=C\hat{h}(\zeta)$$

for some constant C, which is equivalent to

$$\int_0^1 \psi(r)r^{2p\zeta-1}dr=C\int_0^1 h(t)t^{\zeta-1}dt.$$

Now, by a change of variables $t=r^{2p}$, we obtain

$$\int_0^1 \psi(r)r^{2p\zeta-1}dr=\int_0^1 h(r^{2p})r^{2p\zeta-1}2p\,dr.$$

Thus $\psi(r)=2ph(r^{2p})$, and so ψ is bounded. Hence the operator $T_{e^{i\theta}\psi}$ is a genuine Toeplitz operator and a p-th root of $T_{e^{ip\theta}\phi}$.

3.2. ***p-th roots of $T_{e^{ip\theta}\phi}$, where $\hat{\phi}(z)$ is a proper rational fraction.*** Such functions are plenty. For example, take $\Phi(r) = r^a \ln(r)^b$, where $a > 0$ and b is a nonnegative integer. By integration by parts we see that $\hat{\Phi}(z) = (-1)^b b!/(a+z)^{b+1}$.

Assume we are given a radial function $\phi(r)$ such that $\hat{\phi}(r)$ is a proper rational function. Recall that if there is a radial function ψ such that $(T_{e^{i\theta}\psi})^p = T_{e^{ip\theta}\phi}$, then we have Equation (3), which is

$$\hat{\psi}(z+2p) = \hat{\psi}(z)\frac{(z+1)\hat{\phi}(z+p+1)}{(z+2p-1)\hat{\phi}(z+p-1)} \quad \text{for } \operatorname{Re} z > 0.$$

Here we are assuming $\hat{\phi}(z) = P(z)/Q(z)$, where

$$P(z) = \prod_{j=1}^{m}(z+a_j) \quad \text{and} \quad Q(z) = \prod_{k=1}^{n}(z+b_k)$$

with $1 \le m < n$. So,

$$\begin{aligned}\hat{\psi}(z+2p) &= \hat{\psi}(z)\frac{(z+1)}{(z+2p-1)}\frac{P(z+p+1)Q(z+p-1)}{P(z+p-1)Q(z+p+1)}\\ &= \frac{(z+1)}{(z+2p-1)}\prod_{j=1}^{m}\frac{z+a_j+p+1}{z+a_j+p-1}\prod_{k=1}^{n}\frac{z+b_k+p-1}{z+b_k+p+1}\end{aligned}$$

Let $\lambda(\zeta) = \hat{\psi}(2p\zeta)$. Then the equality above becomes

$$\begin{aligned}\frac{\lambda(\zeta+1)}{\lambda(\zeta)} &= \frac{(2p\zeta+1)}{(2p\zeta+2p-1)}\prod_{j=1}^{m}\frac{2p\zeta+a_j+p+1}{2p\zeta+a_j+p-1}\prod_{k=1}^{n}\frac{2p\zeta+b_k+p-1}{2p\zeta+b_k+p+1}\\ &= \frac{F(\zeta+1)G(\zeta)}{F(\zeta)G(\zeta+1)},\end{aligned}$$

where

$$F(\zeta) = \frac{\Gamma(\zeta+A_0)}{\Gamma(\zeta+A_0')}\prod_{k=1}^{n}\frac{\Gamma(\zeta+B_k)}{\Gamma(\zeta+B_k')} \quad \text{and} \quad G(\zeta) = \prod_{j=1}^{m}\frac{\Gamma(\zeta+A_j')}{\Gamma(\zeta+A_j)},$$

$$\begin{aligned} A_0 &= \frac{1}{2p}, & A_0' &= \frac{2p-1}{2p},\\ A_j &= \frac{a_j+p+1}{2p}, & A_j' &= \frac{a_j+p-1}{2p},\\ B_k &= \frac{b_k+p-1}{2p}, & B_k' &= \frac{b_k+p+1}{2p} \quad \text{for } 1 \le j \le m \text{ and } 1 \le k \le n.\end{aligned}$$

Note that any quotient of two Gamma functions, say,

$$\frac{\Gamma(\zeta+\alpha)}{\Gamma(\zeta+\gamma)}=\beta(\zeta+\alpha,\gamma-\alpha)\Gamma(\gamma-\alpha)$$

is a bounded holomorphic function in the right half-plane if α and $\gamma-\alpha$ are positive. Hence both $F(\zeta)$ and $G(\zeta)$ are bounded holomorphic functions in the right half-plane if we assume all A_j, A'_j, B_k, B'_k are positive. We will assume that.

