UNIQUENESS OF THE FOLIATION OF CONSTANT MEAN CURVATURE SPHERES IN ASYMPTOTICALLY FLAT 3-MANIFOLDS

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This paper studies the constant mean curvature surface in asymptotically
flat 3-manifolds with general asymptotics. Under some weak conditions,
the foliation of stable spheres of constant mean curvature is shown to be
unique outside some compact set in the asymptotically flat 3-manifold with
positive mass.

1. Introduction

A three-manifold $M$ with a Riemannian metric $g$ and a two-tensor $K$ is called an
initial data set $(M, g, K)$ if $g$ and $K$ satisfy the constraint equations

\[(1-1) \quad R_g - |K|^2_g + (\text{tr}_g(K))^2 = 16\pi \rho \quad \text{and} \quad \text{div}_g(K) - d(\text{tr}_g(K)) = 8\pi J,\]

where $R_g$ is the scalar curvature of the metric $g$, $\text{tr}_g(K)$ denotes $g^{ij}K_{ij}$, $\rho$ is the
observed energy density, and $J$ is the observed momentum density.

Definition 1.1. Let $q \in \left(\frac{1}{2}, 1\right]$. An initial data set $(M, g, K)$ is called an asymptotically flat (AF) manifold if there is a compact subset $\tilde{K} \subset M$ and coordinates $\{x^i\}$
with the following properties: $M \setminus \tilde{K}$ is diffeomorphic to $\mathbb{R}^3 \setminus B_1(0)$; $\rho$ and $J$ from (1-1) satisfy $\rho(x) = O(|x|^{-2-2q})$ and $J(x) = O(|x|^{-2-2q})$; and

$$g_{ij}(x) = \delta_{ij} + h_{ij}(x) \quad \text{with} \quad h_{ij}(x) = O_5(|x|^{-q})K_{ij}(x) = O_1(|x|^{-1-q}),$$

where $f = O_k(|x|^{-q})$ means $\partial^l f = O(|x|^{-l-q})$ for $l = 0, \ldots, k$. We call $M \setminus \tilde{K}$ an end of the AF manifold $(M, g, K)$; we will only consider AF manifolds with
one end. The mass of this end is defined as

$$m = \lim_{r \to \infty} \frac{1}{16\pi} \int_{|x|=r} (h_{ij,j} - h_{jj,i}) v^i_g \, d\mu_g,$$

where $v_g$ and $d\mu_g$ are the unit normal vector and volume form with respect to the
metric $g$. From [Bartnik 1986], we know the mass is well defined when $q > \frac{1}{2}$.

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Given a function $f$, let $f^{\text{odd}}(x) = f(x) - f(-x)$ and $f^{\text{even}}(x) = f(x) + f(-x)$.

**Definition 1.2.** An AF manifold $(M, g, K)$ is said to satisfy the Regge–Teitelboim condition, and is called an AF-RT manifold, if $\rho$ and $J$ satisfy

$$\rho^{\text{odd}}(x) = O(|x|^{-3-2q}), \quad J^{\text{odd}}(x) = O(|x|^{-3-2q})$$

and $g, K$ satisfy the asymptotically even/odd conditions

$$h^{\text{odd}}_{ij}(x) = O_2(|x|^{-1-q}), \quad K^{\text{even}}_{ij}(x) = O_1(|x|^{-2-q}).$$

For AF-RT manifolds, the center of mass $C$ is defined by

$$C^\alpha = \frac{1}{16\pi m} \lim_{r\to\infty} \left( \int_{|x|=r} x^\alpha (h_{ij,i} - h_{ii,j}) v^j_g d\mu_g - \int_{|x|=r} (h_i v^i_g - h_{ii} v^\alpha_g) d\mu_g \right).$$

From [Huang 2009], we know the center of mass is well defined.

Let $\Sigma$ be a surface of constant mean curvature (CMC). We say that $\Sigma$ is stable if the second variation operator has only nonnegative eigenvalues when restricted to the functions with 0 mean value, that is,

$$\int_\Sigma (|A|^2 + \text{Ric}(v_g, v_g)) f^2 d\mu \leq \int_\Sigma |\nabla f|^2 d\mu$$

for $f$ a function with $\int_\Sigma f d\mu = 0$, where $A$ is the second fundamental form and $\text{Ric}(v_g, v_g)$ is the Ricci curvature in the normal direction with respect to the metric $g$.

We discuss the existence and uniqueness of CMC spheres that separate the compact part from infinity in AF-RT manifolds. The following two theorems are due to Lan-Hsuan Huang [2010]:

**Theorem 1.3** (existence). If $(M, g, K)$ is AF-RT with $q \in \left( \frac{1}{2}, 1 \right]$ and $m > 0$, there exists a foliation by spheres $\{\Sigma_R\}$ with constant mean curvature $H(\Sigma_R) = 2/R + O(R^{-1-q})$ in the exterior region of $M$. Each leaf $\Sigma_R$ is a $(c_0 R^{1-q})$-graph over $S_R(C)$ and is strictly stable.

Set $r(x) = \left( \sum (x^i)^2 \right)^{1/2}$. For a CMC sphere $\Sigma$ separating infinity from $\tilde{K}$, define

$$r_0(\Sigma) = \inf \{ r(x) \mid x \in \Sigma \}, \quad r_1(\Sigma) = \sup \{ r(x) \mid x \in \Sigma \}.$$

**Theorem 1.4** (uniqueness). Assume that $(M, g, K)$ is AF-RT with $q \in \left( \frac{1}{2}, 1 \right]$ and $m > 0$. There exist $\sigma_1$ and $C_1$ such that, if $\Sigma$ is a topological sphere of constant mean curvature $H = H(\Sigma_R)$ for some $R \geq \sigma_1$, and moreover $\Sigma$ is stable and satisfies $r_1(\Sigma) \leq C_1 r_0^{1/a}$ for some $a \in \left( \frac{5-q}{2(2+q)}, 1 \right]$, then $\Sigma = \Sigma_R$. 
Uniqueness is the harder problem. In Theorem 1.4, Huang needs the assumption
\(r_1 \leq C_1 r_0^{1/a}\) for the radius of the surface. To get a sharper uniqueness result as in [Qing and Tian 2007], we consider metrics of the following form:

**Definition 1.5.** An AF-RT manifold \((M, g, K)\) with mass \(m\) is called an \((m, k, \varepsilon)\)-AF-RT manifold, where \(k > 2\) and \(\varepsilon > 0\), if the metric \(g\) can be expressed as

\[
(1-2) \quad g_{ij} = \delta_{ij} + h_{ij} = \delta_{ij} + h^1_{ij}(\theta)/r + Q,
\]

with \(h^1_{ij}(\theta) \in C^5(S^2)\), \(Q = O_5(|x|^{-2})\), and

\[
(1-3) \quad \|h^1_{ij}(\theta) - \delta_{ij}(\theta)\|_{W^{k,2}(S^2)} \leq \varepsilon.
\]

Here \(\theta = (\theta_1, \theta_2)\) is the coordinate on \(S^2 \subset \mathbb{R}^3\).

**Remark 1.6.** From (1-3), we know that the mass \(m\) has positive two-side bounds. We can certainly consider the case \(\|h^1_{ij}(\theta) - C \delta_{ij}(\theta)\|_{W^{k,2}(S^2)} \leq \varepsilon\) for any positive \(C\), but here we only assume \(C = 1\) without lost of generality.

Our main uniqueness result is this:

**Theorem 1.7.** For any \(k > 2\) there exists \(\varepsilon > 0\), depending only on \(k\), with the following property. For any \((m, k, \varepsilon)\)-AF-RT manifold \((M, g, K)\), there exists a compact \(\tilde{K}\) and a constant \(C > 0\) such that, for any constant \(H > 0\) sufficiently small, there is a unique stable CMC sphere \(\Sigma\) separating \(\tilde{K}\) from infinity and such that \(H(\Sigma) = H\) and \(\log r_1(\Sigma) \leq Cr_0(\Sigma)^{1/4}\).

**Remark 1.8.** This is an improvement on Huang’s result, as can be seen by comparing \(r_1 \leq C_1 r_0^{1/a}\) with \(\log r_1(\Sigma) \leq Cr_0(\Sigma)^{1/4}\) (since \(\log r_1\) grows much more slowly than any positive power of \(r_1\)).

**Remark 1.9.** In the proof of Theorem 1.7, the RT condition is needed only for the existence theorem in [Huang 2010].

**Remark 1.10.** Here I can only deal with the case when \(q = 1\). When \(q \in \left(\frac{1}{2}, 1\right)\) it seems that \(\|h_{ij}(\theta) - \delta_{ij}(\theta)\|_{W^{k,2}(S^2)} \leq \varepsilon\) is not a proper condition.

The paper is organized much like [Qing and Tian 2007]: In Section 2 we give an a priori estimate on stable CMC spheres based on Simon’s identity. In Section 3, we introduce blow-down analysis in three different scales. In Section 4 we recall the asymptotic analysis from [Qing and Tian 1997] and prove a technical lemma. In Section 5 we introduce asymptotically harmonic coordinates. In Section 6 we introduce the notion of the center of mass and prove Theorem 1.7.
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2. Curvature estimates

In this section and the next we assume that \((M, g, K)\) is AF-RT with \(q \in (\frac{1}{2}, 1]\). Let \(\Sigma\) be a CMC sphere in the asymptotically flat end \((M \setminus \tilde{K}, g)\), and assume \(\Sigma\) separates the compact part from infinity. First we have the following estimate, similar to [Huisken and Yau 1996, Lemma 5.2].

Lemma 2.1. Let \(X = x^i (\partial / \partial x^i)\) be the Euclidean coordinate vector field and

\[ r = (\sum (x^i)^2)^{1/2}. \]

With respect to the metric \(g\), let \(v\) be the outward normal vector field, \(d\mu\) be the volume form of \(\Sigma\). Then we have the estimate

\[ \int_{\Sigma} (X, v)^2 r^{-4} d\mu \leq H^2 |\Sigma|. \]

Moreover for each \(a \geq a_0 > 2\) and \(r_0\) sufficiently large,

\[ \int_{\Sigma} r^{-a} d\mu \leq C(a_0) r_0^{2-a} H^2 |\Sigma|. \]

Proof. Because the mean curvature \(H\) is constant, for some smooth vector field \(Y\) on \(\Sigma\), we have the divergence formula

\[ \int_{\Sigma} \text{div}_\Sigma Y d\mu = H \int_{\Sigma} \langle Y, v \rangle d\mu. \]

Choose \(Y = X r^{-a}\), \(a \geq 2\) and let \(e_\alpha\) be the orthonormal basis on \(\Sigma\), \(\alpha = 1, 2\). Supposing \(e_\alpha = a^i_\alpha (\partial / \partial x^i)\), it is obvious that \(a^i_\alpha\) is bounded because the manifold is asymptotically flat. Then

\[ \text{div}_\Sigma Y = \text{div}_\Sigma (X r^{-a}) = \langle \nabla e_\alpha (X r^{-a}), e_\alpha \rangle \]

\[ = r^{-a} \text{div}_\Sigma X - ar^{-a-2} a^i_\alpha a^j_\alpha x^i x^j + O(r^{-a-q}) \]

\[ = r^{-a} \text{div}_\Sigma X - ar^{-a-2} |X^\tau|^2 + O(r^{-a-q}), \]

where \(X^\tau\) is the tangent projection of \(X\). Also,

\[ |\text{div}_\Sigma X - 2| = O(r^{-q}). \]
Note that $|X^r|^2 = r^2 - \langle X, v \rangle^2 + O(r^{-q})$. Combining all of these,

\begin{equation}
(2-1) \quad \left| (2-a) \int_{\Sigma} r^{-a} \, d\mu + a \int_{\Sigma} \langle X, v \rangle^2 r^{-a-2} \, d\mu - H \int_{\Sigma} \langle X, v \rangle r^{-a} \, d\mu \right| \leq C \int_{\Sigma} r^{-a-q} \, d\mu.
\end{equation}

Choosing $a = 2$, from Hölder’s inequality, we have

\begin{equation}
(2-2) \quad \int_{\Sigma} \langle X, v \rangle^2 r^{-4} \, d\mu \leq \frac{1}{4} H^2 |\Sigma| + C \int_{\Sigma} r^{-2-q} \, d\mu.
\end{equation}

Then choosing $a = 2 + q$ gives

$$\int_{\Sigma} r^{-2-q} \, d\mu \leq 4 r_0^{-q} \left( \int_{\Sigma} \langle X, v \rangle^2 r^{-4} \, d\mu + H^2 |\Sigma| + C \int_{\Sigma} r^{-2-q} \, d\mu \right).$$

This combined with (2-2) implies

$$\int_{\Sigma} \langle X, v \rangle^2 r^{-4} \, d\mu \leq H^2 |\Sigma|.$$

Again from (2-1), we have for $a \geq a_0 > 2$,

$$\int_{\Sigma} r^{-a} \leq C(a_0 - 2)^{-1} r_0^{2-a} H^2 |\Sigma|.$$

Now we can derive the integral estimate for $|\hat{A}|$ from the stability of the surface as in [Huisken and Yau 1996, Proposition 5.3]:

**Lemma 2.2.** Suppose $\Sigma$ is a stable CMC sphere in an asymptotically flat manifold. For $r_0$ sufficiently large,

$$\int_{\Sigma} |\hat{A}|^2 \, d\mu \leq C r_0^{-q},$$

$$H^2 |\Sigma| \leq C,$$

$$\int_{\Sigma} H^2 \, d\mu = 16\pi + O(r_0^{-q}).$$

**Proof.** Since $\Sigma$ is stable,

$$\int_{\Sigma} |\nabla f|^2 \, d\mu \geq \int_{\Sigma} (|A|^2 + \text{Ric}(v, v)) f^2 \, d\mu$$

for any function $f$ with $\int_{\Sigma} f \, d\mu = 0$, where $A$ is the second fundamental form of $\Sigma$ and Ric is the Ricci curvature of $M$.

