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REMARKS ON A KÜNNETH FORMULA FOR FOLIATED DE RHAM COHOMOLOGY

MÉLANIE BERTELSON

The validity of the Künneth formula for foliated cohomology, that is, for the tangential de Rham cohomology of a foliated manifold, is investigated. The main difficulty encountered is the non-Hausdorff nature of the foliated cohomology spaces, forbidding the completion of the tensor product. We present versions of the Künneth formula when both factors have Hausdorff foliated cohomology and when one factor has finite-dimensional foliated cohomology and a compact underlying manifold. We also give a counterexample to an alternative version of the Künneth formula. The proof of the second result involves a right inverse for the foliated de Rham differential.

Introduction

The tangential de Rham cohomology or *foliated cohomology* of a foliated manifold $(M, \mathscr{F})$ is the cohomology of the complex obtained by forming the quotient of the Fréchet space of ordinary smooth forms on the manifold by those who vanish along the leaves of the foliation. Our initial interest for this cohomology comes from the observation that its vanishing in degree two may, under certain circumstances, be an obstruction to existence of a foliated symplectic structure, or equivalently, a regular Poisson structure whose underlying foliation is $\mathscr{F}$ (see [Bertelson 2001]). Among the tools for computing de Rham cohomology is the Künneth formula which asserts that the cohomology space of a product is isomorphic to the completed tensor product of the cohomology spaces of the factors via the map

$$\varphi : \bigoplus_{p+q=n} H^p(M) \otimes H^q(N) \to H^n(M \times N), \quad a \otimes b \mapsto a \wedge b.$$

This map indeed induces a map on foliated cohomology but because these spaces do not generally enjoy the Hausdorff separation property, the completion of the tensor product may not even be defined.

The results obtained so far and shown in the present paper are:

MSC2000: primary 46A32, 46E10, 53C12, 53D17; secondary 46A63, 46M18, 46A04.
Keywords: foliation, tangential de Rham cohomology, Künneth, tensor product, right inverse.

(1) The Künneth formula is valid when the foliated cohomology spaces of both factors are Hausdorff. This is a consequence of a result of Grothendieck, described in [Schwartz 1954]. We have nevertheless included a relatively detailed proof in Section 2.

(2) It is also valid when the foliated cohomology of one of the factors is finite-dimensional and the underlying manifold of that same factor is compact. Notice that it is not necessary to complete the tensor product in that case. This result was already known when one of the factors is a one-leaf foliation [El Kacimi-Alaoui 1983; Moore and Schochet 2006]. Our proof requires the construction of a right inverse for the foliated de Rham differential. It is based on results in the theory of splitting of exact sequences of Fréchet spaces [Meise and Vogt 1997; Vogt 2004].

(3) In the simple case where one of the factors, say $(M, \mathcal{F})$, has a non-Hausdorff foliated cohomology and the other factor, say $(N, \mathcal{G})$, is a manifold foliated by its points, a natural alternative version to the Künneth formula would involve $C^\infty(N, H^*(\mathcal{F}))$ in place of the completed tensor product. Nevertheless, we have constructed on the torus $\mathbb{T}^2$ foliated by Liouville slope lines a smooth family of exact forms — representing thus the zero element in $C^\infty(N, H^*(\mathcal{F}))$ — which is not the coboundary of any continuous family of functions — corresponding therefore to a nonzero element in $H^*(\mathcal{F} \times \mathcal{G})$.

Many relevant questions remain unanswered:

- Does a more sophisticated version of the Künneth formula, involving some type of higher order functors, hold in a non-Hausdorff situation?
- Does the foliated de Rham differential still admit a right inverse when the assumption of compactness or finite-dimensionality of the foliated cohomology is relaxed?

Finally, the results of this paper may apply to other cohomologies. We have in mind the Poisson cohomology of a Poisson manifold (not surprisingly, the tangential Poisson cohomology of a regular Poisson structure is isomorphic to the foliated cohomology of the induced foliation). For instance, the Künneth formula for Poisson cohomology is valid when the cohomology spaces are Hausdorff.

1. Preliminaries

Let $(M, \mathcal{F})$ be a foliated manifold, that is, a smooth Hausdorff second countable manifold M endowed with a smooth foliation $\mathcal{F}$. The space of smooth p-forms, $p \geq 0$, is denoted by $\Omega^p(M)$ (a smooth 0-form is just a smooth function) and the space of all forms by $\Omega^*(M)$. The weak C^∞ topology provides $\Omega^p(M)$ (and $\Omega^*(M)$) with the structure of a nuclear Fréchet space. We are interested in the nuclear property because it guarantees uniqueness of the completion of the tensor product with any other Fréchet space.

Recall that a Fréchet space is a locally convex, metrizable, complete topological vector space. We will not attempt to explain the nuclear property here, but rather refer to Sections 47 and 50 [Trèves 1967], henceforth, abbreviated as [T]. For our purpose it is sufficient to know that the set of smooth functions on an open subset of $\mathbb{R}^n$ is nuclear (Corollary of Theorem 51.5 in [T, p. 530]), that a product of nuclear spaces is nuclear and that a Hausdorff projective limit of nuclear spaces is nuclear (Proposition 50.1 in [T, p. 514]). Indeed, $\Omega^p(M)$ is the projective limit of the spaces $\Omega^p(\phi_\alpha(U_\alpha))$, where (U_α, ϕ_α) runs through an atlas on M. We will occasionally write TVS for topological vector space.

Consider the space $\Omega^p(M, \mathcal{F}) = \{\omega \in \Omega^p(M) : \omega|_F = 0 \ \forall \text{ leaf } F\}$ of forms vanishing along the leaves of $\mathcal{F}$. It is a closed subspace of $\Omega^p(M)$. Thus the quotient $\Omega^p(M)/\Omega^p(M, \mathcal{F})$ is a Fréchet nuclear space as well (see [T, p. 85 and Proposition 50.1, p. 514]). It is the space of *foliated p-forms*. The de Rham differential $d : \Omega^*(M) \to \Omega^{*+1}(M)$ which is a continuous linear map induces the *foliated de Rham differential* $d_{\mathcal{F}} : \Omega^*(\mathcal{F}) \to \Omega^{*+1}(\mathcal{F})$ with like properties. The space of $d_{\mathcal{F}}$-closed (respectively $d_{\mathcal{F}}$-exact) foliated p-forms is denoted by $\mathcal{Z}^p(\mathcal{F})$ (respectively $\mathcal{B}^p(\mathcal{F})$). The cohomology $H^*(\mathcal{F}) = \mathcal{Z}^*(\mathcal{F})/\mathcal{B}^*(\mathcal{F})$ is called the *foliated (de Rham) cohomology* of $(M, \mathcal{F})$.

Remark 1.1. The (ordinary) de Rham differential is always a homomorphism, that is, the image of an open subset of $\Omega^p(M)$ under d consists of a relative open subset of $d(\Omega^p(M))$. This is a consequence of the fact that a form is exact if and only if its integral over any closed cycle vanishes, showing that exact forms are a closed subset which by the open mapping theorem for metrizable and complete topological vector spaces implies that d is open (Theorem 17.1 in [T, p. 170]). In contrast, the differential $d_{\mathcal{F}}$ need not be a homomorphism, as illustrated by Example 1.2 which describes the Liouville slopes foliations on the torus $\mathbb{T}^2$. Observe that assuming that $d_{\mathcal{F}}$ is open is equivalent to assuming that $\mathcal{B}^*(\mathcal{F})$ is closed (by the open mapping theorem for one direction and the observation that the image by a homomorphism of a complete metrizable TVS is a closed space for the other direction) or that the cohomology $H^*(\mathcal{F})$ is Hausdorff.

In this connection, the following examples are useful to keep in mind.

Example 1.2 (Kronecker foliations). Consider the foliation of $\mathbb{R}^2$ by parallel lines of slope $\alpha \in \mathbb{R}$. Invariant under the action of $\mathbb{Z}^2$ by translations, this foliation induces a foliation, denoted $\mathcal{F}_\alpha$, on the torus $\mathbb{T}^2$. The leaves are circles when α is a rational number and are dense lines otherwise. The foliated de Rham cohomology of $\mathcal{F}_\alpha$ for α irrational depends on the type of irrational number considered. More specifically, it is infinite-dimensional and non-Hausdorff (with a one-dimensional

Hausdorff quotient) when α is a Liouville number (see Definition 1.3), and one-dimensional and Hausdorff otherwise. The proof of this well-known fact can be found in [Haefliger 1980; Moore and Schochet 2006] and will appear implicitly in Section 4.

Definition 1.3. A Liouville number α is an irrational number that is *well approximated* by rational numbers. More precisely, for all integers $p \geq 1$, there exist relatively prime integers m, n with $n > 1$ such that

$$\left|\alpha - \frac{m}{n}\right| < \frac{1}{|n|^p}.$$

A typical example of such a number is Liouville's constant $\sum_{k=1}^{\infty} 10^{-k!}$. Liouville numbers are transcendental because an algebraic number α of degree $p \geq 2$ admits a constant c such that

$$\left|\alpha - \frac{m}{n}\right| > \frac{c}{|n|^p},$$

for all integers m, n with $n > 0$. On the other hand e and π, for instance, are not Liouville, as are uncountably many transcendental numbers. The set of Liouville numbers is a countable intersection of open dense sets and has measure zero. A non-Liouville number is sometimes called a *generic number*.

Example 1.4. Let $(M, \mathcal{F})$ be a foliation that has a vanishing k-cycle, that is, a smooth foliated map $v : (S^k \times [0, 1], \mathcal{F}_\pi) \to (M, \mathcal{F})$, where S^k is a sphere of dimension k and $\mathcal{F}_\pi$ is the foliation by the fibers of the canonical projection $\pi : S^k \times [0, 1] \to [0, 1]$, such that the image of $S^k \times \{t\}$ is homotopically trivial in its leaf for each t except $t = 0$. A p-dimensional foliation from which a point is removed carries a vanishing $(p-1)$-cycle. We explain hereafter, in the specific case of a punctured foliation $(M, \mathcal{F}) = (N - \{q\}, \mathcal{G}|_{N-\{q\}})$, how the presence of the vanishing $(p-1)$-cycle implies that $H^p(\mathcal{F})$ is non-Hausdorff and infinite-dimensional. The argument can certainly be extended to a larger class of vanishing cycles.

Observe that our vanishing cycle can be "filled", in the sense that there exists a foliated map $\bar{v} : (D^p \times [0, 1] - \operatorname{int} D^p \times \{0\}, \mathcal{F}_\pi) \to (M, \mathcal{F})$ that extends v. Let Ω be a foliated volume form on $(N, \mathcal{G})$ and let f be a smooth function on M approaching infinity near the puncture. Then $f\Omega$ is a foliated closed p-form on M than cannot be foliated exact. Indeed, suppose on the contrary that $f\Omega = d_{\mathcal{F}}\alpha$. Then, by Stokes' theorem,

$$\int_{\bar{v}(D^p \times \{t\})} f\Omega = \int_{v(S^{p-1} \times \{t\})} \alpha.$$

Clearly, as t approaches 0, the right-hand side converges to $\int_{v(S^{p-1} \times \{0\})} \alpha$ while the left-hand side diverges, yielding a contradiction. Besides, it is not too difficult to construct an example of a nonexact p-form of this type that is the limit of a sequence of exact forms, showing that the set of foliated exact forms is not closed in the set of foliated closed forms.

2. Künneth formula when the cohomology is Hausdorff

The main result of the present section, that is, a Künneth formula for foliated cohomology when the foliated cohomology of each factor is Hausdorff, is not original, as it is essentially a consequence of a theorem due to Grothendieck and described in [Schwartz 1954]. (A proof in terms of sheaves can also be found in the literature, namely in [Bredon 1997].) Nevertheless we give a relatively detailed explanation of the proof, with systematic references to the book [Trèves 1967] (referred to as [T]) for the background functional analysis, believing that some readers might find it useful to have the proof expressed in a language familiar to differential geometers with references from just one very well-written book.

Let $(M, \mathscr{F})$ and $(N, \mathscr{G})$ be two foliated manifolds both having the property that the foliated de Rham differential is a homomorphism. Consider the (algebraic) tensor product $\Omega^p(\mathscr{F}) \otimes \Omega^q(\mathscr{G})$. There are two natural ways to construct a topology on the tensor product of two locally convex Hausdorff topological vector spaces, namely the ε and the π topology (Sections 42 and 43 in [T]), thus yielding two different completions of the tensor product. However, when one of the factors is Fréchet nuclear, both topologies coincide (Theorem 50.1 in [T, p. 511]). So in our case we can ignore this issue and write $\Omega^p(\mathscr{F}) \mathbin{\widehat{\otimes}} \Omega^q(\mathscr{G})$ for the completion — with respect to this unique natural topology — of the tensor product of $\Omega^p(\mathscr{F})$ with $\Omega^q(\mathscr{G})$. The tensor product of two continuous linear maps $f_1 : E_1 \to F_1$ and $f_2 : E_2 \to F_2$ between nuclear Fréchet spaces is a continuous linear map

$$f_1 \otimes f_2 : E_1 \otimes E_2 \to F_1 \otimes F_2 \subset F_1 \mathbin{\widehat{\otimes}} F_2,$$

by Proposition 43.6 in [T, p. 439]; this induces a continuous linear map $f_1 \mathbin{\widehat{\otimes}} f_2 : E_1 \mathbin{\widehat{\otimes}} E_2 \to F_1 \mathbin{\widehat{\otimes}} F_2$ between the completions. In general, the completion of a Hausdorff locally convex TVS E is denoted by $\hat{E}$ and the extension of a continuous linear map $u : E \to F$ to the completions by $\hat{u} : \hat{E} \to \hat{F}$ [T, Theorem 5.1, p. 39].

Consider the tensor product complex $(\Omega^*(\mathscr{F}) \mathbin{\widehat{\otimes}} \Omega^*(\mathscr{G}), d)$ defined as

$$\big(\Omega^*(\mathscr{F}) \mathbin{\widehat{\otimes}} \Omega^*(\mathscr{G})\big)^n \overset{\text{def}}{=} \bigoplus_{p+q=n} \Omega^p(\mathscr{F}) \mathbin{\widehat{\otimes}} \Omega^q(\mathscr{G}),$$

with differential $d = d_{\mathscr{F}} \mathbin{\widehat{\otimes}} 1 + \varepsilon \mathbin{\widehat{\otimes}} d_{\mathscr{G}}$, where $\varepsilon(\omega) = (-1)^p \omega$ when ω is a foliated form of degree p. It follows from general considerations that $\Omega^*(\mathscr{F}) \mathbin{\widehat{\otimes}} \Omega^*(\mathscr{G})$ is a nuclear Fréchet space (Proposition 50.1 in [T, p. 514]) as well and that d is a homomorphism. The latter assertion is a consequence of Proposition 43.9 in [T, p. 441] and the fact that the sum of two homomorphisms is a homomorphism.

There is a natural map φ between and $\Omega^*(\mathcal{F})\,\widehat{\otimes}\,\Omega^*(\mathcal{G})$ and $\Omega^*(\mathcal{F}\times\mathcal{G})$, given by extension of the map

$$\begin{aligned} \underline{\varphi}:\Omega^*(\mathcal{F})\otimes\Omega^*(\mathcal{G}) &\to \Omega^*(\mathcal{F}\times\mathcal{G}) \\ \sum_{i=1}^{I}\alpha_i\otimes\beta_i &\mapsto {p_M}^*(\alpha_i)\wedge {p_N}^*(\beta_i), \end{aligned}$$

where p_M and p_N denote the projections of $M\times N$ onto M and N. It is clearly a cochain map ($\varphi\circ d = d_{\mathcal{F}\times\mathcal{G}}\circ\varphi$), and so induces a map on foliated cohomology.

Theorem 2.1 (Künneth formula). *The map* φ *is an isomorphism on cohomology*:

$$H^n(\mathcal{F}\times\mathcal{G})\cong\big(H^*(\mathcal{F})\,\widehat{\otimes}\,H^*(\mathcal{G})\big)^n.$$

This is a direct consequence of the following two results:

Theorem 2.2 (Grothendieck; see [Schwartz 1954]). *Let* (E^*, d_E), (F^*, d_F) *be two differential complexes of Fréchet spaces and homomorphisms. Suppose that the* E^p*'s are nuclear. Consider the differential complex* $(E^*\,\widehat{\otimes}\,F^*, d)$ *with* $d = d_E\,\widehat{\otimes}\,1+\varepsilon\,\widehat{\otimes}\,d_F$. *Then* $H^*(E\,\widehat{\otimes}\,F)\cong H^*(E)\,\widehat{\otimes}\,H^*(F)$.

Proposition 2.3. *The differential complexes*

$$\big(\Omega^*(\mathcal{F})\,\widehat{\otimes}\,\Omega^*(\mathcal{G}), d\big)\quad \textit{and}\quad \big(\Omega^*(\mathcal{F}\times\mathcal{G}), d_{\mathcal{F}\times\mathcal{G}}\big)$$

are isomorphic under the map φ.

The proof of Theorem 2.2 relies mostly on the next two lemmas:

Lemma 2.4 [Grothendieck 1955]. *Let* E, F, G *and* H *be four Fréchet spaces with either* E, F *and* G *nuclear or* H *nuclear. Let* $u : E\to F$ *and* $v : F\to G$ *be linear homomorphisms such that*

$$0\to E\overset{u}{\to}F\overset{v}{\to}G\to 0$$

is a short exact sequence. Then the sequence

$$0\to E\,\widehat{\otimes}\,H\xrightarrow{u\widehat{\otimes}\mathrm{id}}F\,\widehat{\otimes}\,H\xrightarrow{v\widehat{\otimes}\mathrm{id}}G\,\widehat{\otimes}\,H\to 0$$

is a short exact sequence of Fréchet spaces and linear homomorphisms as well.

Proof. That $u\,\widehat{\otimes}\,\mathrm{id}$ is one-to-one and $v\,\widehat{\otimes}\,\mathrm{id}$ is onto follows from Propositions 43.6 and 43.9, respectively, in [T, pp. 440–441]. Exactness at $F\,\widehat{\otimes}\,H$ is argued as follows. Firstly, observe that

$$0\to E\otimes H\xrightarrow{u\otimes\mathrm{id}}F\otimes H\xrightarrow{v\otimes\mathrm{id}}G\otimes H\to 0$$

is a short exact sequence of homomorphisms. Indeed, the corollary of Proposition 43.7 in [T, p. 441] implies that $u\otimes\mathrm{id}$ is a homomorphism. As for $v\otimes\mathrm{id}$, it suffices to know that a basis of neighborhoods of 0 for the π-topology consists of the convex hulls of sets of type $U \otimes V = \{u \otimes v : u \in U \text{ and } v \in V\}$, where U (respectively V) is a balanced neighborhood of 0 in the first factor (respectively second factor). In other words, U is a neighborhood of 0 such that $\lambda u \in U$, for all $|\lambda| \leq 1$, $u \in U$. Therefore, $G \otimes H \cong F \otimes H / u \otimes \mathrm{id}(E \otimes H)$. Secondly, it is not difficult to prove if E is a metrizable TVS and if $N \subset E$ is a closed subspace then

$$\widehat{E/N} \cong \hat{E}/\hat{N},$$

where $\hat{N}$ denotes the closure of N in the completion $\hat{E}$ of E. □

Lemma 2.5. *Let (A^*, d_A), (B^*, d_B) and (C^*, d_C) be three differential complexes of metrizable complete TVS's and homomorphisms and let*

$$0 \to A^* \xrightarrow{f} B^* \xrightarrow{g} C^* \to 0$$

be a short exact sequence of differential complexes with f, g continuous maps (*hence homomorphisms by the open mapping theorem*)*. Then the usual long exact sequence*

$$\cdots \to H^*(A) \xrightarrow{f_*} H^*(B) \xrightarrow{g_*} H^*(C) \xrightarrow{\nu} H^{*+1}(A) \to \cdots$$

is well-defined with f_*, g_* and ν homomorphisms.

Proof. Since d_A, d_B and d_C are homomorphisms, all spaces involved (that is, cocycles, coboundary and quotients of the formers by the latter) are complete metrizable spaces. The open mapping theorem implies that any surjective continuous linear map between those spaces will be a homomorphism. The only thing that requires a proof is therefore the continuity of ν. This is easily seen by chasing open sets in the diagram providing the construction of ν, namely,

$$\begin{tikzcd}
0 \arrow[r] & A^{p+1} \arrow[r, "f"] & B^{p+1} \arrow[r, "g"] & C^{p+1} \arrow[r] & 0 \\
0 \arrow[r] & A^p \arrow[r, "f"] \arrow[u, "d_A"'] & B^p \arrow[r, "g"] \arrow[u, "d_B"'] & C^p \arrow[r] \arrow[u, "d_C"'] & 0.
\end{tikzcd}$$

This completes the proof. □

Proof of Theorem 2.2. We introduce some notation:

$$\mathfrak{X}^p = E^p \cap \operatorname{Ker} d_E, \qquad \mathfrak{B}^p = E^p \cap \operatorname{Im} d_E, \qquad H^p = \mathfrak{X}^p / \mathfrak{B}^p,$$
$$\mathfrak{X}'^p = F^p \cap \operatorname{Ker} d_F, \qquad \mathfrak{B}'^p = F^p \cap \operatorname{Im} d_F, \qquad H'^p = \mathfrak{X}'^p / \mathfrak{B}'^p.$$

Now consider the following exact sequences of linear homomorphisms:

$$0 \to \mathfrak{X}^p \to E^p \to \mathfrak{B}^{p+1} \to 0, \qquad 0 \to \mathfrak{X}'^p \to F^p \to \mathfrak{B}'^{p+1} \to 0,$$
$$0 \to \mathfrak{B}^p \to \mathfrak{X}^p \to H^p \to 0, \qquad 0 \to \mathfrak{B}'^p \to \mathfrak{X}'^p \to H'^p \to 0.$$

By Lemma 2.4, they induce the following other exact sequences of linear homomorphisms (obtained by tensoring with a fixed space and the identity map), where we have omitted the superscripts *:

$$0 \to (\mathcal{Z} \mathbin{\widehat{\otimes}} F)^n \to (E \mathbin{\widehat{\otimes}} F)^n \to (\mathcal{B} \mathbin{\widehat{\otimes}} F)^{n+1} \to 0 \tag{1}$$

$$0 \to (\mathcal{Z} \mathbin{\widehat{\otimes}} \mathcal{Z}')^n \to (\mathcal{Z} \mathbin{\widehat{\otimes}} F)^n \to (\mathcal{Z} \mathbin{\widehat{\otimes}} \mathcal{B}')^{n+1} \to 0 \tag{2}$$

$$0 \to (\mathcal{Z} \mathbin{\widehat{\otimes}} \mathcal{B}')^n \to (\mathcal{Z} \mathbin{\widehat{\otimes}} \mathcal{Z}')^n \to (\mathcal{Z} \mathbin{\widehat{\otimes}} H')^n \to 0 \tag{3}$$

$$0 \to (\mathcal{B} \mathbin{\widehat{\otimes}} \mathcal{Z}')^n \to (\mathcal{B} \mathbin{\widehat{\otimes}} F)^n \to (\mathcal{B} \mathbin{\widehat{\otimes}} \mathcal{B}')^{n+1} \to 0 \tag{4}$$

$$0 \to (\mathcal{B} \mathbin{\widehat{\otimes}} \mathcal{B}')^n \to (\mathcal{B} \mathbin{\widehat{\otimes}} \mathcal{Z}')^n \to (\mathcal{B} \mathbin{\widehat{\otimes}} H')^n \to 0 \tag{5}$$

$$0 \to (\mathcal{B} \mathbin{\widehat{\otimes}} H')^n \to (\mathcal{Z} \mathbin{\widehat{\otimes}} H')^n \to (H \mathbin{\widehat{\otimes}} H')^n \to 0 \tag{6}$$

The first one is also an exact sequence of differential complexes when $(\mathcal{Z} \mathbin{\widehat{\otimes}} F)^*$ (respectively $(\mathcal{B} \mathbin{\widehat{\otimes}} F)^*$) is endowed with $d' = \varepsilon \mathbin{\widehat{\otimes}} d_F$ (respectively $d'' = -\varepsilon \mathbin{\widehat{\otimes}} d_F$), yielding, by Lemma 2.5, the long exact sequence

$$\cdots \to H^*(\mathcal{Z} \mathbin{\widehat{\otimes}} F) \to H^*(E \mathbin{\widehat{\otimes}} F) \to H^{*+1}(\mathcal{B} \mathbin{\widehat{\otimes}} F) \to H^{*+1}(\mathcal{Z} \mathbin{\widehat{\otimes}} F) \to \cdots. \tag{7}$$

Moreover, the sequences (2) and (3) imply that $H^*(\mathcal{Z} \mathbin{\widehat{\otimes}} F) \cong (\mathcal{Z} \mathbin{\widehat{\otimes}} H')^*$. Indeed the sequence (2) identifies $(\mathcal{Z} \mathbin{\widehat{\otimes}} \mathcal{Z}')^*$ (respectively $(\mathcal{Z} \mathbin{\widehat{\otimes}} \mathcal{B}')^*$) as being the kernel (respectively the image) of the differential d' (the ε does not affect that conclusion since all maps are graded). Moreover, sequence (3) says that the quotient of $(\mathcal{Z} \mathbin{\widehat{\otimes}} \mathcal{Z}')^*$ by $(\mathcal{Z} \mathbin{\widehat{\otimes}} \mathcal{B}')^*$ is isomorphic to $(\mathcal{Z} \mathbin{\widehat{\otimes}} H')^*$. Similarly (4) and (5) imply that $H^*(\mathcal{B} \mathbin{\widehat{\otimes}} F) \cong (\mathcal{B} \mathbin{\widehat{\otimes}} H')^*$. With these isomorphisms, the sequence (7) becomes

$$\cdots \to (\mathcal{Z} \mathbin{\widehat{\otimes}} H')^* \to H^*(E \mathbin{\widehat{\otimes}} F) \to (\mathcal{B} \mathbin{\widehat{\otimes}} H')^{*+1} \xrightarrow{\nu} (\mathcal{Z} \mathbin{\widehat{\otimes}} H')^{*+1} \to \cdots. \tag{8}$$

We will prove that ν is the map induced by the natural inclusion $\mathcal{B}^* \to \mathcal{Z}^*$. Indeed, consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & (\mathcal{Z} \mathbin{\widehat{\otimes}} F)^{n+1} \arrow[r] & (E \mathbin{\widehat{\otimes}} F)^{n+1} \arrow[r] & (\mathcal{B} \mathbin{\widehat{\otimes}} F)^{n+1} \arrow[r] & 0 \\
0 \arrow[r] & (\mathcal{Z} \mathbin{\widehat{\otimes}} F)^{n} \arrow[r] \arrow[u, "d'"'] & (E \mathbin{\widehat{\otimes}} F)^{n} \arrow[r] \arrow[u, "d"'] & (\mathcal{B} \mathbin{\widehat{\otimes}} F)^{n} \arrow[r] \arrow[u, "d''"'] & 0.
\end{tikzcd}$$

Pick $\sum_{i=1}^k b_i \otimes z_i$ in $\mathcal{B}^p \otimes \mathcal{Z}'^q$. Letting $b_i = d_E x_i$ for some x_i in E^{p-1}, then we have $d(\sum_{i=1}^k x_i \otimes z_i) = \sum_{i=1}^k b_i \otimes z_i$. This shows that ν and $i \mathbin{\widehat{\otimes}} \mathrm{id}$ coincide on the subspace $\mathcal{B}^p \otimes H'^q$. Therefore they coincide on all of $\mathcal{B} \mathbin{\widehat{\otimes}} H'$.

Since ν is an injective map (again by [T, Proposition 43.7, p. 440]), the long exact sequence (8) is equivalent to the short exact sequence

$$0 \to (\mathcal{B} \mathbin{\widehat{\otimes}} H')^* \to (\mathcal{Z} \mathbin{\widehat{\otimes}} H')^* \to H^*(E \mathbin{\widehat{\otimes}} F) \to 0.$$

Hence $H^*(E \mathbin{\widehat{\otimes}} F) \cong (\mathcal{Z} \mathbin{\widehat{\otimes}} H')^*/(\mathcal{B} \mathbin{\widehat{\otimes}} H')^*$ and the latter space is isomorphic to $H \mathbin{\widehat{\otimes}} H'$, as the sequence (6) shows. □

Proof of Proposition 2.3. The proof is notationally heavy but conceptually quite simple. First observe that the continuous map $\underline{\varphi} : \Omega^*(\mathcal{F}) \otimes \Omega^*(\mathcal{G}) \to \Omega^*(\mathcal{F} \times \mathcal{G})$ is injective. We will prove hereafter that it is a homomorphism with dense image, implying that its extension $\varphi : \Omega^*(\mathcal{F}) \mathbin{\widehat{\otimes}} \Omega^*(\mathcal{G}) \to \Omega^*(\mathcal{F} \times \mathcal{G})$ is an isomorphism.