Therefore, by [Louhichi 2007, Lemma 6, page 1468], λ is a constant times the quotient of $m+n+1$ Gamma functions in the numerator and about the same in the denominator, as follows:

$$\lambda(\zeta)=C\frac{\Gamma(\zeta+A_0)}{\Gamma(\zeta+A'_0)}\prod_{j=1}^{m}\frac{\Gamma(\zeta+A_j)}{\Gamma(\zeta+A'_j)}\prod_{k=1}^{n}\frac{\Gamma(\zeta+B_k)}{\Gamma(\zeta+B'_k)}. \tag{6}$$

Based on the argument of the previous subsection, we would like to write each quotient of two Gamma functions as a constant times a Beta function. In order to do that, we must assume that all A_j and B_k are positive for every $0\le j\le m$ and $1\le k\le n$. Moreover, we observe that

$$A'_0-A_0=\frac{p-1}{p},\quad A'_j-A_j=-\frac{1}{p},\quad B'_k-B_k=\frac{1}{p}.$$

So each quotient of two Gamma functions in Equation (6) can be written as a constant times a Beta function except those involving A_j for $1\le j\le m$. We fix this matter by noting that $\Gamma(\zeta+A'_j+1)=(\zeta+A'_j)\Gamma(\zeta+A'_j)$, and so here $A'_j+1-A_j=(p-1)/p$. Hence, Equation (6) becomes

$$\frac{\lambda(\zeta)}{\prod_{j=1}^{m}(\zeta+A'_j)}=C\frac{\Gamma(\zeta+A_0)}{\Gamma(\zeta+A'_0)}\prod_{j=1}^{m}\frac{\Gamma(\zeta+A_j)}{\Gamma(\zeta+A'_j+1)}\prod_{j=1}^{n}\frac{\Gamma(\zeta+B_j)}{\Gamma(\zeta+B'_j)}.$$

As in the previous subsection, this quotient of $m+n+1$ Gamma functions on the numerator and the same in the denominator, respectively would be the Mellin transform of the convolution product of $m+n+1$ functions of type (a_i,b_i). Let us call it h. By Lemma A, we have

$$h(r)\ll r^{A}(1-r)^{B-1}\Big(\ln\Big(\frac{e}{r}\Big)\Big)^{m+n},$$

where $A=\min\{A_j\}$, which is definitely positive, and B is given by

$$A'_0-A_0+\sum_{j=1}^{m}A'_j+1-A_j+\sum_{k=1}^{n}B'_k-B_k=(m+1)\frac{p-1}{p}+\frac{n}{p}=m+1+\frac{n-m-1}{p}.$$

Therefore we obtain

$$h(r) \ll r^A (1-r)^{m+(n-m-1)/p} \left(\ln\left(\frac{e}{r}\right)\right)^{m+n} = r^A (1-r)^{m+\upsilon} \left(\ln\left(\frac{e}{r}\right)\right)^{m+n},$$

where $\upsilon = (n-m-1)/p$ is a nonnegative number. Using Lemma B, we see that h has all derivatives of order not exceeding m and they satisfy the inequality

$$r^j h^{(j)}(r) \ll r^A (1-r)^{m-j+\upsilon} \left(\ln\left(\frac{e}{r}\right)\right)^{m+n}.$$

Further the function ψ, were it to exist, would satisfy the equation

$$\hat{\psi}(2p\zeta) = C\left(\prod_{j=1}^{m} (\zeta + A'_j)\right)\hat{h}(\zeta). \tag{7}$$

Now it is easy to check by integration by parts the identity

$$\zeta \hat{h}(\zeta) = -\mathcal{M}\left(r\frac{dh}{dr}\right)(\zeta)$$

provided h vanishes at 1 and rh' is bounded in (0, 1). Thus in the current case, letting $h' = Dh$, where $D = d/dr$, we can see

$$(\zeta + A'_j)\hat{h}(\zeta) = \mathcal{M}((A'_j - rD)h)(\zeta)$$

and

$$\left(\prod_{j=1}^{m} (\zeta + A'_j)\right)\hat{h}(\zeta) = \mathcal{M}\left(\prod_{j=1}^{m} (A'_j - rD)h\right)(\zeta).$$

Let us set

$$H(r) = \left(\prod_{j=1}^{m} (A'_j - rD)h\right)(r),$$

which allows us to rewrite Equation (7) as

$$\int_0^1 \psi(r) r^{2p\zeta-1} dr = C \int_0^1 H(t) t^{\zeta-1}\, dt.$$

Now, by a change of variables $t = r^{2p}$, we obtain

$$\int_0^1 \psi(r) r^{2p\zeta-1} dr = C \int_0^1 H(r^{2p}) r^{2p\zeta-1} 2p dr.$$

Thus $\psi(r) = 2pCH(r^{2p})$, and hence is bounded and the operator $T_{e^{i\theta}\psi}$ is a genuine Toeplitz operator and a p-th root of $T_{e^{ip\theta}\phi}$.