Choose $\psi$ to be a conformal map of degree 1 from $\Sigma$ to the standard $S^2$ in $\mathbb{R}^3$. Each component $\psi_i$ of $\psi$ can be chosen such that $\int \psi_i \, d\mu = 0$ [Li and Yau 1982].
For each $\psi_i$, \[ \int_\Sigma |\nabla \psi_i|^2 \, d\mu = \frac{8\pi}{3}. \]

Since $\sum \psi_i^2 = 1$ we conclude that \[ \int_\Sigma |A|^2 + \text{Ric}(v, v) \, d\mu \leq 8\pi. \]

From the Gauss equation

\begin{equation}
\frac{1}{2}|A|^2 + \text{Ric}(v, v) - \frac{1}{2}R + K = \frac{1}{2}H^2,
\end{equation}

we have \[ |A|^2 + \text{Ric}(v, v) = \frac{1}{2}|\hat{A}|^2 + \frac{3}{4}H^2 + \frac{1}{2}R - K, \]
where $K$ is the Gauss curvature of $\Sigma$ and $\hat{A}$ is defined as $\hat{A}_{ij} = A_{ij} - (H/2)g_{ij}$.

Then \[ \int_\Sigma \frac{1}{2}|\hat{A}|^2 + \frac{3}{4}H^2 |\Sigma| \leq 12\pi + r_0^{-q} H^2 |\Sigma| \]
because $R = O(r^{-2-2q})$, from the constraint equation (1-1). So $H^2 |\Sigma| \leq 16\pi$.

Using the Gauss equation in a different way, we have \[
\int_\Sigma |\hat{A}|^2 \, d\mu = \int_\Sigma |A|^2 - \frac{H^2}{2} \, d\mu \\
= \frac{1}{2} \int_\Sigma |A|^2 + \text{Ric}(v, v) \, d\mu + \frac{1}{2} \int_\Sigma R - 3 \text{Ric}(v, v) - 2K \, d\mu \\
\leq \int_\Sigma r^{-2-q} \, d\mu \\
= O(r_0^{-q}).
\]

Then again from the Gauss equation (2-3), \[ \int_\Sigma H^2 \, d\mu = 4 \int_\Sigma K \, d\mu + O(r_0^{-q}) = 16\pi + O(r_0^{-q}). \]

\textbf{Lemma 2.3.} \textit{Suppose $M$ is a constant mean curvature surface in an asymptotically flat end $(\mathbb{R}^3 \setminus B_1(0), g)$. Then}
\[ \int_\Sigma H_e^2 \, d\mu_e = 16\pi + O(r_0^{-q}). \]

\textit{Proof.} We follow the calculation of Huisken and Ilmanen [2001] to obtain
\[ g_{ij} = \delta_{ij} + h_{ij}. \]
Suppose that

\[ g_{ij}|\Sigma = f_{ij}, \quad \delta_{ij}|\Sigma = \varepsilon_{ij}, \]

where \( f_{ij} \) and \( \varepsilon_{ij} \) are the corresponding inverse matrices. Let \( v, \omega, A, H, d\mu \) represent the normal vector, the dual form of \( v \), the second fundamental form, the mean curvature and the volume form of \( \Sigma \) in the metric \( g \) and \( v_e, \omega_e, A_e, H_e, \mu_e \) represent the corresponding ones in Euclidean metric. Easy calculation gives

\[
\begin{align*}
(2-4) \quad f_{ij} - \varepsilon_{ij} &= -f^{ik}h_{kl}f_{lj} \pm C|h|^2, \\
(2-5) \quad g_{ij} - \delta_{ij} &= -g^{ik}h_{kl}g_{lj} \pm C|h|^2, \\
(2-6) \quad \omega = \frac{\omega_e}{|\omega_e|}, \quad v^i = g^{ij}\omega_j, \\
(2-7) \quad (\omega_e)_i = \omega_i \pm C|P|, \quad v^i_e = v^i + C|h|, \quad 1 - |\omega_e| = \frac{1}{2}h_{ij}v^i v^j, \\
(2-8) \quad \Gamma^{k}_{ij} &= \frac{1}{2}g^{kl}(\bar{\nabla}_i h_{jl} + \bar{\nabla}_j h_{il} - \bar{\nabla}_l h_{ij}) \pm C|h| \pm C|\bar{\nabla}h|,
\end{align*}
\]

where \( \Gamma^{k}_{ij} \) is the Christoffel symbol for \( \bar{\nabla} - \bar{\nabla}_e \) and we denote the gradient for the metrics \( g \) and \( \delta \) by \( \nabla \) and \( \nabla_e \).

We have the formula

\[
|\omega_e|gA_{ij} = (A_e)_{ij} - (\omega_e)_{k}\Gamma^k_{ij}.
\]

This implies

\[
H - H_e = f_{ij}A_{ij} - \varepsilon_{ij}(A_e)_{ij}
\]

\[
= (f_{ij} - \varepsilon_{ij})A_{ij} + \varepsilon_{ij}A_{ij}(1 - |\omega_e|g) + \varepsilon_{ij}(|\omega_e|gA_{ij} - (A_e)_{ij}).
\]

From \((2-4), (2-5), (2-7),\)

\[
\varepsilon_{ij}A_{ij}(1 - |\omega_e|g) = \frac{1}{2}Hv^i v^j h_{ij} \pm C|h|^2|A|.
\]

Using \((2-4)-(2-9),\) we obtain

\[
\varepsilon_{ij}(|\omega_e|gA_{ij} - (A_e)_{ij}) = -\varepsilon_{ij}(\omega_e)_{k}\Gamma^k_{ij}
\]

\[
= -\frac{1}{2}f^{ij} \omega_k g^{kl}(\bar{\nabla}_i h_{jl} + \bar{\nabla}_j h_{il} - \bar{\nabla}_l h_{ij}) \pm C|h| |\bar{\nabla}h|
\]

\[
= -f^{ij} v^l \bar{\nabla}_i h_{jl} + \frac{1}{2}f^{ij} v^l \bar{\nabla}_i h_{ij} \pm C|h| |\bar{\nabla}h|.
\]

At last,

\[
(2-10) \quad H - H_e = \frac{1}{2}f^{ik}h_{kl}f^{lj}A_{ij} + \frac{1}{2}Hv^i v^j h_{ij} - f^{ij} v^l \bar{\nabla}_i h_{jl}
\]

\[
+ \frac{1}{2}f^{ij} v^l \bar{\nabla}_i h_{ij} \pm C|h| |\bar{\nabla}h| \pm C|h|^2|A|,
\]\
and
\[
\int \Sigma H_e^2 d\mu_e = (1 + O(r_0^{-q})) \int \Sigma H_e^2 d\mu
\]
\[
\leq (1 + O(r_0^{-q}))(\int \Sigma H^2 d\mu + \int \Sigma (H_e - H)^2 + 2|H(H_e - H)| d\mu)
\]
\[
\leq (1 + O(r_0^{-q}))(16\pi + O(r_0^{-q}) + \int \Sigma (H_e - H)^2
\]
\[
+ \left(\int \Sigma H^2 d\mu\right)^{1/2} \left(\int \Sigma (H_e - H)^2 d\mu\right)^{1/2})
\]
\[
\int (H_e - H)^2 d\mu \leq \int O(|x|^{-2q})|A|^2 + H^2 O(|x|^{-2q}) + O(|x|^{-2-2q}) d\mu
\]
\[
\leq \int O(|x|^{-2q})H^2 + O(|x|^{-2q})|\hat{A}|^2 + O(|x|^{-2-2q}) d\mu
\]
\[
= O(r_0^{-2q}),
\]
so we have
\[
\int \Sigma H_e^2 d\mu_e \leq 16\pi + O(r_0^{-q}).
\]

On the other hand, by Euler’s formula,
\[
K_e = \frac{1}{4}H_e^2 - \frac{1}{2}|\hat{A}_e|^2.
\]
So we have
\[
\int \Sigma H_e^2 d\mu_e \geq 16\pi
\]
which implies
\[
\int \Sigma H_e^2 d\mu_e = 16\pi + O(r_0^{-q}).
\]

Based on [Michael and Simon 1973] we have the following Sobolev inequality.

**Lemma 2.4.** Suppose \(\Sigma\) is a constant mean curvature surface in an asymptotically flat end \((\mathbb{R}^3 \setminus B_1(0), g)\) with \(r_0(\Sigma)\) sufficiently large and that \(\int \Sigma H^2 \leq C\). Then
\[
(2-11) \quad \left(\int \Sigma |f|^2 d\mu\right)^{1/2} \leq C \left(\int \Sigma |\nabla f| d\mu + \int \Sigma |H| |f| d\mu\right).
\]

**Proof.** Note that this is valid for a surface in Euclidean space. So by the uniform equivalence of the metrics \(g\) and \(\delta\), we have
\[
\left(\int |f|^2 d\mu\right)^{1/2} \leq C \left(\int |f|^2 d\mu_e\right)^{1/2} \leq C \left(\int |\nabla f| + H|f| + |H - H_e||f| d\mu\right).
\]
To bound the last term on the right, we use
\[
\int |H - H_e| |f| \, d\mu \leq \int O(|x|^{-q})|A||f| + O(|x|^{-q})H |f| + O(|x|^{-q})|f| \, d\mu
\leq O(r_0^{-q}) \int H |f| + \left(\int |\hat{A}|^2 \, d\mu\right)^{1/2} O(r_0^{-q}) \|f\|_{L^2}
\]
\[+ O(r_0^{-q}) \|f\|_{L^2}.
\]
So we can choose \(r_0\) sufficiently large to get the desired result. \(\Box\)

**Lemma 2.5.** Suppose \(\Sigma\) is a constant mean curvature surface in an asymptotically flat end \((\mathbb{R}^3 \setminus B_1(0), g)\) with \(r_0(\Sigma)\) sufficiently large. Then
\[
C_1 H^{-1} \leq \text{diam}(\Sigma) \leq C_2 H^{-1},
\]
where \(\text{diam}(\Sigma)\) denotes the diameter of \(\Sigma\) in the Euclidean space \(\mathbb{R}^3\). In particular, if the surface \(\Sigma\) separates infinity from the compact part, then
\[
C_1 H^{-1} \leq r_1(\Sigma) \leq C_2 H^{-1}.
\]

**Proof.** We already know that
\[
\int_\Sigma H_e^2 \, d\mu_e = 16\pi + O(r_0^{-q}).
\]
Then from [Simon 1993, Lemma 1.1],
\[
\sqrt{\frac{2|\Sigma|_e}{F(\Sigma)}} \leq \text{diam}(\Sigma) \leq C \sqrt{|\Sigma|_e} F(\Sigma),
\]
where \(F(\Sigma) = \frac{1}{2} \int_\Sigma H_e^2\) is the Willmore functional and \(|\Sigma|_e\) is the volume of \(\Sigma\) with respect to the Euclidean metric. Since the Euclidean metric is uniformly equivalent to \(g\), we get the result. \(\Box\)

To get the pointwise estimate for \(\hat{A}\), we use Simon’s identity (2-12) below and Moser’s iteration argument.