To prove that φ is a homomorphism, recall that the following subsets of $\Omega^p(\mathcal{F})$ form a basis of neighborhoods of 0:

$$\begin{aligned}\mathcal{U}(r, \varepsilon, \{(U_i, \phi_i)\}, \{K_i\}) = \{\omega \in \Omega^p(\mathcal{F}) : |D^a \omega_{i, j_1 \cdots j_p}(x)| \le \varepsilon \\ \forall \text{ multi-index } a = (a_1, \ldots, a_{\dim M}) \text{ with } |a| \le r, \\ \forall\, 1 \le i \le n, \forall\, x \in K_i\},\end{aligned}$$

where r is some nonnegative integer, $\varepsilon > 0$, $\{(U_i, \phi_i) : 1 \le i \le n\}$ is a finite collection of foliated charts and K_i is a compact subset of U_i for each $1 \le i \le n$. The functions $\omega_{i, j_1 \cdots j_p}$, $1 \le j_1 < \cdots < j_p \le \dim M$, denote the tangential coordinates of ω with respect to the chart (U_i, ϕ_i) and $D^a \omega_{i, j_1 \cdots j_p}$ is the a–th derivative

$$D^a \omega_{i, j_1 \cdots j_p} = \frac{\partial^{a_1}}{\partial x_1} \cdots \frac{\partial^{a_{\dim M}}}{\partial x_{\dim M}} (\omega_{i, j_1 \cdots j_p}).$$

We want to verify that if U is a neighborhood of 0 in $\Omega^p(\mathcal{F}) \otimes \Omega^q(\mathcal{G})$, then $\varphi(U) \supset O \cap \varphi(\Omega^p(\mathcal{F}) \otimes \Omega^q(\mathcal{G}))$ for some neighborhood of 0 in $\Omega^{p+q}(\mathcal{F} \times \mathcal{G})$. Now a neighborhood of 0 in $\Omega^p(\mathcal{F}) \otimes \Omega^q(\mathcal{G})$ can be chosen of the type

$$\mathcal{U}(U^o, V^o) = \left\{ \sum_{i=1}^{I} \alpha_i \otimes \beta_i : \sup_{\substack{x' \in U^o \\ y' \in V^o}} \left| \sum_{i=1}^{I} \langle x', \alpha_i \rangle \langle y', \beta_i \rangle \right| \le 1 \right\},$$

where U and V are neighborhoods of 0 in $\Omega^p(\mathcal{F})$ and $\Omega^q(\mathcal{G})$, respectively, and where U^o denotes the polar of U, that is, the subset

$$U^o = \{x' \in \Omega^p(\mathcal{F})' : |\langle x', u \rangle| \le 1 \ \forall u \in U\}$$

of the dual $\Omega^p(\mathcal{F})'$ of $\Omega^p(\mathcal{F})$ (same for V^o). We say a few words about this issue in the next paragraph.

The (algebraic) tensor product $E \otimes F$ of two locally convex Hausdorff TVS's is isomorphic to $B(E'_\sigma, F'_\sigma)$, the vector space of continuous bilinear forms on the product $E'_\sigma \times F'_\sigma$ of the duals of E and F each endowed with its respective weak topology (topology of pointwise convergence) (Proposition 42.4 in [T, p. 432]). The latter space can be naturally realized as a subspace of a complete locally convex Hausdorff TVS, namely the space $\mathcal{B}_\varepsilon(E'_\sigma, F'_\sigma)$ of *separately continuous* bilinear forms on $E'_\sigma \times F'_\sigma$ with the ε-topology, or topology of uniform convergence on products of equicontinuous subsets of E' and F' (Definition 43.1 and

Proposition 42.3 in [T, pp. 430, 434]). When endowed with the topology induced by $\mathcal{B}_\varepsilon(E'_\sigma, F'_\sigma)$, the tensor product of E and F is denoted by

$$E \otimes_\varepsilon F.$$

The topology on $\mathcal{B}_\varepsilon(E'_\sigma, F'_\sigma)$ can be defined by the following basis of neighborhoods of 0:

$$\mathcal{U}(A, B) = \{\phi \in \mathcal{B}_\varepsilon(E'_\sigma, F'_\sigma) : |\phi(A, B)| \leq 1\},$$

where A (respectively B) is an equicontinuous subset of E' (respectively F'). The reason for the restriction to equicontinuous sets (rather than just bounded sets) is explained in [T, pp. 427–428]. Now any equicontinuous subset of E' is contained in the polar U^o of some neighborhood U of 0 (Proposition 32.7 in [T, p. 341]). Thus, a basis of neighborhoods of 0 is also given by the sets $\mathcal{U}_\delta(U^o, V^o)$, where U (respectively V) runs through a basis of neighborhoods of 0 in E (respectively F).

Returning to the proof that φ is a homomorphism, we make the claim that if $U = \mathcal{U}(r, \varepsilon, \{(U_i, \phi_i)\}, \{K_i\})$ and $V = \mathcal{U}(s, \delta, \{(V_k, \psi_k)\}, \{L_k\})$, then $\varphi(\mathcal{U}(U^o, V^o)) \supset \operatorname{Im}\varphi \cap O$, where $O = \mathcal{U}(\max\{r, s\}, \varepsilon\delta, \{(U_i \times V_k, \phi_i \times \psi_k)\}, \{K_i \times L_k\})$. For that purpose, it will be useful to observe that the set $\mathcal{U}(r, \varepsilon, \{(U_i, \phi_i)\}, \{K_i\})$ is the polar of the following subset of the dual of $\Omega^p(\mathcal{F})$:

$$\begin{aligned}&\mathcal{A}(r, \varepsilon, \{(U_i, \phi_i)\}, \{K_i\})\\ &\qquad = \big\{\ell_{a,\varepsilon,i,j_1\cdots j_p,x}(\omega) = (1/\varepsilon)\partial^a \omega_{i,j_1\cdots j_p}(x) :\\ &\qquad\qquad |a| \leq r, i = 1, \dots, n, 1 \leq j_1 < \cdots < j_p \leq \dim M, x \in K_i\big\}.\end{aligned}$$

Thus $U = A^o$. On the other hand, a locally convex Hausdorff TVS E is isomorphic to the dual of its weak dual, that is, $E \cong (E'_\sigma)'$ (Proposition 35.1 in [T, p. 361]), and if $U = A^o$, then $U^o = (A^o)^o$ coincides with the closed convex balanced hull of A (that is, the closure of the convex hull of $\bigcup_{\{\lambda:|\lambda|\leq 1\}} \lambda A$), denoted by ΓA (Proposition 35.3 in [T, p. 362]). Furthermore, one verifies directly from the definitions involved that $\mathcal{U}(A, B) = \mathcal{U}(\Gamma A, \Gamma B)$. Thus, $\mathcal{U}(U^o, V^o) = \mathcal{U}(A, B)$ with $A = \mathcal{A}(r, \varepsilon, \{(U_i, \phi_i)\}, \{K_i\})$ and $B = \mathcal{A}(s, \delta, \{(V_k, \psi_k)\}, \{L_k\})$.

Now let $\theta = \sum_{t=1}^T \alpha_t \otimes \beta_t \in \Omega^p(\mathcal{F}) \otimes \Omega^q(\mathcal{G})$ be such that $\varphi(\theta)$ belongs to O, that is, for all multi-indices c with $|c| \leq rs$, for all $i, k, j_1 < \cdots < j_p, l_1 < \cdots < l_q$ and for all $(x, y) \in K_i \times L_k$, one of the following holds:

$$\left| D^c\Big(\sum_{t=1}^T (\alpha_t)_{i,j_1\cdots j_p}(\beta_t)_{k,l_1\cdots l_q}\Big)(x, y)\right| \leq \varepsilon\delta,$$

$$\left|\sum_{t=1}^T \frac{1}{\varepsilon} D^a(\alpha_t)_{i,j_1\cdots j_p}(x)\frac{1}{\delta} D^b(\beta_t)_{k,l_1\cdots l_q}(y)\right| \leq 1,$$

where $c = (a_1, \dots, a_{\dim M}, b_1, \dots, b_{\dim N})$. Equivalently,

$$\left|\sum_{t=1}^{T} \langle \ell_{a,\varepsilon,i,j_1\cdots j_p,x}, \alpha_t\rangle\langle \ell_{b,\delta,k,l_1\cdots l_q,y}, \beta_t\rangle\right| \le 1,$$

which means that $\theta \in \mathcal{U}(A, B)$, thus proving that φ is a homomorphism.

It remains to prove that the image of φ is dense in $\Omega^n(\mathcal{F}\times\mathcal{G})$. This is essentially a consequence of the fact that polynomial functions are dense in the space of smooth functions on the Euclidean space, implying that if X and Y are open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, then the tensor product $C_c^\infty(X) \otimes C_c^\infty(Y)$ of the spaces of smooth functions with compact supports on X and Y is dense in the space $C^\infty(X \times Y)$ of smooth functions on $X \times Y$ (see [T, Theorem 39.2, p. 409 and Corollary 1, p. 159]). Let $\omega \in \Omega^n(\mathcal{F} \times \mathcal{G})$ and consider U a neighborhood of ω of the type $\omega+\mathcal{U}(r, \varepsilon, \{(U_i \times V_k, \phi_i \times \psi_k), \{K_i \times L_k\}\})$. For each tangential component $\omega_{i,k,j_1\cdots j_p,l_1\cdots l_q}$, denoted hereafter $\omega_{i,k,J,L}$, of ω with respect to the chart $\phi_i \times \psi_k$, pick functions $f^n_{i,k,J,L} \in C_c^\infty(U_i)$ and $g^n_{i,k,J,L} \in C_c^\infty(V_k)$, $n = 1, \dots, N$, such that

$$\sum_{n=1}^{N} f^n_{i,k,J,L}\, g^n_{i,k,J,L}$$

lies in $\omega_{i,k,J,L} + \mathcal{U}(r, \varepsilon, K_i \times L_k) \subset C^\infty(U_i \times V_k)$. Then the form

$$\sum_{n,J,L} f^n_{i,k,J,L}\, g^n_{i,k,J,L}\, dx^J \wedge dx^L$$

belongs to $U \cap \varphi(\Omega^p(\mathcal{F}) \otimes \Omega^q(\mathcal{G}))$. □

3. Künneth formula when one of the factors is finite-dimensional

Another natural question is whether the Künneth formula holds in the case where the tensor product does not need to be completed, that is, when one of the factors, say $H^*(\mathcal{F})$, is finite-dimensional. The answer is positive provided the ambient manifold of that factor is compact. There is no assumption on the second factor. This statement was already well-known when $\mathcal{F}$ is a one-leaf foliation (see [El Kacimi-Alaoui 1983; Moore and Schochet 2006]). We use the fact that under the previous assumptions, the foliated de Rham differential $d_{\mathcal{F}}$ admits a right inverse, which is implied by results in the theory of splitting of exact sequences of Fréchet spaces appearing in [Meise and Vogt 1997; Vogt 2004].

Proposition 3.1. *Let $(M, \mathcal{F})$ and $(N, \mathcal{G})$ be foliated manifolds. Suppose $H^*(\mathcal{F})$ is finite-dimensional and M is compact. Then, as topological vector spaces,*

$$H^*(\mathcal{F} \times \mathcal{G}) \cong H^*(\mathcal{F}) \otimes H^*(\mathcal{G}).$$

Although we may not anymore quote theorems about coincidence of the ε- and π-topologies on $H^*(\mathcal{F}) \otimes H^*(\mathcal{G})$ since $H^*(\mathcal{G})$ might not be Hausdorff, one may verify directly that both these topologies coincide with the direct sum topology that appears when $H^*(\mathcal{F}) \otimes H^*(\mathcal{G})$ is identified with a finite direct sum $\bigoplus_{i=1}^n H^*(\mathcal{G})$ via the choice of a basis of $H^*(\mathcal{F})$.

Remark 3.2. The assumption that $H^*(\mathcal{F})$ is finite-dimensional implies that it is Hausdorff as well. This follows from the well-known fact that for a continuous linear map $A : E \to F$ between Fréchet spaces, if the range is finite-codimensional it is also closed, itself a consequence of the open mapping theorem. Indeed, let $f_1, \dots, f_n$ be a basis of an algebraic complement to $\operatorname{Im} A$ in F. Define the map

$$A' : E/\operatorname{Ker} A \oplus \mathbb{R}^n \to F, \quad ([e], a_1, \dots, a_n) \mapsto A(e) + a_1 f_1 + \cdots + a_n f_n.$$

It is continuous as the sum of two continuous maps and bijective. Hence it is an isomorphism. Since $\operatorname{Im} A$ is the image of the closed subspace $E/\operatorname{Ker} A$, it is closed as well. We thank the referee for pointing this out to us.

Proof of Proposition 3.1. The idea is to replace the complex $(\Omega^*(\mathcal{F}), d_{\mathcal{F}})$ by a homotopy equivalent finite-dimensional complex (V, d_V). It is then easy to prove that $H^*(V \otimes \Omega^*(\mathcal{G}))$ coincides with $H^*(\Omega^*(\mathcal{F}) \widehat{\otimes} \Omega^*(\mathcal{G})) = H^*(\mathcal{F} \times \mathcal{G})$. Besides, it is well-known that $H^*(V \otimes \Omega^*(\mathcal{G})) = H^*(V) \otimes H^*(\Omega^*(\mathcal{G}))$ as vector spaces and it is not difficult to be convinced that this equality holds for the topologies as well. So we are done. To obtain an equivalence with a finite-dimensional complex we need a right inverse for the foliated differential $d_{\mathcal{F}}$, that is, a continuous linear map $\varphi : \mathcal{B}^{*+1}(\mathcal{F}) \to \Omega^*(\mathcal{F})$ such that $d_{\mathcal{F}} \circ \varphi = \mathrm{id}$. This is the content of the Lemma 3.3 below. We assume this fact and proceed with the present proof.

The complex $(\Omega^*(\mathcal{F}), d_{\mathcal{F}})$ is denoted hereafter by (Ω, d), $\operatorname{Ker} d_{\mathcal{F}}$ by $\mathcal{Z}$ and $\operatorname{Im} d_{\mathcal{F}}$ by $\mathcal{B}$. Consider closed foliated forms (of pure degree) $\alpha_1, \dots, \alpha_n$ representing a basis $\{[\alpha_1], \dots, [\alpha_n]\}$ of $H^*(\mathcal{F})$. The subset $V = \{\alpha_1, \dots, \alpha_n\}$ endowed with the zero differential ($d_V = 0$) is a finite-dimensional subcomplex of (Ω, d) with cohomology $H^*(\mathcal{F})$. It is thus (algebraically) homotopy equivalent to (Ω, d) (see [Spanier 1966, Theorem 7.4.10, p. 192]). We show next that the homotopy, its inverse and the equivalence may be chosen continuous when d admits a right inverse.

We first need to set up some notation.

- The natural inclusion $V \to \Omega$ is denoted by i.
- The map $\varphi : \mathcal{B} \to \Omega$ denotes a continuous linear right inverse to d.
- The cohomology class of a closed form β is denoted by $[\beta]$.
- Since $H^*(\mathcal{F})$ is finite-dimensional and Hausdorff (see Remark 3.2), the linear map $e : H^*(\mathcal{F}) \to V$ such that $e([\alpha_i]) = \alpha_i$ is continuous; it is a right inverse for the natural projection $\mathcal{Z}^*(\mathcal{F}) \to H^*(\mathcal{F})$ with values in V.

Define $\sigma : \Omega \to V$ and $D : \Omega \to \Omega$ by

$$\sigma(\beta) = e[\beta - \varphi(d\beta)],$$
$$D(\beta) = -\varphi\big((\beta - \varphi(d\beta)) - i \circ e[\beta - \varphi(d\beta)]\big).$$

The maps σ and D are clearly continuous. It is only necessary to verify that σ is a cochain map, that $\sigma \circ i = \mathrm{id}_V$ and that $i \circ \sigma = \mathrm{id}_\Omega + D \circ d + d \circ D$. The first two assertions are obvious and the third holds since

$$\begin{aligned}\big(D \circ d + d \circ D\big)(\beta) &= -\varphi(d\beta) - d \circ \varphi\big((\beta - \varphi(d\beta)) - i \circ e[\beta - \varphi(d\beta)]\big)\\ &= -\beta + i \circ e[\beta - \varphi(d\beta)] = -\beta + i \circ \sigma(\beta).\end{aligned}$$

Now the continuous cochain maps i and σ induce continuous cochain maps $i \mathbin{\widehat{\otimes}} \mathrm{id} : V \mathbin{\widehat{\otimes}} \Omega^*(\mathcal{G}) \to \Omega^*(\mathcal{F}) \mathbin{\widehat{\otimes}} \Omega^*(\mathcal{G})$ and $\sigma \mathbin{\widehat{\otimes}} \mathrm{id} : \Omega^*(\mathcal{F}) \mathbin{\widehat{\otimes}} \Omega^*(\mathcal{G}) \to V \mathbin{\widehat{\otimes}} \Omega^*(\mathcal{G})$ such that

$$(\sigma \mathbin{\widehat{\otimes}} \mathrm{id}) \circ (i \mathbin{\widehat{\otimes}} \mathrm{id}) = \mathrm{id},$$
$$(i \mathbin{\widehat{\otimes}} \mathrm{id}) \circ (\sigma \mathbin{\widehat{\otimes}} \mathrm{id}) = \mathrm{id} + D' \circ d_{\mathcal{F}\times\mathcal{G}} + d_{\mathcal{F}\times\mathcal{G}} \circ D',$$

where $D' = D \mathbin{\widehat{\otimes}} \mathrm{id}$ and $d_{\mathcal{F}\times\mathcal{G}} = d_{\mathcal{F}} \mathbin{\widehat{\otimes}} \mathrm{id} + \varepsilon \mathbin{\widehat{\otimes}} d_{\mathcal{G}}$. (The last assertion follows from the fact that $\varepsilon \circ D + D \circ \varepsilon = 0$.) Furthermore, the continuous cochain maps $i \mathbin{\widehat{\otimes}} \mathrm{id}$ and $\sigma \mathbin{\widehat{\otimes}} \mathrm{id}$ induce continuous maps on cohomology that are inverse to one another. This shows that $H^*(V \mathbin{\widehat{\otimes}} \Omega^*(\mathcal{G})) \cong H^*(\mathcal{F} \times \mathcal{G})$ as TVS's. Of course, $V \mathbin{\widehat{\otimes}} \Omega^*(\mathcal{G})$ is the same as $V \otimes \Omega^*(\mathcal{G})$ since $V \otimes E$ is complete when V is finite-dimensional and E is complete.

Finally, the fact that $H^*(V \otimes \Omega^*(\mathcal{G})) \cong V \otimes H^*(\mathcal{G})$ follows from considering the short exact sequences

$$0 \longrightarrow V \otimes \mathcal{Z}^*(\mathcal{G}) \longrightarrow V \otimes \Omega^*(\mathcal{G}) \overset{\varepsilon\otimes d_{\mathcal{G}}}{\longrightarrow} V \otimes \mathcal{B}^{*+1}(\mathcal{G}) \longrightarrow 0,$$
$$0 \longrightarrow V \otimes \mathcal{B}^*(\mathcal{G}) \longrightarrow V \otimes \mathcal{Z}^*(\mathcal{G}) \longrightarrow V \otimes H^*(\mathcal{G}) \longrightarrow 0.$$

Observe that $\varepsilon \otimes d_{\mathcal{G}}$ is continuous but not open. Likewise $V \otimes \mathcal{B}^{*+1}(\mathcal{G})$ is not complete and $V \otimes H^*(\mathcal{G})$ is not Hausdorff. Nevertheless, the first sequence tells us that the kernel of the differential $\varepsilon \otimes d_{\mathcal{G}}$ on $V \otimes \Omega^*(\mathcal{G})$ is $V \otimes \mathcal{Z}^*(\mathcal{G})$ and that its image is $V \otimes \mathcal{B}^{*+1}(\mathcal{G})$. Besides, the topology induced on $V \otimes \mathcal{Z}^*(\mathcal{G})$ (respectively $V \otimes \mathcal{B}^{*+1}(\mathcal{G})$) from its embedding in $V \otimes \Omega^*(\mathcal{G})$ coincides with the tensor product topology (Proposition 43.7 in [T, p. 440]). Finally, since the maps in the second sequence are homomorphisms (remembering that $V \otimes H^*(\mathcal{G})$ carries the direct sum topology), the quotient $V \otimes \mathcal{Z}^*(\mathcal{G}) / V \otimes \mathcal{B}^*(\mathcal{G})$ is isomorphic to $V \otimes H^*(\mathcal{G})$. $\square$

Lemma 3.3. *If M is compact and $H^*(\mathcal{F})$ is finite-dimensional then $d_{\mathcal{F}}$ admits a continuous linear right inverse.*

Proof. The proof relies on the following result.

Theorem 3.4 [Meise and Vogt 1997, Splitting theorem 30.1, p. 378]. *Let* E, F, G *be Fréchet–Hilbert spaces and let* $0 \to F \to G \to E \to 0$ *be a short exact sequence of continuous linear maps. If* E *has property* (DN) *and* F *has property* (Ω), *then the sequence splits.*

We explain hereafter why Theorem 3.4 can be applied to the short exact sequence

$$0 \to \mathscr{Z}^*(\mathscr{F}) \to \Omega^*(\mathscr{F}) \to \mathscr{B}^*(\mathscr{F}) \to 0.$$

The assumption that the spaces are Fréchet–Hilbert is automatically satisfied for nuclear Fréchet spaces (see [Meise and Vogt 1997, Definition, p. 370 and Lemma 28.1, p. 344]). We mention the definitions of properties (DN) and (Ω) for completeness but we will only need here the fact that they are stable under performing certain operations. Let E be a Fréchet space endowed with a countable fundamental systems $\{\|\cdot\|_k : k \geq 1\}$ of seminorms (that is, for all $x \in E$, $x \neq 0$, there exists k such that $\|x\|_k > 0$ and for all k_1, k_2, there exists k_3 and a constant C such that $\max\{\|\cdot\|_{k_1}, \|\cdot\|_{k_2}\} \leq C\|\cdot\|_{k_3}$). The property (DN) is satisfied by E if and only if it supports a continuous norm $\|\cdot\|$ on E such that for any seminorm $\|\cdot\|_k$ there exists a constant C and a seminorm $\|\cdot\|_K$ such that

$$\|x\|_k^2 \leq C\|x\|\|x\|_K \quad \text{for all } x \in E.$$

The property (Ω) is satisfied by E if and only if for each $p \geq 1$ there exists a $q \geq 1$ so that for every $k \geq 1$, there exists a $0 < \theta < 1$ and a constant C such that

$$\|y\|_q^* \leq C\|y\|_p^{*\,1-\theta}\|y\|_k^{*\theta} \quad \text{for all } y \in E',$$

where $\|y\|_k^*$ means $\sup\{|y(x)| : \|x\|_k \leq 1\}$.

Both properties are satisfied by the Schwartz space

$$s = \left\{(x_j)_{j\geq 1} : \sum_{j=1}^{\infty} |x_j|^2 j^{2k} < \infty \; \forall\, k \geq 1\right\}$$

[Meise and Vogt 1997, Example 29.5(1), p. 363, Lemma 29.2, p. 359 and Lemma 29.11, p. 368]. Any space of C^∞-sections of a finite-dimensional vector bundle E over a compact manifold is isomorphic, as topological vector space, to s (see [Valdivia 1982])[1].

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[1] The reference [Valdivia 1982] contains a proof of the fact that for a compact manifold M, the space $C^\infty(M) \cong s$ which can easily be adjusted to the case of $C^\infty(M, E)$. Indeed, a finite partition of unity $\{\theta_i : i = 1, \dots, n\}$ subordinated to a cover of M by trivializing open subsets allows us to identify the space of smooth sections of the bundle E with a finite direct sum $\bigoplus_i C_c^\infty(C_i, \mathbb{R}^d)$, where C_i is the support of θ_i, where d is the rank of E and where $C_c^\infty(C_i, \mathbb{R}^d)$ is the set of smooth functions with compact support in C_i. Because each $C_c^\infty(C_i, \mathbb{R})$ is isomorphic to s [Valdivia 1982, (5), p. 536] and $s \oplus s \cong s$ [Valdivia 1982, (5), p. 327], we reach our conclusion.

Thus the space of foliated forms $\Omega^*(\mathcal{F})$ enjoys the properties (DN) and (Ω). Property (DN) is inherited by closed subspaces (see [Meise and Vogt 1997, Lemma 29.2, p. 359]). So $\mathcal{B}^*(\mathcal{F})$ has property (DN). To see that $\mathcal{Z}^*(\mathcal{F})$ has property (Ω), we use the fact that $H^*(\mathcal{F})$ is finite-dimensional. Indeed, property (Ω) is inherited by quotients by closed subspaces (see [Meise and Vogt 1997, Lemma 29.11(2), p. 368]) so that $\mathcal{B}^*(\mathcal{F})$, which is isomorphic to $\Omega^{*-1}(\mathcal{F})/\mathcal{Z}^{*-1}(\mathcal{F})$, has property (Ω). Since $H^*(\mathcal{F})$ is finite-dimensional, the natural projection $\mathcal{Z}^*(\mathcal{F}) \to H^*(\mathcal{F})$ admits a right inverse so that $\mathcal{Z}^*(\mathcal{F}) \cong \mathcal{B}^*(\mathcal{F}) \oplus H^*(\mathcal{F})$ and can be thought of as a quotient of $\Omega^{*-1}(\mathcal{F}) \oplus H^*(\mathcal{F})$ (by $\mathcal{Z}^{*-1}(\mathcal{F}) \oplus \{0\}$), which is itself also isomorphic to the Schwartz space s when $* \geq 1$. Finally, $\mathcal{Z}^0(\mathcal{F})$ has property (Ω) because it is finite-dimensional (it is thus a Banach space). □

Remark 3.5. When the manifold M is not compact, the space $\Omega^*(\mathcal{F})$ is isomorphic to $s^{\mathbb{N}}$ (argument similar to the compact case with a locally finite partition of unity subordinated to an open cover of M by foliated chart domains) rather than s. So that it does not anymore enjoy properties (DN) nor (Ω), but rather, so-called properties $(\mathrm{DN}_{\mathrm{loc}})$ and (Ω_{loc}). One might nevertheless reach a similar conclusion, using the following splitting theorem due to Vogt, if one could prove that $d_{\mathcal{F}}$ is a SK-homomorphism when it is a homomorphism (this does not hold in general, but might be true for $d_{\mathcal{F}}$).

Theorem 3.6 [Vogt 2004, Theorem 3.5, p. 820]. *Let* $0 \to F \to G \to E \to 0$ *be an exact sequence of nuclear Fréchet spaces, A an SK-homomorphism. If E has property* $(\mathrm{DN}_{\mathrm{loc}})$ *and F property* (Ω_{loc}), *then the sequence splits.*

4. Counterexample

One would like to understand what happens when neither of the situations encountered above occurs. Let $(M, \mathcal{F})$ and $(N, \mathcal{G})$ be two foliated manifolds. Suppose that both foliated cohomologies are infinite-dimensional with one of them non-Hausdorff. Then the tensor product $H^*(\mathcal{F}) \otimes H^*(\mathcal{G})$ cannot be completed. There is nevertheless a case where an alternative to completion could be proposed, that is, when one of the foliations, say $\mathcal{G}$, is a foliation by points $\mathcal{G} = \mathcal{F}_N$. Then $H^*(\mathcal{G})$ coincides with $C^\infty(N)$ and one is tempted to replace $H^*(\mathcal{F}) \mathbin{\widehat{\otimes}} C^\infty(N)$, which does not make sense here, by $C^\infty(N, H^*(\mathcal{F}))$, since these two spaces coincide when $H^*(\mathcal{F})$ is Hausdorff (Theorem 44.1 in [T, p. 449]). It is therefore natural to wonder whether the trivial map $H^*(\mathcal{F} \times \mathcal{F}_N) \to C^\infty(N, H^*(\mathcal{F}_N))$ yields an isomorphism or not:

$$H^*(\mathcal{F} \times \mathcal{F}_N) \overset{?}{\cong} C^\infty(N, H^*(\mathcal{F})). \tag{9}$$

The answer is negative. Indeed, the torus $\mathbb{T}^2$ endowed with a Liouville foliation (see Example 1.2) supports a smooth family of foliated exact forms which is not the coboundary of any smooth, nor even continuous, family of forms. This smooth family represents thus both the zero element in $C^\infty(N, H^*(\mathcal{F}))$ and a nonzero element in $H^*(\mathcal{F} \times \mathcal{F}_N)$. I do not see any obvious theoretical reason for such a family to exist; both the space $H^*(\mathcal{F} \times \mathcal{F}_N)$ and the space $C^\infty(N, H^*(\mathcal{F}))$ are non-Hausdorff; somehow $H^*(\mathcal{F} \times \mathcal{F}_N)$ is "more separated" than $C^\infty(N, H^*(\mathcal{F}))$.

Let x, y denote standard coordinates on the torus $\mathbb{T}^2$. The leaves of the foliation $\mathcal{F}_\alpha$ are the orbits of the vector field $X = \partial_x + \alpha\partial_y$. Any foliated 1-form is automatically closed and can be written $f\,\overline{dx}$, with f in $C^\infty(\mathbb{T}^2)$ and $\overline{dx}$ the image of the closed form $dx \in \Omega^1(\mathbb{T}^2)$ in $\Omega^1(\mathcal{F}_\alpha)$. It is exact when $f\,\overline{dx} = \overline{dg}$ for some g in $C^\infty(\mathbb{T}^2)$, which is equivalent to $f = Xg$. Besides, we may consider the Fourier expansions of the functions f and g:

$$f = \sum_{m,n\in\mathbb{Z}} f_{m,n} e^{2\pi i(mx+ny)} \quad \text{and} \quad g = \sum_{m,n\in\mathbb{Z}} g_{m,n} e^{2\pi i(mx+ny)}.$$

The equation $f = Xg$ is equivalent to the sequence of equations

$$f_{m,n} = 2\pi i(m+\alpha n)\, g_{m,n}, \quad m, n \in \mathbb{Z},$$

which of course implies $f_{0,0} = 0$.