4. Proof of Lemma A for two functions

We start by proving Lemma A for functions f and g of type (a,b) and (c,d) respectively, where a, b, c and d are all positive. A similar thing was discussed in [Čučković and Rao 1998, pages 210-212] but with less generality since the goal was different.

Let $h(r) = (f *_M g)(r)$. By definition of the Mellin convolution, it is easy to see that

$$h(r) \ll \int_r^1 \left(\frac{r}{t}\right)^a \left(1-\frac{r}{t}\right)^{b-1} t^c (1-t)^{d-1}\,\frac{dt}{t},$$

which after changing variables as $\dfrac{t-r}{1-r} = u$ and using the consequent identities

$$t = r+u-ru, \quad t-r = u(1-r), \quad 1-t = (1-u)(1-r), \quad dt = (1-r)du$$

while keeping r fixed, leads to

$$\begin{aligned}
h(r) &\ll \int_r^1 \left(\frac{r}{t}\right)^a \left(1-\frac{r}{t}\right)^{b-1} t^c (1-t)^{d-1}\,\frac{dt}{t}\\
&= \int_r^1 \left(\frac{r}{t}\right)^a \left(\frac{t-r}{t}\right)^{b-1} t^c (1-t)^{d-1}\,\frac{dt}{t}\\
&= \int_0^1 r^a t^{-a} u^{b-1}(1-r)^{b-1} t^{-b+1} t^c (1-u)^{d-1}(1-r)^{d-1}(1-r)\frac{du}{t}\\
&= r^a(1-r)^{b+d-1}\int_0^1 t^{c-a-b}u^{b-1}(1-u)^{d-1}du.
\end{aligned}$$

We have the following cases.

- $c-a-b \geq 0$. Since $0 \leq t \leq 1$, we have

$$h(r) \ll r^a(1-r)^{b+d-1},$$

  and hence h is of type $(a, b+d)$.

- $c-a-b < 0$. Assuming $c-a > 0$ and noting that $t \geq u$, we obtain

$$\begin{aligned}
h(r) &\ll r^a(1-r)^{b+d-1}\int_0^1 u^{c-a-b}u^{b-1}(1-u)^{d-1}\,du\\
&\leq r^a(1-r)^{b+d-1}\int_0^1 u^{c-a-1}(1-u)^{d-1}\,du\\
&= r^a(1-r)^{b+d-1}B(c-a,d),
\end{aligned}$$

  and therefore h is of type $(a, b+d)$.

Now in case $c = a$, consider any number ϵ in $0 < \epsilon \le b$. Noticing that $t \ge r$ and $u > 0$, we have

$$\begin{aligned} h(r) &\ll r^a(1-r)^{b+d-1}\int_0^1 t^{-b}u^{b-1}(1-u)^{d-1}\,du \\ &\ll r^a(1-r)^{b+d-1}\int_0^1 t^{-\epsilon}t^{\epsilon-b}u^{b-1}(1-u)^{d-1}\,du \\ &\le r^a(1-r)^{b+d-1}\int_0^1 r^{-\epsilon}u^{\epsilon-b}u^{b-1}(1-u)^{d-1}\,du \\ &\le r^a(1-r)^{b+d-1}r^{-\epsilon}\int_0^1 u^{\epsilon-1}(1-u)^{d-1}\,du \\ &\le r^a(1-r)^{b+d-1}B(\epsilon,d)r^{-\epsilon}. \end{aligned}$$

Now since $\epsilon B(\epsilon, d) = \Gamma(\epsilon+1)\Gamma(d)/\Gamma(\epsilon+d)$ is holomorphic as a function of ϵ in a neighborhood of the interval $(0, b)$, there exists a constant C such that $\epsilon B(\epsilon, d) \le C$ on that interval, and therefore

$$h(r) \le Cr^a(1-r)^{b+d-1}r^{-\epsilon}\epsilon^{-1} \quad \text{for every } 0 < \epsilon \le b.$$

Here we emphasize the fact that C does not depend on r and ϵ as long as $0 < r < 1$ and $0 < \epsilon \le b$. For a fixed but arbitrary r, let $E(\epsilon) = r^{-\epsilon}\epsilon^{-1}$ and $m(r) = \min_{(0,b]} E(\epsilon)$. Then

$$h(r) \le Cr^a(1-r)^{b+d-1}m(r). \tag{8}$$

Moreover the function E decreases in the interval $(0, -1/\ln r)$ and increases in the interval $(-1/\ln r, +\infty)$. Further $-1/\ln r \le b$ if and only if $r \le e^{-1/b}$. Thus Equation (8) implies that

$$\begin{aligned} h(r) &\ll r^a(1-r)^{b+d-1}m(r) \le r^a(1-r)^{b+d-1}e\ln(1/r) && \text{if } r \le e^{-1/b}, \\ h(r) &\ll r^a(1-r)^{b+d-1}r^{-b}b^{-1} \le r^a(1-r)^{b+d-1}e/b && \text{if } r > e^{-1/b}, \end{aligned}$$