**Lemma 2.6 [Schoen et al. 1975].** Suppose \(N\) is a hypersurface in a Riemannian manifold \((M, g)\). Then the second fundamental form satisfies the identity
\[
(2-12) \quad \Delta A_{ij} = \nabla_i \nabla_j H + HA_{ik}A_{jk} - |A|^2A_{ij} + H R_{3i3j} - A_{ij}R_{3k3k} + A_{jk}R_{kli} + A_{ik}R_{kjl} - 2A_{lk}R_{iljk} + \nabla_j R_{3k3k} + \nabla_k R_{3ijk},
\]
where \(R_{ijkl}\) and \(\nabla\) are the curvature and gradient operator of \((M, g)\).
From this we easily deduce for CMC surfaces the inequality
\[-|\dot{A}|^4 + CH|\dot{A}|^3 + CH^2|\dot{A}|^2 + C|\dot{A}|^2|x|^{-2-q} + CH|\dot{A}|x|^{-2-q} + C|\dot{A}|x|^{-3-q}.

We also need an inequality for $\nabla \dot{A}$ because we also want to estimate the higher derivative:
\[-|\nabla \dot{A}|^4 |\nabla \dot{A}| \leq C |\nabla \dot{A}|^2 (|\dot{A}|^2 + H|\dot{A}| + H^2 + O(|x|^{-2-q}))
+ |\nabla \dot{A}|(|\dot{A}|^2 + H|\dot{A}| + H^2)O(|x|^{-2-q}) + (|\dot{A}| + H)O(|x|^{-3-q}) + O(|x|^{-4-q}).

Then we can get the pointwise estimates for $\dot{A}$ and $\nabla \dot{A}$.

**Theorem 2.7** [Qing and Tian 2007]. Suppose that $(\mathbb{R}^3 \setminus B_1(0), g)$ is an asymptotically flat end. Then there exist positive numbers $\sigma_0, \delta_0$ such that for any constant mean curvature surface in the end which separates infinity from the compact part, we have

\[(2-13) \quad |\dot{A}|^2(x) \leq C|x|^{-2} \int_{B_{\delta_0|x|}(x)} |\dot{A}|^2 d\mu + C|x|^{-2-2q} \leq C|x|^{-2}r_0^{-q}
\]
and
\[(2-14) \quad |\nabla \dot{A}|^2(x) \leq C|x|^{-4} \int_{B_{\delta_0|x|}(x)} |\nabla \dot{A}|^2 d\mu + C|x|^{-4-2q} \leq C|x|^{-4}r_0^{-q/2},
\]
provided that $r_0 \geq \sigma_0$.

**Proof.** In the Sobolev inequality (2-11), take $f = u^2$. Then

\[
\left( \int_{\Sigma} u^4 d\mu \right)^{1/2} \leq C \left( 2 \int_{\Sigma} |u| |\nabla u| d\mu + \int_{\Sigma} Hu^2 d\mu \right)
\leq C \left( \int_{\Sigma} u^2 \right)^{1/2} \left( \int_{\Sigma} |\nabla u|^2 d\mu \right)^{1/2} + C \left( \int_{\text{supp}(u)} H^2 d\mu \right)^{1/2} \left( \int_{\Sigma} u^4 d\mu \right)^{1/2}.
\]

To proceed, we need some auxiliary results.

**Lemma 2.8.** For any $\varepsilon > 0$, we can find a uniform $\delta_0$ sufficiently small such that

\[
\int_{B_{\delta_0|x|}(x)} H^2 d\mu \leq \varepsilon \quad \text{for any } x \in \Sigma.
\]

**Proof.** The metric $g$ is equivalent to Euclidean metric $\delta$. Thus we need only to prove that there exists $C$ such that

\[|B_{\delta_0|x|}(x)|_e \leq C\delta_0^2 |x|^2,\]
because then

$$H^2 |B_{\delta_0|x|}(x)|_e \leq C \delta_0^2 |x|^2 H^2 \leq C \delta_0^2.$$  

From the proof of Lemma 1.1 in [Simon 1993] we know that, for any $x \in \Sigma$, if $B_\sigma(x)$ denotes the Euclidean ball of radius $\sigma$ with center $x$ in $\mathbb{R}^3$ and $\Sigma_\sigma = \Sigma \cap B_\sigma(x)$, then there exists $C$ such that for $0 < \sigma \leq \rho < \infty$,

$$\sigma^{-2} |\Sigma_\sigma|_e \leq C (\rho^{-2} |\Sigma_\rho| + F(\Sigma_\rho)),$$

where $F(\Sigma_\rho)$ is the Willmore functional. The constant $C$ does not depend on $\Sigma, \sigma, \rho$.

Letting $\rho \to \infty$, $\rho^{-2} |\Sigma_\rho| \to 0$ gives

$$\sigma^{-2} |\Sigma_\sigma|_e \leq C F(\Sigma) \leq C.$$

This proves the lemma. \hfill \Box

So if supp$(u) \subset B_{\delta_0|x|}(x)$, we have the scaling invariant Sobolev inequality

$$\left( \int_{\Sigma} u^4 d\mu \right)^{1/2} \leq C \left( \int_{\Sigma} u^2 \right)^{1/2} \left( \int_{\Sigma} |\nabla u|^2 d\mu \right)^{1/2}.$$

**Lemma 2.9** [Qing and Tian 2007, Lemma 2.6]. Suppose a nonnegative function $v \in L^2$ solves $-\Delta v \leq f v + h$ on $B_{2R}(x_0)$, where

$$\int_{B_{2R}(x_0)} f^2 d\mu \leq CR^{-2}$$

and $h \in L^2(B_{2R}(x_0))$. Also, suppose that

$$\left( \int_{\Sigma} u^4 d\mu \right)^{1/2} \leq C \left( \int_{\Sigma} u^2 \right)^{1/2} \left( \int_{\Sigma} |\nabla u|^2 d\mu \right)^{1/2}$$

holds for all $u$ with support inside $B_{2R}(x_0)$. Then

$$\sup_{B_R(x_0)} v \leq CR^{-1} \|v\|_{L^2(B_{2R}(x_0))} + CR \|h\|_{L^2(B_{2R}(x_0))}.$$

Then we find that

$$-\Delta |\hat{A}| \leq (|\hat{A}|^2 + H^2 + H |\hat{A}| + C |x|^{-2-q} |\hat{A}| + CH |x|^{-2-q} + C |x|^{-3-q}$$

$$= f_1 |\hat{A}| + h_1$$
\[-\Delta |\nabla \hat{A}| \leq C|\nabla \hat{A}|(|\hat{A}|^2 + H|\hat{A}| + H^2 + O(|x|^{-3}))
+ ((|\hat{A}|^2 + H|\hat{A}| + H^2)O(|x|^{-3}) + (|\hat{A}| + H)O(|x|^{-4}) + O(|x|^{-5}))
= f_2|\nabla \hat{A}| + h_2.\]

As in Theorem 2.5 of [Qing and Tian 2007] we have \(\|f_i\|_{L^2(B_{2\delta_0}|x|)(x))} \leq C|x|^{-2}\) for \(i = 1, 2\). Further, it is easy to show that
\[\|h_1\|_{L^2(B_{2\delta_0}|x|)(x))} = O(|x|^{-4-2q})\quad\text{and}\quad\|h_2\|_{L^2(B_{2\delta_0}|x|)(x))} = O(|x|^{-6-2q}).\]

At last we know that
\[(2-15)\quad \int_{B_{\delta_0}|x|} |\hat{A}|^2 d\mu \leq C|x|^{-2}r_0^{-q},\]
and
\[(2-16)\quad \int_{B_{\delta_0}|x|} |\nabla \hat{A}|^2 d\mu \leq |x|^{-2}\left(\int_{B_{\delta_0}|x|} |\hat{A}|^2 d\mu \right)^{1/2} \leq |x|^{-2}r_0^{-q/2}.\]

The first inequality follows from Lemma 2.2 and the second one from Simon’s identity (2-12). This concludes the proof of Theorem 2.7.

3. Blow down analysis

Now like [Qing and Tian 2007], we blow down the surface in three different scales. First we consider
\[\tilde{N} = \frac{1}{2}HN = \left\{ \frac{1}{2}Hx \mid x \in N \right\}.\]
Suppose there is a sequence of constant mean curvature surfaces \(\{N_i\}\) such that
\[\lim_{i \to \infty} r_0(N_i) = \infty.\]
We know that
\[\lim_{i \to \infty} \int_{N_i} H_e^2 d\mu_e = 16\pi.\]
Hence, by the curvature estimates established in the previous section combined with the proof of [Simon 1993, Theorem 3.1], we have:

**Lemma 3.1.** Suppose that \(\{N_i\}\) is a sequence of constant mean curvature surfaces in a given asymptotically flat end \((\mathbb{R}^3 \setminus B_1(0), g)\) and that
\[\lim_{i \to \infty} r_0(N_i) = \infty.\]
Also, suppose that \(N_i\) separates infinity from the compact part. Then, there is a subsequence of \(\{\tilde{N}_i\}\) which converges in Gromov–Hausdorff distance to a round
sphere $S^2_1(a)$ of radius 1 and centered at $a \in \mathbb{R}^3$. Moreover, the convergence is $C^{2,\alpha}$ away from the origin.

Then, we use a smaller scale $r_0$ to blow down the surface

$$\hat{N} = r_0(N)^{-1} N = \{r_0^{-1} x \mid x \in N\}.$$ 

**Lemma 3.2.** Suppose that $\{N_i\}$ is a sequence of constant mean curvature surfaces in a given asymptotically flat end $(\mathbb{R}^3 \setminus B_1(0), g)$ and that $\lim_{i \to \infty} r_0(N_i) = \infty$. Also, suppose that

$$\lim_{i \to \infty} r_0(N_i) H(N_i) = 0.$$

Then there is a subsequence of $\{\hat{N}_i\}$ converging to a 2-plane at distance 1 from the origin. Moreover the convergence is in $C^{2,\alpha}$ in any compact set of $\mathbb{R}^3$.

We must understand the behavior of the surfaces $N_i$ in the scales between $r_0(N_i)$ and $H^{-1}(N_i)$. We consider the scale $r_i$ such that

$$\lim_{i \to \infty} r_0(N_i) \frac{r_i}{r_i} = 0, \quad \lim_{i \to \infty} r_i H(N_i) = 0$$

and blow down the surfaces

$$\overline{N}_i = r_i^{-1} N = \{r_i^{-1} x \mid x \in N\}.$$ 

**Lemma 3.3.** Suppose that $\{N_i\}$ is a sequence of constant mean curvature surfaces in a given asymptotically flat end $(\mathbb{R}^3 \setminus B_1(0), g)$ and that

$$\lim_{i \to \infty} r_0(N_i) = \infty.$$ 

Also, suppose that $r_i$ are such that

$$\lim_{i \to \infty} r_0(N_i) \frac{r_i}{r_i} = 0, \quad \lim_{i \to \infty} r_i H(N_i) = 0.$$ 

Then there is a subsequence of $\{\overline{N}_i\}$ converging to a 2-plane at the origin in Gromov–Hausdorff distance. Moreover the convergence is $C^{2,\alpha}$ in any compact subset away from the origin.

### 4. Asymptotic analysis

In this section and the next two we assume that $(M, g, K)$ is an $(m, k, \varepsilon)$-AF-RT manifold, with $q = 1$. First we revise [Qing and Tian 1997, Proposition 2.1], proving a different version. Set

$$\|u\|_{1,i}^2 = \int_{[(i-1)L,iL] \times S^1} |u|^2 + |\nabla u|^2 \, dt \, d\theta,$$ 

where $(t, \theta)$ is the standard column coordinate.
Lemma 4.1. Suppose \( u \in W^{1,2}(\Sigma, \mathbb{R}^k) \) satisfies
\[
\Delta u + A \cdot \nabla u + B \cdot u = h
\]
in \( \Sigma \), where \( \Sigma = [0, 3L] \times S^1 \) for \( L \) large. Then there exists a positive number \( \delta_0 \) such that if
\[
\|h\|_{L^2(\Sigma)} \leq \delta_0 \max_{1 \leq i \leq 3} \|u\|_{1,i} \quad \text{and} \quad \|A\|_{L^\infty(\Sigma)} \leq \delta_0 \|B\|_{L^\infty(\Sigma)} \leq \delta_0,
\]
the following conditions are satisfied:
(a) \( \|u\|_{1,3} \leq e^{-(1/2)L} \|u\|_{1,2} \) implies \( \|u\|_{1,2} < e^{-(1/2)L} \|u\|_{1,1} \).
(b) \( \|u\|_{1,1} \leq e^{-(1/2)L} \|u\|_{1,2} \) implies \( \|u\|_{1,2} < e^{-(1/2)L} \|u\|_{1,3} \).
(c) If both
\[
\int_{L \times S^1} u \, d\theta \quad \text{and} \quad \int_{2L \times S^1} u \, d\theta \leq \delta_0 \max_{1 \leq i \leq 3} \|u\|_{1,i},
\]
then either \( \|u\|_{1,2} < e^{-(1/2)L} \|u\|_{1,1} \) or \( \|u\|_{1,2} < e^{-(1/2)L} \|u\|_{1,3} \).

