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**We consider the linear heat equation on a manifold that evolves under the Ricci flow. The gradient estimates for positive solutions as well as Li–Yau type inequalities are given in this paper. Both the case where $M$ is a complete manifold without boundary and the case where $M$ is compact are considered. We have also obtained the Harnack inequalities for the heat equation on $M$ by previous results.**

## 1. Introduction

The heat equation is a classical subject that has been extensively studied and has lead to many important results, especially in studies of differential geometry. One of the important techniques used in studying the heat equation is the differential Harnack inequality developed by Li and Yau [1986]. This is also applied to Ricci flow by Hamilton [1993], and plays an important role in solving the Poincaré conjecture.

We consider the positive solutions of the linear heat equation on a manifold $M$ that evolves under the Ricci flow. A series of gradient estimates are obtained for such solutions, including several Li–Yau-type inequalities. The manifold $M$ considered here is a complete manifold without boundary.

Let $M$ be a manifold without boundary, and $(M, g(x,t))_{t\in[0,T]}$ be a complete solution to the Ricci flow

$$\frac{\partial}{\partial t} g(x,t) = -2\,\mathrm{Ric}(x,t), \quad x \in M, t \in [0,T]. \tag{1}$$

We assume that its curvature remains uniformly bounded for all $t \in [0,T]$. Consider a positive function $u(x,t)$ defined on $M \times [0,T]$ solving the equation

$$\frac{\partial u}{\partial t} = \Delta u - q(x,t)u, \quad x \in M, \quad t \in [0,T], \tag{2}$$

where $\Delta$ stands for the Laplacian given by $g(x,t)$ and $q(x,t)$ is a $C^2$ function defined on $M \times [0,T]$. Noticing that $\Delta$ depends on the parameter $t$, we study the

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linear heat equation (2) along with the Ricci flow (1). Equation (1) provides us with additional information about the coefficients of the operator $\Delta$ appearing in (2), but is itself fully independent of (2).

## 2. Gradient estimates

Firstly, we introduce a cutoff function on $B_{\rho,T}$. the notation $B_{\rho,T}$ stands for the set $\{(\chi, t) \in M \times [0, T] \mid \operatorname{dist}(\chi, x_0, t) < \rho\}$, which satisfies the basic analytical results stated in the following lemma.

**Lemma 2.1.** *Given $\tau \in (0, T]$, there exists a smooth function*

$$\overline{\Psi} : [0, \infty) \times [0, T] \to \mathbb{R}$$

*satisfying the following requirements*:

1. *The support of $\overline{\Psi}(r, t)$ is a subset of $[0, \rho] \times [0, T]$, and $0 \le \overline{\Psi}(r, t) \le 1$ in $[0, \rho] \times [0, T]$.*

2. *The equalities*

$$\overline{\Psi}(r, t) = 1 \quad \text{and} \quad \frac{\partial \overline{\Psi}}{\partial r}(r, t) = 0$$

*hold on $[0, \rho/2] \times [\tau, T]$ and $[0, \rho/2] \times [0, T]$, respectively.*

3. *The estimate*

$$\left| \frac{\partial \overline{\Psi}}{\partial t} \right| \le \frac{\overline{C}\, \overline{\Psi}^{1/2}}{\tau}$$

*is satisfied on $[0, \infty) \times [0, T]$ for some $\overline{C} > 0$, and $\overline{\Psi}(r, 0) = 0$ for all $\tau \in [0, \infty)$.*

4. *The inequalities*

$$-\frac{C_\alpha \overline{\Psi}^\alpha}{\rho} \le \frac{\partial \overline{\Psi}}{\partial r} \le 0 \quad \text{and} \quad \left| \frac{\partial^2 \overline{\Psi}}{\partial r^2} \right| \le \frac{C_\alpha \overline{\Psi}^\alpha}{\rho^2}$$

*hold on $[0, \infty) \times [0, T]$ for every $a \in (0, 1)$, with a constant $C_\alpha$ dependent on $a$.*

This lemma was first introduced in [Bailesteanu et al. 2010]. In the following part of this section, we establish Li–Yau-type inequalities for system (1)–(2) and obtain a local and a global estimate. To this end, we must introduce an auxiliary function to apply the maximum principle on it. The following lemma deals with the evolution equation of the auxiliary function.