Combining these two results, we obtain

$$\begin{aligned} h(r) &\ll r^a(1-r)^{b+d-1}(e\ln(1/r) + e/b) \\ &\ll r^a(1-r)^{b+d-1}\ln(e/r) \quad \text{for all } 0 < r < 1. \end{aligned}$$

5. Lemma A for the Mellin convolution product of more than two functions

In this context we can assume that the function f_i, of type (a_i, b_i), is given by

$$f_i(x) = x^{a_i}(1-x)^{b_i-1} \quad \text{for } 1 \le i \le n.$$

The Mellin convolution product of these n functions is defined by a repeated integral

$$
(9)\quad h(r)=\int_r^1\int_{r/x_1}^1\int_{r/x_1x_2}^1\cdots\int_{r/x_1x_2\cdots x_{n-2}}^1 f_1(x_1)f_2(x_2)\cdots f_{n-1}(x_{n-1})f_n\Big(\frac{r}{x_1\cdots x_{n-1}}\Big)\frac{dx_{n-1}}{x_{n-1}}\cdots\frac{dx_3}{x_3}\frac{dx_2}{x_2}\frac{dx_1}{x_1}
$$

As in the case of two functions where we changed variables as $u=(t-r)/(1-r)$, we change variables so that the new integral is over the unit cube I^{n-1}, where limits of integration do not depend on other variables. Let $y_0=1$ and inductively define $y_i=\prod_{j=1}^i x_i$ for $i\geq 1$. Now we change variables as

$$
x_i=\frac{r}{y_{i-1}}+\Big(1-\frac{r}{y_{i-1}}\Big)\xi_i \quad \text{for } i\geq 1,
$$

so that the limits for each ξ_i are 0 and 1. Further we note

$$
y_i-r=x_iy_{i-1}-r=(y_{i-1}-r)\xi_i \quad \text{for } i\geq 0.
$$

Set $\eta_0=1$ and $\eta_i=\prod_{j=1}^i \xi_i$ for $i\geq 1$. It is easy to show, by induction on i, that

$$
y_i-r=(1-r)\eta_i \quad \text{for all } i\geq 1.
$$

Further

$$
(10)\qquad (1-x_i)=(1-\xi_i)\Big(1-\frac{r}{y_{i-1}}\Big)=\frac{(1-\xi_i)(1-r)\eta_{i-1}}{y_{i-1}} \quad \text{for all } i\geq 1.
$$

Thus

$$
f_i(x_i)=x_i^{a_i}(1-x_i)^{b_i-1}=\Big(\frac{y_i}{y_{i-1}}\Big)^{a_i}\Big(\frac{(1-\xi_i)(1-r)\eta_{i-1}}{y_{i-1}}\Big)^{b_i-1} \quad \text{for } 1\leq i\leq n-1.
$$

But for $i=n$, we have

$$
\begin{aligned}
f_n(x_n)&=\Big(\frac{r}{y_{n-1}}\Big)^{a_n}\Big(1-\frac{r}{y_{n-1}}\Big)^{b_n-1}\\
&=\Big(\frac{r}{y_{n-1}}\Big)^{a_n}\Big(\frac{y_{n-1}-r}{y_{n-1}}\Big)^{b_n-1}=\Big(\frac{r}{y_{n-1}}\Big)^{a_n}\Big(\frac{(1-r)\eta_{n-1}}{y_{n-1}}\Big)^{b_n-1}.
\end{aligned}
$$

Writing the product of functions in (9) in terms of ξ_i, η_i, r and y_i for $1\leq i\leq n-1$ yields

$$
(11)\qquad r^{a_n}\prod_{i=1}^{n-1}\eta_i^{b_{i+1}-1}\prod_{i=1}^{n-1}y_i^{a_i-a_{i+1}-b_{i+1}+1}\prod_{i=1}^{n-1}(1-\xi_i)^{b_i-1}\prod_{i=1}^{n}(1-r)^{b_i-1}.
$$

Using equalities of (10), we calculate the differential form

$$\bigwedge_{i=1}^{n-1}\frac{dx_i}{x_i}=\bigwedge_{i=1}^{n-1}\frac{(1-r)\eta_{i-1}d\xi_i}{x_i y_{i-1}}=(1-r)^{n-1}\prod_{i=1}^{n-1}\frac{\eta_{i-1}}{y_i}\bigwedge_{i=1}^{n-1}d\xi_i. \tag{12}$$