Proof. Supposing \( u \in W^{1,2}(\Sigma) \) and \( u \) is harmonic, we can deduce that \( u \) satisfies (a), (b) and this variant condition:
(c’) If both
\[
\int_{L \times S^1} u \, d\theta \quad \text{and} \quad \int_{2L \times S^1} u \, d\theta = 0,
\]
then either \( \|u\|_{1,2} < e^{-(1/2)L} \|u\|_{1,1} \) or \( \|u\|_{1,2} < e^{-(1/2)L} \|u\|_{1,3} \).

A harmonic function \( u \) can be written as
\[
u = a_0 + b_0 t + \sum_{n=1}^{\infty} (e^{it} (a_n \cos n \theta + b_n \sin n \theta) + e^{-it} (a_{-n} \cos n \theta + b_{-n} \sin n \theta)).
\]
Then it follows that for \( i = 1, 2, 3, \)
\[
\|u\|_{1,i}^2 = 2\pi \left( (a_0^2 + b_0^2) L + a_0 b_0 L^2 (2i - 1) + \frac{1}{3} b_0^2 L^3 (3i^2 - 3i + 1) \right)
\]
\[
+ \frac{\pi}{2} \sum_{n=1}^{\infty} \left( \frac{e^{2nL-1}}{n} \left(e^{2(i-1)nL}(a_n^2 + b_n^2) + e^{-2niL}(a_{-n}^2 + b_{-n}^2) \right)
+ 4L(a_n a_{-n} + b_n b_{-n}) \right)
\]
\[
+ \frac{\pi}{2} \sum_{n=1}^{\infty} \left( \frac{e^{2nL-1}}{n} \left(e^{2(i-1)nL}(n^2 a_n^2 + n^2 b_n^2) + e^{-2niL}(n^2 a_{-n}^2 + n^2 b_{-n}^2) \right)
+ 4L(n^2 a_n a_{-n} + n^2 b_n b_{-n}) \right).
\]
If \( L \) is fixed and sufficiently large, then
\[
\|u\|_{1,2}^2 < \frac{1}{2} (e^L \|u\|_{1,3}^2 + e^{-L} \|u\|_{1,1}^2),
\]
which implies (a). We get (b) in the same way. For (c’), we have $a_0 = b_0 = 0$, so then
\[ \|u\|_{1,2}^2 < \frac{1}{2} e^{-L} (\|u\|_{1,3}^2 + \|u\|_{1,1}^2) \]
which implies (c’).

The second step is to pass to limits. If the proposition were false, then one would have a sequence $\delta_k \to 0$ and a sequence of solutions $u_k$, each violating (a), (b), or (c), with $\|h_k\|_{L^2} \leq \delta_k \max_{1 \leq i \leq 3} \|u_k\|_{1,i}$, $\|A_k\|_{\infty} \leq \delta_k$ and $\|B_k\|_{\infty} \leq \delta_k$ solving
\[ \Delta u_k + A_k \cdot \nabla u_k + B_k \cdot u_k = h_k. \]

We may assume $\max_{1 \leq i \leq 3} \|u_k\|_{1,i} = 1$; otherwise we can normalize them. So we know $\|u_k\|_{1,2} > C > 0$, because $u_k$ violates (a), (b), or (c). We know that there is a subsequence that converges to some harmonic function $u \in W^{1,2}(\Sigma)$ weakly. From the interior $W^{2,p}$ estimate we know the convergence is strongly $W^{1,2}$ in $I_2$, which implies that $u$ is not trivially zero.

Because $u_i \rightharpoonup u$ weakly in $W^{1,2}(\Sigma)$ sense, we know $u_i \to u$ in the $W^{1,2}(I_1)$ and $W^{1,2}(I_3)$ senses. Then
\[ \liminf_{i \to \infty} \|u_i\|_{1,1} \geq \|u\|_{1,1}, \quad \lim_{i \to \infty} \|u_i\|_{1,2} = \|u\|_{1,2}, \quad \liminf_{i \to \infty} \|u_i\|_{1,3} \geq \|u\|_{1,3}. \]
Then $u_i$ converges to some nontrivial harmonic function $u$ which violates one of (a), (b), or (c’), proving the lemma.

Given a surface $N$ in $\mathbb{R}^3$, recall from, for example, [Kenmotsu 2003, (8.5)], that
\[ \Delta e v + |\nabla e v|^2 v = \nabla e H_e, \]
where $v$ is the Gauss map from $N \to S^2$.

**Lemma 4.2.** For the constant mean curvature surfaces in the asymptotically flat end $(\mathbb{R}^3 \setminus B_1(0), g)$, we have
\[ |\nabla e H_e|(x) \leq C |x|^{-2} r_0^{-1}. \]

**Proof.** Because the metric $g$ and the Euclidean metric are uniformly equivalent, we must prove that
\[ |\nabla H_e|(x) \leq C |x|^{-2} r_0^{-1}. \]
From (2-10), (2-13), (2-14), and Lemma 2.5 (now $q = 1$), we know that
\[ |\nabla H_e| \leq |\nabla h_{ij}| |A| + |h_{ij}| |A|^2 + |h_{ij}| |\nabla \hat{A}_{ij}| + H |A| |h_{ij}| + H \nabla h_{ij} + |A| |\nabla h_{ij}| + |\nabla^2 h| \leq |x|^{-2} r_0^{-1}, \]
which completes the proof. □
Suppose \( \Sigma \) is a constant mean curvature surface in the asymptotically flat end. Set

\[
A_{r_1, r_2} = \{ x \in \Sigma \mid r_1 \leq |x| \leq r_2 \},
\]

and let \( A_{r_1, r_2}^0 \) be the standard annulus in \( \mathbb{R}^2 \). We are concerned with the behavior of \( v \) on \( A_{Kr_0(\Sigma), sH^{-1}(\Sigma)} \) of \( \Sigma \) where \( K \) is fixed large and \( s \) is fixed small. The lemma below gives us good coordinates on the surface.

**Lemma 4.3.** Suppose \( \Sigma \) is a constant mean curvature surface in a given asymptotically flat end \( (\mathbb{R}^3 \setminus B_1(0), g) \). Then, for any \( \varepsilon > 0 \) and \( L \) large, there are \( M \), \( s \) and \( K \) such that, if \( r_0 \geq M \) and \( Kr_0(\Sigma) < r < sH^{-1}(\Sigma) \), then \( (r^{-1}A_{r,e^Lr}, r^{-2}g_e) \) may be represented as \( (A_{1,e^L}, \tilde{g}) \) and

\[
\| \tilde{g} - |dx|^2 \|_{C^1(A_{1,e^L})} \leq \varepsilon.
\]

In other words, in the cylindrical coordinates \( (S^1 \times [\log r, L + \log r, \tilde{g}_c]) \),

\[
\| \tilde{g}_c - (dt^2 + d\theta^2) \|_{C^1(S^1 \times [\log r, L + \log r])} \leq \varepsilon.
\]

**Proof.** Suppose this is not true. Then we can assume that such \( K \) (or such \( s \)) cannot be found. Then by Lemma 3.2, for some \( \varepsilon_0 > 0 \), there is a sequence \( \Sigma_n \) with \( r_0(\Sigma_n) \to \infty \) and \( \tilde{l}_n \to \infty \) such that

\[
((Kr_0e^{\tilde{l}_nL})^{-1}A_{Kr_0e^{\tilde{l}_nL}, Kr_0e^{(\tilde{l}_n+1)L}}, (Kr_0e^{\tilde{l}_nL})^{-2}g_e)
\]

is not \( \varepsilon_0 \) close to \( (A_{1,e^L}, \tilde{g}) \).

By Lemma 3.1,

\[
\frac{Kr_0e^{\tilde{l}_nL}}{sH^{-1}(\Sigma_n)} \to 0
\]

must hold because we have chosen \( s \) sufficiently small.

So if we assume \( r_n = Kr_0e^{\tilde{l}_nL} \), then

\[
\lim_{n \to \infty} \frac{r_n}{Kr_0} = \infty, \quad \lim_{n \to \infty} \frac{r_n}{sH^{-1}} = 0.
\]

Blowing down the surface using \( r_n \) gives a contradiction with Lemma 3.3. \( \square \)

Now consider the cylindrical coordinates \( (t, \theta) \) on \( (S^1 \times [\log Kr_0, \log sH^{-1}]) \). The tension field satisfies

\[
|\tau(v)| = r^2|\nabla_e H_e| \leq Cr_0^{-1}
\]

for \( t \in [\log Kr_0, \log sH^{-1}] \). Thus,

\[
\int_{S^1 \times [t, t+L]} |\tau(v)|^2 dt d\theta \leq Cr_0^{-2}.
\]

Let \( I_i \) be \( S^1 \times [\log Kr_0 + (i-1)L, \log Kr_0 + iL] \), and \( N_i \) be \( I_{i-1} \cup I_i \cup I_{i+1} \). On \( \Sigma_n \), assume \( \log(sH^{-1}) - \log(Kr_0) = l_nL \). As in [Qing and Tian 1997], first we prove:
Lemma 4.4. For each $i \in [3, l_n - 2]$, there exists a geodesic $\gamma$ such that

$$\int_{I_i} |\tilde{\nabla}(v - \gamma)|^2 \, dt \, d\theta \leq C(e^{-iL} + e^{-(l_n-i)L})s^2 + Cr_0^{-1},$$

where $\tilde{\nabla}$ is the gradient on $S^1 \times [\log(Kr_0), \log(sH^{-1})]$.

Lemma 4.5. By Theorem 2.7,

$$[v]_{C^\alpha(I_i)} \leq \|\tilde{\nabla} v\|_{L^\infty} \leq C(r_0^{-1/2} + s).$$

Thus if $r_0$ sufficiently large and $s$ sufficiently small, then $[v]_{C^\alpha(N_i)}$ is very small.

Proof of Lemma 4.4. To apply the Lemma 4.1 to prove this lemma we choose points $P$ and $Q$ on $S^2$ (the image of Gauss map) satisfying

$$\left| P - \frac{1}{2\pi} \int_{(i-1)L \times S^1} v \, d\theta \right| \leq C \max_{(i-1)L \times S^1} |v - P|^2,$$

$$\left| Q - \frac{1}{2\pi} \int_{iL \times S^1} v \, d\theta \right| \leq C \max_{iL \times S^1} |v - Q|^2.$$ 

Note that $S^2$ is compact and smooth, so by Lemma 4.5 we can always find such $P$ and $Q$ that are very close. So there is a unique geodesic $\gamma_i$ connecting $P$ and $Q$ whose velocity is sufficiently small.

If we write down the equation satisfied by $v - \gamma_i$ on $S^1 \times [\log(Kr_0), \log(sH^{-1})]$,

$$\tilde{\Delta} u + A \cdot \tilde{\nabla} u + B \cdot u = \tau,$$

where $u = v - \gamma_i$, we have

$$|A| \leq C(|\tilde{\nabla} v| + |\tilde{\nabla} \gamma_i|) \leq \delta_0,$$

(4-2) $$|B| \leq C \min\{|\tilde{\nabla} v|^2, |\tilde{\nabla} \gamma_i|^2\} \leq \delta_0.$$ 

If Lemma 4.1(c) cannot be used, the only reason is that

$$\|v - \gamma_i\|_{1,i} \leq C\|\tau\|_{L^2(N_i)},$$

and so

$$\int_{I_i} |\tilde{\nabla}(v - \gamma_i)|^2 \, dt \, d\theta \leq Cr_0^{-2},$$

which implies (4-1).