Lemma 4.1. *Suppose that $(f_t)_{t\in\mathbb{R}}$ is a family of functions on $\mathbb{T}^2$. It is a smooth family of smooth functions if and only if each function $t \mapsto (f_t)_{m,n}$ is smooth and for all compact intervals I in $\mathbb{R}$ and integers $a \geq 0$, $j \geq 1$, there exists a constant $c = c(I, a, j)$ such that*

$$\sup_{t\in I} |\partial_t^a (f_t)_{m,n}| \leq \frac{c}{(|m|+|n|)^j}.$$

To see the necessity of this condition it suffices to combine integration by parts in order to get rid of the derivatives with respect to x and y with the fact that the Fourier coefficient $(\partial_x^k \partial_y^l \partial_t^a f_t)_{m,n}$ is bounded by a constant depending only on I, k, l and a.

With these preliminaries in mind, we are ready to construct a family f_t of functions on $\mathbb{T}^2$ with the following properties:

- (i) f_t is a smooth family of smooth functions.
- (ii) For each value of the parameter t, there is a smooth solution to $f_t = Xg_t$.
- (iii) There no smooth, or even continuous, family of smooth functions g_t solving $f_t = Xg_t$.

Since α is a Liouville number, for each integer $p > 1$, there exists a pair of integers (m_p, n_p) such that

$$|m_p + \alpha n_p| \leq \frac{1}{(|m_p|+|n_p|)^p}.$$

Without loss of generality assume that $(m_p, n_p) \neq (m_q, n_q)$ for $p \neq q$ and that $n_p \geq p$. Now define

$$(f_t)_{m,n} = \begin{cases} (m_p + \alpha n_p)(|m_p| + |n_p|)\rho\big(s_p\big(t - \frac{1}{p}\big)\big) & \text{if } (m,n) = (m_p, n_p), \\ 0 & \text{otherwise,} \end{cases}$$

where ρ is a bump function supported in the interval $[-1, 1]$ that achieves its maximum value 1 at 0 and where $s_p = p(p+1)$. The function $\rho\big(s_p\big(t - \frac{1}{p}\big)\big)$ has its support contained in

$$\Big[\frac{1}{p} - \frac{1}{2p(p+1)}, \frac{1}{p} + \frac{1}{2p(p+1)}\Big].$$

We verify that the $(f_t)_{m,n}$'s are the Fourier coefficients of a family f_t enjoying properties (i), (ii) and (iii).

(i) For smoothness of f_t we use the criterion described in Lemma 4.1.

$$\begin{aligned} |\partial_t^a (f_t)_{m_p,n_p}| &\leq \big|(m_p + \alpha n_p)\big|(|m_p| + |n_p|) \sup_{t \in I} |\partial_t^a \rho(t)||s_p|^a \\ &\leq \frac{c_a |s_p|^a}{(|m_p| + |n_p|)^{p-1}} \leq \frac{c'_a}{(|m_p| + |n_p|)^{p-1-2a}} \leq \frac{c''_{a,j}}{(|m_p| + |n_p|)^j}. \end{aligned}$$

The second inequality on the last line follows from the fact that s_p is a polynomial of degree 2 in p and the assumption $n_p \geq p$, while the last inequality is a consequence of the fact that $p - 1 - 2a \to \infty$ when $p \to \infty$.

(ii) The coefficients $(g_t)_{m,n} = (f_t)_{m,n}/(m + \alpha n)$ define a smooth function for each value of t. Indeed, for a fixed t_0,

$$(g_{t_0})_{m_p,n_p} = (|m_p| + |n_p|)\rho(s_p(t_0 - 1/p)) = 0$$

for all p except perhaps one since the supports of the various functions $\rho(s_p(t_0 - \frac{1}{p}))$ are disjoint. The Fourier series of the function g_{t_0} has thus only one term.

(iii) The function g_t is not smooth, nor even continuous, near $t = 0$. Indeed, the coefficients $(g_t)_{m,n}$ are not uniformly bounded on any interval I around 0:

$$\sup_{t \in I} |(g_t)_{m_p,n_p}| = (|m_p| + |n_p|) \sup_{t \in I} |\rho(s_p(t_0 - 1/p))| = (|m_p| + |n_p|)$$

as soon as p is sufficiently large for $\frac{1}{p}$ to belong to I.

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K-GROUPS OF THE QUANTUM HOMOGENEOUS SPACE $\mathrm{SU}_q(n)/\mathrm{SU}_q(n-2)$

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Dedicated to Prof. K. R. Parthasarathy on his 75th birthday

Quantum Stiefel manifolds were introduced by Vainerman and Podkolzin, who classified the irreducible representations of the C^*-algebras underlying such manifolds. We compute the K-groups of the quantum homogeneous spaces $\mathrm{SU}_q(n)/\mathrm{SU}_q(n-2)$ for $n \geq 3$. In the case $n = 3$, we show that K_1 is a free $\mathbb{Z}$-module, and the fundamental unitary for quantum SU(3) is part of a basis for K_1.

1. Introduction

Quantization of mathematical theories is a major theme of research today. The theories of quantum groups and noncommutative geometry are two prime examples in this program. Both these programs started in the early 1980s. In the setting of operator algebras, the theory of quantum groups was initiated independently in [Woronowicz 1987] and [Vaksman and Soibelman 1988], for the case of quantum SU(2). Later Woronowicz studied the family of compact quantum groups and obtained Tannaka-type duality theorems [Woronowicz 1988]. The notion of quantum subgroups and quantum homogeneous spaces soon followed [Podleś 1995].

The noncommutative differential geometry program of Alain Connes [1985] also started in the 1980s. In his interpretation, geometric data is encoded in elliptic operators or, more generally, in specific unbounded K-cycles, which he called spectral triples. It is natural to expect that, for compact quantum groups and their homogeneous spaces, there should be associated canonical spectral triples. Chakraborty and Pal [2003] showed that indeed that is the case for quantum SU(2). In fact for odd-dimensional quantum spheres, one can construct finitely summable spectral triples that display Poincaré duality [Chakraborty and Pal 2010].

In this connection, a natural question is, are these examples somewhat singular or can one in general construct finitely summable spectral triples with further

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properties like Poincaré duality, on quantum groups associated with Lie groups or their homogeneous spaces? Even though there are suggestions to construct such spectral triples [Neshveyev and Tuset 2010], their nontriviality as a K-cycle is not known. In fact, there are suggestions that, for quantum groups and their homogeneous spaces, one should look for a type-III formulation of noncommutative geometry. On this formulation also, there are currently two points of view, that of Alain Connes and Henri Moscovici [2008], and that of Carey–Phillips–Rennie [2010]. Therefore, to understand the true nature of the interplay between noncommutative geometry and quantum homogeneous spaces, it makes sense to take a closer look at these algebras.

The underlying C^*-algebras of these compact quantum groups were analyzed by Soibelman [1990] (also [Levendorskii and Soibelman 1991]) who described their irreducible representations. Exploiting their findings, Sheu went on to obtain composition sequences for these algebras. He initially obtained the results for $\mathrm{SU}_q(3)$ [Sheu 1991], and later extended them to the general $\mathrm{SU}_q(n)$ [Sheu 1997].

In this hierarchy of exploration, the next thing to look for would be K-groups; that is what we are looking for. But, instead of concentrating on quantum groups, we consider the quantum analogs of the Stiefel manifolds $\mathrm{SU}(n)/\mathrm{SU}(n-m)$, introduced by Podkolzin and Vainerman [1999]. Those authors have already described the structure of irreducible representations of the quantum Stiefel manifolds $\mathrm{SU}_q(n)/\mathrm{SU}_q(n-m)$. We take up the case of $\mathrm{SU}_q(n)/\mathrm{SU}_q(n-2)$ when $n \geq 3$. We obtain the composition sequences for these algebras and then, utilizing them, we compute the K-groups. More importantly, as we remarked earlier, applications towards noncommutative geometry require an explicit understanding of generators for these K-groups; during our calculation we also achieve that. Specializing to the case $n = 3$, we get the K-groups of quantum $\mathrm{SU}(3)$.

We should remark that these K-groups can be computed using the variant of KK-theory introduced by Nagy in [2000]. In fact, it is shown in [Nagy 1998] that $\mathrm{SU}_q(n)$ and $\mathrm{SU}(n)$ are KK-equivalent, but here we produce explicit generators, which is essential to test the nontriviality of K-cycles by computing the K-theory–K-homology pairing. To our knowledge, there are not many instances of K-theory calculations for compact quantum groups. Other than the paper by Nagy, there is another related work by McClanahan [1992], where he computes the K-groups of the universal C^*-algebra generated by the elements of a unitary matrix, and shows that the associated K_1 is generated by the defining unitary itself. This raises the question whether something similar holds for compact matrix quantum groups, namely, whether the defining unitary of a compact matrix quantum group is nontrivial in K_1. For quantum $\mathrm{SU}(2)$, this was remarked by Connes [2004]. Here, we not only prove that the defining unitary of quantum $\mathrm{SU}(3)$ is nontrivial, the K_1 is a free $\mathbb{Z}$-module, and the fundamental unitary for quantum $\mathrm{SU}(3)$ is part of a basis for K_1.

2. The quantum Stiefel manifolds and their irreducible representations

The quantum Stiefel manifold $S_q^{n,m}$ was introduced in [Podkolzin and Vainerman 1999]. Throughout, we assume that $q \in (0,1)$. Recall that the C^*-algebra $C(\mathrm{SU}_q(n))$ is the universal unital C^*-algebra generated by n^2 elements u_{ij} satisfying the conditions

$$\sum_{k=1}^{n} u_{ik}u_{jk}^* = \delta_{ij}, \qquad \sum_{k=1}^{n} u_{ki}^*u_{kj} = \delta_{ij},$$
$$\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_n=1}^{n} E_{i_1 i_2 \ldots i_n} u_{j_1 i_1} \ldots u_{j_n i_n} = E_{j_1 j_2 \ldots j_n},$$

where

$$E_{i_1 i_2 \ldots i_n} := \begin{cases} 0 & \text{if } i_1, i_2, \ldots i_n \text{ are not distinct,} \\ (-q)^{\ell(i_1, i_2, \ldots, i_n)} & \text{otherwise,} \end{cases}$$

and where $\ell(\sigma)$ denotes the length of a permutation σ on $\{1,2,\ldots,n\}$. The C^*-algebra $C(\mathrm{SU}_q(n))$ has a compact quantum group structure with comultiplication given by

$$\Delta(u_{ij}) := \sum_k u_{ik} \otimes u_{kj}.$$

Let $1 \le m \le n-1$. Call v_{ij} the generators of $\mathrm{SU}_q(n-m)$. The map $\varphi : C(\mathrm{SU}_q(n)) \to C(\mathrm{SU}_q(n-m))$ defined by

$$\varphi(u_{ij}) := \begin{cases} v_{ij} & \text{if } 1 \le i,\ j \le n-m, \\ \delta_{ij} & \text{otherwise.} \end{cases} \tag{2-1}$$

is a surjective unital C^*-algebra homomorphism such that $\Delta \circ \varphi = (\varphi \otimes \varphi)\Delta$. In this way, the quantum group $\mathrm{SU}_q(n-m)$ is a subgroup of the quantum group $\mathrm{SU}_q(n)$. The C^*-algebra of the quotient $\mathrm{SU}_q(n)/\mathrm{SU}_q(n-m)$ is defined as

$$C\big(\mathrm{SU}_q(n)/\mathrm{SU}_q(n-m)\big) := \big\{a \in C(\mathrm{SU}_q(n)) :\ (\varphi \otimes 1)\Delta(a) = 1 \otimes a\big\}.$$

We refer to [Podkolzin and Vainerman 1999] for the proof of the following:

Proposition 2.1. *The C^*-algebra $C(\mathrm{SU}_q(n)/\mathrm{SU}_q(n-m))$ is generated by the last m rows of the matrix (u_{ij}), that is, by the set $\{u_{ij} : n-m+1 \le i \le n\}$.*

In [Podkolzin and Vainerman 1999], the quotient space $\mathrm{SU}_q(n)/\mathrm{SU}_q(n-m)$ is called a quantum Stiefel manifold and is denoted by $S_q^{n,m}$. We will use the same notation.

Before proceeding further, let us fix some notations. Let $\mathbb{N}$ be the set of nonnegative integers. Consider the number operator N and the left shift S on $\ell^2(\mathbb{N})$

defined on the standard orthonormal basis $\{e_n : n \geq 0\}$ by

$$Se_n := e_{n-1} \quad \text{and} \quad Ne_n := ne_n.$$

Note that N is an unbounded selfadjoint operator. We denote by τ the C^*-algebra generated by S. The C^*-algebra τ is nothing but the Toeplitz algebra.

The irreducible representations of the C^*-algebra $C(S_q^{n,m})$ was described in [Podkolzin and Vainerman 1999]. First, we recall the irreducible representations of $C(\mathrm{SU}_q(n))$ as in [Soibelman 1990]. The one-dimensional representations of $C(\mathrm{SU}_q(n))$ are parametrized by the torus $\mathbb{T}^{n-1}$. We consider $\mathbb{T}^{n-1}$ as a subset of $\mathbb{T}^n$ under the inclusion $(t_1, t_2, \dots t_{n-1}) \to (t_1, t_2, \dots, t_{n-1}, t_n)$, where $t_n := \prod_{i=1}^{n-1} \bar{t}_i$. For $t := (t_1, t_2, \dots, t_n) \in \mathbb{T}^{n-1}$, let $\tau_t : C(\mathrm{SU}_q(n)) \to \mathbb{C}$ be defined as

$$\tau_t(u_{ij}) := t_{n-i+1}\delta_{ij}.$$

Then, τ_t is a $*$-algebra homomorphism. The set $\{\tau_t : t \in \mathbb{T}^{n-1}\}$ is a complete set of mutually inequivalent one-dimensional representations of $C(\mathrm{SU}_q(n))$.

Denote the transposition $(i, i+1)$ by s_i. The map $\pi_{s_i} : C(\mathrm{SU}_q(n)) \to B(\ell^2(\mathbb{N}))$, defined on the generators u_{rs} by

$$\pi_{s_i}(u_{rs}) := \begin{cases} \sqrt{1-q^{2N+2}}S & \text{if } r = i,\ s = i, \\ -q^{N+1} & \text{if } r = i,\ s = i+1, \\ q^N & \text{if } r = i+1,\ s = i, \\ S^*\sqrt{1-q^{2N+2}} & \text{if } r = i+1,\ s = i+1, \\ \delta_{ij} & \text{otherwise,} \end{cases}$$

is a $*$-algebra homomorphism. For any two representations φ and ξ of $C(\mathrm{SU}_q(n))$, let $\varphi * \xi := (\varphi \otimes \xi)\Delta$. For $\omega \in S_n$, let $\omega = s_{i_1}s_{i_2}\dots s_{i_k}$ be a reduced expression. Then, the representation $\pi_\omega := \pi_{s_{i_1}} * \pi_{s_{i_2}} * \cdots * \pi_{s_{i_k}}$ is an irreducible representation. Up to unitary equivalence, the representation π_ω is independent of the reduced expression. For $t \in \mathbb{T}^{n-1}$ and $\omega \in S_n$ let $\pi_{t,\omega} := \tau_t * \pi_\omega$. We refer to [Soibelman 1990] for the proof of the following:

Theorem 2.2. $\{\pi_{t,\omega} : t \in \mathbb{T}^{n-1}, \omega \in S_n\}$ *is a complete set of mutually inequivalent irreducible representations of* $C(\mathrm{SU}_q(n))$.

The irreducible representations of $C(S_q^{n,m})$ were studied in [Podkolzin and Vainerman 1999]. We recall them here. Embed $\mathbb{T}^m$ into $\mathbb{T}^{n-1}$ via the map

$$t = (t_1, t_2, \dots, t_m) \to (t_1, t_2, \dots, t_m, 1, 1, \dots, 1, t_n),$$

where $t_n := \prod_{i=1}^{m} \bar{t}_i$. For a permutation $\omega \in S_n$, let ω^s be the permutation in the coset $S_{n-m}\omega$ with the least possible length. We denote the restriction of the representation $\pi_{t,\omega}$ to the subalgebra $C(S_q^{n,m})$ by $\pi_{t,\omega}$ itself.

Theorem 2.3 [Podkolzin and Vainerman 1999]. *The set* $\{\pi_{t,\omega^s} : t \in \mathbb{T}^m,\ \omega \in S_n\}$ *is a complete set of mutually inequivalent irreducible representations of* $C(S_q^{n,m})$.

3. Composition sequences

In this section, we derive certain exact sequences analogous to that of [Sheu 1997, Theorem 4]. We then apply the six-term sequence in K-theory to compute the K-groups of $C(S_q^{n,2})$.

Lemma 3.1. *Let* $t \in \mathbb{T}^m$ *and* $\omega := s_{n-1}s_{n-2}\dots s_{n-k}$. *The image of* $C(S_q^{n,m})$ *under the homomorphism* $\pi_{t,\omega}$ *contains the algebra of compact operators* $\mathcal{K}(\ell^2(\mathbb{N}^k))$.

Proof. Since $\pi_{t,\omega}(C(S_q^{n,m})) = \pi_\omega(C(S_q^{n,m}))$, it is enough to show that $\mathcal{K}(\ell^2(\mathbb{N}^k)) \subset \pi_\omega(C(S_q^{n,m}))$. We prove this result by induction on n. Since

$$\pi_\omega(u_{nn}) := S^*\sqrt{1-q^{2N+2}} \otimes 1,$$

it follows that $S \otimes 1 \in \pi_\omega(C(S_q^{n,m})$. Hence, $\mathcal{K}(\ell^2(\mathbb{N})) \otimes 1 \subset \pi_w(C(S_q^{n,m})$, and the result is true when $n = 2$.

Next, observe that $(p \otimes 1)\pi_\omega(u_{n,i}) := p \otimes \pi_{\omega'}(v_{n-1,i})$ for $1 \le i \le n-1$, where $\omega' := s_{n-2}s_{n-3}\dots s_{n-k}$ and (v_{ij}) denotes the generators of $C(\mathrm{SU}_q(n-1))$. Hence, $\pi_\omega(C(S_q^{n,m}))$ contains the algebra $p \otimes \pi_{\omega'}(C(S_q^{n-1,m}))$. Now, by the induction hypothesis, it follows that $\pi_\omega(C(S_q^{n,m}))$ contains $p \otimes \mathcal{K}(\ell^2(\mathbb{N}^{k-1}))$. Since $\pi_\omega(C(S_q^{n,m}))$ contains both $\mathcal{K}(\ell^2(\mathbb{N})) \otimes 1$ and $p \otimes \mathcal{K}(\ell^2(\mathbb{N}^{k-1}))$, it follows that $\pi_\omega(C(S_q^{n,m}))$ contains the algebra of compact operators, which completes the proof. □

Let w be a word on $s_1, s_2, \dots, s_n$, say, $w := s_{i_1}s_{i_2}\dots s_{i_n}$ (not necessarily a reduced expression). Define $\psi_w := \pi_{s_{i_1}} * \pi_{s_{i_2}} * \dots \pi_{s_{i_r}}$ and, for $t \in \mathbb{T}^n$, let $\psi_{t,w} := \tau_t * \psi_w$. Observe that the image of $\psi_{t,w}$ is contained in $\tau^{\otimes r}$. We prove that, if w' is a subword of w, then $\psi_{t,w'}$ factors through $\psi_{t,w}$.

Proposition 3.2. *Let* $w = w_1 s_k w_2$ *be a word on* $s_1, s_2, \dots, s_n$. *Denote the word* $w_1 w_2$ *by* w' *and let* $t \in \mathbb{T}^m$ *be given. There exists a* $*$*-homomorphism*

$$\varepsilon : \psi_{t,w}(C(S_q^{n,m})) \to \psi_{t,w'}(C(S_q^{n,m}))$$

such that $\psi_{t,w'} = \varepsilon \circ \psi_{t,w}$.

Proof. If $\ell(u)$ denotes the length of a word u on $s_1, s_2, \dots, s_n$, then $\psi_{t,w}(C(S_q^{n,m}))$ is contained in $\tau^{\otimes \ell(w_1)} \otimes \tau \otimes \tau^{\otimes \ell(w_2)}$. Let ε denote the restriction of $1 \otimes \sigma \otimes 1$ to $\psi_{t,w}(C(S_q^{n,m}))$, where $\sigma : \tau \to \mathbb{C}$ is the homomorphism for which $\sigma(S) = 1$.

$$\psi_{t,w}(u_{rs}) = \sum_{j_1,j_2} \psi_{t,w_1}(u_{rj_1}) \otimes \pi_{s_k}(u_{j_1j_2}) \otimes \psi_{w_2}(u_{j_2s}).$$

Since $\sigma(\pi_{s_k}(u_{j_1 j_2}))=\delta_{j_1 j_2}$, it follows that

$$\varepsilon\circ\psi_{t,w}(u_{rs})=\sum_j \psi_{t,w_1}(u_{rj})\otimes\psi_{w_2}(u_{js})=\psi_{t,w'}(u_{rs}).$$

This completes the proof. □

Let w be a word on $s_1, s_2, \dots s_n$. Then, for $n-m+1\le i\le n$ and $1\le j\le n$, the map $\mathbb{T}^m : t\to\psi_{t,w}(u_{ij})\in\tau^{\otimes\ell(w)}$ is continuous. Thus, we get a homomorphism $\chi_w : C(S_q^{n,m})\to C(\mathbb{T}^m)\otimes\tau^{\otimes\ell(w)}$ such that $\chi_w(a)(t)=\psi_{t,w}(a)$ for all $a\in C(S_q^{n,m})$.

Remark 3.3. Clearly, for a word w on $s_1, s_2, \dots s_n$, the representations $\psi_{t,w}$ factors through χ_w. One can also prove, as in Proposition 3.2, that if w' is a subword of w, then $\chi_{w'}$ factors through χ_w.

Let us introduce some notation. Denote by $\omega_{j,i}$ the permutation $s_j s_{j-1}\dots s_i$ for $j\ge i$. If $j<i$, let $\omega_{j,i}:=1$. For $1\le k\le n$, let $\omega_k:=\omega_{n-m,1}\omega_{n-m+1,1}\dots\omega_{n-1,n-k+1}$.

Theorem 3.4. *The homomorphism* $\chi_{\omega_n} : C(S_q^{n,m})\to C(\mathbb{T}^m)\otimes\tau^{\otimes\ell(\omega_n)}$ *is faithful.*

Proof. If $\omega_0\in S_n$ then ω_0^s(the representative in $S_{n-m}\omega_0$ with the shortest length) is a subword of ω_n. By Remark 3.3, it follows that every irreducible representation of $C(S_q^{n,m})$ factors through χ_{ω_n}. Hence, χ_{ω_n} is faithful. This completes the proof. □

For $1\le k\le n$, let $C(S_q^{n,m,k}):=\chi_{\omega_k}(C(S_q^{n,m}))$. Then,

$$C(S_q^{n,m,k})\subset C(S_q^{n,m,1})\otimes\tau^{\otimes(k-1)}.$$

For $2\le k\le n$, let σ_k denote the restriction of $(1\otimes 1^{\otimes(k-2)}\otimes\sigma)$ to $C(S_q^{n,m,k})$. The image of σ_k is $C(S_q^{n,m,k-1})$. We determine the kernel of σ_k in the next proposition. We need the following two lemmas.

Lemma 3.5. *The algebra* $\chi_{\omega_{n-1,n-k}}(C(S_q^{n,1}))$ *contains* $C^*(t_1)\otimes\mathcal{K}(\ell^2(\mathbb{N}^k))$, *which is isomorphic to* $C(\mathbb{T})\otimes\mathcal{K}(\ell^2(\mathbb{N}^k))$.

Proof. Note that $\chi_{\omega_{n-1,n-k}}(u_{nn})=t_1\otimes S^*\sqrt{1-q^{2N+2}}\otimes 1$. Hence it follows that the operator

$$1\otimes\sqrt{1-q^{2N+2}}\otimes 1=\chi_{\omega_{n-1,n-k}}(u_{nn}^*u_{nn})$$

lies in the algebra $\chi_{\omega_{n-1,n-k}}(C(S_q^{n,1}))$. As $\sqrt{1-q^{2N+2}}$ is invertible, $t_1\otimes S^*\otimes 1\in\chi_{\omega_{n-1,n-k}}(C(S_q^{n,1}))$. Thus, the projection $1\otimes p\otimes 1$ is in the algebra $C(S_q^{n,1,k+1})$. Observe that, for $1\le s\le n-1$, one has

$$(1\otimes p\otimes 1)\,\chi_{\omega_{n-1,n-k}}(u_{ns})=t_1\otimes p\otimes\pi_{\omega_{n-2,n-k}}(v_{n-1,s}),\tag{3-1}$$

where (v_{ij}) are the generators of $C(\mathrm{SU}_q(n-1))$. If $n=2$, then $k=1$, and what we have shown is that $C(S_q^{2,1,2})$ contains $t_1\otimes S^*$ and $t_1\otimes p$. Hence, $C^*(t_1)\otimes\mathcal{K}$ is contained in the algebra $C(S_q^{2,1,2})$.

We can now complete the proof by induction on n. Equation (3-1) shows that $C^*(t_1)\otimes p\otimes\mathcal{K}^{\otimes(k-1)}$ is contained in the algebra $C(S_q^{n,1,k+1})$. Also, $t_1\otimes S^*\otimes 1\in C(S_q^{n,1,k+1})$. It follows that $C^*(t_1)\otimes\mathcal{K}^{\otimes k}$ is contained in the algebra $C(S_q^{n,1,k+1})$. This completes the proof. □

Lemma 3.6. *Given $1\le s\le n$, there exist compact operators x_s, y_s such that $x_s\pi_{\omega_{n-1,n-k}}(u_{js})y_s=\delta_{js}(p\otimes p\otimes\cdots\otimes p)$, where $p:=1-S^*S$.*

Proof. Let $1\le s\le n$ be given. Note that the operator

$$\omega_{n-1,n-k}(u_{ss})=z_1\otimes z_2\otimes\cdots\otimes z_k,$$

where $z_i\in\{1,\sqrt{1-q^{2N+2}}S,\,S^*\sqrt{1-q^{2N+2}}\}$. Define x_i, y_i by

$$x_i:=\begin{cases}p & \text{if } z_i=1,\\ p & \text{if } z_i=\sqrt{1-q^{2N+2}}S,\\ (1-q^2)^{-\frac12}pS & \text{if } z_i=S^*\sqrt{1-q^{2N+2}};\end{cases}$$

$$y_i:=\begin{cases}p & \text{if } z_i=1,\\ (1-q^2)^{-\frac12}S^*p & \text{if } z_i=\sqrt{1-q^{2N+2}}S,\\ p & \text{if } z_i=S^*\sqrt{1-q^{2N+2}}.\end{cases}$$

Then, $x_iz_iy_i=p$ for $1\le i\le k$. Now, let $x_s:=x_1\otimes x_2\otimes\ldots x_k$ and $y_s:=y_1\otimes y_2\otimes\ldots y_k$. Then,

$$x_s\,\chi_{\omega_{n-1,n-k}}(u_{ss})=\underbrace{p\otimes p\otimes\cdots\otimes p}_{k\text{ times}}.$$

Let $j\ne s$ be given. Then, $\chi_{\omega_{n-1,n-k}}(u_{js})=a_1\otimes a_2\otimes\cdots\otimes a_k$ where

$$a_i\in\{1,\sqrt{1-q^{2N+2}}\,S,\,S^*\sqrt{1-q^{2N+2}},\,-q^{N+1},\,q^N\}.$$

Since $j\ne s$, there exists an i such that $a_i\in\{q^N,-q^{N+1}\}$. Let r be the largest integer for which $a_r\in\{q^N,-q^{N+1}\}$. Then, $z_r\ne 1$ and hence $x_ra_ry_r=0$. Thus, $x_s\,\chi_{\omega_{n-1,n-k}}(u_{js})y_s=0$, which completes the proof. □

Proposition 3.7. *Let $2\le k\le n$. Then, $C(S_q^{n,m,1})\otimes\mathcal{K}(\ell^2(\mathbb{N}))^{\otimes(k-1)}$ is contained in the algebra $C(S_q^{n,m,k})$. Moreover, the kernel of the homomorphism σ_k is exactly $C(S_q^{n,m,1})\otimes\mathcal{K}(\ell^2(\mathbb{N}))^{\otimes(k-1)}$. We have the exact sequence*

$$0\longrightarrow C(S_q^{n,m,1})\otimes\mathcal{K}^{\otimes(k-1)}\longrightarrow C(S_q^{n,m,k})\xrightarrow{\sigma_k}C(S_q^{n,m,k-1})\longrightarrow 0.$$

Proof. First, we prove that $C(S_q^{n,m,1})\otimes\mathcal{K}^{\otimes(k-1)}$ is contained in $C(S_q^{n,m,k})$. For $a\in C(S_q^{n,1})$, one has $\chi_{\omega_k}(a)=1\otimes\chi_{\omega_{n-1,n-k+1}}(a)$, and it follows from Lemma 3.5 that $C(S_q^{n,m,k})$ contains $1\otimes\mathcal{K}(\ell^2(\mathbb{N}^{k-1}))$. Let $n-m+1\le r\le m$ and $1\le s\le n$ be given. Note that

$$\chi_{\omega_k}(u_{rs}) = \sum_{j=1}^{n} \chi_{\omega_1}(u_{rj}) \otimes \pi_{\omega_{n-1,n-k+1}}(u_{js}).$$

By Lemma 3.6, there exist $x_s, y_s \in C(S_q^{n,m,k})$ such that

$$x_s\, \chi_{\omega_k}(u_{rs})\, y_s := \chi_{\omega_1}(u_{rs}) \otimes p^{\otimes(k-1)},$$

where $p^{\otimes(k-1)} := p \otimes p \otimes \cdots \otimes p$. Thus, we have shown that $C(S_q^{n,m,k})$ contains $1 \otimes \mathcal{K}^{\otimes(k-1)}$ and $C(S_q^{n,m,1}) \otimes p^{\otimes(k-1)}$. Hence, $C(S_q^{n,m,k})$ contains $C(S_q^{n,m,1}) \otimes \mathcal{K}^{\otimes(k-1)}$.