From (9), (10), and (11) we derive

$$h(r)=r^{a_n}(1-r)^{b_1+\cdots+b_n-1}\int_{I^{n-1}}\eta_{n-1}^{b_n-1}\prod_{i=1}^{n-2}\eta_i^{b_{i+1}} \times\prod_{i=1}^{n-1}y_i^{a_i-a_{i+1}-b_{i+1}}\prod_{i=1}^{n-1}(1-\xi_i)^{b_i-1}\bigwedge_{i=1}^{n-1}d\xi_i. \tag{13}$$

Let us assume that the a_i are arranged in decreasing order. Then

$$\eta_i^{b_{i+1}}y_i^{a_i-a_{i+1}-b_{i+1}}\le 1$$

since $\eta_i\le y_i\le 1$. Therefore

$$h(r)\le r^{a_n}(1-r)^{b_1+\cdots+b_n-1}\int_{I^{n-1}}\eta_{n-1}^{b_n-1}y_{n-1}^{a_{n-1}-a_n-b_n}\prod_{i=1}^{n-1}(1-\xi_i)^{b_i-1}\bigwedge_{i=1}^{n-1}d\xi_i.$$

Here four cases have to be discussed.

Case 1: $a_{n-1}-a_n-b_n\ge 0$. In this case

$$\begin{aligned}h(r)&\le r^{a_n}(1-r)^{b_1+\cdots+b_n-1}\int_{I^{n-1}}\eta_{n-1}^{b_n-1}\prod_{i=1}^{n-1}(1-\xi_i)^{b_i-1}\bigwedge_{i=1}^{n-1}d\xi_i\\ &\le r^{a_n}(1-r)^{b_1+\cdots+b_n-1}\prod_{i=1}^{n-1}\frac{\Gamma(b_n)\Gamma(b_i)}{\Gamma(b_n+b_i)}.\end{aligned}$$

Case 2: $a_{n-1}-a_n-b_n<0$ and $a_{n-1}\ne a_n$. Then

$$\begin{aligned}h(r)&\le r^{a_n}(1-r)^{b_1+\cdots+b_n-1}\int_{I^{n-1}}y_{n-1}^{a_{n-1}-a_n-b_n}\eta_{n-1}^{b_n-1}\prod_{i=1}^{n-1}(1-\xi_i)^{b_i-1}\bigwedge_{i=1}^{n-1}d\xi_i\\ &\le r^{a_n}(1-r)^{b_1+\cdots+b_n-1}\int_{I^{n-1}}\eta_{n-1}^{a_{n-1}-a_n-b_n}\eta_{n-1}^{b_n-1}\prod_{i=1}^{n-1}(1-\xi_i)^{b_i-1}\bigwedge_{i=1}^{n-1}d\xi_i\\ &\le r^{a_n}(1-r)^{b_1+\cdots+b_n-1}\int_{I^{n-1}}\eta_{n-1}^{a_{n-1}-a_n-1}\prod_{i=1}^{n-1}(1-\xi_i)^{b_i-1}\bigwedge_{i=1}^{n-1}d\xi_i\\ &\le r^{a_n}(1-r)^{b_1+\cdots+b_n-1}\prod_{i=1}^{n-1}\frac{\Gamma(a_{n-1}-a_n)\Gamma(b_i)}{\Gamma(a_{n-1}-a_n+b_i)}.\end{aligned}$$

Case 3: $a_{n-1} = a_n$. Choose an arbitrary $0 < \epsilon \le b_n$. Note that $y_{n-1} \ge r$ and $\eta_{n-1} > 0$. Then

$$\begin{aligned} h(r) &\le r^{a_n}(1-r)^{b_1+\cdots+b_n-1} \int_{I^{n-1}} y_{n-1}^{-b_n} \eta_{n-1}^{b_n-1} \prod_{i=1}^{n-1} (1-\xi_i)^{b_i-1} \bigwedge_{i=1}^{n-1} d\xi_i \\ &\le r^{a_n}(1-r)^{b_1+\cdots+b_n-1} \int_{I^{n-1}} y_{n-1}^{-\epsilon} y_{n-1}^{\epsilon-b_n} \eta_{n-1}^{b_n-1} \prod_{i=1}^{n-1} (1-\xi_i)^{b_i-1} \bigwedge_{i=1}^{n-1} d\xi_i \\ &\le r^{a_n}(1-r)^{b_1+\cdots+b_n-1} \int_{I^{n-1}} r^{-\epsilon} \eta_{n-1}^{\epsilon-b_n} \eta_{n-1}^{b_n-1} \prod_{i=1}^{n-1} (1-\xi_i)^{b_i-1} \bigwedge_{i=1}^{n-1} d\xi_i \\ &\le r^{a_n}(1-r)^{b_1+\cdots+b_n-1} \int_{I^{n-1}} r^{-\epsilon} \eta_{n-1}^{\epsilon-1} \prod_{i=1}^{n-1} (1-\xi_i)^{b_i-1} \bigwedge_{i=1}^{n-1} d\xi_i \\ &= r^{a_n}(1-r)^{b_1+\cdots+b_n-1} r^{-\epsilon} \prod_{i=1}^{n-1} \frac{\Gamma(\epsilon)\Gamma(b_i)}{\Gamma(\epsilon+b_i)}. \end{aligned}$$