If Lemma 4.1(c) can be used, then applying it for $u = v - \gamma_i$ over $N_i$, we have one of the following:

$$\|u\|_{1,i} < e^{-(1/2)L} \|u\|_{1,i-1},$$

$$\|u\|_{1,i} < e^{-(1/2)L} \|u\|_{1,i+1}.$$
Suppose the first one happens (without loss of generality). Then we may push this relation to the left because \((4-2)\) holds regardless of \(t\)’s position. If the theorem can be used on \(N_{j+1}\) but not on \(N_j\) for some \(j \geq 2\), then
\[
\|u\|_{1,i} < e^{-(1/2)(i-j)L}\|u\|_{1,j} \leq Ce^{-(1/2)(i-j)L}r_0^{-1} \leq Cr_0^{-1}.
\]
If the theorem can be used until \(I_2\), then
\[
e^{L/2}\|u\|_{1,2} \leq \|u\|_{1,1}
\]
\[
= \left(\int_{I_1} u^2 \, dt \, d\theta\right)^{1/2} + \left(\int_{I_1} |\tilde{\nabla}u|^2 \, dt \, d\theta\right)^{1/2}
\]
\[
\leq \left(\int_{I_2} u^2 \, dt \, d\theta\right)^{1/2} + \left(\int_{I_1} (u(t, \theta) - u(t + L, \theta))^2 \, dt \, d\theta\right)^{1/2}
\]
\[
+ \left(\int_{I_1} |\tilde{\nabla}u|^2 \, dt \, d\theta\right)^{1/2}.
\]
So we have
\[
(e^{L/2} - 1)\|u\|_{1,2} \leq \left(\int_{I_1} \left(\int_0^L \frac{\partial u}{\partial t}(t+s, \theta) \, ds\right)^2 \, dt \, d\theta\right)^{1/2} + \left(\int_{I_1} |\tilde{\nabla}u|^2 \, dt \, d\theta\right)^{1/2}
\]
\[
\leq \int_0^L \left(\int_{I_1} \left|\frac{\partial u}{\partial t}(t+s, \theta)\right|^2 \, dt \, d\theta\right)^{1/2} \, ds + \left(\int_{I_1} |\tilde{\nabla}u|^2 \, dt \, d\theta\right)^{1/2}
\]
\[
\leq C\left(\int_{I_1 \cup I_2} |\tilde{\nabla}u|^2 \, dt \, d\theta\right)^{1/2}
\]
\[
\leq C\left(\int_{I_1 \cup I_2} |\tilde{\nabla}v|^2 \, dt \, d\theta\right)^{1/2} + C\left(\int_{I_1 \cup I_2} |\tilde{\nabla}y|^2 \, dt \, d\theta\right)^{1/2}
\]
\[
\leq C(r_0^{-1/2} + s).
\]
This implies the estimate
\[
\|u\|_{1,i} \leq Ce^{(-(i-2)/2)L}\|u\|_{1,2} \leq Ce^{-(i/2)L}(r_0^{-1/2} + s).
\]
If \(\|u\|_{1,i} < e^{-(1/2)L}\|u\|_{1,i+1}\) happens, we will have similarly
\[
\|u\|_{1,i} \leq Ce^{-(i/2)L}(r_0^{-1/2} + s).
\]
Finally we get
\[
\|u\|_{1,i} \leq Ce^{-(i/2)L} + e^{-(l_n-i)/2L}r_0^{-1/2} + Cr_0^{-1/2},
\]
which implies \((4-1)\). \qed

Then to get the energy decay, we use the Hopf differential
\[
\Phi = |\partial_t v|^2 - |\partial_\theta v|^2 - 2\sqrt{-1}\partial_t v \cdot \partial_\theta v.
\]
We know that the $L^1$ norm of $\Phi$ is invariant under conformal change of the coordinates. Now $(t, \theta)$ is the coordinate of $A_{K r_0 e^{(i-2)L},Kr_0 e^{(i+1)L}}$. We find another coordinate for it: Set $r_i = K r_0 e^{iL}$. Then

$$(r^{-1}_i A_{K r_0 e^{(i-2)L},Kr_0 e^{(i+1)L}}, r^{-2}_i g)$$

can be represented as $(A_{e^{-2L},e^L}, \tilde{g})$, where

$$\| \tilde{g} - |dx|^2 \|_{C^1(A_{e^{-2L},e^L})} \leq \varepsilon.$$ 

Assume this Euclidean coordinate is $(x, y)$, so

$$\int_{S^1 \times [\log K r_0 + (i-1)L, \log K r_0 + i L]} |\Phi| \, dt \, d\theta = \int_{A_{e^{-L},e^L}} |\Phi| \, dx \, dy.$$ 

To estimate the right hand side, we use the Cauchy integral formula on $\Omega' = A_{e^{-L},e^L}$, and set $\Omega' = A_{e^{-L},e^L}$. Then for any $z \in \Omega'$,

$$\Phi(v)(z) = \frac{1}{2\pi \sqrt{-1}} \int_{\partial \Omega} \frac{\Phi(w)}{w - z} \, dw + \frac{1}{2\pi \sqrt{-1}} \int_{\Omega} \frac{\partial \Phi(w)}{\partial \overline{w}} \frac{dw \wedge d\overline{w}}{w - z}.$$ 

We know

$$|\partial_x v|, |\partial_y v| \leq C K r_0 e^{iL} |A| \leq C K r_0 e^{iL} (|x|^{-1} r_0^{-1/2} + r_1^{-1})$$

$$\leq C (r_0^{-1/2} + s e^{-2(l_n-i)L}),$$

so we have

$$\left| \frac{1}{2\pi \sqrt{-1}} \int_{\partial \Omega} \frac{\Phi(w)}{w - z} \, dw \right| \leq C (r_0^{-1} + s^2 e^{-2(l_n-i)L}).$$

For the second term, notice that by easy calculation

$$\frac{\partial \Phi(w)}{\partial \overline{w}} = \partial v \cdot \overline{\tau}(v),$$

where $\overline{\tau}(v)$ is the tension field under this coordinate. Now,

$$|\overline{\tau}(v)| \leq (K r_0 e^{iL})^2 |\nabla e H e| \leq C r_0^{-1}$$

so we have

$$\frac{1}{2\pi \sqrt{-1}} \int_{\Omega} \frac{\partial \Phi(w)}{\partial \overline{w}} \frac{dw \wedge d\overline{w}}{w - z} \leq C r_0^{-1}.$$ 

Then we get

$$\int_{\Omega'} |\Phi| \leq C (r_0^{-1} + s^2 e^{-2(l_n-i)L}).$$
By direct calculation,
\[
\int_{S^1 \times [Kr_0 e^{(i-1)L}, Kr_0 e^{iL}]} |\partial_t v|^2 \, dt \, d\theta \\
\leq \int_{S^1 \times [Kr_0 e^{(i-1)L}, Kr_0 e^{iL}]} |\Phi| \, dt \, d\theta + \int_{S^1 \times [Kr_0 e^{(i-1)L}, Kr_0 e^{iL}]} |\partial_\theta v|^2 \, dt \, d\theta,
\]
and we can get the estimate of
\[
\int_{S^1 \times [Kr_0 e^{(i-1)L}, Kr_0 e^{iL}]} |\partial_\theta v|^2 \, dt \, d\theta
\]
directly by (4-1). So we get the estimate
\[
\int_{S^1 \times [Kr_0 e^{(i-1)L}, Kr_0 e^{iL}]} |\tilde{\nabla} v|^2 \, dt \, d\theta \leq C(e^{-iL} + e^{-(l_n-i)L})s^2 + Cr_0^{-1}.
\]

**Proposition 4.6.** Suppose that \{\Sigma_n\} is a sequence of constant mean curvature surfaces in a given asymptotically flat end \((\mathbb{R}^3 \setminus B_1(0), g)\) and that
\[
\lim_{i \to \infty} r_0(\Sigma_n) = \infty.
\]
Also, suppose that
\[
\lim_{n \to \infty} r_0(\Sigma_n) H(\Sigma_n) = 0.
\]
Then there exists a large number \(K\), a small number \(s\) and a number \(n_0\) such that, when \(n \geq n_0\),
\[
\max_{I_i} |\tilde{\nabla} v| \leq C(e^{-(i/2)L} + e^{-(l_n-i)/2L})s + Cr_0^{-1/2},
\]
where
\[
I_i = S^1 \times [\log(Kr_0(\Sigma_n)) + (i - 1)L, \log(Kr_0(\Sigma_n)) + iL],
\]
\[
i \in [0, l_n] \log(Kr_0(\Sigma_n)) + l_n L = \log(s H^{-1}(\Sigma_n)).
\]

**Proof.** We use the interior estimate of the elliptic equation
\[
\tilde{\Delta} v + |\tilde{\nabla} v|^2 v = \tau.
\]
We know \(\|\tilde{\nabla} v\|_{\infty} \leq C(r_0^{-1/2} + s)\), and \(\|	au\|_{\infty} \leq C r_0^{-1}\). Assume that
\[
I_i \subset \subset \bar{I}_i \subset \subset N_i.
\]
Then for some \(p > 2\),
\[
\sup_{\bar{I}_i} |\tilde{\nabla} v| \leq C \|\tilde{\nabla} v\|_{W^{1,p}(I_i)} \leq C(\|v\|_{L^p(\bar{I}_i)} + r_0^{-1}) \leq C(\|v\|_{L^2(N_i)} + r_0^{-1})
\]
\[
\leq C(e^{-(i/2)L} + e^{-(l_n-i)/2L})s + Cr_0^{-1/2}. \qedhere
\]
This analysis improves our understanding of the blowdowns that we discussed in the previous section. Namely,

**Corollary 4.7.** Assume the same conditions as in Proposition 4.6 and, in addition,

\[
\lim_{r_0 \to \infty} \frac{\log r_1}{r_0^{1/4}} = 0.
\]

Then the limit plane in Lemmas 3.2 and 3.3 are all orthogonal to the same vector \(a\). In fact, we may choose \(s\) small and \(i\) large enough so that

\[|v(x) + a| \leq \varepsilon\]

for all \(x \in \Sigma_n\) and \(|x| \leq sH^{-1}(\Sigma_n)\).

**Proof.** We want to prove that

\[\text{Osc}_{B_{sH^{-1}} \cap \Sigma_n} v\]

is sufficiently small if \(r_0(\Sigma_n)\) is large and \(s\) is small. We already know that

\[\text{Osc}_{B_{Kr_0} \cap \Sigma_n} v\]

is very small from Lemma 3.2, so we need only to prove that

\[\text{Osc}_{(B_{sH^{-1}} \setminus B_{Kr_0}) \cap \Sigma_n} v\]

is small.

From Proposition 4.6 above we find that

\[
\text{Osc}_{(B_{sH^{-1}} \setminus B_{Kr_0}) \cap \Sigma_n} v \leq \sum_{i=1}^{l_n} \text{Osc}_{I_i} v \leq C \sum_{i=1}^{l_n} \sup_{I_i} |\tilde{V} v| \\
\leq C \sum_{i=1}^{l_n} e^{-i/2L} + e^{-(ln-i)/2L}s + r_0^{-1/2} \leq Cs + ln r_0^{-1/2}.
\]

From the inequalities \(C^{-1}r_1 \leq H^{-1} \leq Cr_1\), we have

\[l_n r_0^{-1/2} = L^{-1}(\log(sH^{-1}) - \log(Kr_0)) r_0^{-1/2} \leq C \frac{\log r_1}{r_0^{1/2}} \to 0\]

as \(r_0 \to \infty\), which proves the corollary. \(\square\)
Corollary 4.8. Assume the same conditions as in Proposition 4.6. Let \( v_n = v(p_n) \) for some \( p_n \in I_{l_n/2} \). Then for \( i \in [0, \frac{1}{2} l_n] \),

\[
\sup_{I_i} |v - v_n| \leq C \left( e^{-\left(\frac{1}{2}\right)iL} + e^{-\left(\frac{1}{4}\right)l_nL} \right)s + l_n r_0^{-1/2},
\]

and for \( i \in [\frac{1}{2} l_n, l_n] \),

\[
\sup_{I_i} |v - v_n| \leq C \left( e^{-\left(\frac{1}{4}\right)l_nL} + e^{-\left(\frac{1}{2}\right)(l_n-i)L} \right)s + l_n r_0^{-1/2}.
\]

5. Harmonic coordinates

We assume that the metric \( g \) can be expanded in the coordinate \( \{x^i\} \) as

\[
g_{ij} = \delta_{ij} + h_{ij} = \delta_{ij} + \frac{h_{ij}(\theta)}{r} + Q,
\]

where \( \theta \) is the coordinate on the unit sphere \( S^2 \), \( h_{ij}(\theta) \) is a function extended constantly along the radial direction, and \( Q \) satisfies

\[
\sup r^{2+k} |\partial^k Q| \leq C
\]

for \( k = 0, 1, \ldots, 5 \).