**Lemma 2.2.** *Suppose $(M, g(x, t))_{t \in [0,T]}$ is a complete solution to the Ricci flow* (1). *Assume that $-k_1 g(x, t) \le \operatorname{Ric}(x, t) \le k_2 g(x, t)$ for some $k_1, k_2 > 0$ and all $(x, t) \in B_{\rho,T}$. Suppose $u : M \times [0, T] \to \mathbb{R}$ is a smooth positive function satisfying the heat equation* (2), *and $q(x, t)$ is a $C^2$ function defined on $M \times [0, T]$. Given*

$\alpha \geq 1$, *define* $f = \log u$ *and* $F = t(|\nabla f|^2 - \alpha f_t - \alpha q)$. *The estimate*

$$\left(\Delta - \frac{\partial}{\partial t}\right)F \geq -2\langle \nabla f, \nabla F\rangle - Ft^{-1} - 2k_1 \alpha t\, |\nabla f|^2 + \frac{2a\alpha t}{n}(|\nabla f|^2 - f_t - q)^2 - \frac{\alpha t n}{2b}\max\{k_1^2, k_2^2\} - t\alpha\Delta q - 2t(\alpha-1)\nabla f \nabla q \tag{3}$$

*holds for any* $a, b > 0$, *such that* $a + b = 1/\alpha$.

*Proof.* We begin with finding a convenient bound on $\Delta F$. Notice that

$$\Delta F = t(2f_{ji}^2 + 2f_j f_{jii} - \alpha\Delta(f_t) - a\Delta q), \quad x \in M, \quad t \in [0, T].$$

By the assumption on the Ricci curvature of $M$, it follows that

$$f_j f_{jii} = f_j f_{iij} + R_{ij} f_i f_j \geq \langle \nabla f, \nabla \Delta f\rangle - k_1 |\nabla f|^2$$

at an arbitrary point $(x, t) \in B_{\rho,T}$. Using (1), we get that

$$\Delta(f_t) = (\Delta f)_t - 2\sum_{i,j=1}^{n} R_{ij} f_{ij}.$$

Thus, the estimate

$$\Delta F \geq t[2f_{ji}^2 + 2\langle \nabla f, \nabla \Delta f\rangle - 2k_1 |\nabla f|^2 - \alpha(\Delta f)_t + 2\alpha R_{ij} f_{ij} - \alpha\Delta q]$$

holds at $(x, t) \in B_{\rho,T}$. The next step is to find a suitable bound on those terms in the right side involving $f_{ij}$ by completing the square. Specifically, we find that

$$\begin{aligned}\sum_{i,j=1}^{n}\left(f_{ij}^2 + \alpha R_{ij} f_{ij}\right) &= \sum_{i,j=1}^{n}[(a\alpha + b\alpha) f_{ij}^2 + \alpha R_{ij} f_{ij})] \\ &= \sum_{i,j=1}^{n}\left(a\alpha f_{ij}^2 + \alpha\left(\sqrt{b} f_{ij} + \frac{R_{ij}}{2\sqrt{b}}\right)^2 - \frac{\alpha}{4b}R_{ij}^2\right) \\ &\geq \sum_{i,j=1}^{n}\left(a\alpha f_{ij}^2 - \frac{\alpha}{4b}R_{ij}^2\right)\end{aligned}$$

at $(x, t) \in B_{\rho,T}$ for any $a, b > 0$, such that $a + b = 1/\alpha$. Using the assumptions in the lemma and the standard inequality

$$\sum_{i,j=1}^{n} f_{ij}^2 \geq \frac{(\Delta f)^2}{n},$$

we obtain the estimate

$$\sum_{i,j=1}^{n}(f_{ij}^2 + \alpha R_{ij} f_{ij}) \geq \frac{a\alpha}{n}(\Delta f)^2 - \frac{\alpha}{4b}\max\{k_1^2, k_2^2\}, \quad (x, t) \in B_{\rho,T}.$$

Obviously, we conclude that

$$(4)\quad \Delta F \geq t\Big[\frac{2a\alpha}{n}(\Delta f)^2 + 2\langle \nabla f, \nabla \Delta f\rangle - 2k_1|\nabla f|^2 - \alpha(\Delta f)_t - \alpha\Delta q - \frac{\alpha n}{2b}\max\{k_1^2, k_2^2\}\Big]$$
$$= \frac{2a\alpha t}{n}(|\nabla f|^2 - f_t - q)^2 - 2t\nabla f\nabla(|\nabla f|^2 - f_t - q) - 2k_1 t|\nabla f|^2 - t\alpha\Delta q + \alpha t(|\nabla f|^2 - f_t - q)_t - \frac{\alpha n t}{2b}\max\{k_1^2, k_2^2\}$$

in the set $B_{\rho,T}$.