Similarly to Section 2, this product of quotients of Gamma functions is meromorphic on the interval $[0, b_n]$ except at zero, where it has a pole of order $n-1$, and so there exists a constant C such that

$$\prod_{i=1}^{n-1} \frac{\Gamma(\epsilon)\Gamma(b_i)}{\Gamma(\epsilon+b_i)} \le C\epsilon^{1-n}.$$

Hence

$$h(r) \ll r^{a_n}(1-r)^{b_1+\cdots+b_n-1} r^{-\epsilon} \epsilon^{1-n}.$$

Now there are two subcases: If $r \le e^{-1/b_n}$, then $(\ln(1/r))^{-1} \le b_n$. In this case, we choose $\epsilon = (\ln(1/r))^{-1}$ and obtain

$$h(r) \ll r^{a_n}(1-r)^{b_1+\cdots+b_n-1} e(\ln(1/r))^{n-1}. \tag{14}$$

On the other hand, if $r \ge e^{-1/b_n}$, we choose $\epsilon = b_n$ and obtain

$$\begin{aligned} h(r) &\ll r^{a_n}(1-r)^{b_1+\cdots+b_n-1} r^{-b_n} b_n^{1-n} \\ &\le r^{a_n}(1-r)^{b_1+\cdots+b_n-1} e b_n^{1-n}. \end{aligned} \tag{15}$$

Combining (14) and (15) yields

$$\begin{aligned} h(r) &\ll r^{a_n}(1-r)^{b_1+\cdots+b_n-1} (e(\ln\left(1/r\right))^{n-1} + e b_n^{1-n}) \\ &\ll r^{a_n}(1-r)^{b_1+\cdots+b_n-1} (\ln(1/r))^{n-1} \quad \text{for all } 0 < r < 1. \end{aligned}$$

This is enough for our purposes but we can get a more refined estimate as mentioned in the second case of Lemma A. Thus we reach the final case:

Case 4: there exists k such that $a_k > a_{k+1} = \cdots = a_n = a$. Let $F(r)$ be the Mellin convolution product of $f_1, f_2, \ldots, f_{k+1}$ and $G(r)$ be the Mellin convolution product of the rest, namely $f_{k+2} \ldots, f_n$. From the previous discussion it is clear that

$$F(r) \ll r^a (1-r)^{b_1+\cdots+b_{k+1}-1}$$

and

$$G(r) \ll r^a (1-r)^{b_{k+2}+\cdots+b_n-1} (\ln(e/r))^{n-k-2}.$$

Let $b = b_1 + \cdots + b_{k+1}$, $d = b_{k+2} + \cdots + b_n$ and $n-k-1 = l$. The case $l = 1$ has been treated previously. So assume $l > 1$. We see that

$$\begin{aligned} h(r) &= (F *_M G)(r) \\ &\ll \int_r^1 (r/t)^a (1-r/t)^{b-1} t^a (1-t)^{d-1} \Big(\ln\Big(\frac{e}{t}\Big)\Big)^{l-1} \frac{dt}{t} \\ &\le r^a \int_r^1 (t-r)^{b-1} t^{-b} (1-t)^{d-1} \Big(\ln\Big(\frac{e}{t}\Big)\Big)^{l-1} \frac{dt}{t}. \end{aligned}$$

Now the change of variables $t = u + r - ur$ leads to $t - r = u(1-r)$, $1 - t = (1-u)(1-r)$, $dt = (1-r)du$ and

$$\begin{aligned} h(r) &\ll r^a \int_0^1 (1-r)^{b-1} u^{b-1} t^{-b} (1-r)^{d-1} (1-u)^{d-1} \Big(\ln\Big(\frac{e}{t}\Big)\Big)^{l-1} (1-r)\, du \\ &\le r^a (1-r)^{b+d-1} \int_0^1 u^{b-1} t^{-b} (1-u)^{d-1} \Big(\ln\Big(\frac{e}{t}\Big)\Big)^{l-1} du. \end{aligned}$$