First, note that

\[
\Delta_g x^k = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left( \sqrt{g} g^{ij} \frac{\partial}{\partial x^j} x^k \right)
\]

\[
= \frac{\partial}{\partial x^i} g^{ik} + \frac{1}{2} g^{ik} g^{mn} g_{mni} = -g^{mn} \Gamma_{mn}^k = O(|x|^{-2}).
\]

Our aim is to find an asymptotically harmonic coordinate, that is, a coordinate \( y^i \) such that \( \Delta_g y^k = O(|x|^{-3}) \):

\[
\Delta_g x^k = -g^{jl} g^{ik} \frac{1}{2} \left( \frac{\partial}{\partial x^j} h_{li} + \frac{\partial}{\partial x^l} h_{ji} - \frac{\partial}{\partial x^i} h_{jl} \right)
\]

\[
= -g^{jl} g^{ik} \frac{1}{2} \left( r^{-2} \left( h_{li,j}^1(\theta) - h_{li}^1(\theta) \frac{x^j}{r} \right) + \left( h_{ji,l}^1(\theta) - h_{ji}^1(\theta) \frac{x^l}{r} \right) \right)
\]

\[
- \left( h_{jl,i}^1(\theta) - h_{jl}^1(\theta) \frac{x^i}{r} \right)) \right) + \partial Q
\]

\[
= -g^{jl} g^{ik} \frac{1}{2} r^{-2} f_{lij}^1(\theta) + O(|x|^{-3}).
\]

We also know that \( g^{ij} = \delta^{ij} - h_{ij}^1(\theta)/r + O(r^{-2}) \).

Then

\[
\Delta_g x^k = -\frac{1}{2} r^{-2} f_{jk}^1(\theta) + O(r^{-3}).
\]

Suppose \( 0 = \xi_0 > \xi_1 \geq \xi_2 \geq \cdots \) are the eigenvalues of \( \Delta |_{S^2} \), and \( A_n(\theta) \) are the corresponding orthonormal eigenvectors.
Set
\[ y^k = x^k + \sum_{n=0}^{\infty} f_n^k(r) A_n(\theta). \]

We have
\[ \Delta_g y^k = \Delta_g x^k + \sum_{n=0}^{\infty} \Delta_{\mathbb{R}^3} (f_n^k(r) A_n(\theta)) + \sum_{n=0}^{\infty} (\Delta_g - \Delta_{\mathbb{R}^3}) (f_n^k(r) A_n(\theta)). \]

Solve the equation
\[ \Delta_g x^k + \sum_{n=0}^{\infty} \Delta_{\mathbb{R}^3} (f_n^k(r) A_n(\theta)) = O(|x|^{-3}). \]

Assume
\[ \frac{1}{2} f_{jk}^1(\theta) = \sum_{n=0}^{\infty} \lambda_n^k A_n(\theta). \]

So we have
\[ \sum_{n=0}^{\infty} \Delta_{\mathbb{R}^3} (f_n^k(r) A_n(\theta)) = r^{-2} \sum_{n=0}^{\infty} \lambda_n^k A_n(\theta), \]
\[ \frac{1}{r^2} (2r f_n^{k''} + r^2 f_n^{k'''} + f_n^k(r) \xi_n) = \lambda_n^k, \quad n = 0, \ldots, \infty, \]
\[ f_0^k = \lambda_0^k \log r, \]
\[ f_n^k = \frac{\lambda_n^k}{\xi_n}, \quad n > 0, \]

and this solution satisfies
\[ \sum_{n=0}^{\infty} (\Delta_g - \Delta_{\mathbb{R}^3}) (f_n^k(r) A_n(\theta)) = O(|x|^{-3}). \]

So if
\[ (5-1) \quad y^k = x^k + \frac{1}{2 \sqrt{\pi}} \lambda_0^k \log r + \sum_{n=1}^{\infty} \frac{\lambda_n^k}{\xi_n} A_n(\theta), \]
then we must have
\[ \Delta y^k = O(|x|^{-3}). \]

Note that
\[ \Delta |S^2| \sum_{n=1}^{\infty} \frac{\lambda_n^k}{\xi_n} A_n(\theta) = \sum_{n=1}^{\infty} \lambda_n^k A_n(\theta) = \frac{1}{2} f_{jkj}^1(\theta) - \frac{1}{2} f_{jkj}^1(\theta) \]

where \( f_{jkj}^1(\theta) \) is its mean value on the unit sphere.
Set
\[ g_k^1(\theta) = \sum_{n=1}^{\infty} \frac{\lambda_n^k}{\xi_n^k} A_n(\theta) = \Delta^{-1}\left( \frac{1}{2} f^1_{jk} \theta - \frac{1}{2} f^1_{jk} \theta \right), \]
(5-2)
\[
\frac{\partial y^k}{\partial x^i} = \delta_{ik} + \frac{\lambda_0^k}{2\sqrt{\pi}} r \frac{x^i}{r} + g_k^1(\theta) \frac{1}{r},
\]
\[
\frac{\partial x^i}{\partial y^k} = \delta_{ik} + O(|x|^{-1}).
\]
So we get
\[
\tilde{g}_{ij} = g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = \delta_{ij} + O(|x|^{-1}).
\]

We now define \( \tilde{h}_{ij} \) by
\[
\tilde{g}_{ij} = \delta_{ij} + \tilde{h}_{ij}
\]
and discuss its ellipticity. We have
\[
\tilde{h}_{ij} = h_{ij} - \frac{1}{2r\sqrt{\pi}} \left( \lambda_0^j x^j + \lambda_0^i x^i \right) - \frac{\left( g_{i,j}^1(\theta) + g_{j,i}^1(\theta) \right)}{r},
\]
where \( g_{i,j}^1(\theta) \) denotes the constant extension along the radial direction of function \( (\partial g_i^1(\theta)/\partial x^j)\big|_{S^2} \).

**Example 5.1.** For the metric \( g_{ij} = \delta_{ij} + \delta_{ij}/r \), we have
\[
\Delta_g x^k = -\frac{1}{2} \frac{x^k}{r^3} + O(|x|^{-3}).
\]
On \( S^2 \), we have \( \Delta|_{S^2} x^k = -2x^k \). So if we let
\[
y^k = x^k - \frac{1}{4} \frac{x^k}{r},
\]
then \( \Delta_g y^k = O(|x|^{-3}) \). Thus,
\[
\frac{\partial y^k}{\partial x^i} = \delta_{ki} - \frac{1}{4} \left( \frac{\delta_{ki}}{r} - \frac{x^k x^i}{r^3} \right),
\]
\[
\tilde{h}_{ij} = \frac{3\delta_{ij}}{2r} - \frac{x^i x^j}{2r^3} + O(r^{-2}).
\]

**Lemma 5.2.** Suppose in some coordinate \( \{x^i\} \) that \( g_{ij} = \delta_{ij} + h_{ij}^1(\theta)/r + Q \). Then for any \( m > 2 \) there exists \( \epsilon > 0 \) such that if \( ||h_{ij}^1(\theta) - \delta_{ij}(\theta)||_{W^{m,2}(S^2)} \leq \epsilon \), then in the asymptotically harmonic coordinate \( \{y^i\} \) from above, we have
\[
\tilde{g}_{ij} = \delta_{ij} + \tilde{h}_{ij},
\]
where \( \tilde{h}_{ij} = O(|y|^{-1}) \) s and \( |y|\tilde{h}_{ij} \) is uniformly elliptic.
Proof. We know easily from (5-2) that $\tilde{h}_{ij} = O(|x|^{-1})$ and that $\lim_{|x| \to \infty} |y|/|x| = 1$. Then $\tilde{h}_{ij} = O(|y|^{-1})$. So we need only to prove that $|y|\tilde{h}_{ij}$ is uniformly elliptic.

First, from $\|h^1_{ij}(\theta) - \delta_{ij}(\theta)\|_{W^{m,2}(S^2)} \leq \varepsilon$,

$$\left\| \frac{1}{2} f_{jki}^1(\theta) - \frac{1}{2} x^k \right\|_{W^{m-1,2}(S^2)} \leq C \varepsilon.$$ 

Note that $\frac{1}{2} f_{jki}^1(\theta) = \sum_{n=0}^{\infty} \lambda_n^k A_n(\theta)$ and $x^k$ is an eigenvector of $\Delta_{S^2}$, so we can assume that $A_1(\theta) = C_k x^k \big|_{S^2}$ without loss of generality. Now,

$$\|\lambda_0^k A_0(\theta) + (\lambda_1^k C_k - \frac{1}{2}) x^k + \sum_{n=2}^{\infty} \lambda_n^k A_n(\theta)\|_{W^{m-1,2}(S^2)} \leq \varepsilon,$$

so we get

$$|\lambda_0^k| \leq \varepsilon, \quad \lambda_1^k C_k - \frac{1}{2} \leq \varepsilon, \quad \sum_{n=2}^{\infty} (|\xi_n|(m-1)/2 \lambda_n^k)^2 \leq \varepsilon.$$ 

By (5-1),

$$\frac{\partial y^k}{\partial x^l} = \delta_{lk} + \frac{\lambda_0^k}{2\sqrt{\pi}} \frac{1}{r} x^i - \frac{1}{2} \left( \frac{\lambda_1^k C_k - \frac{1}{2}}{r^3} - \frac{x^i x^k}{r^3} \right) + \sum_{n=2}^{\infty} \frac{\lambda_n^k}{\xi_n} \frac{\partial A_n(\theta)}{\partial x^i},$$

where the last term on the right can be estimated, for some $p > 0$, as

$$\left| \sum_{n=2}^{\infty} \frac{\lambda_n^k}{\xi_n} \frac{\partial A_n(\theta)}{\partial x^i} \right| \leq \sum_{n=2}^{\infty} \frac{|\lambda_n^k|}{|\xi_n|} \frac{|\nabla_{S^2} A_n(\theta)|}{r}$$

$$\leq \sum_{n=2}^{\infty} \frac{|\lambda_n^k|}{|\xi_n|} \frac{\|A_n(\theta)\|_{W^{2+p,2}}}{r}$$

$$\leq \sum_{n=2}^{\infty} \frac{|\lambda_n^k|}{|\xi_n|} \frac{|\xi_n|^{1+p/2} \|A_n(\theta)\|_{L^2}}{r}$$

$$\leq \frac{1}{r} \sum_{n=2}^{\infty} \frac{|\lambda_n^k|}{|\xi_n|} \frac{|\xi_n|(m-1)/2}{|\xi_n|^{p-m+1}/2}$$

$$\leq \frac{1}{r} \left( \sum_{n=2}^{\infty} (|\lambda_n^k|/|\xi_n|)^{(m-1)/2} \right)^{1/2} \left( \sum_{n=2}^{\infty} |\xi_n|^{p-m+1} \right)^{1/2}.$$ 

Let $p = (m - 2)/2$. Then from $\xi_n = O(n)$ we have

$$\sum_{n=2}^{\infty} |\xi_n|^{p-m+1} \leq C,$$
so
\[ \sum_{n=2}^{\infty} \frac{\lambda_n^k}{\xi_n} \partial A_n(\theta) \leq \frac{C\varepsilon}{r}. \]

Then we have
\[ \frac{\partial y^k}{\partial x^i} = \delta_{ik} - \frac{1}{4} \left( \frac{\delta_{ik}}{r} - \frac{x^i x^k}{r^3} \right) + \frac{C\varepsilon}{r}, \]
so we can deduce that
\[ \tilde{h}_{ij} = h_{ij} + \frac{\delta_{ij}}{2} \frac{x^i x^j}{r^2} + C\varepsilon. \]

It follows from \( \| h^1_{ij}(\theta) - \delta_{ij}(\theta) \|_{W^{2,2}(S^2)} \leq \varepsilon \) that \( rh_{ij} \) is uniformly elliptic. The eigenvalues of \( (x^i x^j)/r^2 \) are between 0 and 1, so \( |y|\tilde{h}_{ij} \) is uniformly elliptic, from the fact that \( \lim_{r \to \infty} \frac{|y|}{r} = 1 \) and \( \varepsilon \) is sufficiently small. \( \square \)

So all the analysis in Sections 2–4 can be done in the asymptotically harmonic coordinate \( \{y^i\} \).