This gives a convenient bound for $\Delta F$. Now we consider the derivative of $F$ in $t \in [0, T]$. It is not hard to compute that

$$\frac{\partial F}{\partial t} = \frac{F}{t} + t(|\nabla f|^2 - \alpha f_t - \alpha q)_t.$$

Subtracting this from (4), we see that the inequality

$$\Big(\Delta - \frac{\partial}{\partial t}\Big)F \geq \frac{2a\alpha t}{n}(|\nabla f|^2 - f_t - q)^2 - 2t\nabla f\nabla(|\nabla f|^2 - f_t - q) - 2k_1 t|\nabla f|^2 - t\alpha\Delta q - \frac{\alpha n t}{2b}\max\{k_1^2, k_2^2\} - \frac{F}{t} + (\alpha - 1)t(|\nabla f|^2)_t$$

holds in the set $B_{\rho,T}$, with $t > 0$. We need the estimate on $|\nabla f|_t^2$ in order to obtain (3) from this inequality. The Ricci flow equation (1) and the assumptions of the lemma imply

$$|\nabla f|_t^2 = 2\nabla f\nabla(f_t) + 2\,\mathrm{Ric}(\nabla f, \nabla f) \geq 2\nabla f\nabla(f_t) - 2k_1|\nabla f|^2$$

at $(x, t) \in B_{\rho,T}$. As a consequence,

$$\Big(\Delta - \frac{\partial}{\partial t}\Big)F \geq \frac{2a\alpha t}{n}(|\nabla f|^2 - f_t - q)^2 - 2k_1\alpha|\nabla f|^2 - t\alpha\Delta q - \frac{\alpha n t}{2b}\max\{k_1^2, k_2^2\} - \frac{F}{t} - 2\nabla f\nabla F - 2(\alpha - 1)t\nabla f\nabla q$$

in $B_{\rho,T}$. The desired assertion follows. □

Now we can consider the local space-time gradient estimate with Lemma 2.2. In the following part, $n$ is the dimension of $M$.

**Theorem 2.3.** *Let $(M, g(x,t))_{t\in[0,T]}$ be a complete solution to the Ricci flow* (1). *Suppose $-k_1 g(x,t) \leq \mathrm{Ric}(x,t) \leq k_2 g(x,t)$ for some $k_1, k_2 > 0$ and all $(x,t) \in B_{\rho,T}$. Suppose $u : M \times [0,T] \to \mathbb{R}$ is a smooth positive function solving the heat equation* (2), *and $q(x,t)$ is a $C^2$ function defined on $M \times [0,T]$, $|\nabla q| \leq \gamma$, $|\Delta q| \leq \theta$. There exists a constant $C'$ that depends only on the dimension of $M$ and* *satisfies the estimate*

$$(5)\quad \frac{|\nabla u|^2}{u^2}-\alpha\frac{u_t}{u}-\alpha q$$
$$\leq C^{'}\alpha^2\Big(\frac{\alpha^2}{\rho^2(\alpha-1)}+\frac{1}{t}+\max\{k_1,k_2\}\Big)+\frac{nk_1\alpha^3}{\alpha-1}+\frac{\gamma(\alpha-1)}{\alpha}\sqrt{\frac{2\alpha k_1}{n}}+2\alpha\sqrt{\alpha n\theta}$$

*for all* $\alpha > 1$ *and all* $(x,t)\in B_{\rho/2,T}$ *with* $t\neq 0$.

*Proof.* We will use the same notation, $f=\log u$ and $F=t(|\nabla f|^2-\alpha f_t-\alpha q)$, as in Lemma 2.2 and denote $\max\{k_1,k_2\}$ as $\bar{k}$. For a fixed $\tau\in(0,T]$ and fixed $\overline{\Psi}(r,t)$ satisfying the conditions in Lemma 2.2, define $\Psi: M\times[0,T]\to\mathbb{R}$ by setting

$$\Psi(x,t)=\overline{\Psi}(\mathrm{dist}(x,x_0,t),t).$$

We will establish the inequality in Theorem 2.3 at $(x,\tau)$ for $x\in M$, such that $\mathrm{dist}(x,x_0,\tau)<\rho/2$. This will complete the proof.

From Lemma 2.2, some straightforward computations lead to

$$(6)\quad \Big(\Delta-\frac{\partial}{\partial t}\Big)(\Psi F)$$
$$\geq -2\nabla f\nabla(\Psi F)+2F\nabla f\nabla\Psi+\Big(\frac{2a\alpha t}{n}(|\nabla f|^2-f_t-q)^2-2k_1\alpha t\,|\nabla f|^2$$
$$-\frac{\alpha n t}{2b}\bar{k}^2-t\alpha\Delta q-\frac{F}{t}-2(\alpha-1)t\nabla f\nabla q\Big)\Psi$$
$$+(\Delta\Psi)F+2\frac{\nabla\Psi}{\Psi}\nabla(\Psi F)-2\frac{|\nabla\Psi|^2}{\Psi}F-\frac{\partial\Psi}{\partial t}F.$$