Noting that $t \ge u$ and $r > 0$, and choosing an arbitrary $0 < \epsilon \le b$ implies

$$\begin{aligned} h(r) &\ll r^a (1-r)^{b+d-1} \int_0^1 u^{b-1} t^{-\epsilon} t^{\epsilon-b} (1-u)^{d-1} \Big(\ln\Big(\frac{e}{t}\Big)\Big)^{l-1} du \\ &\le r^a (1-r)^{b+d-1} \int_0^1 u^{b-1} r^{-\epsilon} u^{\epsilon-b} (1-u)^{d-1} \Big(\ln\Big(\frac{e}{t}\Big)\Big)^{l-1} du \\ &\le r^a (1-r)^{b+d-1} r^{-\epsilon} \int_0^1 u^{b-1} u^{\epsilon-b} (1-u)^{d-1} \Big(\ln\Big(\frac{e}{u}\Big)\Big)^{l-1} du \\ &\le r^a (1-r)^{b+d-1} r^{-\epsilon} \int_0^1 u^{\epsilon-1} (1-u)^{d-1} \Big(\ln\Big(\frac{e}{u}\Big)\Big)^{l-1} du. \end{aligned}$$

Let $H_j(\epsilon) = \int_0^1 u^{\epsilon-1}(1-u)^{d-1}(\ln u)^j du$. This is the j-th derivative of the beta function $B(\epsilon, d)$ as a function of ϵ, and $B(\epsilon, d)$ is holomorphic on $(-1, \infty)$ except at zero where it has a simple pole with residue 1. This is easy to verify. So $\epsilon^{j+1} H_j(\epsilon)$ will be holomorphic on the interval $(-1, \infty)$. Observing that

$$\int_0^1 u^{\epsilon-1} (1-u)^{d-1} \Big(\ln\Big(\frac{e}{u}\Big)\Big)^{l-1} du$$

is a linear sum of the derivatives of order less than or equal to $l-1$ of the Beta function, we find

$$\epsilon^l \int_0^1 u^{\epsilon-1}(1-u)^{d-1}\left(\ln\left(\frac{e}{u}\right)\right)^{l-1} du$$

is bounded by a constant C in the interval $[0, b]$. Thus

$$h(r) \ll r^a(1-r)^{b+d-1} r^{-\epsilon}\epsilon^{-l}.$$

Now arguing as in Case 3, if $r \le e^{-1/b}$, we choose $\epsilon = 1/\ln(1/r)$ and get

$$h(r) \ll r^a(1-r)^{b+d-1} e(\ln(1/r))^l$$

and if $r > e^{-1/b}$, we let $\epsilon = b$, and have

$$h(r) \ll r^a(1-r)^{b+d-1} e/b^l.$$

Combining these two cases, we obtain

$$h(r) \ll r^a(1-r)^{b+d-1}(\ln(e/r))^l.$$

This fully proves Lemma A. □

6. Proof of Lemma B

We recall (13):

$$h(r) = r^{a_n}(1-r)^{b_1+\cdots+b_n-1}\int_{I^{n-1}} \eta_{n-1}^{b_n-1}\prod_{i=1}^{n-2}\eta_i^{b_{i+1}} \times \prod_{i=1}^{n-1} y_i^{a_i-a_{i+1}-b_{i+1}} \prod_{i=1}^{n-1}(1-\xi_i)^{b_i-1}\bigwedge_{i=1}^{n-1} d\xi_i.$$

To make the differentiation easier, we introduce some notation. Let

$$A = a_n, \qquad B = b_1 + \cdots + b_n,$$

$$\eta = (\eta_1, \ldots, \eta_{n-1}), \qquad \xi = (\xi_1, \ldots, \xi_{n-1}), \qquad y = (y_1, \ldots, y_{n-1}),$$

$$\alpha_i = a_i - a_{i+1} - b_{i+1} \qquad \text{for } 1 \le i \le n-1,$$

$$\beta_i = b_{i+1} \qquad \text{for } 1 \le i \le n-2, \qquad \beta_{n-1} = b_n - 1,$$

$$\beta = (\beta_1, \ldots, \beta_{n-1}), \qquad G(\xi) = \prod_{i=1}^{n-1}(1-\xi_i)^{b_i-1}, \qquad d\xi = \bigwedge_{i=1}^{n-1} d\xi_i, \qquad J = I^{n-1}.$$