**Lemma 5.3.** In the asymptotically harmonic coordinate \( \{y^i\} \), we have
\[ -\frac{1}{2} \Delta_g \log |\tilde{g}| = R(g) + O(|y|^{-4}). \]

**Proof.** From direct calculation we have
\[ R(g) = \tilde{g}^{jk} \tilde{g}^{il} \tilde{g}_{ml} \left( \frac{\partial \tilde{\Gamma}^{m}_{jk}}{\partial y^i} - \frac{\partial \tilde{\Gamma}^{m}_{ik}}{\partial y^j} \right) + O(|y|^{-4}), \]
\[ \tilde{g}^{jk} \tilde{g}^{il} \tilde{g}_{ml} \frac{\partial \tilde{\Gamma}^{m}_{jk}}{\partial y^i} = \tilde{g}^{il} \tilde{g}_{ml} \frac{\partial (\tilde{g}^{jk} \tilde{\Gamma}^{m}_{jk})}{\partial y^i} + O(|y|^{-4}) \]
\[ = -\tilde{g}^{il} \tilde{g}_{ml} \frac{\partial \Delta_g y^m}{\partial y^i} + O(|y|^{-4}) = O(|y|^{-4}), \]
\[ -\tilde{g}^{jk} \tilde{g}^{il} \tilde{g}_{ml} \frac{\partial \tilde{\Gamma}^{m}_{ik}}{\partial y^j} = -\frac{1}{2} \tilde{g}^{jk} \tilde{g}^{ip} \frac{\partial^2 \tilde{g}_{ij}}{\partial y^j \partial y^k} + O(|y|^{-4}) \]
\[ = -\frac{1}{2} \Delta_g \log |\tilde{g}| + O(|y|^{-4}), \]
which proves the lemma. \( \square \)

**Corollary 5.4.** If in addition \( R = O(|x|^{-3-\tau}) \) for some \( \tau > 0 \), then in the asymptotically harmonic coordinate \( \{y^i\} \), we have
\[ \sum_{i=1}^{3} \tilde{h}_{ii} = 8m(g)/|y| + o(|y|^{-1-\tau/2}). \]

**Proof.** First we know that
\[ \lim_{|x| \to \infty} \frac{|y|}{|x|} = 1. \]
Then from Lemma 5.3, in the coordinate \( \{ y^i \} \) we have
\[
\Delta_g \log |\tilde{g}| = O(|y|^{-3-\tau}).
\]
We know that
\[
\log |\tilde{g}| = O(|y|^{-1}).
\]
From the theory of harmonic functions in \( \mathbb{R}^n \), there exists a constant \( C \) such that
\[
\log |\tilde{g}| = \frac{C}{|y|} + o(|y|^{-1-\tau/2}).
\]
From Bartnik’s result, we know the mass is invariant under the change of coordinates because \( R(g) \in L^1 \):
\[
m(g) = \lim_{R \to \infty} \frac{1}{16\pi} \int_{s_R} \left( -\frac{1}{2} \tilde{h}_{ij,j} - \tilde{h}_{jj,i} \right) v^i_g d\mu.
\]
Now,
\[
\tilde{g}_{ik,k} - \frac{1}{2} \tilde{g}_{kk,i} = \tilde{g}^{ij} \tilde{g}^{kl} \left( \tilde{g}_{jk,l} - \frac{1}{2} \tilde{g}_{kl,j} \right) + O(|y|^{-3}) = -\Delta_g y^i + O(|y|^{-3}) = O(|y|^{-3}),
\]
therefore
\[
m(g) = \lim_{R \to \infty} \frac{1}{16\pi} \int_{s_R} \left( -\frac{1}{2} \tilde{h}_{jj,i} \right) v^i_g d\mu
\]
\[
= -\lim_{R \to \infty} \frac{1}{32\pi} \int_{s_R} \frac{\partial \log |\tilde{g}|}{\partial y^i} v^i_g d\mu
\]
\[
= \lim_{R \to \infty} \frac{1}{32\pi} \int_{s_R} \frac{C y^i}{|y|^3} v^i_g d\mu
\]
\[
= \frac{C}{8}.
\]
So we get the result by easy calculation. \( \square \)

**Remark 5.5.** We can replace the constraint equation by the condition
\[
R = O(|x|^{-3-\tau}) \quad \text{for some } \tau > 0.
\]

**6. Proof of Theorem 1.7**

Now let’s prove Theorem 1.7. First note that, if it were false, we could find a sequence \( \Sigma_n \) of stable constant mean curvature surfaces with \( r_0(\Sigma_n) \to \infty \) and \( \log r_1(\Sigma_n) \leq \frac{1}{n} r_0(\Sigma_n)^{1/4} \); but the \( \Sigma_n \) do not belong to the foliation constructed by the standard method. So \( r_1 \leq C r_0 \) cannot hold with a uniform \( C \), by Huang’s uniqueness theorem.
Recall that, for any surface $\Sigma$ embedded in $\mathbb{R}^3$ and any given vector $b \in \mathbb{R}^3$, 
\[ \int_{\Sigma} H_e \langle v_e \cdot b \rangle_e \, d\mu_e = 0, \]
where $H_e$ and $v_e$ denote the mean curvature and normal vector field with respect to the Euclidean metric.

On the other hand, if $\Sigma$ is a constant mean curvature surface in the asymptotically flat end, then 
\[ \int_{\Sigma} H \langle v_e \cdot b \rangle_e \, d\mu_e = 0. \]
So we have 
\[ \int_{\Sigma} (H - H_e) \langle v_e \cdot b \rangle_e \, d\mu_e = 0. \]
Now we want to prove that, for a sequence of stable constant mean curvature spheres $\Sigma_n$ with $r_0(\Sigma_n) \to \infty$ and $\log r_1(\Sigma_n) \leq \frac{1}{n} r_0(\Sigma_n)^{1/4}$, if there does not exist a uniform constant $C$ such that $r_1 \leq Cr_0$ for every $\Sigma_n$, then there exists a subsequence (also denoted by $\Sigma_n$) and a constant vector $b$, such that 
\[ \limsup_{n \to \infty} \int_{\Sigma_n} (H - H_e) \langle v_e, b \rangle_e \, d\mu_e < 0. \]
Then Theorem 1.7 follows from this contradiction.

From now on, our calculation is in the asymptotically harmonic coordinate $\{x^i\}$.

We have calculated $H - H_e$, so 
\[ \int_{\Sigma} (H - H_e) \langle v_e \cdot b \rangle_e \, d\mu_e \]
\[ = \int_{\Sigma} \left( -f^i k h_{kl} f^{lj} A_{ij} + \frac{1}{2} H v^i h_{ij} h_{ij} - f^{ij} v^l \nabla_i h_{lj} \right. \]
\[ \left. + \frac{1}{2} f^{ij} v^l \nabla_i h_{lj} \pm C |h| \mid \nabla h \mid \pm C |h|^2 |A| \right) \langle v_e \cdot b \rangle_e \, d\mu_e. \]
For the sequence of constant mean curvature surfaces $\Sigma_n$ chosen above, we have 
\[ \lim_{n \to \infty} r_0(\Sigma_n) = \infty, \quad \lim_{n \to \infty} H(\Sigma_n) r_0(\Sigma_n) = 0, \]
and 
\[ \lim_{n \to \infty} \frac{\log r_1(\Sigma_n)}{r_0(\Sigma_n)^{1/4}} = 0 \]
because all the radius conditions are preserved when the coordinates turn into asymptotically harmonic coordinates.

So we can choose $s$ sufficiently small and $K$ sufficiently large with $s H^{-1} > Kr_0$ for $r_0$ sufficiently large.
We know that
\[ |h| = O(|x|^{-1}), \quad |
\bar{\nabla}h| = O(|x|^{-2}), \quad |A| \leq CH + C|\hat{A}|. \]

From the estimate
\[ |\hat{A}| \leq r_0^{-1/2}O(|x|^{-1}), \]
we have
\[
\left| \int_{\Sigma} (\pm C|h| |\nabla h| \pm C|h|^2|A|) \langle v_e \cdot b \rangle_e \, d\mu_e \right| \leq C \int_{\Sigma} (H|\nabla h|^{-2} + |\nabla h|^{-3}) = O(r_0^{-1})
\]
by the estimates in Section 2.

Now we calculate other terms in (6-2):
\[
\int_{\Sigma} (H - H_e) \langle v_e \cdot b \rangle_e \, d\mu_e = \int_{\Sigma} -\frac{1}{2} f^{ik} h_{kl} f^{ij} A_{ij} v^m b^m + f^{ij} v^l h_{jl} A_{ik} f^{km} b^m
\]
\[
+ \frac{1}{2} f^{ij} (\nabla_i h_{jl}) v^l v^m b^m + \frac{1}{2} f^{ij} v^l (\nabla_i h_{jl}) v^m b^m + O(r_0^{-1})d\bar{\mu}.
\]

Note that
\[ A_{ij} = \hat{A}_{ij} + \frac{f_{ij}}{2} H, \quad \sup |\hat{A}| \leq r_0^{-1/2}O(|x|^{-1}), \]

hence
\[
\int_{\Sigma} (H - H_e) \langle v_e \cdot b \rangle_e \, d\mu_e = \int_{\Sigma} -\frac{H}{4} f^{kl} h_{kl} v^m b^m + \frac{H}{4} f^{jm} h_{jl} v^l b^m
\]
\[
+ \frac{1}{2} f^{ij} (\nabla_i h_{jl}) v^l v^m b^m - \frac{1}{2} f^{ij} (\nabla_i h_{jl}) v^l v^m b^m
\]
\[
\pm C \int_{\Sigma} |x|^{-2}r_0^{-1/2} + O(r_0^{-1}).
\]

In this case we calculate
\[
\int_{\Sigma} |x|^{-2}r_0^{-1/2} \, d\mu_e.
\]
We divide the integral into three parts:

\[
\int_\Sigma |x|^{-2}r_0^{-1/2} = \int_{\Sigma \cap B_{sH^{-1}}(0)} + \int_{\Sigma \cap B_{K^{(0)}}(0)} + \int_{\Sigma \cap (B_{sH^{-1}} \setminus B_{K^{(0)}})} |x|^{-2}r_0^{-1/2}.
\]

Then by the blowdown results in Section 3,

\[
\int_{\Sigma \cap B_{sH^{-1}}(0)} |x|^{-2}r_0^{-1/2} \, d\mu_e = \int_{\Sigma \cap B'(0)} |\tilde{x}|^{-2}r_0^{-1/2} \, d\tilde{\mu} \leq Cr_0^{-1/2}
\]

\[
\int_{\Sigma \cap B_{K^{(0)}}(0)} |x|^{-2}r_0^{-1/2} \, d\mu_e = \int_{\Sigma \cap B_k(0)} |\tilde{x}|^{-2}r_0^{-1/2} \, d\tilde{\mu} \leq Cr_0^{-1/2}
\]

\[
\int_{\Sigma \cap (B_{sH^{-1}} \setminus B_{K^{(0)}})} |x|^{-2}r_0^{-1/2} \, d\mu_e = \sum_{i=0}^n \int_{\Sigma \cap (B_{K^{(0)e^{iL}}} \setminus B_{K^{(0)e^{(i-1)L}}})} |x|^{-2}r_0^{-1/2} \, d\mu_e
\]

\[
\leq C \sum_{i=0}^n \int_{B_{e^{iL}} \setminus B_1} |\tilde{x}|^{-2}r_0^{-1/2} \, d\tilde{\mu} \leq C r_0^{-1/2} l_n L,
\]

where \( e^{l_nL} K^{(0)} = sH^{-1} \).

So if

\[
\lim_{r_0 \to 0} \frac{|\log H|}{r_0^{1/2}} = 0,
\]

in other words,

\[
\lim_{r_0 \to 0} \frac{|\log r_1|}{r_0^{1/2}} = 0,
\]

we have

\[
\int_\Sigma |x|^{-2}r_0^{-1/2} \, d\tilde{\mu} \to 0
\]
as \( r_0 \to \infty \).

From the property of the asymptotically harmonic coordinate,

\[
g^{ij} h_{ij} = \frac{8m(g)}{r} + o(r^{-1-\tau/2}),
\]

\[
g^{kl} (g_{ik,l} - \frac{1}{2} g_{kl,i}) = O(|x|^{-3}),
\]

\[
\int_\Sigma -\frac{H}{4} f^{kl} h_{kl} v^m b^m + \frac{H}{4} f^{jm} h_{jl} v^l b^m + \frac{1}{2} f^{ij} (\tilde{\nabla}_i h_{lj} - \tilde{\nabla}_l h_{ij}) v^l v^m b^m
\]

\[
= \int_\Sigma -\frac{H}{4} \tilde{g}^{kl} h_{kl} v^m b^m + \frac{H}{4} \tilde{g}^{jm} h_{jl} v^l b^m
\]

\[
+ \frac{1}{2} \tilde{g}^{ij} (\tilde{\nabla}_i h_{lj} - \tilde{\nabla}_l h_{ij}) v^l v^m b^m + O(|r_0|^{-1})
\]

\[
= -2m(g) \int_\Sigma \left( \frac{H}{r} \langle v_e \cdot b_e \rangle_e + \frac{\langle x \cdot v_e \rangle_e \langle v_e \cdot b \rangle_e}{r^3} \right) + \int_\Sigma \frac{H}{4} \tilde{h}_{ml} v^l b^m + o(1).
\]
Therefore,
\[
\lim_{n \to \infty} \left( -2m(g) \int_{\Sigma_n} \left( \frac{H}{r} (v \cdot b) + \frac{(x \cdot v)e (v \cdot b)e}{r^3} \right) + \int_{\Sigma_n} \frac{H}{4} h_{ml} v^l b^m \right) = 0.
\]

Note that
\[
h_{ml} v^l = (h_{ml} - \frac{\text{tr}(h)}{2} \delta_{ml}) v^l + \frac{\text{tr}(h)}{2} v^m,
\]
where \(\text{tr}(h) = g^{ij} h_{ij}\).