This inequality holds in the part of $B_{\rho,T}$ where $\Psi(x,t)$ is smooth and strictly positive. Let $(x_1,t_1)$ be a maximum point for the function $\Psi F$ in the set

$$\{(x,t)\mid 0\leq t\leq\tau,\, d(x,x_0,t)\leq\rho\}.$$

Then we have

$$(7)\quad 0\geq 2F\nabla f\nabla\Psi+\Big(\frac{2a\alpha t_1}{n}(|\nabla f|^2-f_t-q)^2-2k_1\alpha t_1\,|\nabla f|^2$$
$$-\frac{\alpha n t_1}{2b}\bar{k}^2-t_1\alpha\Delta q-\frac{F}{t_1}-2(\alpha-1)t_1\nabla f\nabla q\Big)\Psi$$
$$+(\Delta\Psi)F-2\frac{|\nabla\Psi|^2}{\Psi}F-\frac{\partial\Psi}{\partial t}F$$

at $(x_1,t_1)$. We will now use (7) to show that a certain quadratic expression in $\Psi F$ is nonpositive. The desired result will then follow.

We recall Lemma 2.1 and apply the Laplacian comparison theorem to obtain

$$\frac{|\nabla\Psi|^2}{\Psi} \le \frac{C_{1/2}^2}{\rho^2},$$

$$\Delta\Psi \ge -\frac{C_{1/2}\Psi^{1/2}}{\rho^2} - \frac{C_{1/2}\Psi^{1/2}}{\rho}(n-1)\sqrt{k_1}\coth\bigl(\sqrt{k_1}\rho\bigr) \ge -\frac{d_1}{\rho^2} - \frac{d_1\Psi^{1/2}}{\rho}\sqrt{k_1}$$

at the point $(x_1, t_1)$, where $d_1$ is a positive constant depending on $n$. There exists $C > 0$ such that

$$-\frac{\partial\Psi}{\partial t} \ge -\frac{\overline{C}\Psi^{1/2}}{\tau} - C_{1/2}\overline{k}\Psi^{1/2}.$$

Using these observations along with (6), we get the estimate

$$\begin{aligned} 0 \ge -2F\,|\nabla f|\,|\nabla\Psi| + \Bigl(\frac{2a\alpha t_1}{n}(|\nabla f|^2 - f_t - q)^2 - 2k_1\alpha t_1\,|\nabla f|^2 \\ - t_1\alpha\Delta q - \frac{\alpha n t_1}{2b}\overline{k}^2 - \frac{F}{t_1} - 2(\alpha-1)t_1\nabla f\nabla q\Bigr)\Psi \\ + d_2\Bigl(-\frac{1}{\rho^2} - \frac{\Psi^{1/2}}{\rho}\sqrt{k_1} - \frac{\Psi^{1/2}}{\tau} - \overline{k}\Psi^{1/2}\Bigr)F \end{aligned}$$

at $(x_1, t_1)$, where

$$d_2 = \{3d_1, C_{1/2}, 3C_{1/2}^2, \overline{C}\}.$$

Multiplying by $t\Psi$ and making a few elementary manipulations, we obtain

$$\begin{aligned} (8)\quad 0 \ge& -2t_1F\frac{C_{1/2}\Psi^{3/2}}{\rho}\,|\nabla f| \\ &+ \frac{2t_1^2}{n}\Bigl(a\alpha\Psi^2(|\nabla f|^2 - f_t - q)^2 - nk_1\alpha\Psi^2\,|\nabla f|^2 \\ &\qquad - \frac{n^2\alpha}{4b}\overline{k}^2\Psi^2 - \frac{n}{2}\alpha\Psi^2\Delta q - n(\alpha-1)\nabla f\nabla q\Psi^2\Bigr) \\ &+ d_2\Bigl(-\frac{1}{\rho^2} - \frac{1}{\rho}\sqrt{k_1} - \frac{1}{\tau} - \overline{k}\Bigr)t_1(\Psi F) - \Psi F \end{aligned}$$

at $(x_1, t_1)$. Our next step is to estimate the first two terms on the right side. In order to finish, we introduce the following notations.

Define

$$y = \Psi\,|\nabla f|^2 \quad\text{and}\quad z = \Psi(f_t + q).$$

It is clear that

$$y^{1/2}(y - \alpha z) = \frac{\Psi^{3/2}F\,|\nabla f|}{t}$$

when $t \neq 0$, which yields

$$
\begin{aligned}
(9)\quad & -2t_1 F \frac{C_{1/2}\Psi^{3/2}}{\rho}|\nabla f| + \frac{2t_1^2}{n}\Big(a\alpha\Psi^2(|\nabla f|^2 - f_t - q)^2 \\
&\qquad - nk_1\alpha\Psi^2|\nabla f|^2 - \frac{n^2\alpha}{4b}\overline{k}^2\Psi^2 - \frac{n}{2}\alpha\Psi^2\Delta q - n(\alpha-1)\nabla f\nabla q\Psi^2\Big) \\
&\geq \frac{2t^2}{n}\Big(a\alpha(y-z)^2 - nk_1\alpha y - \frac{n^2\alpha}{4b}\overline{k}^2\Psi^2 \\
&\qquad - \frac{nC_{1/2}}{\rho}y^{1/2}(y-\alpha z) - \frac{n}{2}\alpha\Psi^2\Delta q - n(\alpha-1)\nabla f\nabla q\Psi^2\Big).
\end{aligned}
$$