Clearly, σ_k vanishes on $C(S_q^{n,m,1}) \otimes \mathcal{K}^{\otimes(k-1)}$. Let π be an irreducible representation of $C(S_q^{n,m,k})$ which vanishes on the ideal $C(S_q^{n,m,1}) \otimes \mathcal{K}^{\otimes(k-1)}$. Then, $\pi \circ \chi_{\omega_k}$ is an irreducible representation of $C(S_q^{n,m})$ and hence $\pi \circ \chi_{\omega_k} = \pi_{t,\omega}$ for $t \in \mathbb{T}^m$ and some ω of the form $\omega_{n-m,i_1}\omega_{n-m+1,i_2} \cdots \omega_{n-1,i_{n-m}}$. Since $\pi \circ \chi_{\omega_k}(u_{n,n-k+1}) = 0$, it follows that $\pi_{t,w}(u_{n,n-k+1}) = 0$. However, $\pi_{t,\omega}(u_{n,n-k+1}) = t_n(1 \otimes \pi_{\omega_{n-1,i_{n-m}}}(u_{n,n-k+1}))$ and hence $i_{n-m} > n-k+1$. In other words, ω is a subword of ω_{k-1}. Therefore, $\pi \circ \chi_{\omega_k}$ factors through $\chi_{\omega_{k-1}}$ and so there exists a representation ρ of $C(S_q^{n,m,k-1})$ such that $\pi \circ \chi_{\omega_k} = \rho \circ \chi_{\omega_{k-1}}$. Since $\chi_{\omega_{k-1}} = \sigma_k \circ \chi_{\omega_k}$, it follows that $\pi = \rho \circ \sigma_k$.

We have shown that every irreducible representation of $C(S_q^{n,m,k})$ which vanishes on the ideal $C(S_q^{n,m,1}) \otimes \mathcal{K}^{\otimes(k-1)}$ factors through σ_k. Hence, the kernel of σ_k is exactly the ideal $C(S_q^{n,m,1}) \otimes \mathcal{K}^{\otimes(k-1)}$. This completes the proof. □

We apply the six-term exact sequence in K-theory to the exact sequence in Proposition 3.7 to compute the K-groups of $C(S_q^{n,2,k})$ for $1 \le k \le n$. In the next section, we briefly recall the product operation in K-theory.

4. The operation *P*

The algebras that we consider will be nuclear. So, no problem arises with regard to tensor products. Let A and B be C^*-algebras. We have the product maps

$$K_0(A) \otimes K_0(B) \to K_0(A \otimes B), \qquad K_1(A) \otimes K_0(B) \to K_1(A \otimes B),$$
$$K_0(A) \otimes K_1(B) \to K_1(A \otimes B), \qquad K_1(A) \otimes K_1(B) \to K_0(A \otimes B).$$

The first map is defined as $[p] \otimes [q] \to [p \otimes q]$; the second one, as $[u] \otimes [p] \to [u \otimes p + 1 - 1 \otimes p]$; and the third one likewise. The fourth map is defined using Bott periodicity and the first product; we describe it briefly, referring the reader to [Connes 1981, Appendix] for details.

Let $h : \mathbb{T}^2 \to P_1(\mathbb{C}) := \{p \in \mathrm{Proj}(M_2(\mathbb{C})) : \mathrm{trace}(p) = 1\}$ be a degree-one map. Given unitaries $u \in M_p(A)$ and $v \in M_q(B)$, the product $[u] \otimes [v]$ is given by

$[h(u, v)] - [e_0]$, where

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(M_{pq}(A \otimes B))$$

and $h(u, v)$ is the matrix with entries $h_{ij}(u \otimes 1, 1 \otimes v)$. We denote the image of $[x] \otimes [y]$ by $[x] \otimes [y]$ itself. Let A be a unital commutative C^*-algebra. Then, the multiplication $m : A \otimes A \to A$ is a C^*-algebra homomorphism. Hence, we get a map at the K-theory level from $K_1(A) \otimes K_1(A)$ to $K_0(A)$.

Suppose U and V are two commuting unitaries in a C^*-algebra B. If $A := C^*(U, V)$, then A is commutative. Define

$$P(U, V) := K_0(m)([U] \otimes [V]),$$

which is an element in $K_0(A)$. By composing with the inclusion map, we can think of it as an element in $K_0(B)$. From the formula of [Connes 1981] that we just recalled, the following properties are clear:

(1) If U and V are commuting unitaries in A, and p is a rank-one projection in $\mathcal{K}$, then we have $P(U \otimes p + 1 - 1 \otimes p, V \otimes p + 1 - 1 \otimes p) := P(U, V) \otimes p$.

(2) If U and V are commuting unitaries, and p is a projection that commutes with U and V, then $P(U, Vp + 1 - p) = P(Up + 1 - p, Vp + 1 - p)$.

(3) If $\varphi : A \to B$ is a unital homomorphism, and U and V are commuting unitaries in A, then $K_0(\varphi)(P(U, V)) = P(\varphi(U), \varphi(V))$.

(4) If U is a unitary in A, then $P(U, U) = 0$. Since $P_1(\mathbb{C})$ is simply connected, the matrix $h(U, U)$ is path-connected to a rank-one projection in $M_2(\mathbb{C})$. Hence, $P(U, U) = 0$.

We need the following lemma in the six-term computation. Let $z_1 \otimes 1$ and $1 \otimes z_2$ be the generating unitaries of $C(\mathbb{T}) \otimes C(\mathbb{T})$. Then, $K_0(C(\mathbb{T}^2))$ is isomorphic to $\mathbb{Z}^2$ and is generated by 1 and $P(z_1 \otimes 1, 1 \otimes z_2)$.

Lemma 4.1. *Consider the exact sequence*

$$0 \longrightarrow C(\mathbb{T}) \otimes \mathcal{K} \longrightarrow C(\mathbb{T}) \otimes \tau \longrightarrow C(\mathbb{T}) \otimes C(\mathbb{T}) \longrightarrow 0$$

and the six-term sequence in K-theory

$$\begin{tikzcd}
K_0(C(\mathbb{T}) \otimes \mathcal{K}) \arrow[r] & K_0(C(\mathbb{T}) \otimes \tau) \arrow[r] & K_0(C(\mathbb{T}) \otimes C(\mathbb{T})) \arrow[d, "\delta"] \\
K_1(C(\mathbb{T}) \otimes C(\mathbb{T})) \arrow[u, "\partial"] & K_1(C(\mathbb{T}) \otimes \tau) \arrow[l] & K_1(C(\mathbb{T}) \otimes \mathcal{K}). \arrow[l]
\end{tikzcd}$$

The subgroup generated by $\delta(P(z_1 \otimes 1, 1 \otimes z_2))$ coincides with the group generated by $z_1 \otimes p + 1 - 1 \otimes p$, which is $K_1(C(\mathbb{T}) \otimes \mathcal{K}) \cong \mathbb{Z}$.

Proof. The Toeplitz map $\varepsilon : \tau \to C(\mathbb{T})$ induces an isomorphism at the K_0-level. Thus, by the Künneth theorem, it follows that the image of $K_0(1 \otimes \varepsilon)$ is $K_0(C(\mathbb{T})) \otimes K_0(C(\mathbb{T}))$, which is the subgroup generated by [1]. The inclusion $0 \to \mathcal{K} \to \tau$ induces the zero map at the K_0 level and hence, again by the Künneth theorem, the inclusion $0 \to C(\mathbb{T}) \otimes \mathcal{K} \to C(\mathbb{T}) \otimes \tau$ induces the zero map at the K_1-level. Thus, the image of δ is $K_1(C(\mathbb{T}) \otimes \mathcal{K})$, which completes the proof. ☐

Corollary 4.2. *Let* $0 \longrightarrow I \longrightarrow A \xrightarrow{\varphi} B \longrightarrow 0$ *be a short exact sequence of* C^**-algebras. Consider the six-term sequence in K-theory*

$$\begin{tikzcd}
K_0(I) \arrow[r] & K_0(A) \arrow[r, "K_0(\varphi)"] & K_0(B) \arrow[d, "\delta"'] \\
K_1(B) \arrow[u, "\partial"] & K_1(A) \arrow[l, "K_1(\varphi)"] & K_1(I). \arrow[l]
\end{tikzcd}$$

Suppose that U *and* V *are two commuting unitaries in* B *such that there exists a unitary* X *and an isometry* Y *with* $\varphi(X) = U$ *and* $\varphi(Y) = V$. *If* X *and* Y *commute, then the subgroup generated by* $\delta(P(U, V))$ *coincides with the subgroup generated by the unitary* $X(1 - YY^*) + YY^*$ *in* $K_1(I)$.

Proof. Since $C(\mathbb{T})$ is the universal C^*-algebra generated by a unitary, and τ is the universal C^*-algebra generated by an isometry, there exists homomorphisms $\Phi : C(\mathbb{T}) \otimes \tau \to A$ and $\Psi : C(\mathbb{T}) \otimes C(\mathbb{T}) \to B$ such that

$$\Phi(z_1 \otimes 1) := X, \quad \Phi(1 \otimes S^*) := Y, \quad \Psi(z_1 \otimes 1) := U, \quad \Psi(1 \otimes z_2) := V.$$

We have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C(\mathbb{T}) \otimes \mathcal{K} \arrow[r] \arrow[d, "\Phi"] & C(\mathbb{T}) \otimes \tau \arrow[r] \arrow[d, "\Phi"] & C(\mathbb{T}) \otimes C(\mathbb{T}) \arrow[r] \arrow[d, "\Psi"] & 0 \\
0 \arrow[r] & I \arrow[r] & A \arrow[r, "\varphi"] & B \arrow[r] & 0.
\end{tikzcd}$$

By the functoriality of δ and P, it follows that

$$\delta(P(U, V)) = K_1(\Phi)(\delta(P(z_1 \otimes 1, 1 \otimes z_2))).$$

By Lemma 4.1, it follows that the subgroup generated by $\delta(P(U, V))$ is the subgroup generated by $\Phi(z_1 \otimes p + 1 - 1 \otimes p)$ in $K_1(I)$. Note that $\Phi(z_1 \otimes p + 1 - 1 \otimes p) = X(1 - YY^*) + YY^*$. This completes the proof. ☐

5. K-groups of $C(S_q^{n,2,k})$ for $k < n$

In this section, we compute the K-groups of $C(S_q^{n,2,k})$ for $1 \leq k < n$, by applying the six-term sequence in K-theory to the exact sequence in Proposition 3.7. We fix some notation. If q is a projection in $\ell^2(\mathbb{N})$ then q_r denotes the projection $q \otimes q \otimes \cdots \otimes q$ (r factors) in $\ell^2(\mathbb{N}^r)$. We define the unitaries U_k, V_k, u_k, v_k by

$$\begin{aligned}
U_k &:= t_1 \otimes 1_{n-2} \otimes p_{k-1} + 1 - 1 \otimes 1_{n-2} \otimes p_{k-1},\\
V_k &:= t_2 \otimes p_{n-2} \otimes 1_{k-1} + 1 - 1 \otimes p_{n-2} \otimes 1_{k-1},\\
u_k &:= t_1 \otimes p_{n-2} \otimes p_{k-1} + 1 - 1 \otimes p_{n-2} \otimes p_{k-1},\\
v_k &:= t_2 \otimes p_{n-2} \otimes p_{k-1} + 1 - 1 \otimes p_{n-2} \otimes p_{k-1}.
\end{aligned}$$

Note that the operators U_k, V_k, u_k, v_k lies in the algebra $C(S_q^{n,2,k})$. Indeed,

$$\begin{aligned}
U_k &= 1_{\{1\}}(u_{n,n-k+1}u_{n,n-k+1}^*)\, u_{n,n-k+1} + 1 - 1_{\{1\}}(u_{n,n-k+1}u_{n,n-k+1}^*),\\
V_k &= 1_{\{1\}}(u_{n-1,1}u_{n-1,1}^*)\, u_{n-1,1} + 1 - 1_{\{1\}}(u_{n-1,1}u_{n-1,1}^*),\\
u_k &= 1_{\{1\}}(u_{n,n-k+1}u_{n,n-k+1}^*u_{n-1,1}u_{n-1,1}^*)\, u_{n,n-k+1} + 1\\
&\qquad\qquad - 1_{\{1\}}(u_{n,n-k+1}u_{n,n-k+1}^*u_{n-1,1}u_{n-1,1}^*),\\
v_k &= 1_{\{1\}}(u_{n,n-k+1}u_{n,n-k+1}^*u_{n-1,1}u_{n-1,1}^*)u_{n-1,1} + 1\\
&\qquad\qquad - 1_{\{1\}}(u_{n,n-k+1}u_{n,n-k+1}^*u_{n-1,1}u_{n-1,1}^*).
\end{aligned}$$

Note that the unitaries U_n, u_n and v_n lie in the algebra $C(S_q^{n,2,n})$. We start with the computation of the K-groups of $C(S_q^{n,2,1})$.

Lemma 5.1. *The K-groups $K_0(C(S_q^{n,2,1}))$ and $K_1(C(S_q^{n,2,1}))$ are both isomorphic to $\mathbb{Z}^2$. In fact, $[U_1]$ and $[V_1]$ form a $\mathbb{Z}$-basis for $K_1(C(S_q^{n,2,1}))$, while $[1]$ and $P(u_1, v_1)$ form a $\mathbb{Z}$-basis for $K_0(C(S_q^{n,2,1}))$.*

Proof. First, note that $C(S_q^{n,2,1})$ is generated by $t_1 \otimes 1_{n-2}$ and $t_2 \otimes \pi_{\omega_{n-2,1}}(u_{n-1,j})$ for $1 \le j \le n-1$. The C^*-algebra generated by $\{t_2 \otimes \pi_{\omega_{n-2,1}}(u_{n-1,j}) : 1 \le j \le n-1\}$ is isomorphic to $C(S_q^{2n-3})$. Hence, $C(S_q^{n,2,1})$ is isomorphic to $C(\mathbb{T}) \otimes C(S_q^{2n-3})$. Also, $K_0(C(S_q^{2n-3})$ and $K_1(C(S_q^{2n-3}))$ are both isomorphic to $\mathbb{Z}$, with $[1]$ generating $K_0(C(S_q^{2n-3}))$, and $[t_2 \otimes p_{n-2} + 1 - 1 \otimes p_{n-2}]$ generating $K_1(C(S_q^{2n-3}))$.

Now, by the Künneth theorem for the tensor product of C^*-algebras (see [Blackadar 1986]), it follows that $C(S_q^{n,2,1})$ has both K_1 and K_0 isomorphic to $\mathbb{Z}^2$, with $[U_1]$ and $[V_1]$ generating $K_1(C(S_q^{n,2,1}))$, and $[1]$ and

$$P(t_1 \otimes 1_{n-2}, t_2 \otimes p_{n-2} + 1 - 1 \otimes p_{n-2})$$

generating $K_0(C(S_q^{n,2,1}))$. The projection $1 \otimes p_{n-2} = 1_{\{1\}}(\chi_{\omega_{n-2,1}}(u_{n-1,1}u_{n-1,1}^*))$ is in $C(S_q^{n,2,1})$ and commutes with the unitaries $t_1 \otimes 1_{n-2}$ and $t_2 \otimes p_{n-2} + 1 - 1 \otimes p_{n-2}$. Hence,

$$P(t_1 \otimes 1_{n-2}, t_2 \otimes p_{n-2} + 1 - 1 \otimes p_{n-2}) = P(u_1, v_1).$$

This completes the proof. □

Proposition 5.2. *Let* $1 \leq k < n$. *The* K-*groups* $K_0(C(S_q^{n,2,k}))$ *and* $K_1(C(S_q^{n,2,k}))$ *are both isomorphic to* $\mathbb{Z}^2$ *and, in particular,* $[U_k]$ *and* $[V_k]$ *form a* $\mathbb{Z}$-*basis for* $K_1(C(S_q^{n,2,k}))$, *while* $[1]$ *and* $P(u_k, v_k)$ *form a* $\mathbb{Z}$-*basis for* $K_0(C(S_q^{n,2,k}))$.

Proof. We prove this result by induction on k. The case $k = 1$ is just Lemma 5.1. Assume the result to be true for k. From Proposition 3.7, we have the short exact sequence

$$0 \longrightarrow C(S_q^{n,2,1}) \otimes \mathcal{K}^{\otimes(k)} \longrightarrow C(S_q^{n,2,k+1}) \xrightarrow{\sigma_{k+1}} C(S_q^{n,2,k}) \longrightarrow 0,$$

which gives rise to the following six-term sequence in K-theory:

$$\begin{tikzcd}
K_0(C(S_q^{n,2,1}) \otimes \mathcal{K}^{\otimes k}) \arrow[r] & K_0(C(S_q^{n,2,k+1})) \arrow[r, "K_0(\sigma_{k+1})"] & K_0(C(S_q^{n,2,k})) \arrow[d, "\delta"] \\
K_1(C(S_q^{n,2,k}) \arrow[u, "\partial"] & K_1(C(S_q^{n,2,k+1})) \arrow[l, "K_1(\sigma_{k+1})"'] & K_1(C(S_q^{n,2,1}) \otimes \mathcal{K}^{\otimes k}). \arrow[l]
\end{tikzcd}$$

To compute the six-term sequence, we determine δ and ∂. Since $\sigma_{k+1}(V_{k+1}) = V_k$, it follows that $\partial([V_k]) = 0$. Since $C(S_q^{n,2,k+1})$ contains the algebra $C(S_q^{n,2,1}) \otimes \mathcal{K}^{\otimes k}$, it follows that the operator

$$\tilde{X} := t_1 \otimes 1_{n-2} \otimes \underbrace{q^N \otimes q^N \otimes \dots q^N}_{(k-1)\text{ times}} \otimes S^*$$

is in the algebra $C(S_q^{n,2,1})$; indeed, the difference $X - \chi_{\omega_{k+1}}(u_{n,n-k+1})$ lies in the ideal $C(S_q^{n,2,1}) \otimes \mathcal{K}^{\otimes k}$. Let

$$X := 1_{\{1\}}(\tilde{X}^* \tilde{X})\tilde{X} + 1 - 1_{\{1\}}(\tilde{X}^* \tilde{X}).$$

Then, X is an isometry such that $\sigma_{k+1}(X) = U_k$ and hence

$$\partial([U_k]) = [1 - X^*X] - [1 - XX^*].$$

Thus, $\partial([U_k]) = -[1 \otimes 1_{n-2} \otimes p_k]$. The image of ∂ is the subgroup of $K_0(C(S_q^{n,2,1}) \otimes \mathcal{K}^{\otimes k})$ generated by $[1 \otimes 1_{n-2} \otimes p_k]$, while its kernel is $[V_k]$.

Next, we compute δ. Since $\sigma_{k+1}(1) = 1$, it follows that $\delta([1]) = 0$. Let

$$Y := (1 \otimes p_{n-2} \otimes 1_k)(1 \otimes 1_{n-2} \otimes p_{k-1} \otimes 1)\tilde{X} + 1 - 1 \otimes p_{n-2} \otimes p_{k-1} \otimes 1.$$

Since $1 \otimes p_{n-2} \otimes 1 = 1_{\{1\}}(\chi_{\omega_k}(u_{n-1,1}^* u_{n-1,1}))$ and $1 \otimes 1_{n-2} \otimes p_{k-1} = 1_{\{1\}}(\tilde{X}^* \tilde{X})$, it follows that $Y \in C(S_q^{n,2,k+1})$. Also,

$$Y = t_1 \otimes p_{n-2} \otimes p_{k-1} \otimes S^* + 1 - 1 \otimes p_{n-2} \otimes p_{k-1} \otimes 1.$$

Note that Y is an isometry such that $\sigma_{k+1}(Y) = u_k$. One has $\sigma_{k+1}(v_{k+1}) = v_k$. Observe that Y and v_{k+1} commute. By Lemma 4.1, it follows that the image of δ is the subgroup generated by $[v_{k+1}(1 - YY^*) + YY^*] = [V_1 \otimes p_k + 1 - 1 \otimes p_k]$.

This computation with the six-term sequence implies that $K_0(C(S_q^{n,2,k+1}))$ is isomorphic to $\mathbb{Z}^2$ and is generated by $P(u_1, v_1) \otimes p_k = P(u_k, v_k)$ and $[1]$. Also, the group $K_1(C(S_q^{n,2,k+1}))$ is isomorphic to $\mathbb{Z}^2$ and is generated by $[V_{k+1}]$ and $[U_1 \otimes p_k + 1 - 1 \otimes p_k] = [U_{k+1}]$. This completes the proof. □

6. *K*-groups of $C(S_q^{n,2})$

In this section, we compute the K-groups of $C(S_q^{n,2})$. We start with a few observations.

Lemma 6.1. *In the permutation group S_n, one has $\omega_{n-2,1}\omega_{n-1,1} = \omega_{n-1,1}\omega_{n-1,2}$.*

Proof. First, note that $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, and $s_i s_j = s_j s_i$ if $|i - j| \geq 2$. Hence, $\omega_{n-1,k}\omega_{n-1,1} = \omega_{n-1,k+1}\omega_{n-1,1} s_{k+1}$. The result follows by induction on k. □

We denote the representation $\chi_{\omega_{n-1,1}} * \pi_{\omega_{n-1,2}}$ by $\tilde{\chi}_{\omega_n}$. Since $\omega_{n-1,1}\omega_{n-1,2}$ is a reduced expression for ω_n, the representations $\tilde{\chi}_{\omega_n}$ and χ_{ω_n} are equivalent. Let U be a unitary such that $U\chi_{\omega_n}(\,\cdot\,)U^* = \tilde{\chi}_{\omega_n}(\,\cdot\,)$. It is clear that

$$\tilde{\chi}_{\omega_n}(C(S_q^{n,2})) \subset C(\mathbb{T}^m) \otimes \tau \otimes \tau^{\otimes \ell(\omega_{n-1})}.$$

Let $\tilde{\sigma}_n$ denote the restriction of $1 \otimes \sigma \otimes 1^{\otimes(2(n-2)}$ to $\tilde{\chi}_{\omega_n}(C(S_q^{n,2}))$. Since

$$\tilde{\sigma}_n(\tilde{\chi}_{\omega_n}(u_{ij})) = \chi_{\omega_{n-1}}(u_{ij}),$$

we have the commutative diagram

$$\begin{tikzcd}
\chi_{\omega_n}(C(S_q^{n,2})) \arrow[rr, "U(\cdot)U^*"] \arrow[dr, "\sigma_n"'] & & \tilde{\chi}_{\omega_n}(C(S_q^{n,2})). \arrow[dl, "\tilde{\sigma}_n"] \\
& C(S_q^{n,2,n-1}) &
\end{tikzcd}$$

Lemma 6.2. *There exists a coisometry $X \in \chi_{\omega_n}(C(S_q^{n,2}))$ such that $\sigma_n(X) = V_{n-1}$ and $X^*X = 1 - 1_{\{1\}}(\chi_{\omega_n}(u_{n1}^* u_{n1}))$.*

Proof. From the commutative diagram above, it is enough to show that there exists a coisometry $\widetilde{X} \in \tilde{\chi}_{\omega_n}(C(S_q^{n,2}))$ such that

$$\widetilde{\sigma}_n(X) = V_{n-1} \quad \text{and} \quad X^*X = 1 - 1_{\{1\}}(\tilde{\chi}_{\omega_n}(u_{n1}^* u_{n1}).$$

Note that

$$\widetilde{\chi}_{\omega_n}(u_{n-1,1}^* u_{n-1,1} - q^2 u_{n1} u_{n1}) = 1 \otimes 1 \otimes \underbrace{q^{2N} \otimes q^{2N} \otimes \dots q^{2N}}_{(n-2)\text{ times}} \otimes 1_{n-2}.$$

Hence the projection $1\otimes 1\otimes p_{n-2}\otimes 1_{n-2} = 1_{\{1\}}(\widetilde{\chi}_{\omega_n}(u^*_{n-1,1}u_{n-1,1} - q^2u^*_{n1}u_{n1}))$ is in the algebra $\widetilde{\chi}_{\omega_n}(C(S_q^{n,2}))$. Now let $Y := (1\otimes 1\otimes p_{n-2}\otimes 1_{n-2})\widetilde{\chi}_{\omega_n}(u_{n-1,1})$. Then,

$$Y := t_2\otimes\sqrt{1-q^{2N+2}}\,S\otimes p_{n-2}\otimes 1_{n-2}.$$

Hence, the operator $Z := t_2\otimes S\otimes p_{n-2}\otimes 1_{n-2}$ is in the algebra $\widetilde{\chi}_{\omega_n}(C(S_q^{n,2}))$. Now, let $\widetilde{X} := Z + 1 - ZZ^*$. Then, $\widetilde{X}$ is a coisometry such that $\widetilde{\sigma}_n(\widetilde{X}) = V_{n-1}$ and $\widetilde{X}^*\widetilde{X} = 1 - 1\otimes p_{n-1}\otimes 1_{n-2}$, which is $1 - 1_{\{1\}}(\widetilde{\chi}_{\omega_n}(u^*_{n1}u_{n1}))$. This completes the proof. □

Observe that the operator

$$\tilde{Z} := t_1\otimes 1_{n-1}\otimes\underbrace{q^N\otimes q^N\otimes\dots q^N}_{(n-2)\text{ times}}\otimes S^*$$

lies in the algebra $C(S_q^{n,2,n})$, since the difference $\tilde{Z} - \chi_{\omega_n}(u_{n,2})$ lies in the ideal $C(S_q^{n,2,1})\otimes\mathcal{K}^{\otimes(n-1)}$. Let $Z := 1_{\{1\}}(\tilde{Z}^*\tilde{Z})\,\tilde{Z}$ and $Y_n := Z + 1 - Z^*Z$. Then,

$$Z_n = t_1\otimes 1_{n-2}\otimes p_{n-2}\otimes S^*, \tag{6-1}$$

$$Y_n = t_1\otimes 1_{n-2}\otimes p_{n-2}\otimes S^* + 1 - 1\otimes 1_{n-2}\otimes p_{n-2}\otimes 1. \tag{6-2}$$

Hence, Y is an isometry and $YY^* = 1 - 1_{\{1\}}(\chi_{\omega_n}(u^*_{n1}u_{n1}))$. If X is a coisometry in $C(S_q^{n,2,n})$ such that $\sigma_n(X) = v_{n-1}$ and $X^*X := 1 - 1_{\{1\}}(\chi_{\omega_n}(u^*_{n1}u_{n1}))$, then XY is a unitary. (The existence of such an X was shown in Lemma 6.2.)

Proposition 6.3. *The K-groups $K_0(C(S_q^{n,2})$ and $K_1(C(S_q^{n,2})$ are both isomorphic to $\mathbb{Z}^2$. In particular,*

(1) *the projections* $[1]$ *and* $P(u_n, v_n)$ *generate* $K_0(C(S_q^{n,2}))$;

(2) *the unitaries U_n and XY_n generate $K_1(C(S_q^{n,2}))$, where X is a coisometry in $C(S_q^{n,2})$ such that $\sigma_n(X) = V_{n-1}$ and $X^*X = 1 - 1_{\{1\}}(u^*_{n1}u_{n1})$, while Y_n is as in* (6-2)

Proof. By Proposition 3.7, we have the exact sequence

$$0\longrightarrow C(S_q^{n,2,1})\otimes\mathcal{K}^{\otimes(n-1)}\longrightarrow C(S_q^{n,2,n})\xrightarrow{\sigma_n} C(S_q^{n,2,n-1})\longrightarrow 0,$$

which gives rise to the six-term sequence in K-theory

$$\begin{tikzcd}
K_0(C(S_q^{n,2,1})\otimes\mathcal{K}^{\otimes n-1}) \arrow[r] & K_0(C(S_q^{n,2,n})) \arrow[r, "K_0(\sigma_n)"] & K_0(C(S_q^{n,2,k})) \arrow[d, "\delta"] \\
K_1(C(S_q^{n,2,n-1})) \arrow[u, "\partial"] & K_1(C(S_q^{n,2,n})) \arrow[l, "K_1(\sigma_n)"'] & K_1(C(S_q^{n,2,1})\otimes\mathcal{K}^{\otimes n-1}). \arrow[l]
\end{tikzcd}$$

Now, we evaluate ∂ and δ to compute the six-term sequence. Since $[U_{n-1}]$ and $[V_{n-1}]$ generate $K_1(C(S_q^{n,2,n-1}))$, it follows that $[U_{n-1}]$ and $[V_{n-1}U_{n-1}]$ generate $K_1(C(S_q^{n,2,n-1}))$. As XY_n is a unitary with $\sigma_n(XY_n) = V_{n-1}U_{n-1}$, it follows that $\partial([V_{n-1}U_{n-1}]) = 0$. Next, Y_n is an isometry with $\sigma_n(Y_n) = U_{n-1}$. Hence $\partial([U_{n-1}]) = [1 - Y^*Y] - [1 - YY^*]$. Thus, $\partial([U_{n-1}]) = -[1 \otimes 1_{n-2} \otimes p_{n-1}]$.