Assume that the three eigenvalues of \(h_{ml}\) are \(\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0\).

For \(p \in \Sigma\) fixed, choose coordinates properly so
\[
h_{ml} - \frac{\text{tr}(h)}{2} \delta_{ml}
\]
can be written as
\[
\begin{pmatrix}
\lambda_1 - \frac{\text{tr}(h)}{2} & 0 & 0 \\
0 & \lambda_2 - \frac{\text{tr}(h)}{2} & 0 \\
0 & 0 & \lambda_3 - \frac{\text{tr}(h)}{2}
\end{pmatrix}.
\]
Assume \(v = (\tilde{v}^1, \tilde{v}^2, \tilde{v}^3)\) and \((\tilde{v}^1)^2 + (\tilde{v}^2)^2 + (\tilde{v}^3)^2 = 1\). Then
\[
\sum_{i=1}^{3} \left( (\lambda_i - \frac{\text{tr}(h)}{2}) \tilde{v}^i \right)^2 = \frac{(\text{tr}(h))^2}{4} - \sum_{i=1}^{3} \lambda_i (\text{tr}(h) - \lambda_i) (\tilde{v}^i)^2.
\]
Because of uniform ellipticity, there exists \(C > 0\) such that
\[
\frac{\text{tr}(h)}{C} \leq \lambda_3 \leq \lambda_2 \leq \lambda_1 \leq \left( 1 - \frac{1}{C} \right) \text{tr}(h),
\]
so
\[
\lambda_i (\text{tr}(h) - \lambda_i) \geq \frac{1}{C} \left( 1 - \frac{1}{C} \right) (\text{tr}(h))^2.
\]
Hence
\[
\sum_{i=1}^{3} \left( (\lambda_i - \frac{\text{tr}(h)}{2}) \tilde{v}^i \right)^2 \leq \left( \frac{1}{4} - \frac{1}{C} \left( 1 - \frac{1}{C} \right) \right) (\text{tr}(h))^2
\]
\[
\int_{\Sigma_n} \frac{H}{4} h_{ml} v^l b^m = \int_{\Sigma_n} \frac{H}{4} \left( \frac{\text{tr}(h)}{2} (v \cdot b) + \left( h_{ml} - \frac{\text{tr}(h)}{2} \delta_{ml} \right) v^l b^m \right)
\leq \int_{\Sigma_n} \frac{H \text{tr}(h)}{4} \left( \frac{1}{2} (v \cdot b) + \sqrt{\frac{1}{4} - \frac{1}{C} \left( 1 - \frac{1}{C} \right)} \right)
= \int_{\Sigma_n} \frac{H m(g)}{r} \left( (v \cdot b) + 1 - \frac{2}{C} \right).
Thus, as $n \to \infty$,

$$
\int_{\Sigma_n} (H - H_e) \langle v_e \cdot b \rangle_e \\
\leq -m \int_{\Sigma_n} \frac{H}{r} \langle v_e \cdot b \rangle_e + \frac{2}{r^3} \langle x \cdot v_e \rangle_e \langle v_e \cdot b \rangle_e \, d\mu_e + \left(1 - \frac{2}{C}\right)m(g) \int_{\Sigma_n} \frac{H}{r} \, d\mu_e + o(1).
$$

From Lemma 3.1, we have that $(H/2) \Sigma_n$ subconverges to some sphere $S^2_1(a)$ with $|a| = 1$. Now we choose $b = -a$. Then by the calculation in [Qing and Tian 2007],

$$
-m(g) \int_{\Sigma_n} \frac{H}{r} \langle v_e \cdot b \rangle_e \to -\frac{8}{3} \pi m(g)
$$

$$
-m(g) \int_{\Sigma_n} \frac{2}{r^3} \langle x \cdot v_e \rangle_e \langle v_e \cdot b \rangle_e \to -\frac{16}{3} \pi m(g)
$$

$$
\left(1 - \frac{2}{C}\right)m(g) \int_{\Sigma_n} \frac{H}{r} \to \left(1 - \frac{2}{C}\right)8\pi m(g)
$$

as $n \to \infty$.

Since there is a small difference from [Qing and Tian 2007], we prove these convergences again. Notice from Lemma 3.1 that $(H/2) \Sigma_n$ subconverges to some sphere $S_1(a)$ with $|a| = 1$, and the first and third integral converge, respectively, to

$$
-m(g) \int_{S_1(a)} \frac{2}{r} \langle x \cdot v_e \rangle_e \langle v_e \cdot b \rangle_e \, d\mu_e = -\frac{8}{3} \pi m(g) \quad \text{and} \quad \left(1 - \frac{2}{C}\right)m(g) \int_{S_1(a)} \frac{2}{r} = \left(1 - \frac{2}{C}\right)8\pi m(g).
$$

To deal with (6-4), first notice that

$$
\int_{S^2_1(a)} \frac{2}{r^3} \langle x \cdot v_e \rangle_e \langle v_e \cdot b \rangle_e \, d\mu_e = \frac{4}{3} \pi.
$$

Then we break up the integral (6-4) into three parts:

$$
\int_{\Sigma_n} \frac{2}{r^3} \langle x \cdot v_e \rangle_e \langle v_e \cdot b \rangle_e \, d\mu_e
$$

$$
= \int_{\Sigma_n \cap B^c_{sH^{-1}}(0)} + \int_{\Sigma_n \cap B_{Kr_0}(0)} + \int_{\Sigma_n \cap B_{sH^{-1}} \setminus B_{Kr_0}} \frac{2}{r^3} \langle x \cdot v_e \rangle_e \langle v_e \cdot b \rangle_e \, d\mu_e.
$$

Then

$$
\lim_{n \to \infty} \int_{\Sigma_n \cap B^c_{sH^{-1}}(0)} \frac{2}{r^3} \langle x \cdot v_e \rangle_e \langle v_e \cdot b \rangle_e \, d\mu_e = \int_{S^2_1(a) \cap B^c_2} \frac{2}{r^3} \langle x \cdot v_e \rangle_e \langle v_e \cdot b \rangle_e \, d\mu_e,
$$

$$
\lim_{n \to \infty} \int_{\Sigma_n \cap B_{Kr_0}(0)} \frac{2}{r^3} \langle x \cdot v_e \rangle_e \langle v_e \cdot b \rangle_e \, d\mu_e = \int_{P \cap B_K(0)} \frac{2}{r^3} \langle x \cdot v_e \rangle_e \langle v_e \cdot b \rangle_e \, d\mu_e.
$$
where $P$ is the limit plane in Lemma 3.2. From Corollary 4.7, the normal vector of $P$ is $v_e$. Due to an easy calculation,

$$
\int_P \frac{2}{r^3} \langle x \cdot v_e \rangle_e \langle v_e \cdot b \rangle_e \, d\mu_e = 4\pi.
$$

From the divergence theorem,

$$
\int_{\Sigma_n} \frac{2}{r^3} \langle x \cdot v_e \rangle_e \, d\mu_e = 8\pi
$$

for any $n$ and

$$
\int_{S^2(a)} \frac{2}{r^3} \langle x \cdot v_e \rangle_e \, d\mu_e = 4\pi,
$$

because the origin is on the sphere $S^2(a)$. Since

$$
\lim_{n \to \infty} \int_{\Sigma_n \cap B_{sH-1}(0) \setminus B_{Kr_0}(0)} \frac{2}{r^3} \langle x \cdot v_e \rangle_e \, d\mu_e = \int_{S^2(a) \cap B_{K}(0)} \frac{2}{r^3} \langle x \cdot v_e \rangle_e \, d\mu_e,
$$

$$
\lim_{n \to \infty} \int_{\Sigma_n \cap B_{Kr_0}(0)} \frac{2}{r^3} \langle x \cdot v_e \rangle_e \, d\mu_e = \int_{P \cap B_{K}(0)} \frac{2}{r^3} \langle x \cdot v_e \rangle_e \, d\mu_e,
$$

$$
\int_{p} \frac{2}{r^3} \langle x \cdot v_e \rangle_e \, d\mu_e = 4\pi,
$$

we have

(6-4) \hspace{1cm} \lim_{s \to 0, K \to \infty} \limsup_{n \to \infty} \left| \int_{\Sigma_n \cap (B_{sH-1} \setminus B_{Kr_0})} \frac{2}{r^3} \langle x \cdot v_e \rangle_e \, d\mu_e \right| = 0.

Now we want to prove that

(6-5) \hspace{1cm} \lim_{s \to 0, K \to \infty} \limsup_{n \to \infty} \left| \int_{\Sigma_n \cap (B_{sH-1} \setminus B_{Kr_0})} \frac{2}{r^3} \langle x \cdot v_e \rangle_e \langle v_e \cdot b \rangle_e \, d\mu_e \right| = 0.

Use Corollary 4.8 to get (6-5) from (6-4), but there is a small difference from [Qing and Tian 2007]:

$$
\int_{\Sigma_n \cap (B_{sH-1} \setminus B_{Kr_0})} \frac{2}{r^3} \langle x \cdot v_e \rangle_e \langle v_e \cdot b \rangle_e \, d\mu_e
$$

$$
= \langle v_n \cdot b \rangle_e \int_{\Sigma_n \cap (B_{sH-1} \setminus B_{Kr_0})} \frac{2}{r^3} \langle v_e \cdot b \rangle_e \, d\mu_e
$$

$$
+ \int_{\Sigma_n \cap (B_{sH-1} \setminus B_{Kr_0})} \frac{2}{r^3} \langle x \cdot v_e \rangle_e \langle (v_e - v_n) \cdot b \rangle_e \, d\mu_e.
$$
The first term will converge to 0. We deal with the second term using the cylindrical coordinates in Section 4:

\[
\left| \int_{\Sigma_n \cap (B_{sH-1} \setminus B_{Kr_0})} \frac{2}{r^3} (x \cdot v_e) e \langle (v_e - v_n) \cdot b \rangle_e d\mu_e \right|
\]

\[
= \left| \sum_{j=1}^{l_n} \int_{A_{Kr_0}^{(j-1)L,Kr_0L}} \frac{2}{r^3} (x \cdot v_e) e \langle (v_e - v_n) \cdot b \rangle_e d\mu_e \right|
\]

\[
\leq C \sum_{j=1}^{l_n} L \max_{I_j} |v_e - v_n|
\]

\[
= C \sum_{j=1}^{l_n/2} L \max_{I_j} |v_e - v_n| + C \sum_{j=l_n/2+1}^{l_n} L \max_{I_j} |v_e - v_n|.
\]

From Corollary 4.8,

\[
CL \sum_{i=1}^{l_n/2} \sup_{I_i} |v - v_n| + CL \sum_{i=l_n/2+1}^{l_n} \sup_{I_i} |v - v_n| \leq C (ln e^{-(1/4)L_nL} + C) s + l_n^2 r_0^{-1/2}.
\]

But from the condition

\[
\lim_{n \to \infty} \frac{\log r_1(\Sigma_n)}{r_0(\Sigma_n)^{1/4}} = 0,
\]

we know

\[
\lim_{n \to \infty} l_n^2 r_0^{-1/2} = \lim_{n \to \infty} \left( \frac{L^{-1}(\log s H^{-1} - \log Kr_0)}{r_0^{1/4}} \right)^2 = 0,
\]

so (6-5) holds.

Then

\[
0 \leq -\frac{8}{3} \pi m(g) - \frac{16}{3} \pi m(g) + \left( 1 - \frac{2}{C} \right) 8 \pi m(g) = -\frac{16}{C} \pi m(g).
\]

But \( m(g) > 0 \), so

\[
(6-6) \quad \limsup_{n \to \infty} \int_{\Sigma_n} (H - H_e) \langle v_e, b \rangle_e d\mu_e < 0.
\]

This proves Theorem 1.7.

References


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