Let us observe that

$$
(y-z)^2 = \frac{1}{\alpha^2}(y-\alpha z)^2 + \frac{(\alpha-1)^2}{\alpha^2}y^2 + \frac{2(\alpha-1)}{\alpha^2}y(y-\alpha z)
$$

and substitute this into the previous estimate. Regrouping the terms and applying the inequality $mv^2 - nv \geq -n^2/4m$, which is valid for $m, n > 0$ and $v \in \mathbb{R}$, we get

$$
\begin{aligned}
& -2t_1 F \frac{C_{1/2}\Psi^{3/2}}{\rho}|\nabla f| + \frac{2t_1^2}{n}\Big(a\alpha\Psi^2(|\nabla f|^2 - f_t - q)^2 - nk_1\alpha\Psi^2|\nabla f|^2 \\
&\qquad - \frac{n^2\alpha}{4b}\overline{k}^2\Psi^2 - \frac{n}{2}\alpha\Psi^2\Delta q - n(\alpha-1)\nabla f\nabla q\Psi^2\Big) \\
&\geq \frac{2t_1^2}{n}\Big(\frac{a}{\alpha}(y-\alpha z)^2 - 2nk_1\alpha y + nk_1\alpha y - \frac{n^2\alpha}{4b}\overline{k}^2\Psi^2 - \frac{nC_{1/2}}{\rho}y^{1/2}(y-\alpha z) \\
&\qquad - \frac{n}{2}\alpha\Psi^2\theta - n(\alpha-1)\gamma\Psi^{1/2}y^{1/2} + \frac{a}{\alpha}(\alpha-1)^2y^2 + \frac{2a(\alpha-1)}{\alpha}y(y-\alpha z)\Big) \\
&\geq \frac{2t_1^2}{n}\Big(\frac{a}{\alpha}(y-\alpha z)^2 - \frac{n^2\alpha}{4b}\overline{k}^2\Psi^2 \\
&\qquad - \frac{n}{2}\alpha\Psi^2\theta - \frac{n^2 d_2\alpha}{8a\rho^2(\alpha-1)}(y-\alpha z) - \frac{n^2k_1^2\alpha^3}{a(\alpha-1)^2} - \frac{\gamma^2 n(\alpha-1)^2\Psi}{4k_1\alpha}\Big).
\end{aligned}
$$

Because $t(y-\alpha z) = \Psi F$ by definition, (8) now implies that

$$
\begin{aligned}
0 &\geq \frac{2a}{n\alpha}(\Psi F)^2 + \Big(-\frac{nd_2t_1}{\rho^2}\Big(\frac{\alpha}{a(\alpha-1)} + 1 + \rho\sqrt{k_1} + \frac{\rho^2}{\tau} + \rho^2\overline{k}\Big)(\Psi F) \\
&\qquad - \Psi F - \frac{2nk_1^2\alpha^3}{a(\alpha-1)^2}t_1^2 - \frac{\alpha n}{2b}t_1^2\overline{k}^2\Psi^2 - \frac{\gamma^2(\alpha-1)^2t_1^2\Psi}{2k_1\alpha} - \alpha t_1^2\Psi^2\theta \\
&\geq \frac{2a}{n\alpha}(\Psi F)^2 + \Big(-\frac{d_3t_1}{\rho^2}\Big(\frac{\alpha}{a(\alpha-1)} + \frac{\rho^2}{\tau} + \rho^2\overline{k}\Big) - 1\Big)(\Psi F) - \frac{2nk_1^2\alpha^3}{a(\alpha-1)^2}t_1^2 \\
&\qquad - \frac{\alpha n}{2b}t_1^2\overline{k}^2\Psi^2 - \frac{\gamma^2(\alpha-1)^2t_1^2\Psi}{2k_1\alpha} - \alpha t_1^2\Psi^2\theta
\end{aligned}
$$

at $(x_1, t_1)$ with $d_3 = 4nd_2$. The expression in the last two lines is a polynomial in $\Psi F$ of degree 2. Consequently, in accordance with the quadratic formula,

$$\Psi F \le \frac{n\alpha}{2a}\left[\frac{d_3 t_1}{\rho^2}\left(\frac{\alpha}{a(\alpha-1)}+\frac{\rho^2}{\tau}+\rho^2\bar{k}\right)+1+\frac{2k_1\alpha}{\alpha-1}t_1 \right.$$
$$\left. +\sqrt{\frac{a}{b}}t_1\bar{k}\Psi+\frac{\gamma(\alpha-1)t_1}{\alpha}\sqrt{\frac{a}{k_1 n}}+t_1\sqrt{\frac{2a\theta}{n}}\Psi\right]$$

at $(x_1, t_1)$. We will now use this conclusion to obtain a bound on $F(x, \tau)$ for an appropriate range of $x \in M$.