Now, we compute δ. Since $\sigma_n(1) = 1$, it follows that $\delta([1]) = 0$. One observes that $p_{n-2} \otimes S^*\pi_{\omega_{n-1,1}}(u_{j1}) = 0$ if $j > 1$. Hence,

$$Z_n\,\chi_{\omega_n}(u_{n-1,1}) = t_1t_2 \otimes p_{n-2} \otimes p_{n-2} \otimes \sqrt{1-q^{2N+2}},$$

where Z_n is as defined in (6-1). The operator $R_n := t_1t_2 \otimes p_{n-2} \otimes p_{n-2} \otimes 1$ lies in the algebra $C(S_q^{n,2,n})$, since the difference $R_n - Z_n\,\chi_{\omega_n}(u_{n-1,1})$ lies in the ideal $C(\mathbb{T}^2) \otimes \mathcal{K}^{\otimes(2n-3)}$. Hence, the projection $1 \otimes p_{n-2} \otimes p_{n-2} \otimes 1$ lies in the algebra $C(S_q^{n,2,n})$. Now, define

$$\begin{aligned} S_n &:= R_n + 1 - R_nR_n^*, \\ T_n &:= (1 \otimes p_{n-2} \otimes p_{n-2} \otimes 1)Z_n + 1 - 1 \otimes p_{n-2} \otimes p_{n-2} \otimes 1. \end{aligned}$$

Then, S_n is a unitary and T_n is an isometry such that $\sigma_n(S_n) = u_{n-1}v_{n-1}$ and $\sigma_n(T_n) = u_{n-1}$. Moreover, S_n and T_n commute. Note that $P(u_{n-1}, v_{n-1}) = P(u_{n-1}, u_{n-1}v_{n-1})$. By Lemma 4.1, the image of δ is the subgroup generated by $S_n(1 - T_nT_n^*) + T_nT_n^*$ in $K_1(C(S_q^{n,2,1}) \otimes \mathcal{K}^{\otimes(n-1)})$. Now,

$$S_n(1 - T_nT_n^*) + T_nT_n^* = t_1t_2 \otimes p_{n-2} \otimes p_{n-1} + 1 - 1 \otimes p_{n-2} \otimes p_{n-1}.$$

Since $1 \otimes p_{n-2}$ is trivial in $K_0(C(S_q^{2n-3}))$, the unitary $t_1 \otimes p_{n-2} + 1 - 1 \otimes p_{n-2}$ is trivial in $K_1(C(S_q^{n,2,1})) = K_1(C(\mathbb{T}) \otimes C(S_q^{2n-3}))$. One has $[S_n(1-T_nT_n^*)+T_nT_n^*] = [V_1 \otimes p_{n-1} + 1 - 1 \otimes p_{n-1}]$ in $K_1(C(S_q^{n,2,1}) \otimes \mathcal{K}^{\otimes(n-1)})$.

This computation, with the exactness of the six-term sequence, completes the proof. □

7. *K*-groups of quantum SU(3)

In this section, we show that when $n = 3$ the unitary XY_n in Proposition 6.3 can be replaced by the fundamental 3×3 matrix (u_{ij}) of $C(\mathrm{SU}_q(3))$. First, note that for $n = 3$ we have $C(S_q^{n,2}) = C(\mathrm{SU}_q(3))$, since $C(\mathrm{SU}_q(1)) = \mathbb{C}$. The embedding $\mathrm{SU}_q(1) \subseteq \mathrm{SU}_q(3)$ is given by the counit. The quotient $C(\mathrm{SU}_q(3)/\mathrm{SU}_q(1))$ becomes isomorphic with $C(\mathrm{SU}_q(3))$. In [Sheu 1997], the algebra $C(S_q^{3,2,1})$ is denoted $C(U_q(2))$. Then, $C(U_q(2)) = C(\mathbb{T}) \otimes C(\mathrm{SU}_q(2))$. Let $\mathrm{ev}_1 : C(\mathbb{T}) \to \mathbb{C}$ be the evaluation at the point 1. Then, $\varphi = (\mathrm{ev}_1 \otimes 1)\,\sigma_2\sigma_3$, where $\varphi : C(\mathrm{SU}_q(3)) \to C(\mathrm{SU}_q(2))$ is the subgroup homomorphism defined in (2-1).

Proposition 7.1. *The K-group $K_1(C(\mathrm{SU}_q(3))$ is isomorphic to $\mathbb{Z}^2$, generated by the unitary $U_3 := t_1 \otimes p \otimes p + 1 - 1 \otimes p \otimes p$ and the fundamental unitary $U = (u_{ij})$*

Proof. By Proposition 6.3, we know that $K_1(C(\mathrm{SU}_q(3))$ is isomorphic to $\mathbb{Z}^2$ and generated by $[U_3]$ and $[XY_3]$, where X is a coisometry such that $\sigma_3(X) = V_2$ and $X^*X = 1 - 1_{\{1\}}(\chi_{\omega_3}(u_{31}^*u_{31}))$. Observe that $\varphi(X) = t_2 \otimes p + 1 - 1 \otimes p$ and $\varphi(Y_3) = 1$. Hence, $\varphi(XY_3) = t_2 \otimes p + 1 - 1 \otimes p$. Also note that

$$\varphi(U_3) = 0 \quad \text{and} \quad \varphi(U) = \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix},$$

where u denote the fundamental unitary of $C(\mathrm{SU}_q(2))$. Since $K_1(C(\mathrm{SU}_q(2))$ is isomorphic to $\mathbb{Z}$, the proof is complete if we show that $t_2 \otimes p + 1 - 1 \otimes p$ and $[u]$ represent the same element in $K_1(C(\mathrm{SU}_q(2))$; we do this in the next lemma. □

Denote by u_q the 2×2 fundamental unitary $u = (u_{ij})$ of $C(\mathrm{SU}_q(2))$. Consider the representation $\chi_{s_1} : C(\mathrm{SU}_q(2)) \to B(\ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{N}))$. We let the unitary t act on $\ell^2(\mathbb{Z})$ as the right shift, that is, $te_n = e_{n+1}$. Let $\{e_{n,m} : n \in \mathbb{Z}, m \in \mathbb{N}\}$ be the standard orthonormal basis for the Hilbert space $\ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{N})$. For an integer k, denote by P_k the orthogonal projection onto the closed subspace spanned by $\{e_{n,m} : n + m \leq k\}$, and set $F_k := 2P_k - 1$. Note that F_k is a selfadjoint unitary.

Proposition 7.2. *For any integer k, the triple $(\chi_{s_1}, \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{N}), F_k)$ is an odd Fredholm module for $C(\mathrm{SU}_q(2))$, and we have the pairing*

(1) $\langle [u_q], F_k\rangle = -1$,

(2) $\langle t \otimes p + 1 - 1 \otimes p, F_k\rangle = -1$, *where* $p = 1 - S^*S$.

Proof. It is not difficult to show that $C(\mathrm{SU}_q(2))$ is generated by $t \otimes S$ and $t \otimes p$. Now, one can see that $[t\otimes S, P_k] = 0$ and $[t\otimes p, P_k]$ is a finite-rank operator. Hence, the triple $(\chi_{s_1}, \ell^2(\mathbb{Z})\otimes\ell^2(\mathbb{N}), F_k)$ is an odd Fredholm module for $C(\mathrm{SU}_q(2))$. Since $C(\mathrm{SU}_q(2))$ is generated by $t \otimes S$ and $t \otimes p$, it follows that $u_p \in C(\mathrm{SU}_q(2))$ for every $p > 0$. Also, as $p \to 0$, u_p approaches u in norm, where u is given by

$$u := \begin{pmatrix} t \otimes S & 0 \\ \bar{t} \otimes p & \bar{t} \otimes S^* \end{pmatrix}.$$

Hence, $[u_q] = [u]$ in $K_1(C(\mathrm{SU}_q(2)))$. It is now easy to check that $\langle [u], F_k\rangle = -1$ and $\langle [t \otimes p + 1 - 1 \otimes p], F_k\rangle = -1$. This completes the proof. □

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A CLASS OF IRREDUCIBLE INTEGRABLE MODULES FOR THE EXTENDED BABY TKK ALGEBRA

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The baby TKK algebra is a core of the extended affine Lie algebra of type A_1 over a semilattice in $\mathbb{R}^2$. In this paper, we classify the irreducible integrable weight modules for the extended baby TKK algebra under the assumption that its center acts nontrivially.

1. Introduction

Extended affine Lie algebras (EALAs) were first introduced in [Høegh-Krohn and Torrésani 1990] and studied systematically in [Allison et al. 1997; Berman et al. 1996]. They are natural generalizations of finite-dimensional simple Lie algebras and affine Kac–Moody algebras. There are many examples of EALAs, such as toroidal algebras and TKK algebras [Moody et al. 1990; Mao and Tan 2007a; 2007b; Eswara Rao 2004; Tan 1999]. In [Eswara Rao 2004], the author studied the irreducible integrable weight modules of toroidal algebras.

The baby TKK algebra $\hat{\mathscr{G}}(\mathscr{J}(S))$ is the universal central extension of $\mathscr{G}(\mathscr{J}(S))$ obtained by the Tits–Kantor–Koecher construction. Its vertex operator representation and quantum analogue were studied in [Tan 1999; Gao and Jing 2010].

We recall this construction [Allison et al. 1997; Tan 1999]: Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$ be the unit elements in the lattice $\mathbb{Z}^2$. Let S_i for $0 \le i \le 3$ be the cosets of $2\mathbb{Z}^2$ in $\mathbb{Z}^2$ defined by

$$S_0 = 2\mathbb{Z}^2, \quad S_1 = e_1 + 2\mathbb{Z}^2, \quad S_2 = e_2 + 2\mathbb{Z}^2, \quad S_3 = e_1 + e_2 + 2\mathbb{Z}^2. \tag{1-1}$$

Let $S = S_0 \cup S_1 \cup S_2$. For $\sigma \in S$, let x^σ be a symbol. Then we obtain a Jordan algebra $\mathscr{J}(S) = \bigoplus_{\sigma \in S} \mathbb{C}x^\sigma$ with multiplication

$$x^r x^s = \begin{cases} x^{r+s} & \text{if } r, s \in S_0 \cup S_i \text{ and } 0 \le i \le 2, \\ 0 & \text{otherwise.} \end{cases} \tag{1-2}$$

Let $L_{\mathscr{J}(S)}$ be the set of multiplication operators of $\mathscr{J}(S)$ and

$$\mathrm{Inder}(\mathscr{J}(S)) = [L_{\mathscr{J}(S)}, L_{\mathscr{J}(S)}] = \mathrm{span}_{\mathbb{C}}\{[L_a, L_b] : a, b \in \mathscr{J}(S)\}$$

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where $[L_a, L_b]$ is an inner derivation of the Jordan algebra $\mathcal{J}(S)$. Let $\mathfrak{sl}_2(\mathbb{C})$ be the 3-dimensional simple Lie algebra. We use x_+, x_- and $\alpha^\vee$ to denote the Chevalley basis of $\mathfrak{sl}_2(\mathbb{C})$ with relations

$$[x_+, x_-] = \alpha^\vee \quad \text{and} \quad [\alpha^\vee, x_\pm] = \pm 2x_\pm. \tag{1-3}$$

Define a Lie algebra $\mathcal{G}(\mathcal{J}(S)) = \big(\mathfrak{sl}_2(\mathbb{C}) \otimes \mathcal{J}(S)\big) \oplus \operatorname{Inder}(\mathcal{J}(S))$ with multiplication

$$\begin{aligned}
&[A \otimes x^r, B \otimes x^s] = [A, B] \otimes x^r x^s + 2\operatorname{tr}(AB)[L_{x^r}, L_{x^s}],\\
&[D, A \otimes x^r] = A \otimes Dx^r,\\
&\big[D, [L_{x^r}, L_{x^s}]\big] = [L_{Dx^r}, L_{x^s}] + [L_{x^r}, L_{Dx^s}],
\end{aligned}$$

for $A, B \in \mathfrak{sl}_2(\mathbb{C})$, $x^r, x^s \in \mathcal{J}(S)$, and $D \in \operatorname{Inder}(\mathcal{J}(S))$. The Lie algebra $\mathcal{G}(\mathcal{J}(S))$ is a perfect Lie algebra. Its universal central extension $\hat{\mathcal{G}}(\mathcal{J}(S))$ is called the *baby TKK algebra*.

Let $\langle \mathcal{J}(S), \mathcal{J}(S)\rangle$ be the quotient space $(\mathcal{J}(S) \otimes \mathcal{J}(S))/I$, where I is the subspace of $\mathcal{J}(S) \otimes \mathcal{J}(S)$ spanned by all vectors of the form

$$a \otimes b + b \otimes a \quad \text{or} \quad ab \otimes c + bc \otimes a + ca \otimes b$$

for $a, b, c \in \mathcal{J}(S)$. We will use $\langle a, b\rangle$ to denote the element $a \otimes b + I$ in $(\mathcal{J}(S) \otimes \mathcal{J}(S))/I$. In [Tan 1999], the baby TKK algebra $\hat{\mathcal{G}}(\mathcal{J}(S))$ is realized as the vector space

$$\hat{\mathcal{G}}(\mathcal{J}(S)) = \big(\mathfrak{sl}_2(\mathbb{C}) \otimes \mathcal{J}(S)\big) \oplus \langle \mathcal{J}(S), \mathcal{J}(S)\rangle, \tag{1-4}$$

with the Lie bracket given by

$$\begin{aligned}
&[A \otimes a, B \otimes b] = [A, B] \otimes ab + 2\operatorname{tr}(AB)\langle a, b\rangle,\\
&[\langle a, b\rangle, A \otimes c] = A \otimes [L_a, L_b]c,\\
&[\langle a, b\rangle, \langle c, d\rangle] = \big\langle [L_a, L_b]c, d\big\rangle + \big\langle c, [L_a, L_b]d\big\rangle,
\end{aligned} \tag{1-5}$$

for $a, b, c, d \in \mathcal{J}(S)$ and $A, B \in \mathfrak{sl}_2(\mathbb{C})$. A vertex operator representation of $\hat{\mathcal{G}}(\mathcal{J}(S))$ was given in [Tan 1999] on a mixed bosonic-fermionic Fock space.

Let d_1, d_2 be the derivations on the baby TKK algebra $\hat{\mathcal{G}}(\mathcal{J}(S))$ given by

$$\begin{aligned}
&[d_i, A \otimes x^\sigma] = (\sigma \cdot e_i) A \otimes x^\sigma,\\
&[d_i, \langle x^\sigma, x^\tau\rangle] = ((\sigma + \tau) \cdot e_i)\langle x^\sigma, x^\tau\rangle,
\end{aligned} \tag{1-6}$$

for $\sigma, \tau \in S$, $A \in \mathfrak{sl}_2(\mathbb{C})$, $i, j = 1, 2$, where $a \cdot b$ denotes the inner product of $a, b \in \mathbb{R}^2$.

The *extended baby TKK algebra* $\mathcal{L}$ is defined to be

$$\mathcal{L} = \hat{\mathcal{G}}(\mathcal{J}(S)) \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2. \tag{1-7}$$

The center of $\mathscr{L}$ is two-dimensional, denoted by $\mathbb{C}C_1 \oplus \mathbb{C}C_2$, where $C_1 = \langle x^{e_1}, x^{-e_1}\rangle$ and $C_2 = \langle x^{e_2}, x^{-e_2}\rangle$.

In this paper, we study the irreducible integrable weight modules of the extended baby TKK algebra $\mathscr{L}$ such that C_1 acts nonzero while C_2 acts as zero. We identify $\mathfrak{sl}_2(\mathbb{C})$ with the subalgebra $\mathfrak{sl}_2(\mathbb{C}) \otimes 1$ of $\mathscr{L}$. Then, $\mathscr{L}$ has a five-dimensional Cartan subalgebra $\mathbb{C}\alpha^\vee \oplus \mathbb{C}C_1 \oplus \mathbb{C}C_2 \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2$. Let Δ be the root system of $\mathscr{L}$ with respect to this Cartan subalgebra. In Section 2, we will decompose Δ into $\Delta = \Delta_- \cup \Delta_0 \cup \Delta_+$ and, correspondingly, have a "triangular decomposition" of the extended baby TKK algebra $\mathscr{L}$,

$$\mathscr{L} = \mathscr{L}(\Delta_-) \oplus \mathscr{L}(\Delta_0) \oplus \mathscr{L}(\Delta_+), \tag{1-8}$$

where $\mathscr{L}(\Delta_\pm) = \bigoplus_{\beta\in\Delta_\pm} \mathscr{L}_\beta$ and $\mathscr{L}(\Delta_0) = \bigoplus_{\beta\in\Delta_0} \mathscr{L}_\beta$, where $\mathscr{L}_\beta$ denotes the root space for $\beta \in \Delta$. By a highest-weight module we mean a weight module generated by a weight vector that is annihilated by $\mathscr{L}(\Delta_+)$. We show that any irreducible integrable module V for $\mathscr{L}$ with the actions of $C_1 > 0$ and $C_2 = 0$ is a highest-weight module, and we also determine the conditions for a highest weight module to be integrable.

The paper is organized as follows: In Section 2, we recall some results on the structure of the extended baby TKK algebra $\mathscr{L}$, and give the definition of integrable modules of $\mathscr{L}$. We close the section with a lemma about the properties of irreducible integrable modules of $\mathscr{L}$. In Section 3, we study the highest-weight modules of $\mathscr{L}$. Let $\mathscr{H} = \hat{\mathscr{G}}(\mathscr{J}(S)) \oplus \mathbb{C}d_1$ be a subalgebra of $\mathscr{L}$. We define irreducible highest-weight modules, denoted by $V(\bar{\psi})$ and $L(\psi)$, for the Lie algebras $\mathscr{L}$ and $\mathscr{H}$, respectively. We show that the integrability of the $\mathscr{L}$-module $V(\bar{\psi})$ is equivalent to the integrability of the $\mathscr{H}$-module $L(\psi)$. Then, we investigate the conditions for the $\mathscr{H}$-module $L(\psi)$ to be integrable. In Section 4, we prove that every irreducible integrable module of $\mathscr{L}$ with the actions of $C_1 > 0$ and $C_2 = 0$ is isomorphic to a highest-weight module $V(\bar{\psi})$ constructed in Section 3.

We denote by $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Z}_+$, $\mathbb{R}$, $\mathbb{C}$ the sets of integers, nonnegative integers, positive integers, real numbers, and complex numbers, respectively. $U(\mathfrak{g})$ stands for the universal enveloping algebra of a Lie algebra $\mathfrak{g}$. All algebras are over $\mathbb{C}$.

2. Basic concepts

We recall the structure of $\mathscr{L}$ and its root system. Following [Tan 1999], we define

$$x_\pm(\sigma) = x_\pm(m,n) := \begin{cases} x_\pm \otimes x^\sigma & \text{if } \sigma \in S, \\ 0 & \text{if } \sigma \in S_3, \end{cases}$$

$$\alpha^\vee(\sigma) = \alpha^\vee(m,n) := \begin{cases} \alpha^\vee \otimes x^\sigma & \text{if } \sigma \in S, \\ 2\langle x^{e_1}, x^{\sigma - e_1}\rangle & \text{if } \sigma \in S_3, \end{cases}$$

and

$$C_i(\sigma) = C_i(m,n) := \begin{cases} \langle x^{e_i}, x^{\sigma - e_i}\rangle & \text{if } \sigma \in S_0, \\ 0 & \text{if } \sigma \notin S_0, \end{cases}$$

where $i = 1, 2,\ m, n \in \mathbb{Z}$ and $\sigma = (m, n)$. We also define

$$\Omega(\tau) := \begin{cases} 0 & \text{if } \tau \in S_0, \\ -1 & \text{if } \tau \in S_1, \\ 1 & \text{if } \tau \in S_2, \end{cases}$$

for $\tau \in S$. The sets S_0, S_1, S_2, S_3 and S were defined in (1-1).

Proposition 2.1 [Tan 1999]. *The universal central extension $\hat{\mathscr{G}}(\mathscr{J}(S))$ of $\mathscr{G}(\mathscr{J}(S))$ is spanned by the elements $\{x_\pm(\sigma), \alpha^\vee(\tau), C_i(\rho)\}$, for $i = 1, 2,\ \sigma \in S,\ \tau \in \mathbb{Z}^2 = \mathbb{Z}e_1 + \mathbb{Z}e_2$, and $\rho \in S_0$, and satisfies the following relations*:

(R1) *For $\sigma, \tau \in S$,*

$$[x_\pm(\sigma), x_\pm(\tau)] = 0,$$

$$[x_+(\sigma), x_-(\tau)] = \begin{cases} \Omega(\tau)\,\alpha^\vee(\sigma+\tau) & \text{if } \sigma+\tau \notin S, \\ \alpha^\vee(\sigma+\tau) + 2\sum_{i=1,2}(\sigma\cdot e_i)\,C_i(\sigma+\tau) & \text{if } \sigma+\tau \in S. \end{cases}$$

(R2) *For $\sigma \in \mathbb{Z}^2,\ \tau \in S$,*

$$[\alpha^\vee(\sigma), x_\pm(\tau)] = \begin{cases} \pm 2x_\pm(\sigma+\tau) & \text{if } \sigma \in S, \\ 2\Omega(\tau)\,x_\pm(\sigma+\tau) & \text{if } \sigma \notin S. \end{cases}$$

(R3) *For $\sigma, \tau \in \mathbb{Z}^2$,*

$$[\alpha^\vee(\sigma), \alpha^\vee(\tau)] = \begin{cases} 2\Omega(\tau)\,\alpha^\vee(\sigma+\tau) & \text{if } \sigma \notin S, \tau \in S, \\ -4\sum_{i=1,2}(\sigma\cdot e_i)\,C_i(\sigma+\tau) & \text{if } \sigma, \tau \notin S, \\ 4\sum_{i=1,2}(\sigma\cdot e_i)\,C_i(\sigma+\tau) & \text{if } \sigma, \tau \in S \text{ and } \sigma+\tau \in S, \\ 2\Omega(\tau)\,\alpha^\vee(\sigma+\tau) & \text{if } \sigma, \tau \in S \text{ and } \sigma+\tau \notin S. \end{cases}$$

(R4) *$C_i(\sigma)$ are central for $\sigma \in S_0$ and $i = 1, 2$, and satisfy*

$$(\sigma\cdot e_1)\,C_1(\sigma) + (\sigma\cdot e_2)\,C_2(\sigma) = 0. \qquad \square$$

Remark 2.2. We set $\mathfrak{h}_0 = \mathbb{C}\alpha^\vee(0,0) = \mathbb{C}\alpha^\vee$ and the Cartan subalgebra

$$\mathfrak{h} = \mathfrak{h}_0 \oplus \mathbb{C}C_1 \oplus \mathbb{C}C_2 \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2$$

of the baby TKK algebra $\mathscr{L} = \hat{\mathscr{G}}(\mathscr{J}(S)) \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2$. The center $\mathscr{Z}(\mathscr{L})$ of $\mathscr{L}$ is $\mathbb{C}C_1 \oplus \mathbb{C}C_2$.

Remark 2.3. $\mathscr{L}$ contains as a subalgebra the affine Kac–Moody algebra

$$\widetilde{\mathfrak{sl}}_2(\mathbb{C}) = \bigl(\mathfrak{sl}_2(\mathbb{C}) \otimes \bigl(\sum_{n\in\mathbb{Z}} \mathbb{C}x^{ne_1}\bigr)\bigr) \oplus \mathbb{C}C_1 \oplus \mathbb{C}d_1.$$

Definition 2.4. A module M over $\mathscr{L}$ is called a *weight module* if

$$M = \bigoplus_{\lambda\in\mathfrak{h}^*} M_\lambda,$$

where $M_\lambda = \{v \in M : h\,.\,v = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}$. The set $P(M) = \{\lambda \in \mathfrak{h}^* : M_\lambda \neq 0\}$ is called the *weight set* of M. For $\lambda \in P(M)$, M_λ is called a *weight space* associated to λ.

Lemma 2.5. *If M is any irreducible weight module over $\mathscr{L}$, then the actions of C_1 and C_2 are constant.* □

From this lemma, we see that, for any irreducible weight module M over $\mathscr{L}$, the actions of C_1 and C_2 are always linearly dependent. Due to this, in this paper we will consider modules with the actions of C_1 nonzero and $C_2 = 0$.

Define the elements α, δ_i and w_i in $\mathfrak{h}^*$ $(i = 1, 2)$ by

$$\begin{aligned}
&\alpha(\alpha^\vee) = 2, \quad \alpha(d_j) = \alpha(C_j) = 0,\\
&\delta_i(\alpha^\vee) = 0, \quad \delta_i(d_j) = \delta_{ij}, \quad \delta_i(C_j) = 0,\\
&w_i(\alpha^\vee) = 0, \quad w_i(d_j) = 0, \quad w_i(C_j) = \delta_{ij},
\end{aligned}$$

for $j = 1, 2$. Define also

$$\begin{aligned}
\Delta^{\mathrm{Re}} &= \{\pm\alpha + n_1\delta_1 + n_2\delta_2 : (n_1, n_2) \in S\},\\
\Delta^{\mathrm{Im}} &= \{n_1\delta_1 + n_2\delta_2 : (n_1, n_2) \in \mathbb{Z}^2\},\\
\Delta &= \Delta^{\mathrm{Re}} \cup \Delta^{\mathrm{Im}}.
\end{aligned}$$

The elements in Δ^{Re} and Δ^{Im} are called real and imaginary (or isotropic) roots, respectively. Then, $\mathscr{L}$ has a root space decomposition

$$\mathscr{L} = \bigoplus_{\beta\in\Delta} \mathscr{L}_\beta,$$

where $\mathscr{L}_\beta = \{x \in \mathscr{L} : [h, x] = \beta(h)x \text{ for all } h \in \mathfrak{h}\}$ and $\mathscr{L}_0 = \mathfrak{h}$.

Define the coroot $\gamma^\vee = \pm\alpha^\vee + 2n_1C_1 + 2n_2C_2$ for $\gamma = \pm\alpha + n_1\delta_1 + n_2\delta_2 \in \Delta^{\mathrm{Re}}$, and define the reflection r_γ on $\mathfrak{h}^*$ by setting

$$r_\gamma(\lambda) = \lambda - \lambda(\gamma^\vee)\gamma.$$

Let $\mathscr{W}$ be the subgroup of $\mathrm{GL}(\mathfrak{h}^*)$ generated by $\{r_\gamma : \gamma \in \Delta^{\mathrm{Re}}\}$. We call $\mathscr{W}$ the *Weyl group* of $\mathscr{L}$. One can read more about the structure of $\mathscr{W}$ in [Azam 1999].

Set

$$\begin{aligned}
&\Delta_+ = \big((\alpha+\mathbb{N}\delta_1+\mathbb{Z}\delta_2)\cup(-\alpha+\mathbb{Z}_+\delta_1+\mathbb{Z}\delta_2)\cup(\mathbb{Z}_+\delta_1+\mathbb{Z}\delta_2)\big)\cap\Delta,\\
&\Delta_- = \big((\alpha-\mathbb{Z}_+\delta_1+\mathbb{Z}\delta_2)\cup(-\alpha-\mathbb{N}\delta_1+\mathbb{Z}\delta_2)\cup(-\mathbb{Z}_+\delta_1+\mathbb{Z}\delta_2)\big)\cap\Delta,\\
&\Delta_0 = \mathbb{Z}\delta_2.
\end{aligned}$$

Correspondingly, set

$$\mathcal{L}(\Delta_+) = \bigoplus_{\beta\in\Delta_+}\mathcal{L}_\beta,\quad \mathcal{L}(\Delta_-) = \bigoplus_{\beta\in\Delta_-}\mathcal{L}_\beta,\quad \mathcal{L}(\Delta_0) = \bigoplus_{\beta\in\Delta_0}\mathcal{L}_\beta.$$

Then, one has $\Delta = \Delta_-\cup\Delta_0\cup\Delta_+$ and $\mathcal{L} = \mathcal{L}(\Delta_-)\oplus\mathcal{L}(\Delta_0)\oplus\mathcal{L}(\Delta_+)$.

Remark 2.6. The three subspaces $\mathcal{L}(\Delta_\pm)$ and $\mathcal{L}(\Delta_0)$ are all Lie subalgebras of $\mathcal{L}$.

Definition 2.7. A module M for $\mathcal{L}$ is said to be integrable if

(1) M is a weight module,

(2) each weight space of M is finite-dimensional,

(3) for any $\beta\in\Delta^{\mathrm{Re}}$, $x\in\mathcal{L}_\beta$ and $v\in M$, there exists some $k\in\mathbb{Z}_+$ such that $x^k\,.\,v=0$; that is, x acts locally nilpotent on M.

Lemma 2.8. *If M is an irreducible integrable module for $\mathcal{L}$, then*

(1) *the weight set $P(M)$ is $\mathcal{W}$-invariant*;

(2) $\dim M_\lambda = \dim M_{\omega\lambda}$, *for all $\lambda\in P(M)$ and $\omega\in\mathcal{W}$*;

(3) *for any real root γ and weight $\lambda\in P(M)$, $\lambda(\gamma^\vee)\in\mathbb{Z}$*;

(4) *if γ is real, $\lambda\in P(M)$ and $\lambda(\gamma^\vee)>0$, then $\lambda-\gamma\in P(M)$*;

(5) *for $i=1,2$, the action of $2C_i$ on M is a constant integer.*

Proof. Without loss of generality, we take a real root $\gamma=\alpha+n_1\delta_1+n_2\delta_2$ and set $\sigma=n_1e_1+n_2e_2$. Let $\mathfrak{sl}_2(\gamma)=\mathrm{span}_{\mathbb{C}}\{x_+(\sigma),\ x_-(-\sigma),\ \gamma^\vee=\alpha^\vee+2n_1C_1+2n_2C_2\}$, which is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. Set $s_\gamma=\exp(x_-(-\sigma))\cdot\exp(-x_+(\sigma))\cdot\exp(x_-(-\sigma))$. Then, s_γ is well-defined on M. It is easy to check that $s_\gamma M_\lambda\subset M_{r_\gamma\lambda}$ and, hence, $s_\gamma M_\lambda = M_{r_\gamma\lambda}$. Statements (1) and (2) follow from these observations.