With this notation and the multiindex notation as for example $y^\alpha = y_1^{\alpha_1} \cdots y_{n-1}^{\alpha_{n-1}}$, (13) can be written as

$$h(r) = r^A (1-r)^{B-1} \int_J y^\alpha \eta^\beta G(\xi)\, d\xi. \tag{16}$$

Clearly the function $\eta^\beta G(\xi_i)$ is summable $d\xi$, and each $y_i = \eta_i + r(1-\eta_i)$ satisfies $0 < r \le y_i < 1$ for $0 < r < 1$. So one can differentiate under the integral sign with respect to r. But before we do that let us introduce the notation

$$g_1(r) = r^A, \quad g_2(r) = (1-r)^{B-1}, \quad u_i = y_i^{\alpha_i} \quad \text{for } 1 \le i \le n-1.$$

Rewrite (16) as

$$h(r) = \int_J g_1 g_2 u_1 \cdots u_{n-1} \eta^\beta G(\xi)\, d\xi. \tag{17}$$

Now differentiating under the integral sign, we obtain

$$h^{(k)}(r) = \sum \int_J g_1^{(l_1)} g_2^{(l_2)} u_1^{(j_1)} \cdots u_{n-1}^{(j_{n-1})} \eta^\beta G(\xi)\, d\xi, \tag{18}$$

where the sum is over all $(n+1)$-tuples of nonnegative integers $(l_1, l_2, j_1, \dots j_{n-1})$ such that $k = l_1 + l_2 + j_1 + \cdots + j_{n-1}$. Further it easy to check that

$$\begin{aligned} u_i^{(j_i)}(r) &= \alpha_i(\alpha_i - 1)\cdots(\alpha_i - j_i + 1) y_i^{\alpha_i - j_i}(1-\eta_i)^{j_i}, \\ g_1^{(l_1)}(r) &= A(A-1)\cdots(A-l_1+1) r^{A-l_1}, \\ g_2^{(l_2)}(r) &= (B-1)(B-2)\cdots(B-l_2)(-1)^{l_2}(1-r)^{B-l_2-1}. \end{aligned}$$

Since $y_i \ge r$ and $0 \le \eta_i \le 1$, the equalities above imply

$$\begin{aligned} u_i^{(j_i)}(r) &\ll y_i^{\alpha_i} r^{-j_i} \\ g_1^{(l_1)}(r) &\ll g_1(r) r^{-l_1} \\ g_2^{(l_2)}(r) &\ll (1-r)^{B-k-1} = g_2(r)(1-r)^{-k}, \end{aligned}$$

where the last inequality is obtained because $0 \le l_2 \le k$. From these three, we deduce that

$$\begin{aligned} g_2^{(l_2)}(r) g_1^{(l_1)}(r) u_1^{(j_1)}(r) \cdots u_{n-1}^{(j_{n-1})}(r) &\ll g_2(r)(1-r)^{-k} g_1(r) u_1(r) \cdots \\ &\qquad \cdots u_{n-1}(r) r^{-l_1 - j_1 - \cdots - j_{n-1}} \\ &\ll r^{-k}(1-r)^{-k} g_2(r) g_1(r) u_1(r) \cdots u_{n-1}(r). \end{aligned}$$

Multiplying both sides by $\eta^\beta G(\xi) d\xi$ and integrating over J yield

$$h^{(k)}(r) \ll r^{-k}(1-r)^{-k} h(r)$$

and by Lemma A,

$$h(r) \ll r^A (1-r)^{B-1} (\ln(e/r))^{n-1}.$$

Hence we have

$$h^{(k)}(r) \ll r^{A-k} (1-r)^{B-k-1} (\ln(e/r))^{n-1}.$$

This proves Lemma B. □

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References

[Čučković and Rao 1998] Ž. Čučković and N. V. Rao, "Mellin transform, monomial symbols, and commuting Toeplitz operators", *J. Funct. Anal.* **154**:1 (1998), 195–214. MR 99f:47033 Zbl 0936.47015

[Louhichi 2007] I. Louhichi, "Powers and roots of Toeplitz operators", *Proc. Amer. Math. Soc.* **135**:5 (2007), 1465–1475. MR 2007k:47043 Zbl 1112.47023

[Louhichi and Zakariasy 2005] I. Louhichi and L. Zakariasy, "On Toeplitz operators with quasihomogeneous symbols", *Arch. Math.* (*Basel*) **85**:3 (2005), 248–257. MR 2006e:47061 Zbl 1088.47019

[Remmert 1998] R. Remmert, *Classical topics in complex function theory*, Graduate Texts in Mathematics **172**, Springer, New York, 1998. MR 98g:30002 Zbl 0895.30001

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ISSAM LOUHICHI
DEPARTMENT OF MATHEMATICS AND STATISTICS
KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS
DHAHRAN 31261
SAUDI ARABIA
issam@kfupm.edu.sa

N. V. RAO
DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF TOLEDO
MAIL STOP 942
TOLEDO, OHIO 43606-3390
UNITED STATES
rnagise@math.utoledo.edu

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