Recall that $\Psi(x, \tau) = 1$ whenever $\operatorname{dist}(x, x_0, t) < \rho/2$. Also, $(x_1, t_1)$ is a maximum point for $\Psi F$ in the set $\{(x, t) \in M \times [0, \tau] \mid \operatorname{dist}(x, x_0, \tau) < \rho\}$. Hence,

$$\begin{aligned} F(x,\tau) &= (\Psi F)(x,\tau) \le (\Psi F)(x_1,t_1) \\ &\le \frac{n\alpha d_3\tau}{2a\rho^2}\left(\frac{\alpha}{a(\alpha-1)}+\frac{\rho^2}{\tau}+\rho^2\bar{k}\right)+\frac{n\alpha}{2a}+\frac{nk_1\alpha^2}{a(\alpha-1)}\tau \\ &\qquad +\frac{\alpha\tau n\bar{k}}{2}\sqrt{\frac{1}{ab}}+\frac{\gamma(\alpha-1)\tau}{\alpha}\sqrt{\frac{k_1}{an}}+\alpha\tau\sqrt{\frac{2n\theta}{a}} \end{aligned}$$

for all $x \in M$, such that $\operatorname{dist}(x, x_0, \tau) < \rho/2$. Since $\tau \in (0, T]$ is chosen arbitrarily, this formula implies that

$$\begin{aligned} &(|\nabla f|^2-\alpha f_t-\alpha q)(x,t) \\ &\le \frac{\alpha d_4}{a\rho^2}\left(\frac{\alpha}{a(\alpha-1)}+\frac{\rho^2}{t}+\rho^2\bar{k}\right)+\frac{nk_1\alpha^2}{a(\alpha-1)}+\frac{\alpha n\bar{k}}{2}\sqrt{\frac{1}{ab}}+\frac{\gamma(\alpha-1)}{\alpha}\sqrt{\frac{k_1}{an}}+\alpha\sqrt{\frac{2n\theta}{a}}, \end{aligned}$$

$(x, t) \in B_{\rho/2,T}$, with $d_4 = \max\{nd_3, n\}$, as long as $t \neq 0$. If we set $a = 1/(2\alpha)$, note that $b = 1/\alpha - a$, and define the constant $C'$ appropriately, estimate (5) will follow by a straightforward computation. □

Now we consider the case where the manifold $M$ is compact. Assume $M$ has nonnegtive Ricci curvature, we will deduce a global estimate on $u(x, t)$. A related inequality for (1)–(2) can be found in [Hamilton 1993].

**Theorem 2.4.** *Suppose the manifold $M$ is a solution to the Ricci flow* (1). *Assume that* $0 \le \operatorname{Ric}(x, t) \le kg(x, t)$ *for some* $k > 0$ *and all* $(x, t) \in M \times [0, T]$. *Suppose* $u : M \times [0, T] \to \mathbb{R}$ *is a smooth positive function satisfying the heat equation* (2), *and* $q(x, t)$ *is a* $C^2$ *function defined on* $M \times (0, T)$ *and* $|\Delta q| \le \theta$. *The estimate*

$$\frac{|\nabla u|^2}{u^2}-\frac{u_t}{u}-q \le kn+\frac{n}{2t}+2\sqrt{n\theta} \tag{10}$$

*holds for all* $(x, t) \in M \times [0, T]$.

*Proof.* As before, we write $f$ instead of $\log u$. It will be convenient for us to denote $F_1 = t(|\nabla f|^2 - f_t - q)$. Fix $\tau \in (0, T]$, and choose a point $(x_0, t_0) \in M \times [0, \tau]$ where $F_1$ attains its maximum on $M \times [0, \tau]$. Our first step is to show that

$$F_1(x_0, t_0) \leq t_0 kn + \frac{n}{2} + 2\sqrt{n\theta}\, t_0.$$

Then the theorem will follow.