Statement (3): Since $x_+(\sigma)$ and $x_-(-\sigma)$ are nilpotent on any nonzero vector $v\in M_\lambda$, by the representation theory of $\mathfrak{sl}_2(\mathbb{C})$ one sees that $\lambda(\gamma^\vee)$ is an integer.

Statement (4): For any $v_\lambda\in M_\lambda$, $W=U(\mathfrak{sl}_2(\gamma))v_\lambda$ is finite dimensional. As a $(\mathfrak{sl}_2(\gamma)+\mathfrak{h})$-module, the weights of W are $\lambda-p\gamma,\dots,\lambda+q\gamma$, where p,q are nonnegative integers, and $p-q=\lambda(\gamma^\vee)$. Now, if $\lambda(\gamma^\vee)>0$, then $p>0$ and, hence, $\lambda-\gamma\in P(M)$.

Statement (5) follows from (3) and Lemma 2.5. □

3. The highest- and lowest-weight modules

We define highest-weight and lowest-weight modules over $\mathscr{L}$, and construct a class of irreducible highest-weight modules $V(\bar{\psi})$ for $\mathscr{L}$ so that $2C_1$ acts as a positive integer and C_2 acts as zero. Then, we investigate sufficient conditions for $V(\bar{\psi})$ to be integrable.

Definition 3.1. A module M over $\mathscr{L}$ is called a *highest-* (respectively, *lowest-*) *weight module*, if there exists some $0 \neq v \in M$ such that

(1) v is a weight vector; that is, for all $h \in \mathfrak{h}$, we have $h\,.\,v = \lambda(h)v$ for some $\lambda \in \mathfrak{h}^*$;

(2) $\mathscr{L}(\Delta_+)\,.\,v = 0$ (respectively, $\mathscr{L}(\Delta_-)\,.\,v = 0$);

(3) $U(\mathscr{L})\,.\,v = M$.

Let $H = \operatorname{span}_{\mathbb{C}}\{\alpha^\vee(\sigma), C_1(2\sigma), C_2, d_1 : \sigma \in \mathbb{Z}e_2\}$ and ψ be a linear functional on H satisfying $\psi(C_1) \neq 0$ and $\psi(C_2) = 0$. Note that $\mathscr{L}(\Delta_0) = H \oplus \mathbb{C}d_2$ and that $H/\mathbb{C}C_2$ is abelian. Let $\mathbb{C}[t, t^{-1}]$ be the Laurent polynomial ring. Define an associative algebra homomorphism $\bar{\psi}$ by

$$\begin{aligned}\bar{\psi} :\ & U(H) \to \mathbb{C}[t, t^{-1}],\\ & X_1 \dots X_k \mapsto \psi(X_1) \dots \psi(X_k) t^{m_1+\dots+m_k},\end{aligned} \tag{3-1}$$

where X_i is homogeneous in H and $[d_2, X_i] = m_i X_i$ for $1 \leq i \leq k$.

Denote by $A_{\bar{\psi}}$ the image of $\bar{\psi}$ in $\mathbb{C}[t, t^{-1}]$. Since $\mathscr{L}(\Delta_0)$ is $\mathbb{Z}$-graded with respect to d_2, we have a $\mathscr{L}(\Delta_0)$-module structure on $A_{\bar{\psi}}$ defined, for $X \in H$, by

$$X\,.\,t^n = \bar{\psi}(X)t^n \quad \text{and} \quad d_2\,.\,t^n = nt^n.$$

Lemma 3.2 [Rao 1995]. *The $\mathscr{L}(\Delta_0)$-module $A_{\bar{\psi}}$ defined by* (3-1) *is irreducible if and only if each homogeneous element of $A_{\bar{\psi}}$ is invertible in $A_{\bar{\psi}}$.* □

Let $\bar{\psi}$ be given by (3-1) such that $A_{\bar{\psi}}$ is irreducible as an $\mathscr{L}(\Delta_0)$-module, and let $\mathscr{L}(\Delta_+)$ act trivially on $A_{\bar{\psi}}$. Consider the following induced module for $\mathscr{L}$:

$$M(\bar{\psi}) = U(\mathscr{L}) \otimes_{U(\mathscr{L}(\Delta_0)\oplus\mathscr{L}(\Delta_+))} A_{\bar{\psi}}.$$

Let ψ_0 be the restriction of ψ on $\mathfrak{h}_1 = \mathfrak{h}_0 \oplus \mathbb{C}C_1 \oplus \mathbb{C}C_2 \oplus \mathbb{C}d_1$. We extend ψ_0 to a linear functional (still denoted by ψ_0) on $\mathfrak{h}$ by setting $\psi_0(d_2) = 0$.

Proposition 3.3. (1) *$M(\bar{\psi})$ is a highest-weight module over $\mathscr{L}$.*

(2) *The weight set $P(M(\bar{\psi}))$ is a subset of $\psi_0 + \mathbb{Z}\delta_2 - \operatorname{span}_{\mathbb{N}}\Delta_-$. Moreover, $x \in M(\bar{\psi})$ has a weight of form $\psi_0 + n\delta_2$ if and only if $x \in A_{\bar{\psi}}$.*

(3) *$M(\bar{\psi})$ has a unique irreducible quotient $V(\bar{\psi})$.*

Proof. (1) Applying the Poincaré–Birkhoff–Witt (PBW) theorem, we have $M(\bar{\psi}) = U(\mathscr{L}(\Delta_-))\,A_{\bar{\psi}}$. Noting that $1 = t^0 \in A_{\bar{\psi}}$ and $A_{\bar{\psi}}$ is irreducible as $\mathscr{L}(\Delta_0)$-module, we see that $A_{\bar{\psi}} = U(\mathscr{L}(\Delta_0))\,t^0$. Hence, $M(\bar{\psi}) = U(\mathscr{L}(\Delta_-))\,U(\mathscr{L}(\Delta_0))\,(1 \otimes t^0) = U(\mathscr{L})\,(1 \otimes t^0)$. It follows that $M(\bar{\psi})$ is a highest-weight module over $\mathscr{L}$.

(2) This is clear.

(3) Let W_1 and W_2 be two nonzero proper submodules of $M(\bar{\psi})$. Since $A_{\bar{\psi}}$ is irreducible as $\mathscr{L}(\Delta_0)$-module, it follows that $A_{\bar{\psi}} \cap W_i = 0$ for $i = 1, 2$. Now, we check that $(W_1 + W_2) \cap A_{\bar{\psi}} = \{0\}$, that is, $W_1 + W_2$ is still a proper submodule of $M(\bar{\psi})$. If $(W_1 + W_2) \cap A_{\bar{\psi}} \neq \{0\}$, we may write a weight vector $x \in A_{\bar{\psi}}$ as $x = y_1 + y_2$ for some $y_i \in W_i$ for $i = 1, 2$. By (2), we can assume that the weight of x is $\psi_0 + n\delta_2$ for some $n \in \mathbb{Z}$. Then, in at least one of W_1 and W_2, there exists a weight vector of weight $\psi_0 + n\delta_2$, which is again impossible by (2). If M is the sum of all proper submodules of $M(\bar{\psi})$, then $V(\bar{\psi}) = M(\bar{\psi})/M$ is the unique irreducible quotient. □

In the rest of this section, we investigate the conditions for $V(\bar{\psi})$ to be integrable. We will show in next section that any irreducible integrable module of $\mathscr{L}$ with the actions $C_1 > 0$ and $C_2 = 0$ is isomorphic to $V(\bar{\psi})$ for some $\bar{\psi}$.

Let $\mathscr{H} = \hat{\mathscr{G}}(\mathscr{J}(S)) \oplus \mathbb{C}d_1$ be a subalgebra of $\mathscr{L}$. Then, $\mathscr{H} = \mathscr{L}(\Delta_-) \oplus H \oplus \mathscr{L}(\Delta_+)$.

Definition 3.4. A $\mathscr{H}$-module W is called a *highest-weight module* if there exists a nonzero vector $v \in W$ such that

(1) $\mathscr{L}(\Delta_+)\,.\,v = 0$,

(2) $U(\mathscr{H})\,.\,v = W$,

(3) there exists some $\psi \in H^*$ with $\psi(C_2) = 0$ such that $h.v = \psi(h)\,v$ for all h in H.

Let ψ be in H^* with $\psi(C_2) = 0$. We view $\mathbb{C}$ as a one-dimensional $H \oplus \mathscr{L}(\Delta_+)$-module, on which h acts as the scalar $\psi(h)$ for $h \in H$, and $\mathscr{L}(\Delta_+)$ acts trivially. Consider the induced module for $\mathscr{H}$,

$$W(\psi) = U(\mathscr{H}) \otimes_{U(H \oplus \mathscr{L}(\Delta_+))} \mathbb{C}.$$

Clearly, $W(\psi)$ has a unique irreducible quotient denoted by $L(\psi)$, with the highest weight vector $v = 1 \otimes 1$.

Consider any $\bar{\psi}$ defined by (3-1) such that $A_{\bar{\psi}}$ is an irreducible $\mathscr{L}(\Delta_0)$-module. Define a linear map $\mathfrak{X} : A_{\bar{\psi}} \to \mathbb{C}$ by evaluating the polynomials at 1. In other words, $\mathfrak{X}(f(t)) = f(1)$ for all $f(t) \in A_{\bar{\psi}}$. If $\psi = \mathfrak{X} \circ (\bar{\psi}|_H)$, then we get the $L(\psi)$ defined above. One can easily check that the following action gives an $\mathscr{L}$-module structure on the vector space $L(\psi) \otimes \mathbb{C}[t, t^{-1}]$:

$$X\,.\,(a \otimes t^m) = (X\,.\,a) \otimes t^{m+n} \quad \text{and} \quad d_2\,.\,(a \otimes t^m) = ma \otimes t^m \tag{3-2}$$

for $X \in \mathscr{H}$ satisfying $[d_2, X] = nX$, $a \in L(\psi)$, and $m \in \mathbb{Z}$.

Theorem 3.5. *If* $A_{\bar{\psi}}$ *is irreducible as an* $\mathscr{L}(\Delta_0)$*-module, then* $L(\psi)\otimes\mathbb{C}[t,t^{-1}]$ *is completely reducible as an* $\mathscr{L}$*-module, and the component containing* $v\otimes 1$ *is isomorphic to* $V(\bar{\psi})$ *as an* $\mathscr{L}$*-module.*

Proof. First, note that $A_{\bar{\psi}}=\mathbb{C}[t^N,t^{-N}]$ for some nonnegative integer N. Take $G=\{0,1,\dots,N-1\}$ if $N\geq 1$, or $G=\mathbb{Z}$ if $N=0$. We will show that

$$L(\psi)\otimes\mathbb{C}[t,t^{-1}]=\bigoplus_{n\in G}U(\mathscr{L})(v\otimes t^n),$$

and that each $U(\mathscr{L})(v\otimes t^n)$ is irreducible as an $\mathscr{L}$-module.

If $w\otimes t^m\in L(\psi)\otimes\mathbb{C}[t,t^{-1}]$, then there exists some $X\in U(\mathscr{H})$ such that $Xv=w$ in $L(\psi)$. Write $X=\sum_n X_n$, where $[d_2,X_n]=nX_n$. We have $\sum_n X_n\,.\,(v\otimes t^{m-n})=w\otimes t^m$, which implies that $L(\psi)\otimes\mathbb{C}[t,t^{-1}]=\sum_{n\in\mathbb{Z}}U(\mathscr{L})(v\otimes t^n)$.

For $t^r\in A_{\bar{\psi}}$, we have $\bar{\psi}(X')=t^r$ for some $X'\in U(H)$, and then $X'\,.\,(v\otimes t^m)=v\otimes t^{m+r}$. Hence, $U(\mathscr{L})(v\otimes t^m)=U(\mathscr{L})(v\otimes t^{m+r})$ and

$$L(\psi)\otimes\mathbb{C}[t,t^{-1}]=\sum_{n\in G}U(\mathscr{L})(v\otimes t^n). \tag{3-3}$$

Next, we prove that $U(\mathscr{L})(v\otimes t^m)$ is irreducible as an $\mathscr{L}$-module when $m\in G$. Let W be a nonzero $\mathscr{L}$-submodule of $U(\mathscr{L})(v\otimes t^m)$. Consider the linear map

$$\pi: W\to L(\psi),\qquad w\otimes t^m\mapsto w.$$

It is clear that π is a homomorphism of $\mathscr{H}$-modules. Since $L(\psi)$ is irreducible as a $\mathscr{H}$-module, π has to be surjective. Using the fact that W is $\mathbb{Z}$-graded with respect to d_2, it follows that W contains $v\otimes t^n$ for some integer n. Clearly, $v\otimes t^n\in U(\mathscr{L})(v\otimes t^m)$ implies that $v\otimes t^n\in U(\mathscr{L}(\Delta_0))(v\otimes t^m)$. Then, there exists some $Y\in U(H)$ such that $Y(v\otimes t^m)=v\otimes t^n$, which means that $\bar{\psi}(Y)=t^{n-m}\in A_{\bar{\psi}}$. Choose $Z\in U(H)$ such that $\bar{\psi}(Z)=t^{m-n}$. Then, $v\otimes t^m=Z(v\otimes t^n)\in W$ and hence $W=U(\mathscr{L})(v\otimes t^m)$, as required. From the above, we see that $v\otimes t^m\in U(\mathscr{L})(v\otimes t^n)$ if and only if $m-n\in G \pmod N$. Therefore,

$$L(\psi)\otimes\mathbb{C}[t,t^{-1}]=\bigoplus_{n\in G}U(\mathscr{L})(v\otimes t^n). \tag{3-4}$$

Finally, the assertion that "the component containing $v\otimes 1$ is isomorphic to $V(\bar{\psi})$ as an $\mathscr{L}$-module" is clear. □

Proposition 3.6. *If* $\bar{\psi}$ *is defined by* (3-1) *and such that* $\dim A_{\bar{\psi}}=1$, *then at least one of the weight spaces of* $V(\bar{\psi})$ *is infinite-dimensional.*

Proof. Since $\dim A_{\bar{\psi}}=1$, we have that $C_1v\neq 0$ and $C_1(0,2m)v=0$ for all $m\neq 0$. First, we show that $\alpha^\vee(-2,2m)v\neq 0$ in $L(\psi)$ for all $m\in\mathbb{Z}$. Otherwise, we assume that $\alpha^\vee(-2,2m)\,v=0$ for some $n\in\mathbb{Z}$. Then,

$$0=\alpha^\vee(2,-2m)\,\alpha^\vee(-2,2m)\,v=[\alpha^\vee(2,-2m),\alpha^\vee(-2,2m)]\,v=8C_1v,$$

which is a contradiction.

We complete the proof by showing that the set

$$\{\alpha^\vee(-2,2m)\,\alpha^\vee(-2,-2m)\,(v\otimes 1):m>0\}$$

is linearly independent in $V(\bar{\psi})$. Otherwise, we may assume that we have a relation

$$\sum_m b_m\,\alpha^\vee(-2,2m)\,\alpha^\vee(-2,-2m)\,(v\otimes 1)=0$$

with some $b_m\neq 0$. Under the action of $\alpha^\vee(2,2s)$, we obtain

$$\sum_m b_m\big(\alpha^\vee(-2,-2m)\,C_1(0,2(m+s))+\alpha^\vee(-2,2m)\,C_1(0,2(-m+s))\big)\,(v\otimes 1)=0.$$

For any element $s\in\{m:b_m\neq 0\}$, we deduce that $b_s=0$ — a contradiction. □

Proposition 3.7. *Let $\bar{\psi}$ be defined by* (3-1) *and such that $A_{\bar{\psi}}$ is an irreducible $\mathscr{L}(\Delta_0)$-module with* $\dim A_{\bar{\psi}}>1$. *Then, $V(\bar{\psi})$ has finite-dimensional weight spaces with respect to $\mathfrak{h}$ if and only if $L(\psi)$ has finite-dimensional weight spaces with respect to $\mathfrak{h}_1$.*

Proof. Suppose that $V(\psi)$ has finite-dimensional weight spaces with respect to $\mathfrak{h}_1$. Then, $L(\psi)\otimes\mathbb{C}[t,t^{-1}]$ has finite-dimensional weight spaces with respect to $\mathfrak{h}$. By Theorem 3.5, we see that $V(\bar{\psi})$ has finite-dimensional weight spaces with respect to $\mathfrak{h}$.

Suppose now that $V(\bar{\psi})$ has finite-dimensional weight spaces with respect to $\mathfrak{h}$, and consider the $\mathscr{H}$-module homomorphism

$$\begin{aligned}\zeta: L(\psi)\otimes\mathbb{C}[t,t^{-1}]&\to L(\psi),\\ w\otimes t^n&\mapsto w,\end{aligned}\tag{3-5}$$

where $w\in L(\psi)$ and $n\in\mathbb{Z}$. For $k\in\mathbb{Z}$, let ζ_k be the restriction of ζ to $L(\psi)\otimes t^k$. Then, ζ_k is a $\mathscr{H}$-module isomorphism. If $L(\psi)$ has a weight space $L(\psi)_\nu$ satisfying $\dim L(\psi)_\nu=\infty$, then $\zeta_k^{-1}(L(\psi)_\nu)=(L(\psi)\otimes t^k)_\nu$ is infinite-dimensional. Note that G is a finite set. Therefore, there is at least one $n\in G$ such that the weight space $(U(\mathscr{L})(v\otimes t^n))_{\nu'}$ of $U(\mathscr{L})(v\otimes t^n)$ is infinite dimensional, where $\nu'|_{\mathfrak{h}_1}=\nu$ and $\nu'(d_2)=k$. This is a contradiction. □

Now, we investigate the conditions for $L(\psi)$ to be integrable.

Theorem 3.8. *Let $\lambda_1,\dots,\lambda_k;-\mu_1,\dots,-\mu_l$ be nonnegative integers, and take two sets of nonzero distinct complex numbers, $\{a_1,\dots,a_k\}$ and $\{b_1,\dots,b_l\}$.*

If $\psi : H \to \mathbb{C}$ *is a linear map such that*

$$\psi(\alpha^\vee(0,m)) = \sum_{i=1}^{k} \lambda_i a_i^m, \tag{3-6}$$

$$\psi\big(\alpha^\vee(0,2m) - 2C_1(0,2m)\big) = \sum_{i=1}^{l} \mu_i b_i^m, \tag{3-7}$$

$$\psi(C_2) = 0, \tag{3-8}$$

then $L(\psi)$ *is an integrable module for* $\mathcal{H}$.

Conversely, if $L(\psi)$ *is integrable* (*with* $\psi(C_2) = 0$) *for* $\mathcal{H}$, *then* ψ *has to be defined as above.*

Before proving Theorem 3.8, we present several results which we will use later.

Lemma 3.9. *The Lie subalgebra* $\mathcal{L}(\Delta_+)$ *is generated by the set*

$$\{x_+(0,n),\ x_-(1,2n),\ x_-(2,2n+1) : n \in \mathbb{Z}\}. \tag{3-9}$$

Proof. It is straightforward to check. ☐

For $n \in \mathbb{Z}$, we define

$$X_{1,n} = x_+(0,n), \quad X_{2,n} = x_-(1,2n), \quad X_{3,n} = x_-(2,2n+1). \tag{3-10}$$

Recall that an element $X \in \mathcal{H}$ is said to be *locally nilpotent* on $L(\psi)$ if, for any element $w \in L(\psi)$, one has $X^m w = 0$ when $m \gg 0$. For an arbitrary Lie algebra $\mathfrak{g}$, we have the following results:

Proposition 3.10 [Kac 1990]. *Let* $v_1, v_2, \ldots$ *be a system of generators of a* $\mathfrak{g}$-*module* V, *and let* $x \in \mathfrak{g}$ *be such that* $\operatorname{ad} x$ *is locally nilpotent on* $\mathfrak{g}$ *and* $x^{N_i}(v_i) = 0$ *for some positive integers* N_i, $i = 1, 2, \ldots$. *Then* x *is locally nilpotent on* V. ☐

Proposition 3.11 [Moody and Pianzola 1995]. *Let* $\pi : \mathfrak{g} \to \mathrm{gl}(V)$ *be a representation of* $\mathfrak{g}$ *on a vector space* V. *If* $x \in \mathfrak{g}$ *is such that both* $\operatorname{ad} x$ *and* $\pi(x)$ *are locally nilpotent, then, for all* $y \in \mathfrak{g}$,

$$\pi((\exp \operatorname{ad} x)(y)) = (\exp \pi(x))\,\pi(y)\,(\exp \pi(x))^{-1}.$$

☐

Let $\alpha_0 = -\alpha + \delta_1$. Then, $\{\alpha,\ \alpha_0\}$ is a set of simple roots of the affine Kac–Moody algebra $\widetilde{\mathfrak{sl}}_2(\mathbb{C}) = \big(\mathfrak{sl}_2(\mathbb{C}) \otimes (\sum_{k\in\mathbb{Z}} \mathbb{C}x^{ke_1})\big) \oplus \mathbb{C}C_1 \oplus \mathbb{C}d_1$ (see Remark 2.3). Let $\mathcal{W}_{\mathrm{aff}}$ be the subgroup of $\mathcal{W}$ generated by the reflections associated to α and α_0. Then, $\mathcal{W}_{\mathrm{aff}}$ is the Weyl group of $\widetilde{\mathfrak{sl}}_2(\mathbb{C})$.

Lemma 3.12. *If* $\gamma = \pm\alpha + n_1\delta_1 + n_2\delta_2 \in \Delta^{\mathrm{Re}}$ *is a real root, then there exists some* $\omega \in \mathcal{W}_{\mathrm{aff}}$ *such that* $\omega(\gamma) = \alpha + n_2\delta_2$ *or* $\omega(\gamma) = \alpha_0 + n_2\delta_2$. *In any case,* $\omega(\gamma)$ *is still a root in* Δ^{Re}.

Proof. Denote $\gamma' = \gamma - n_2\delta_2$. Since γ' is a real root of the affine Kac–Moody algebra $\widetilde{\mathfrak{sl}}_2(\mathbb{C})$, there exists $\omega \in \mathscr{W}_{\text{aff}}$ such that $\omega(\gamma') = \alpha$ or $\omega(\gamma') = \alpha_0$. We see that $\omega(\gamma') = \alpha$ (respectively, α_0) if n_1 is even (respectively, odd). Thus, $\omega(\gamma) = \alpha + n_2\delta_2$ or $\omega(\gamma) = \alpha_0 + n_2\delta_2$. In either case, $\omega(\gamma)$ is a root in Δ^{Re}. □

Lemma 3.13. *Suppose that, for all $m \in \mathbb{Z}$, both $x_+(\sigma_m)$ and $x_-(\tau_m)$ are nilpotent on the highest-weight vector v in $L(\psi)$, where $\sigma_m = -e_1 + 2me_2$ and $\tau_m = me_2$. Then, $x_\pm(\sigma)$ are locally nilpotent on $L(\psi)$ for all $\sigma = k_1e_1 + k_2e_2 \in S$.*

Proof. Since $x_+(\sigma_m)$ and $x_-(\tau_m)$ are nilpotent on v and locally nilpotent on $\mathscr{L}$ under the adjoint action, they are locally nilpotent on $L(\psi)$ by Proposition 3.10. Thus, $L(\psi)$ is an integrable module (without the finite-dimensional weight-spaces condition) for the $\mathfrak{sl}_2(\mathbb{C})$-copies $\{x_+(-\tau_m), x_-(\tau_m), \alpha^\vee\}$ and $\{x_+(\sigma_m), x_-(-\sigma_m), \alpha^\vee - 2C_1\}$ (we are assuming $C_2 = 0$).

Let $\gamma = \pm\alpha + k_1\delta_1 + k_2\delta_2$ be the root of $x_\pm(\sigma)$ for $\sigma = k_1e_1 + k_2e_2$. By Lemma 3.12, there exists some $\omega \in \mathscr{W}_{\text{aff}}$ such that $\omega(\gamma) = \beta + k_2\delta_2$ for $\beta \in \{\alpha, \alpha_0\}$. Let s_ω be the inner automorphism of $\mathscr{L}$ associated to ω, and take $Y \in \mathscr{L}_{\beta+k_2\delta_2}$ to be a nonzero root vector. Up to a nonzero constant multiple, we have $s_\omega(x_\pm(\sigma)) = Y$. By Proposition 3.11, we know that $x_\pm(\sigma)$ are locally nilpotent on $L(\psi)$. □

Consider the loop algebra $\widehat{\mathfrak{sl}}_2(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}[t, t^{-1}]$. Let $u_1, \dots, u_n$ be nonzero complex numbers and $\xi_1, \dots, \xi_n$ (with $n > 0$) be nonnegative integers. Let B be the $\widehat{\mathfrak{sl}}_2(\mathbb{C})$-module generated by an element w subject to the relations

$$(x_+ \otimes \mathbb{C}[t, t^{-1}]) \,.\, w = 0, \quad (\alpha^\vee \otimes t^m) \,.\, w = \sum_{j=1}^{n} \xi_j u_j^m w, \quad (x_- \otimes 1)^{\sum_j \xi_j + 1} \,.\, w = 0,$$

with $m \in \mathbb{Z}$. We have:

Theorem 3.14 [Chari and Pressley 2001]. (1) *The $\widehat{\mathfrak{sl}}_2(\mathbb{C})$-module B (associated with $u_1, \dots, u_n$ and $\xi_1, \dots, \xi_n$ with $n > 0$) is finite-dimensional.*

(2) *If B' is any finite-dimensional $\widehat{\mathfrak{sl}}_2(\mathbb{C})$-module generated by an element w' such that* $\dim U(\alpha^\vee \otimes \mathbb{C}[t, t^{-1}])\, w' = 1$, *then B' is a quotient of some module B constructed as above.* □

Lemma 3.15. *If ψ is as in Theorem 3.8, then, for all $m \in \mathbb{Z}$, both $x_+(\sigma_m)$ and $x_-(\tau_m)$ are nilpotent on the generator v of $L(\psi)$, where $\sigma_m = -e_1 + 2me_2$ and $\tau_m = me_2$.*

Proof. As $L(\psi)$ is irreducible, it is enough to show that

$$\mathscr{L}(\Delta_+) \,.\, (x_+(\sigma_m))^N v = 0 \quad \text{and} \quad \mathscr{L}(\Delta_+) \,.\, (x_-(\tau_m))^N v = 0 \tag{3-11}$$

for some $N \gg 0$. By Lemma 3.9, $\mathscr{L}(\Delta_+) \,.\, (x_+(\sigma_m))^N v = 0$ is equivalent to

$$X_{1,n}(x_+(\sigma_m))^N v = 0, \tag{3-12}$$

$$X_{2,n}(x_+(\sigma_m))^N v = 0, \tag{3-13}$$

$$X_{3,n}(x_+(\sigma_m))^N v = 0. \tag{3-14}$$

It is easy to see that (3-12) and (3-14) hold for $N \geq 0$. To show (3-13), we set

$$x_n = x_+(\sigma_n), \quad y_n = x_-(-\sigma_{-n}), \quad h_n = \alpha^\vee(0, 2n) - 2C_1(0, 2n), \tag{3-15}$$

for $n \in \mathbb{Z}$. Noting that $C_2 = 0$ on $L(\psi)$, these vectors satisfy

$$[x_a, y_b] = h_{a+b}, \quad [h_c, x_a] = 2x_{c+a}, \quad [h_c, y_b] = -2y_{b+c}.$$

Hence, they form a basis for a loop algebra of type A_1. Denote this subalgebra by $\mathfrak{S}$. In $W(\psi)$, we consider the $\mathfrak{S}$-submodule generated by v. From Theorem 3.14, we know that $(x_+(\sigma_m))^N v$ belongs to a proper submodule of $U(\mathfrak{S})v$ for some $N \gg 0$. Applying the PBW Theorem to $W(\psi)$, we see that (3-13) holds. The proof that $\mathscr{L}(\Delta_+) \,.\, (x_-(\tau_m))^N v = 0$ is similar and is omitted. □

The following proposition gives the first part of Theorem 3.8.

Proposition 3.16. *For ψ as in Theorem 3.8, $L(\psi)$ is integrable as a $\mathscr{K}$-module.*

Proof. By applying Lemmas 3.13 and 3.15, we show that, with respect to $\mathfrak{h}_1$, the weight spaces of $L(\psi)$ are finite-dimensional.

Let ψ_1 be the restriction of ψ on $\mathfrak{h}_1$. Then, the weight set $P(L(\psi))$ is a subset of $\psi_1 - (\mathbb{Z}_+\alpha_0 + \mathbb{Z}_+\alpha)$. Consider any weight space $L(\psi)_{\psi_1 - \eta}$ with $\eta \in \mathbb{Z}_+\alpha_0 + \mathbb{Z}_+\alpha$. From applying the PBW Theorem to $L(\psi)$, the vector space $L(\psi)_{\psi_1 - \eta}$ is spanned by some vectors of the form

$$X(\beta_1, n_1)\, X(\beta_2, n_2) \ldots X(\beta_k, n_k)\, v, \tag{3-16}$$

where $X(\beta_i, n_i)$ is a root vector of $\mathscr{L}(\Delta_-)$ with root $\beta_i + n_i\delta_2$, and the β_i are negative affine roots satisfying $\sum \beta_i = -\eta$. For a fixed η, only finitely many β_i will appear. It suffices to show that, for fixed $\beta_1, \ldots, \beta_k$, the vectors of the form (3-16) span a finite-dimensional vector space.

As a subalgebra of $\mathscr{J}(S)$, the subspace $\mathscr{T} = \bigoplus_{s \in \mathbb{Z}} \mathbb{C} x^{se_2}$ is isomorphic to the Laurent polynomial ring $\mathbb{C}[t, t^{-1}]$. Define

$$p = \sum_{i=0}^{k} \epsilon_i x^{ie_2} = \prod_{j=1}^{k} (x^{e_2} - a_j) \quad \text{and} \quad q = \sum_{i=0}^{l} \epsilon_i' x^{2ie_2} = \prod_{j=1}^{l} (x^{2e_2} - b_j).$$

Let $s = pq$. We use P, Q and S to denote the ideals $p\mathscr{T}$, $q\mathscr{T}$ and $s\mathscr{T}$ of $\mathscr{T}$, respectively. Write $s = \sum_i \epsilon_i'' x^{ie_2}$. By using the definition of ψ, it is straightforward to check the following two identities:

$$\psi(\alpha^\vee \otimes S) = 0. \tag{3-17}$$

$$\psi\Big(\sum_{m=0}^{l} \epsilon'_m h_{m+n}\Big) = 0 \quad \text{for all } n \in \mathbb{Z} \text{ (see (3-15)).} \tag{3-18}$$

First, we show that, for any negative affine root β and all $m \in \mathbb{Z}$, we have $\sum_i \epsilon''_i X(\beta, m+i)v = 0$, where $X(\beta, m+i)$ is a root vector of $\mathscr{L}(\Delta_-)$ with root $\beta + (m+i)\delta_2$. We prove this by induction on the height of $-\beta$. When the height of $-\beta$ is 1, we need

$$\sum_i \epsilon''_i \big(x_- \otimes x^{(m+i)e_2}\big) \,.\, v = 0. \tag{3-19}$$

$$\sum_i \epsilon''_i \big(x_+ \otimes x^{-e_1+(m+i)e_2}\big) \,.\, v = 0. \tag{3-20}$$

Since $L(\psi)$ is irreducible, this is equivalent to both $\sum_i \epsilon''_i(x_- \otimes x^{(m+i)e_2}) \,.\, v$ and $\sum_i \epsilon''_i(x_+ \otimes x^{-e_1+(m+i)e_2}) \,.\, v$ being annihilated by $\mathscr{L}(\Delta_+)$. By Lemma 3.9, it is enough to check that they are annihilated by $X_{1,n}$, $X_{2,n}$ and $X_{3,n}$ for $n \in \mathbb{Z}$. Now, it is clear that

$$X_{2,n} \sum_i \epsilon''_i \big(x_- \otimes x^{(m+i)e_2}\big) \,.\, v = 0 \quad \text{and} \quad X_{3,n} \sum_i \epsilon''_i \big(x_- \otimes x^{(m+i)e_2}\big) \,.\, v = 0.$$

But, by (3-17) and using that $C_2 = 0$ on $L(\psi)$,

$$X_{1,n} \sum_i \epsilon''_i \big(x_- \otimes x^{(m+i)e_2}\big) \,.\, v = \alpha^\vee \otimes (x^{ne_2} s) \,.\, v = 0.$$

Similarly, we can prove (3-20). If the height of $-\beta$ is 2, then $\sum_i \epsilon''_i X(\beta, m+i)v$ is 0, as it is annihilated by $X_{i,n}$ for $i = 1, 2, 3$. Now, we assume that the height of $-\beta$ is 3. Then, $\beta = -\alpha - \delta_1$ or $\alpha - 2\delta_1$. In case $\beta = -\alpha - \delta_1$, one can easily see that

$$X_{j,n} \sum_i \epsilon''_i X(\beta, m+i)v = 0 \quad \text{for } j = 1, 2, 3.$$

So, $\sum_i \epsilon''_i X(\beta, m+i)v = 0$. In case $\beta = \alpha - 2\delta_1$,

$$X_{j,n} \sum_i \epsilon''_i X(\beta, m+i)v = 0 \quad \text{for } j = 1, 2.$$

Thus, $X_{3,n} \sum_i \epsilon''_i X(\beta, m+i)v = 0$ by (3-17) and (3-18). When the height of $-\beta$ is greater than 3, consider

$$X_{j,n} \sum_i \epsilon''_i X(\beta, m+i)v = \sum_i \epsilon''_i [X_{j,n}, X(\beta, m+i)] \,.\, v.$$

Clearly, the negative of the height decreases and hence it is zero by induction, as required.

For the fixed negative affine roots $\gamma_1, \dots, \gamma_l$ $(1 \le j \le l)$, we show that

$$\sum_i \epsilon_i'' X(\gamma_1, n_1) \dots X(\gamma_j, n+i) X(\gamma_{j+1}, n_{j+1}) \dots X(\gamma_l, n_l) \,.\, v = 0,$$

for all integers $n, n_1, \dots, n_l$, using induction on the height of $-(\gamma_{j+1} + \dots + \gamma_l)$. It is clear when $\beta_{j+1}, \dots, \beta_l$ are 0. Now, since

$$\begin{aligned}\sum_i \epsilon_i'' X(\gamma_1, n_1) \dots X(\gamma_j, n+i) X(\gamma_{j+1}, n_{j+1}) \dots X(\gamma_l, n_l) \,.\, v \\ = \sum_i \epsilon_i'' X(\gamma_1, n_1) \dots [X(\gamma_j, n+i), X(\gamma_{j+1}, n_{j+1})] \dots X(\gamma_l, n_l) \,.\, v \\ + \sum_i \epsilon_i'' X(\gamma_1, n_1) \dots X(\gamma_{j+1}, n_{j+1}) X(\gamma_j, n+i) \dots X(\gamma_l, n_l) \,.\, v,\end{aligned}$$

the terms on the right hand side are zero by induction.

Since $\dim(\mathcal{T}/S) < \infty$, for fixed $\beta_1, \dots, \beta_k$, the vectors of the form (3-16) span a finite-dimensional vector space. Therefore, we know that the weight spaces of $L(\psi)$ are finite-dimensional. This completes the proof of this proposition. □

The second part of Theorem 3.8 follows from the next proposition.

Proposition 3.17. *If $L(\psi)$ is integrable as a $\mathcal{K}$-module, with the action $C_2 = 0$, then ψ satisfies the conditions of Theorem 3.8.*

Proof. We consider the affine algebra $\mathfrak{T} = \mathfrak{sl}_2(\mathbb{C}) \otimes \mathcal{T} \oplus \mathbb{C}C_2$. Denote by V the irreducible quotient of $U(\mathfrak{T})v$ of $\mathfrak{T}$. We claim that $\dim V < \infty$. From the integrability of $L(\psi)$, the set

$$\{x_-(0, n) \,.\, v : n \in \mathbb{Z}\}$$

is linearly dependent. So, there exists some nonzero polynomial $f = \sum_i f_i x^{ie_2}$ such that $(x_- \otimes f)v = 0$. Set $F = f\mathcal{T}$. We have $(x_- \otimes F) . v = 0$ and $(\alpha^\vee \otimes F) . v = 0$. The first identity follows since

$$0 = \alpha^\vee(0, m)(x_- \otimes f)v = (x_- \otimes f)\alpha^\vee(0, m)v - 2(x_- \otimes x^{me_2} f)v$$

and $\alpha^\vee(0, m)$ acts on v as a constant. The second identity follows from the first.

It follows that $(\mathfrak{sl}_2(\mathbb{C}) \otimes F \oplus \mathbb{C}C_2) . v = 0$, and we show that $(\mathfrak{sl}_2(\mathbb{C}) \otimes F \oplus \mathbb{C}C_2) . V = 0$. In fact, if we define $W = \{w \in V : (\mathfrak{sl}_2(\mathbb{C}) \otimes F \oplus \mathbb{C}C_2) \,.\, w = 0\}$, then W is a nonzero submodule. Hence $V = W$, since V is irreducible. We deduce that V is an irreducible integrable module for $(\mathfrak{sl}_2(\mathbb{C}) \otimes \mathcal{T} \oplus \mathbb{C}C_2)/(\mathfrak{sl}_2(\mathbb{C}) \otimes F \oplus \mathbb{C}C_2)$. This implies that $\dim V < \infty$. Using Theorem 3.14, we can see that ψ satisfies the condition (3-6) of Theorem 3.8. Similarly, we can prove that ψ satisfies (3-7). □

4. The classification theorem

We classify the irreducible integrable modules for the extended baby TKK algebra $\mathcal{L}$ with actions $C_1 \neq 0$ and $C_2 = 0$.

Proposition 4.1. *If V is an irreducible integrable module for the extended baby TKK algebra $\mathscr{L}$ such that C_1 acts as a positive number and C_2 acts as zero, then V is a highest-weight module.*

Proof. By Lemma 2.8, we may assume that $2C_1$ acts on V as a positive integer, say $2c_1$.

First, we show that, for any fixed $\lambda \in P(V)$, there exists some $\lambda' \in P(V)$ such that $\lambda' + n\alpha$ is not a weight for any positive integer n, and that $\lambda'(d_i) = \lambda(d_i)$ for $i = 1, 2$.

Let $W = \{w \in V : d_i w = \lambda(d_i) w, i = 1, 2\}$. Write $P_1 = \{\mu \in P(V) : V_\mu \subset W\}$. Then, for any $\mu \in P_1$, we can write μ in the form

$$\mu = \bar{\mu} + \lambda(d_1)\delta_1 + \lambda(d_2)\delta_2 + c_1 w_1,$$

where $\bar{\mu} = \mu|_{\mathfrak{h}_0}$. Set $\bar{P}_1 = \{\bar{\mu} : \mu \in P_1\}$. Since W is an integrable module for the Lie subalgebra $\operatorname{span}_{\mathbb{C}}\{x_\pm, \alpha^\vee\}$, with finite-dimensional weight spaces with respect to $\mathfrak{h}_0 = \mathbb{C}\alpha^\vee$, it follows from Weyl's theorem that W can be decomposed as

$$W = \bigoplus_{\bar{\mu} \in \mathfrak{h}_0^*} V(\bar{\mu}),$$

where each $V(\bar{\mu})$ is an irreducible finite-dimensional module for $\operatorname{span}_{\mathbb{C}}\{x_\pm, \alpha^\vee\}$ with highest weight $\bar{\mu}$. Since V is irreducible, for any two weights μ, ν in P_1, we have $\mu - \nu = n\alpha$ for some integer n. Thus, $\bar{P}_1$ belongs to either $\mathbb{Z}\alpha$ or $\frac{1}{2}\alpha + \mathbb{Z}\alpha$. Set

$$\mu = \lambda(d_1)\delta_1 + \lambda(d_2)\delta_2 + c_1 w_1 \quad \text{if } \bar{P}_1 \subset \mathbb{Z}\alpha, \quad \text{or}$$
$$\mu = \tfrac{1}{2}\alpha + \lambda(d_1)\delta_1 + \lambda(d_2)\delta_2 + c_1 w_1 \quad \text{if } \bar{P}_1 \subset (1/2)\alpha + \mathbb{Z}\alpha.$$

By $\mathfrak{sl}_2(\mathbb{C})$-theory, we know that $\bar{\mu}$ is a common weight of the $V(\bar{\nu})$-terms that occur in $W = \bigoplus_{\bar{\nu} \in \mathfrak{h}_0^*} V(\bar{\nu})$. Since V_μ is finite-dimensional, P_1 is a finite set. Take $\lambda' \in P_1$ so that $\bar{\lambda'}(\alpha^\vee)$ is maximal. Then, λ' is the required weight.

Recall that $\{\alpha_0 = -\alpha + \delta_1, \alpha\}$ is a set of simple roots of the affine Kac–Moody Lie algebra

$$\widetilde{\mathfrak{sl}}_2(\mathbb{C}) = \Bigl(\mathfrak{sl}_2(\mathbb{C}) \otimes \Bigl(\sum_{j \in \mathbb{Z}} \mathbb{C}x^{je_1}\Bigr)\Bigr) \oplus \mathbb{C}C_1 \oplus \mathbb{C}d_1.$$

Define a partial order $\preceq$ on $\mathfrak{h}^*$ by setting

$$\lambda \preceq \mu \quad \text{if and only if} \quad \lambda - \mu = n_1\alpha_0 + n_2\alpha \ \text{ for some } n_1, n_2 \in -\mathbb{N}.$$

If λ' is as above and such that $\lambda' + n\alpha$ is not a weight for any positive integer n, then $\lambda'(\alpha^\vee) \geq 0$ by Lemma 2.8. Let $\Pi = \{\alpha + m\delta_1 : m \geq 0\} \cup \{-\alpha + m\delta_1 : m > 0\}$ be the set of positive real roots of $\widetilde{\mathfrak{sl}}_2(\mathbb{C})$, and $\Pi_{\lambda'} = \{\gamma \in \Pi : \lambda'(\gamma^\vee) \leq 0\}$. Since $\lambda'(C_1) > 0$, it follows that $\Pi_{\lambda'}$ is a finite set. Using a similar technique as in the proof of [Chari 1986, Thm 2.4], we get a nonzero weight vector $v \in V_{\lambda' + p\delta_1}$, $p \geq 0$, such that $\mathscr{L}_{r\delta_1} v = 0$ for all $r > 0$, and $\mathscr{L}_\beta v = 0$ for all but finitely many roots $\beta \in \Pi$.

Using an argument similar to the first paragraph of the proof of [Eswara Rao 2004, Prop 2.8], we obtain a weight $\mu \in P(V)$ such that

$$\mu + \eta \notin P(V) \text{ for all } \eta \not\leq 0. \tag{4-1}$$

In particular, $\mu + \beta \notin P(V)$ for all $\beta \in \Pi$.

By Lemma 2.8, we have $\mu(\beta^\vee) \geq 0$ for all $\beta \in \Pi$. In particular, $\mu(\alpha) \geq 0$. To prove that the module V has a highest-weight vector, we divide the argument into two cases: case 1, for $\mu(\alpha) > 0$, and case 2, for $\mu(\alpha) = 0$.

Case 1: Suppose that $\mu(\alpha) > 0$. If $\mu + \beta + m\delta_2 \notin P(V)$ for all integers m such that $\beta + m\delta_2 \in \Delta_+$, then it is clear that $\mathscr{L}(\Delta_+) . v = 0$ for any $0 \neq v \in V_\mu$, and we are done. On the other hand, assume that there exist some $\beta \in \Pi$ and $m_0 \in \mathbb{Z}$ such that $\beta + m_0\delta_2 \in \Delta_+$ and $V_{\mu+\beta+m_0\delta_2} \neq 0$. Let $\nu = \mu + \beta + m_0\delta_2$. We show that ν is a highest weight. That is, $V_{\nu+\gamma+k\delta_2} = 0$ for all $\gamma \in \Pi$ and all $k \in \mathbb{Z}$ such that $\gamma + k\delta_2 \in \Delta_+$. Suppose this is false. Then, $V_{\nu+\gamma+k_0\delta_2} \neq 0$ for some $\gamma \in \Pi$ and $k_0 \in \mathbb{Z}$ such that $\gamma + k_0\delta_2 \in \Delta_+$. Let $\gamma_1 = \beta + (m_0 + k_0)\delta_2$. We divide the argument into three subcases. In each subcase, we will get a contradiction with (4-1).

Subcase 1.1: Suppose $\beta, \gamma \in \{\alpha + m\delta_1 : m \geq 0\}$ or $\beta, \gamma \in \{-\alpha + m\delta_1 : m > 0\}$. We have $(\beta + \gamma)(\beta^\vee) > 0$ and $(\beta + \gamma)(\gamma^\vee) > 0$. If γ_1 is a root in Δ_+, then $(\nu + \gamma + k_0\delta_2)(\gamma_1^\vee) = (\mu + \beta + \gamma)(\beta^\vee) > 0$, which implies that

$$\mu + \gamma = (\nu + \gamma + k_0\delta_2) - \gamma_1 \in P(V),$$

which contradicts (4-1). If γ_1 is not a root, then we take $\gamma_1 - \delta_1$, which is obviously a root in Δ. Similar arguments show that $\mu + \gamma + \delta_1 \in P(V)$, contradicting (4-1) again.

Subcase 1.2: Suppose $\beta = \alpha + m\delta_1$ and $\gamma = -\alpha + n\delta_1$ for some $m \geq 0$ and $n > 0$. If $\gamma_1 \in \Delta_+$, then we have $(\mu + \beta + \gamma + (m_0 + k_0)\delta_2)(\gamma_1^\vee) = \mu(\beta^\vee) > 0$, which implies that

$$\mu + \gamma = (\mu + \beta + \gamma + (m_0 + k_0)\delta_2) - (\beta + (m_0 + k_0)\delta_2) \in P(V).$$

This contradicts (4-1). If $\gamma_1 \notin \Delta_+$, then $(\mu + \beta + \gamma + (m_0 + k_0)\delta_2)((\gamma_1 - \delta_1)^\vee) > 0$, which gives

$$\mu + \gamma + \delta_1 = (\mu + \beta + \gamma + (m_0 + k_0)\delta_2) - (\beta - \delta_1 + (m_0 + k_0)\delta_2) \in P(V).$$

This contradicts (4-1) again.

Subcase 1.3: Suppose $\beta = -\alpha + m\delta_1$ and $\gamma = \alpha + n\delta_1$ for some $m > 0$ and $n \geq 0$. This can be dealt with similarly to Subcase 1.2. This completes the proof of Case 1.

Case 2: Suppose now that $\mu(\alpha^\vee) = 0$. We assume that there exist some $\beta_0 \in \Pi$ and $t \in \mathbb{Z}$ such that $\beta_0 + t\delta_2 \in \Delta_+$ and $V_{\mu+\beta_0+t\delta_2} \neq 0$. Let $\mu_1 = \mu + \beta_0 + t\delta_2$.

If $\mu_1+\beta+m\delta_2 \notin P(V)$ for all integers m such that $\beta+m\delta_2 \in \Delta_+$, then, for any $0 \neq v \in V_{\mu_1}$, we have $\mathscr{L}(\Delta_+) \,.\, v = 0$ and we are done. On the other hand, we assume that there exist some $\beta' \in \Pi$ and $m_1 \in \mathbb{Z}$ such that $\beta'+m_1\delta_2 \in \Delta_+$ and $V_{\mu_1+\beta'+m_1\delta_2} \neq 0$. Let $\nu_1 = \mu_1+\beta'+m_1\delta_2$. We prove that ν_1 is a highest weight. That is, $V_{\nu_1+\gamma+k\delta_2} = 0$ for all $\gamma \in \Pi$ and all $k \in \mathbb{Z}$ such that $\gamma+k\delta_2 \in \Delta_+$. Suppose this is false. Then, $V_{\nu_1+\gamma'+k_1\delta_2} \neq 0$ for some $\gamma' \in \Pi$ and $k_1 \in \mathbb{Z}$ such that $\gamma'+k_1\delta_2 \in \Delta_+$. Let $\gamma_2 = \beta'+(t+m_1+k_1)\delta_2$. We divide the arguments into four subcases. In each subcase, we will get a contradiction with (4-1).

<u>Subcase 2.1</u>: Suppose $\beta', \gamma' \in \{\alpha+m\delta_1 : m \geq 0\}$. In this case, $(\beta'+\gamma')(\beta'^{\vee}) > 0$ and $(\beta'+\gamma')(\gamma'^{\vee}) > 0$. If γ_2 is a root in Δ_+, then

$$(\nu_1+\gamma'+k_1\delta_2)(\gamma_2^{\vee}) = (\mu+\beta_0+\beta'+\gamma')(\beta'^{\vee}) > 0,$$

which implies that

$$\mu+\beta_0+\gamma' = (\nu_1+\gamma'+k_1\delta_2)-\gamma_2 \in P(V).$$

If $\beta_0 \in \{-\alpha+m\delta_1 : m > 0\}$, then we arrive at a contradiction with (4-1). If $\beta_0 \in \{\alpha+m\delta_1 : m \geq 0\}$, then $(\mu+\beta_0+\gamma')(\gamma'^{\vee}) > 0$, which means that $\mu+\beta_0 \in P(V)$ — a contradiction again. If γ_2 is not a root, then we take $\gamma_2-\delta_1$, which is a root in Δ. Similar arguments give a contradiction with (4-1).

<u>Subcase 2.2</u>: Suppose $\beta', \gamma' \in \{-\alpha+m\delta_1 : m\rangle 0\}$. This is very similar to the arguments for Subcase 2.1.

<u>Subcase 2.3</u>: Suppose $\beta' = \alpha+m'\delta_1$ and $\gamma' = -\alpha+n'\delta_1$ for some $m' \geq 0$ and $n' > 0$. We have these two subcases:

<u>Subcase 2.3.1</u>: Suppose $\beta_0 \in \{\alpha+m\delta_1 : m \geq 0\}$. If $\gamma_2 \in \Delta_+$, then

$$(\mu+\beta_0+\beta'+\gamma'+(t+m_1+k_1)\delta_2)(\gamma_2^{\vee}) = (\mu+\beta_0+\beta'+\gamma')(\beta'^{\vee}) > 0.$$

This implies that $\mu+\beta_0+\gamma' \in P(V)$, which is impossible by (4-1). If $\gamma_2 \notin \Delta_+$, we consider $\gamma_2-\delta_1 \in \Delta_+$. Then,

$$\begin{aligned}(\mu+\beta_0+\beta'+\gamma'+(t+m_1+k_1)\delta_2)&((\gamma_2-\delta_1)^{\vee})\\ &= (\mu+\beta_0+\beta'+\gamma')((\beta'-\delta_1)^{\vee}) > 0.\end{aligned}$$

This implies that $\mu+\beta_0+\gamma'+\delta_1 \in P(V)$, which is also impossible.

<u>Subcase 2.3.2</u>: Suppose $\beta_0 \in \{-\alpha+m\delta_1 : m > 0\}$. We denote $\gamma_3 = \gamma'+(t+m_1+k_1)\delta_2$. If $\gamma_3 \in \Delta_+$, then

$$(\mu+\beta_0+\beta'+\gamma'+(t+m_1+k_1)\delta_2)(\gamma_3^{\vee}) = (\mu+\beta_0+\beta'+\gamma')(\gamma'^{\vee}) > 0.$$

So we have $\mu+\beta_0+\beta' \in P(V)$, which is impossible. If $\gamma_3 \notin \Delta_+$, then

$$(\mu+\beta_0+\beta'+\gamma'+(t+m_1+k_1)\delta_2)((\gamma_3-\delta_1)^\vee) \\ =(\mu+\beta_0+\beta'+\gamma')((-\alpha+(n'-1)\delta_1)^\vee)>0.$$

We get $\mu+\beta_0+\beta'+\delta_1 \in P(V)$, which is a contradiction.

Subcase 2.4: Finally, suppose $\beta'=-\alpha+m'\delta_1$ and $\gamma'=\alpha+n'\delta_1$ for some $m'>0$ and $n' \geq 0$. This can be discussed similarly to Subcase 2.3, and thus completes the proof of Case 2.

In every case, there exists some weight vector, say $v \in V$, such that $\mathscr{L}(\Delta_+).v=0$. Therefore, V is a highest-weight module for $\mathscr{L}$. □

Lemma 4.2 [Eswara Rao 2001]. *Any $\mathbb{Z}$-graded simple commutative and associative algebra, with all its homogeneous subspaces finite-dimensional, is isomorphic to a subalgebra $A_{\bar\psi}$ of $\mathbb{C}[t,t^{-1}]$ for some $\bar\psi$ (as defined by* (3-1)*). Furthermore, every nonzero homogeneous element in $A_{\bar\psi}$ is invertible in $A_{\bar\psi}$.* □

Theorem 4.3. *Let V be an irreducible integrable module for the extended baby TKK algebra $\mathscr{L}$ such that C_1 acts as a positive number and C_2 acts as zero. Then, V is isomorphic to $V(\bar\psi)$, for some $\bar\psi$ given in Section 3, such that $A_{\bar\psi}$ is an irreducible $\mathscr{L}(\Delta_0)$-module.*

Proof. By Proposition 4.1, there exists some nonzero weight vector $v \in V$ such that $\mathscr{L}(\Delta_+) \,.\, v=0$. Let M be the $\mathscr{L}(\Delta_0)$-module generated by v. In fact,

$$M=\{w \in V : \mathscr{L}(\Delta_+) \,.\, w=0\}$$

and M is irreducible as an $\mathscr{L}(\Delta_0)$-module by the irreducibility of V. Let $I=\{X \in U(H) : X \,.\, v=0\}$. It is clear that $M \cong U(H)/I$ as $\mathscr{L}(\Delta_0)$-modules. Since $U(H)/(U(H)C_2)$ is commutative and I is an ideal of $U(H)$, we see that $U(H)/I$ is a $\mathbb{Z}$-graded simple commutative and associative algebra. By Lemma 4.2, M is isomorphic to some $A_{\bar\psi}$. It is now clear that V is isomorphic to $V(\bar\psi)$. □

In view of Proposition 4.1, we have:

Corollary 4.4. *If V is an irreducible integrable module for the extended baby TKK algebra $\mathscr{L}$ with $C_1<0$ and $C_2=0$, then V is a lowest-weight module.* □

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DUALITY PROPERTIES FOR QUANTUM GROUPS

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Some duality properties for induced representations of enveloping algebras involve the character $\mathrm{Trad}_{\mathfrak{g}}$. We extend them to deformation Hopf algebras A_h of a noetherian Hopf k-algebra A_0 satisfying $\mathrm{Ext}^i_{A_0}(k, A_0) = \{0\}$ except for $i = d$ where it is isomorphic to k. These duality properties involve the character of A_h defined by right multiplication on the one-dimensional free $k[[h]]$-module $\mathrm{Ext}^d_{A_h}(k[[h]], A_h)$. In the case of quantized enveloping algebras, this character lifts the character $\mathrm{Trad}_{\mathfrak{g}}$. We also prove Poincaré duality for such deformation Hopf algebras in the case where $k[[h]]$ is an A_h-module of finite projective dimension. We explain the relation of our construction with quantum duality.

1. Introduction

Let k be a field of characteristic 0 and set $K = k[[h]]$. Let A_0 be a noetherian algebra. Assume k has a left A_0-module structure such that, for some integer d,

$$\begin{cases} \mathrm{Ext}^i_{A_0}(k, A_0) = \{0\} & \text{if } i \neq d, \\ \mathrm{Ext}^d_{A_0}(k, A_0) \simeq k. \end{cases}$$

It follows from Poincaré duality that any finite-dimensional Lie algebra $\mathfrak{g}$ verifies these assumptions. In this case, $d = \dim \mathfrak{g}$ and the character defined by the right representation of $U(\mathfrak{g})$ on $\mathrm{Ext}^{\dim \mathfrak{g}}_{U(\mathfrak{g})}(k, U(\mathfrak{g}))$ is $\mathrm{Trad}_{\mathfrak{g}}$ [Chemla 1994]. The algebra of regular functions on an affine algebraic Poisson group and the algebra of formal power series also satisfy these hypothesis. Let A_h be a deformation algebra of A_0. Assume that there exists an A_h-module structure on K that reduces modulo h to the A_0-module structure we started with. The main theorem of the paper constructs a new character of A_h that will be denoted by θ_{A_h}.

Theorem 4.1. *With the assumptions made above*:

(a) $\mathrm{Ext}^i_{A_h}(K, A_h) = \{0\}$ *if* $i \neq d$.

(b) $\mathrm{Ext}^d_{A_h}(K, A_h)$ *is a free K-module of dimension one. The right A_h-module structure given by right multiplication lifts that of* A_0 *on* $\mathrm{Ext}^d_{A_0}(k, A_0)$.

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The right A_h-module $\operatorname{Ext}^d_{A_h}(K, A_h)$ will be denoted by Ω_{A_h}. If there is an ambiguity, the integer d will be written d_{A_h}.

Theorem 4.1 applies to universal quantum enveloping algebras, quantization of affine algebraic Poisson groups and quantum formal series Hopf algebras.

Let $\mathfrak{g}$ be a Lie bialgebra. Denote by $F[\mathfrak{g}]$ the formal series Poisson algebra $U(\mathfrak{g})^*$. If $F_h[\mathfrak{g}]$ is a quantum formal series algebra such that $F_h[\mathfrak{g}]/hF_h[\mathfrak{g}]$ is isomorphic to $F[\mathfrak{g}]$ as a Poisson Hopf algebra, we construct a resolution of the trivial $F_h[\mathfrak{g}]$-module that lifts the Koszul resolution of the trivial $F[\mathfrak{g}]$-module k and that behaves well with respect to quantum duality [Drinfeld 1987, Gavarini 2002]. This construction is not explicit, but it allows us to show that if $F_h[\mathfrak{g}]$ and $U_h(\mathfrak{g}^*)$ are linked by quantum duality, the relation $\theta_{F_h[\mathfrak{g}]} = h\theta_{U_h(\mathfrak{g}^*)}$ holds.

As an application of Theorem 4.1, we show Poincaré duality:

Theorem 7.1. *We make the same assumptions as above. Let M be an A_h-module. Assume that K is an A_h-module of finite projective dimension. For all integers i, the K-modules $\operatorname{Ext}^i_{A_h}(K, M)$ and $\operatorname{Tor}^{A_h}_{d_{A_h}-i}(\Omega_{A_h}, M)$ are isomorphic.*

Convention. From now on, we assume that A_h is a deformation Hopf algebra.