If $t_0 = 0$, then $F_1(x, t_0)$ is equal to 0 for every $x \in M$, and estimate (10) becomes evident. Therefore, we can assume $t_0 > 0$ without loss of generality. By Lemma 2.2 and assumptions on the Ricci curvature of $M$, we can deduce that

$$\Big(\Delta - \frac{\partial}{\partial t}\Big) F_1 \geq \frac{2a}{n}\frac{F^2}{t_0} - \frac{F}{t_0} - 2\nabla f \nabla F - \frac{n t_0 k^2}{2(1-a)} - t_0 \Delta q$$

for all $a \in (0, 1)$ at the point $(x_0, t_0)$. Recall that $F_1$ attains its maximum at $(x_0, t_0)$. This tells us that

$$\Delta F_1(x_0, t_0) \leq 0, \qquad \frac{\partial}{\partial t} F_1(x_0, t_0) \geq 0, \qquad \text{and } \nabla F_1(x_0, t_0) = 0.$$

In consequence, the estimate

$$0 \geq \frac{2a}{n}\frac{F^2}{t_0} - \frac{F}{t_0} - \frac{n t_0 k^2}{2(1-a)} - t_0 \theta$$

holds at $(x_0, t_0)$, and the quadratic formula yields

$$F_1 \leq \frac{n}{4a}\bigg(1 + \sqrt{1 + \frac{4 a t_0^2}{1-a} k^2 + \frac{8 a \theta}{n} t_0^2}\,\bigg).$$

Letting $a = (1 + k t_0)/(1 + 2k t_0)$, and substituting this into the above inequality, we arrive at (10).

We now need only a simple argument to complete the proof. Since $(x_0, t_0)$ is the maximum point for $F_1$ on $M \times [0, \tau]$, we are then able to conclude that

$$F_1(x, \tau) \leq F_1(x_0, t_0) \leq t_0 kn + \frac{n}{2} + 2\sqrt{n\theta}\, t_0 \leq \tau kn + \frac{n}{2} + 2\sqrt{n\theta}\, \tau$$

for all $x \in M$. Therefore, the estimate

$$\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} - q \leq kn + \frac{n}{2\tau} + 2\sqrt{n\theta}$$

holds at $(x, \tau)$. Because $\tau \in (0, T]$ can be chosen arbitrarily, the assertion of the theorem follows. □

## 3. Harnack Inequalities

The last goal is to get two Harnack inequalities. These may be viewed as applications of Theorems 2.3 and 2.4.

**Lemma 3.1.** *Let* $(M, g(x,t))_{t\in[0,T]}$ *be a complete solution to the Ricci flow* (1). *Suppose* $u : M \times [0,T] \to \mathbb{R}$ *is a smooth positive function satisfying the heat equation* (2), *and* $q(x,t)$ *is a* $C^2$ *function defined on* $M \times [0,T]$. *Define* $f = \log u$, *and assume that*

$$\frac{\partial f}{\partial t} \geq \frac{1}{A_1}\Big(|\nabla f|^2 - A_2 - \frac{A_3}{t}\Big) - q, \quad x \in M, t \in (0,T]$$

*for some* $A_1, A_2, A_3 > 0$. *Then the inequality*

$$u(x_2,t_2) \geq u(x_1,t_1)\Big(\frac{t_2}{t_1}\Big)^{-A_3/A_1} \exp\Big(-\frac{A_1}{4}\Gamma(x_1,t_{1,}x_2,t_2) - \frac{A_2}{A_1}(t_2-t_1)\Big)$$

*holds for all* $(x_1,t_1) \in M \times (0,T)$, *and* $(x_2,t_2) \in M \times (0,T)$ *such that* $t_1 < t_2$, *where*

$$\Gamma(x_1,t_{1,}x_2,t_2) = \inf \int_{t_1}^{t_2} \Big[\Big|\frac{d}{dt}\gamma(t)\Big|^2 + \frac{4}{A_1}q\Big]dt,$$

*and the infimum is taken over the set* $\Theta(x_1,t_{1,}x_2,t_2)$ *of all the smooth paths* $\gamma : [t_1,t_2] \to M$ *that connect* $x_1$ *to* $x_2$. *We remind the reader that the norm* $|\cdot|$ *depends on* $t$.

*Proof.* Consider a path $\gamma \in \Theta(x_1,t_{1,}x_2,t_2)$. We begin by computing

$$\begin{aligned}\frac{d}{dt}f(\gamma(t),t) &= \nabla f(\gamma(t),t)\frac{d}{dt}\gamma(t) + \frac{\partial}{\partial s}f((\gamma(t),s)\big|_{s=t}\\ &\geq -|\nabla f(\gamma(t),t)|\Big|\frac{d}{dt}\gamma(t)\Big| + \frac{1}{A_1}\Big(|\nabla f|^2 - A_2 - \frac{A_3}{t}\Big) - q\\ &\geq -\frac{A_1}{4}\Big|\frac{d}{dt}\gamma(t)\Big|^2 + \frac{1}{A_1}\Big(-A_2 - \frac{A_3}{t}\Big) - q,\end{aligned}$$

$t \in [t_1,t_2]$.

By the inequality $mv^2 - nv \geq -n^2/(4m)$, valid for $m, n > 0$ and $v \in \mathbb{R}$, we get the last step. It then follows that

$$\begin{aligned}f(x_2,t_2) - f(x_1,t_1) &= \int_{t_1}^{t_2} \frac{d}{dt}f(\gamma(t),t)\,dt\\ &\geq -\frac{A_1}{4}\int_{t_1}^{t_2}\Big|\frac{d}{dt}\gamma(t)\Big|^2 - \frac{A_2}{A_1}(t_2-t_1) - \frac{A_3}{A_1}\ln\frac{t_2}{t_1} - \int_{t_1}^{t_2} q\,dt.\end{aligned}$$

The assertion of the lemma follows by exponentiating. $\square$

We are ready to formulate our Harnack inequalities for (1)–(2).The first one applies on noncompact manifolds. The second one does not, but provides a more explicit estimate.

**Theorem 3.2.** *Let* $(M, g(x,t))_{t\in[0,T]}$ *be a complete solution to the Ricci flow* (1). *Assume that* $-k_1 g(x,t) \le \mathrm{Ric}(x,t) \le k_2 g(x,t)$ *for some* $k_1, k_2 > 0$ *and all* $(x,t) \in M\times[0,T]$. *Suppose* $u : M\times[0,T] \to \mathbb{R}$ *is a smooth positive function solving the heat equation* (2), *and* $q(x,t)$ *is a* $C^2$ *function defined on* $M\times[0,T]$, $|\nabla q| \le \gamma$, *and* $|\Delta q| \le \theta$. *Given* $\alpha > 1$, *the estimate*

$$(11)\quad u(x_2,t_2) \ge u(x_1,t_1)\Big(\frac{t_2}{t_1}\Big)^{-C'\alpha} \exp\Big(-\frac{\alpha}{4}\Gamma(x_1,t_{1,}x_2,t_2) - \frac{1}{\alpha}\Big(C'\alpha^2\bar{k} + \frac{nk_1\alpha^3}{\alpha-1} + \frac{\gamma(\alpha-1)}{\alpha}\sqrt{\frac{2\alpha k_1}{n}} + 2\alpha\sqrt{\alpha n\theta}\Big)(t_2-t_1)\Big)$$

*holds for all* $(x_1,t_1) \in M\times[0,T]$, *and* $(x_2,t_2) \in M\times[0,T]$, *such that* $t_1 < t_2$. *The constant* $C'$ *comes from Theorem 2.3, where*

$$\Gamma(x_1,t_{1,}x_2,t_2) = \inf \int_{t_1}^{t_2} \Big[\Big|\frac{d}{dt}\gamma(t)\Big|^2 + \frac{4}{A_1}q\Big]dt,$$

*and the infimum is taken over the set* $\Theta(x_1,t_{1,}x_2,t_2)$ *of all the smooth paths* $\gamma : [t_1,t_2] \to M$ *that connect* $x_1$ *to* $x_2$.

*Proof.* Letting $\rho$ go to infinity in (5), we conclude that

$$\frac{u_t}{u} \ge \frac{1}{\alpha}\Big(\frac{|\nabla u|^2}{u^2} - C'\alpha^2\Big(\frac{1}{t}+\bar{k}\Big) - \frac{nk_1\alpha^3}{\alpha-1} - \frac{\gamma(\alpha-1)}{\alpha}\sqrt{\frac{2\alpha k_1}{n}} - 2\alpha\sqrt{\alpha n\theta}\Big) - q$$

on $M\times(0,T]$. The desired assertion is now a consequence of Lemma 3.1. □

**Theorem 3.3.** *Suppose the manifold* $M$ *is a solution to the Ricci flow* (1). *Assume that* $0 \le \mathrm{Ric}(x,t) \le kg(x,t)$ *for some* $k > 0$ *and all* $(x,t) \in M\times[0,T]$. *Suppose* $u : M\times[0,T] \to \mathbb{R}$ *is a smooth positive function satisfying the heat equation* (2), *and* $q(x,t)$ *is a* $C^2$ *function defined on* $M\times[0,T]$, $|\Delta q| \le \theta$. *The estimate*

$$u(x_2,t_2) \ge u(x_1,t_1)\Big(\frac{t_2}{t_1}\Big)^{-n/2} \exp\Big(-\frac{\alpha}{4}\Gamma(x_1,t_{1,}x_2,t_2) - \big(kn + 2\sqrt{n\theta}\big)(t_2-t_1)\Big)$$

*holds for all* $(x_1,t_1) \in M\times[0,T]$ *and* $(x_2,t_2) \in M\times(0,T)$ *as long as* $t_1 < t_2$.

*Proof.* Theorem 2.4 implies that

$$\frac{|\nabla u|^2}{u^2} - q - \Big(kn + \frac{n}{2t} + 2\sqrt{n\theta}\Big) \le \frac{u_t}{u}, \quad x \in M, t \in (0,T].$$

We now use Lemma 3.1 to complete the proof. □

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