Brown and Levasseur [1985] and Kempf [1991] showed that, in the semisimple context, the Ext-dual of a Verma module is a Verma module. In [Chemla 1994] we extended this result to the Ext-dual of an induced representation of any Lie superalgebra. In this article, we show that this result can be generalized to quantum groups provided that the quantization is functorial. Such a quantization has been constructed in [Etingof and Kazhdan 1996, 1998a, 1998b, Etingof and Schiffmann 2002]. As the result holds for quantized universal enveloping algebras, for quantized functions algebras and for quantum formal series Hopf algebras, we state it in the more general setting of Hopf algebras.

Corollary 7.3. *Let A_h and B_h be topological Hopf deformations of A_0 and B_0, respectively. We assume that there exists a morphism of Hopf algebras from B_h to A_h such that A_h is a flat B_h^{op}-module. We also assume that B_h satisfies the condition of the Theorem 4.1. Let V be a B_h-module which is a free finite-dimensional K-module. Then, if S_h denotes the antipode of B_h, one has:*

(a) $\operatorname{Ext}^i_{A_h}(A_h \underset{B_h}{\otimes} V, A_h)$ *is* $\{0\}$ *if i is different from d_{B_h}.*

(b) *The right A_h-module* $\operatorname{Ext}^{d_{B_h}}_{A_h}(A_h \otimes_{B_h} V, A_h)$ *is isomorphic to* $(\Omega_{B_h} \otimes V^*) \otimes_{B_h} A_h$, *where $\Omega_{B_h} \otimes V^*$ is endowed with the right B_h-module structure given by*

$$(\omega \otimes f) \cdot u = \lim_{n \to +\infty} \sum_j \theta_{B_h}(u'_{j,n})\omega \otimes f \cdot S_h^2(u''_{j,n})$$

and $\Delta(u) = \lim_{n \to +\infty} \sum_j u'_{j,n} \otimes u''_{j,n}$, *for all $u \in B_h$, all $f \in V^*$, and all $\omega \in \Omega_{B_h}$.*

Proposition 7.4. *Let A_h be a Hopf deformation of A_0, B_h be a Hopf deformation of B_0 and C_h be a Hopf deformation of C_0. We assume that there exists a morphism of Hopf algebras from B_h to A_h and a morphism of Hopf algebras from C_h to A_h such that A_h is a flat B_h^{op}-module and a flat C_h^{op}-module. We also assume that B_h and C_h satisfies the hypothesis of Theorem 4.1. Let V (respectively W) be a B_h-module (respectively C_h-module) which is a free finite dimensional K-module. Then, for all integers n, there is an isomorphism*

$$\operatorname{Ext}_{A_h}^{n+d_{B_h}}\Big(A_h \underset{B_h}{\otimes} V,\, A_h \underset{C_h}{\otimes} W\Big) \simeq \operatorname{Ext}_{A_h}^{n+d_{C_h}}\Big((\Omega_{C_h} \otimes W^*) \underset{C_h}{\otimes} A_h,\, (\Omega_{B_h} \otimes V^*) \underset{B_h}{\otimes} A_h\Big).$$

The right B_h-module structure on $\Omega_{B_h} \otimes V^*$ and the C_h-module structure on $\Omega_{C_h} \otimes W^*$ are as in Corollary 7.3.

Remarks. Proposition 7.4 was already known in the case where $\mathfrak{g}$ is a Lie algebra, $\mathfrak{h}$ and $\mathfrak{k}$ are Lie subalgebras of $\mathfrak{g}$, and A_h, B_h, C_h are the corresponding enveloping algebras. In this case, $d_{B_h} = \dim \mathfrak{h}$ and $d_{C_h} = \dim \mathfrak{k}$. More precisely, Boe and Collingwood [1985] and Gyoja [2000], generalizing a result of G. Zuckerman, proved a part of this theorem (the case where $\mathfrak{h} = \mathfrak{g}$ and $n = \dim \mathfrak{h} = \dim \mathfrak{k}$) under the assumptions that $\mathfrak{g}$ is split semisimple and $\mathfrak{h}$ is a parabolic subalgebra of $\mathfrak{g}$. In [Collingwood and Shelton 1990], such a duality is also proved in a slightly different context (but still under the semisimple hypothesis).

M. Duflo [1987] proved Proposition 7.4 for a $\mathfrak{g}$ general Lie algebra, $\mathfrak{h} = \mathfrak{k}$, $V = W^*$ being one-dimensional representations.

Proposition 7.4 is proved in full generality in the context of Lie superalgebras in [Chemla 1994].

Wet set $A_h^e = A_h \otimes A_h^{op}$. Using the properties of a Hopf algebra [Chemla 2004], we show that all the $\operatorname{Ext}^i_{\widehat{A_h^e}}(A_h, A_h \widehat{\otimes_{k[[h]]}} A_h)$'s are zero except one. More precisely:

Proposition 7.5. *Assume that A_h satisfies the conditions of Theorem 4.1. Assume moreover that $A_0 \otimes A_0^{op}$ is noetherian. Consider $A_h \widehat{\otimes_{k[[h]]}} A_h$ with the following $\widehat{A_h^e}$-module structure: for any α, β, x, y in A_h, $\alpha \cdot (x \otimes y) \cdot \beta = \alpha x \otimes y\beta$.*

(a) *$HH^i_{A_h}(A_h \widehat{\otimes_{k[[h]]}} A_h)$ is zero if $i \neq d_{A_h}$.*

(b) *The $\widehat{A_h^e}$-module $HH^{d_{A_h}}_{A_h}(A_h \widehat{\otimes_{k[[h]]}} A_h)$ is isomorphic to $\Omega_{A_h} \otimes A_h$ with the following $\widehat{A_h^e}$-module structure: for any α, β, x in A_h,*

$$\alpha \cdot (\omega \otimes x) \cdot \beta = \omega \theta_{A_h}(\beta_i') \otimes S(\beta_i'') x S^{-1}(\alpha), \quad \textit{where } \beta = \sum_i \beta_i' \otimes \beta_i''.$$

This result has already been obtained in [Dolgushev and Etingof 2005] for a deformation of the algebra of regular functions on a smooth algebraic affine variety. From Proposition 7.5, as in [van den Bergh 1998], we deduce a duality between Hochschild homology and Hochschild cohomology.

Organization of the paper. In Section 2, we gather all the necessary results about decreasing filtrations, and in Section 3, we recall some basic facts about deformation algebras. The main theorem of the paper, Theorem 4.1, is stated, proved and illustrated by examples in Section 4. In Section 5, we study the behavior of the character θ_{F_h} with respect to quantum duality. Section 6 is devoted to the study of an example. In Section 7, we give applications of our main theorem.

Our study of algebras endowed with a decreasing filtration and filtered modules over such algebras relies on the use of the associated graded algebra and graded module, and on topological arguments. We apply this study to deformation algebras endowed with the h-adic filtration and filtered modules over such algebras. In [Kashiwara and Schapira 2008], a study of the derived category of A_h-modules is carried out using the right derived functor of the functor $M \mapsto M/(hM)$.

2. Decreasing filtrations

In this section, we give results about decreasing filtrations. These results are proved in [Schneiders 1994] in the framework of increasing filtrations. Most of our proofs are obtained by adjusting those of Schneiders.

Let $GA = \bigoplus_{t\in\mathbb{Z}} G_t A$ be a $\mathbb{Z}$-graded algebra. Let $GM = \bigoplus_{t\in\mathbb{Z}} G_t M$ and $GN = \bigoplus_{t\in\mathbb{Z}} G_t N$ be two graded GA-modules. A morphism of graded GA-modules from GM to GN is a morphism of GA-modules $f : GM \to GN$, such that $f(G_t M) \subset G_t N$. The group of morphisms of graded GA-modules from GM to GN will be denoted by $\mathrm{Hom}_{GA}(GM, GN)$.

For $r \in \mathbb{Z}$ and a graded GA-module GM, define the shifted graded GA-module $GM(r)$ to be the GA-module GM with the grading defined by $G_t M(r) = G_{t+r} M$. Denote by $\underline{\mathrm{Hom}}_{GA}(GM, GN)$ the graded group defined by setting

$$G_t \underline{\mathrm{Hom}}_{GA}(GM, GN) = \mathrm{Hom}_{GA}(GM, GN(t)).$$

The i-th right derived functor of the functor $\underline{\mathrm{Hom}}_{GA}(-, N)$ will be denoted by $\underline{\mathrm{Ext}}^i_{GA}(-, N)$.

A graded GA-module GL is finite free if there are integers $d_1, \dots, d_n$ such that

$$GL \simeq \bigoplus_{i=1}^{n} GA(-d_i).$$

A graded GA-module GM is of finite type if there exists a finite free graded GA-module GL and an exact sequence in the category of graded GA-modules $GL \to GM \to 0$.

A graded ring GA is noetherian if any graded GA-submodule of a graded GA-module of finite type is of finite type.

Henceforth, all the GA-modules we consider will be graded, so we refer to graded GA-modules simply as GA-modules.

We are now going to consider a k-algebra endowed with a decreasing filtration $\cdots \subset F_{t+1}A \subset F_tA \subset \cdots \subset F_1A \subset F_0A = A$. The order of an element a, $o(a)$, is the biggest t such that $a \in F_tA$. The principal symbol of a is the image of a in $F_{o(a)}/F_{o(a)+1}$. It will be denoted by $[a]$.

A filtered module over FA is the data of an A-module M and a family $(F_tM)_{t\in\mathbb{Z}}$ of k-subspaces, such that

$$\bigcup_{t\in\mathbb{Z}} F_tM = M, \quad F_{t+1}M \subset F_tM, \quad F_tA \cdot F_lM \subset F_{t+l}M.$$

We will assume that $F_tM = M$ for $t << 0$. The principal symbol of an element of M is defined. We endow such a module with the topology for which a basis of neighborhoods is $(F_tM)_{t\in\mathbb{Z}}$. The topological space M is Hausdorff if and only if $\bigcap_{t\in\mathbb{Z}} F_tM = \{0\}$. If M is Hausdorff, the topology defined by the filtration is that of the metric given by

$$d(x, y) = \|x - y\| = 2^{-\sup\{j\in\mathbb{Z}\,|\,x-y\in F_jM\}} \quad \text{for all } (x, y) \in FM.$$

Example. Let k be a field and set $K = k[[h]]$. If V is a K-module, it is endowed with the following decreasing filtration $\cdots \subset h^nV \subset h^{n-1}V \subset \cdots \subset hV \subset V$. The topology induced by this filtration is the h-adic topology.

Lemma 2.1 [Schwartz 1986, page 245]. *Let N be a Hausdorff filtered module. Let P be a submodule of N which is closed in N. Let p be the canonical projection from N to N/P.*

(a) *The topology defined by the filtration $p(F_tN)$ on N/P is the quotient topology. N/P is Hausdorff and its topology is defined by the distance*

$$d(\bar{x}, \bar{y}) = \|\bar{x} - \bar{y}\|, \quad \textit{where } \|\bar{x}\| = \inf\{\|a\|, a \in \bar{x}\}.$$

(b) *If N is complete, then N/P is complete for the quotient topology.*

Let FM and FN be two filtered FA-modules. $Fu : FM \to FN$, a filtered morphism, is a morphism $u : M \to N$ of the underlying A-modules, such that $u(F_tM) \subset F_tN$. It is continuous if we endow M and N with the topology defined by the filtrations. Denote the morphism $u_{|F_tM} : F_tM \to F_tN$ by F_tu. Denote the group of filtered morphisms from FM to FN by $\mathrm{Hom}_{FA}(FM, FN)$. The kernel of Fu is the kernel of u filtered by the family $\mathrm{Ker}\,Fu \cap F_tM$. If M is complete and N is Hausdorff, then $\mathrm{Ker}\,Fu$, endowed with the induced topology is complete.

A graded ring $GA = \bigoplus_{t\in\mathbb{N}} F_tA/F_{t+1}A$ is associated to a filtered ring FA. A graded GA-module $GM = \bigoplus_{t\in\mathbb{Z}} F_tM/F_{t+1}M$ is associated to a filtered FA-module FM. If x is in F_tM, we will write $\sigma_t(x)$ for the class of x in $F_tM/F_{t+1}M$. We will denote by $Gu : GM \to GN$ the morphism of GA-modules induced by Fu.

An arrow $Fu : FM \to FN$ is strict if it satisfies

$$u(F_t M) = u(M) \cap F_t N.$$

An exact sequence of FA-modules is a sequence $FM \xrightarrow{Fu} FN \xrightarrow{Fv} FP$, such that $\operatorname{Ker} F_t v = \operatorname{Im} F_t u$. It follows from this definition that Fu is strict. If, moreover, Fv is strict, we say that it is a strict exact sequence.

Proposition 2.2. (a) *Consider $Fu : FM \to FN$ and $Fv : FN \to FP$ two filtered FA-morphisms such that $Fv \circ Fu = 0$. If the sequence $FM \xrightarrow{Fu} FN \xrightarrow{Fv} FP$ is strict exact, then $GM \xrightarrow{Gu} GN \xrightarrow{Gv} GP$ is exact.*

(b) *Conversely, assume that FM is complete for the topology defined by the filtration and FN is Hausdorff for the topology defined by the filtration. If the sequence $GM \xrightarrow{Gu} GN \xrightarrow{Gv} GP$ is exact, then the sequence $FM \xrightarrow{Fu} FN \xrightarrow{Fv} FP$ is strict exact.*

Corollary 2.3. *Let FA be a filtered k-algebra and let FM and FN be two FA-modules. Let $Fu : FM \to FN$ be a morphism of FA-modules. Then it follows that $G \operatorname{Ker} Fu \subset \operatorname{Ker} GFu$ and $\operatorname{Im} GFu \subset G \operatorname{Im} Fu$. Assume moreover that FM is complete and FN is Hausdorff. Then the following conditions are equivalent:*

(a) *Fu is strict.*

(b) *$G \operatorname{Ker} Fu = \operatorname{Ker} GFu$.*

(c) *$\operatorname{Im} GFu = G \operatorname{Im} Fu$.*

Proposition 2.4. *Let $(M^\bullet, d^\bullet)$ be a complex of complete FA-modules. $H^i(M^\bullet)$ is filtered as follows:*

$$F_t H^i(M^\bullet) = \frac{\operatorname{Ker} d_i \cap F_t M^i + \operatorname{Im} d_{i-1}}{\operatorname{Im} d_{i-1}} \simeq \frac{\operatorname{Ker} d_i \cap F_t M^i}{\operatorname{Im} d_{i-1} \cap F_t M^{i-1}}.$$

If d_i and d_{i-1} are strict, then $GH^i(M^\bullet)$ is isomorphic to $H^i(GM^\bullet)$

Remark. The isomorphism from $G_t H^i(M^\bullet)$ to $H^i(G_t M^\bullet)$ associates $\operatorname{cl}(\sigma_t(x))$ to $\sigma_t \operatorname{cl}(x)$.

For any $r \in \mathbb{Z}$ and for any FA-module FM, we define the shifted module $FM(r)$ as the module M endowed with the filtration $(F_{t+r} M)_{t \in \mathbb{Z}}$.

An FA-module module is finite free if it is isomorphic to an FA-module of the type $\bigoplus_{i=1}^p FA(-d_i)$, where $d_1, \dots, d_p$ are integers. An FA-module FM is of finite type if there exists a strict epimorphism $FL \to FM$, where FL is a finite free FA-module. This means that we can find $m_1 \in F_{d_1} M, \dots, m_p \in F_{d_p} M$, such that any $m \in F_d M$ may be written as

$$m = \sum_{i=1}^p a_{d-d_i} m_i, \quad \text{where } a_{d-d_i} \in F_{d-d_i} A.$$

Proposition 2.5. *Let FA be a filtered k-algebra and FM be an FA-module.*

(a) *If FM is an FA-module of finite type generated by $(s_1, \dots, s_r)$, then GM is a GA-module of finite type generated by $([s_1], \dots, [s_r])$. Conversely, assume that FA is complete for the topology given by the filtration, and FM is an FA-module which is Hausdorff for the topology defined by the filtration. If GM is a GA-module of finite type generated by $([s_1], \dots, [s_r])$, then FM is an FA-module of finite type generated by $(s_1, \dots, s_r)$.*

(b) *If FM is a finite free FA-module, then GM is a finite free GA-module. Conversely, assume that FA is complete for the topology given by the filtration, and FM is an FA-module that is Hausdorff for the topology defined by the filtration. If GM is a finite free GA-module, then FM is a finite free FA-module.*

Definition 2.6. A filtered k-algebra is said to be (filtered) noetherian if it satisfies one of the following equivalent conditions:

- Any filtered submodule (not necessarily a strict submodule) of a finite-type FA-module is of finite type.
- Any filtered ideal (not necessarily a strict ideal) of FA is of finite type.

Proposition 2.7. *Let FA be a filtered complete k-algebra and GA its associated graded algebra. If GA is graded noetherian, then FA is filtered noetherian.*

Proof of Proposition 2.7. We assume that GA is a noetherian algebra. We need to prove that a filtered submodule FM' of a finitely generated FA-module FM is finitely generated.

First we assume that FM is Hausdorff. For this case, the proof is identical to that of [Schneiders 1994].

We no longer assume that FM is Hausdorff. As FM is a finite-type FA-module, there exists a strict exact sequence

$$FL = \bigoplus_{i=1}^{n} FA(-d_i) \xrightarrow{p} FM \to 0.$$

We may apply the first case to the submodule of FL, $p^{-1}(FM')$, endowed with the filtration

$$F_t[p^{-1}(M')] = p_t^{-1}(F_t M') = p^{-1}(F_t M') \cap F_t L.$$

The general case follows easily. □

Proposition 2.8. *Assume that FA is noetherian for the topology given by the filtration. Any FA-module of finite type has an infinite resolution by finite free FA-modules.*

Remark. The sequence $\cdots \to GL_s \to GL_{s-1} \to \cdots \to GL_0 \to GM \to 0$ is a resolution of the GA-module GM for such a resolution of FM.

Proposition 2.9. *Assume FA is noetherian and complete. If FN is a finite-type FA-module, then it is complete.*

Proof of Proposition 2.9. Assume that FN is Hausdorff. Let FN be a finite-type Hausdorff FA-module. We have $FL = \bigoplus_{i=1}^{n} FA(-d_i) \xrightarrow{p} FN \to 0$, a strict exact sequence. The filtration on FN is given by $p(F_tL)$. Let us endow the kernel K of p with the induced topology. We have $0 \to FK \to FL \to FN \to 0$, a strict exact sequence. As N is Hausdorff, $K = p^{-1}(\{0\})$ is closed in FL. The filtered FA-module FN is isomorphic to FL/K, endowed with the quotient topology. Hence, FN is complete (see Lemma 2.1).

We no longer assume that FN is Hausdorff. From the first case, FK, endowed with the induced topology, is complete and therefore closed in FL. We have $FN \simeq FL/K$, so the FA-module FN is Hausdorff. □

Remark. Proposition 2.9 is proved in [Kashiwara and Schapira 2008] in the case of an A_h-module (A_h being a deformation algebra) endowed with the h-adic filtration.

3. Deformation algebras

In this section k will be a field of characteristic 0 and we will set $K = k[\![h]\!]$.

Definition 3.1. A topologically free K-algebra A_h is a topologically free K-module together with a K-bilinear (multiplication) map $A_h \times A_h \to A_h$, making A_h into an associative algebra.

Let A_0 be an associative k-algebra. A deformation of A_0 is a topologically free K-algebra A_h such that $A_0 \simeq A_h/hA_h$ as algebras.

Remark. If A_h is a deformation algebra of A_0, we may endow it with the h-adic filtration. We then have

$$GA_h = \bigoplus_{i\in\mathbb{N}} \frac{h^i A_h}{h^{i+1} A_h} \simeq A_0[h]$$

as $k[h]$-algebras. From Proposition 2.7, we deduce that a deformation algebra of a noetherian algebra is noetherian.

Definition 3.2. A deformation of a Hopf algebra $(A, \iota, \mu, \epsilon, \Delta, S)$ over a field k is a topological Hopf algebra $(A_h, \iota_h, \mu_h, \epsilon_h, \Delta_h, S_h)$ over the ring $k[\![h]\!]$, such that

(i) A_h is isomorphic to $A_0[\![h]\!]$ as a $k[\![h]\!]$-module, and

(ii) A_h/hA_h is isomorphic to A_0 as a Hopf algebra.

Example 3.3 (QUEA: quantized universal enveloping algebras). Let $\mathfrak{g}$ be a Lie bialgebra. A Hopf algebra deformation of $U(\mathfrak{g})$, $U_h(\mathfrak{g})$, such that $U_h(\mathfrak{g})/(hU_h(\mathfrak{g}))$ is isomorphic to $U(\mathfrak{g})$ as a coPoisson Hopf algebra, is called a quantization of $U(\mathfrak{g})$.

Quantizations of Lie bialgebras have been constructed in [Etingof and Kazhdan 1996].

Example 3.4 (quantization of affine algebraic Poisson groups). A quantization of an affine algebraic Poisson group $(G, \{,\})$ is a Hopf algebra deformation $\mathcal{F}_h(G)$ of the Hopf algebra $\mathcal{F}(G)$ of regular functions on G, such that $\mathcal{F}_h(G)/(h\mathcal{F}_h(G))$ is isomorphic to $(\mathcal{F}(G), \{\,,\})$ as a Poisson Hopf algebra.

Etingof and Kazhdan [1998b] have constructed quantizations of affine algebraic Poisson groups. (See also [Chari and Pressley 1994] for the case of G simple.)

Example 3.5 (QFSHA: quantum formal series Hopf algebras). The vector space dual $U(\mathfrak{g})^*$ of the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra can be identified with an algebra of formal power series and has a natural Hopf algebra structure, provided we interpret the tensor product $U(\mathfrak{g})^* \otimes U(\mathfrak{g})^*$ in a suitable, completed sense. If $\mathfrak{g}$ is a Lie bialgebra, then $U(\mathfrak{g})^*$ is a Hopf Poisson algebra.

A quantum formal series Hopf algebra is a topological Hopf algebra B_h over $k[\![h]\!]$, such that $B_h/(hB_h)$ is isomorphic to $U(\mathfrak{g})^*$ as a topological Poisson Hopf algebra, for some finite-dimensional Lie bialgebra.

Proposition 3.6 [Kashiwara and Schapira 2008, Theorem 2.6]. *Let A_h be a deformation algebra of A_0 and let M be an A_h-module. If*

(i) *M has no h-torsion,*

(ii) *$M/(hM)$ is a flat A_0-module, and*

(iii) $M = \varprojlim_n M/(h^n M)$,

then M is a flat A_h-module.

4. A quantization of the character trad

Theorem 4.1. *Let A_0 be a noetherian k-algebra and let A_h be a deformation of A_0. Assume that k has a left A_0-module structure such that there exists an integer d, such that*

$$\begin{cases} \operatorname{Ext}^i_{A_0}(k, A_0) = \{0\} & \text{if } i \neq d, \\ \operatorname{Ext}^d_{A_0}(k, A_0) \simeq k. \end{cases}$$

Assume that K is endowed with an A_h-module structure, which reduces modulo h to the A_0-module structure on k that we started with. Then:

(a) $\operatorname{Ext}^i_{A_h}(K, A_h)$ *is zero if $i \neq d$.*

(b) $\mathrm{Ext}^d_{A_h}(K, A_h)$ *is a free* K*-module of dimension* 1, *and a right* A_h*-module under right multiplication. It is a lift of the right* A_0*-module structure* (*given by right multiplication*) *on* $\mathrm{Ext}^d_{A_0}(k, A_0)$.

Notation. We denote by Ω_{A_h} the right A_h-module $\mathrm{Ext}^d_{A_h}(k, A_h)$,i and by and the character defined by this action θ_{A_h}.

Remark. Kashiwara and Schapira [2008, Section 6] make a similar construction in the setup of DQ-algebroids. In [Chemla 2004], it is shown that a result similar to Theorem 4.1 holds for $U_q(\mathfrak{g})$ ($\mathfrak{g}$ semisimple).

Example 4.2. Poincaré duality gives us the following result for any finite dimensional Lie algebra.

$$\begin{cases} \mathrm{Ext}^i_{U(\mathfrak{g})}(k, U(\mathfrak{g})) = \{0\} & \text{if } i \neq 0, \\ \mathrm{Ext}^{\dim \mathfrak{g}}_{U(\mathfrak{g})}(k, U(\mathfrak{g})) \simeq \Lambda^{\dim \mathfrak{g}}(\mathfrak{g}^*). \end{cases}$$

The character defined by the right action of $U(\mathfrak{g})$ on $\mathrm{Ext}^{\dim \mathfrak{g}}_{U(\mathfrak{g})}(k, U(\mathfrak{g}))$ is $\mathrm{trad}_{\mathfrak{g}}$ [Chemla 1994]. Thus, the character defined by Theorem 4.1 is a quantization of the character $\mathrm{trad}_{\mathfrak{g}}$.

- If $\mathfrak{g}$ is a complex semisimple algebra, as $H^1(\mathfrak{g}, k) = \{0\}$ [Hilton and Stammbach 1997, page 247], there exists a unique lift of the trivial representation of $U_h(\mathfrak{g})$, hence the representation $\Omega_{U_h(\mathfrak{g})}$ is the trivial representation.
- Let $\mathfrak{a}$ be a k-Lie algebra. Denote by $\mathfrak{a}_h$ the Lie algebra obtained from $\mathfrak{a}$ by multiplying the bracket of $\mathfrak{a}$ by h. Thus, it is true that for any elements X and Y of $\mathfrak{a}_h \simeq \mathfrak{a}$, one has $[X, Y]_{\mathfrak{a}_h} = h[X, Y]_{\mathfrak{a}}$. Denote by $\widehat{U(\mathfrak{a}_h)}$ the h-adic completion of $U(\mathfrak{a}_h)$. Then $\widehat{U(\mathfrak{a}_h)}$ is a Hopf deformation of $(\mathfrak{a}^{ab}, \delta = 0)$. The character $\theta_{\widehat{U(\mathfrak{a}_h)}}$ defined by the theorem in this case is given by

  $$\theta_{\widehat{U(\mathfrak{a}_h)}}(X) = h\,\mathrm{trad}_{\mathfrak{a}}(X) \quad \text{for all } X \in \mathfrak{a}.$$

  Thus, even if $\mathfrak{g}$ is unimodular, the character defined by the right action of $U_h(\mathfrak{g})$ on $\Omega_{U_h(\mathfrak{g})} \simeq \bigwedge^{\dim \mathfrak{g}}(\mathfrak{g}^*)[[h]]$ might not be trivial.
- We consider the following Lie algebra: $\mathfrak{a} = \bigoplus_{i=1}^5 ke_i$ with nonzero bracket $[e_2, e_4] = e_1$. Consider $k[[h]]$-Lie algebra structure on $\mathfrak{a}[[h]]$ defined by the nonzero brackets $[e_3, e_5] = he_3$ and $[e_2, e_4] = 2e_1$. Then $\widehat{U(\mathfrak{a}[[h]])}$ is a quantization of $U(\mathfrak{a})$. It is easy to see that

  $$\theta_{\widehat{U(\mathfrak{a}[[h]])}}(e_i) = \begin{cases} 0 & \text{if } i \neq 5, \\ -h & \text{if } i = 5. \end{cases}$$

Example 4.3. Theorem 4.1 also applies to quantization of affine algebraic Poisson groups. If G is an affine algebraic Poisson group with neutral element e, we take

k to be given by the counit of the Hopf algebra $\mathcal{F}(G)$. By [Altman and Kleiman 1970], we have $\mathrm{Ext}^i_{\mathcal{F}(G)}(k, \mathcal{F}(G)) = \{0\}$ if $i \neq \dim G$, while

$$\mathrm{Ext}^{\dim G}_{\mathcal{F}(G)}(k, \mathcal{F}(G)) \simeq \textstyle\bigwedge^{\dim G}(\mathcal{M}_e/\mathcal{M}_e^2)^*, \quad \text{where } \mathcal{M}_e = \{f \in \mathcal{F}(G) \mid f(e) = 0\}.$$

Let $\mathfrak{g}$ be a real Lie algebra. The algebra of regular functions on $\mathfrak{g}^*$, $\mathcal{F}(\mathfrak{g}^*)$, is isomorphic to $S(\mathfrak{g})$ and is naturally equipped with a Poisson structure given by the following: if X and Y are in $\mathfrak{g}$, then $\{X, Y\} = [X, Y]$. In the example above, $\widehat{U(\mathfrak{g}_h)}$ is a quantization of the Poisson algebra $\mathcal{F}(\mathfrak{g}^*)$. $\mathcal{F}(\mathfrak{g}^*)$ acts trivially on $\mathrm{Ext}^{\dim \mathfrak{g}}_{\mathcal{F}(\mathfrak{g}^*)}(k, \mathcal{F}(\mathfrak{g}^*))$, whereas the action of $\mathcal{F}_h(\mathfrak{g}^*) \simeq \widehat{U(\mathfrak{g}_h)}$ on $\mathrm{Ext}^{\dim \mathfrak{g}}_{\mathcal{F}_h(\mathfrak{g}^*)}(k, \mathcal{F}_h(\mathfrak{g}^*))$ is not trivial.

Example 4.4. Theorem 4.1 also applies to quantum formal series Hopf algebras.

Proof of Theorem 4.1. Let us consider a resolution of the A_h-module K by filtered finite free A_h-modules

$$\cdots \xrightarrow{\partial_{i+1}} FL^i \xrightarrow{\partial_i} \cdots \xrightarrow{\partial_2} FL^1 \xrightarrow{\partial_1} FL^0 \to K \to \{0\},$$

with $FL^i = \bigoplus_{k=1}^{d_i} FA_h(-m_{j,i})$, so that the graded complex

$$\ldots GL^i \xrightarrow{G\partial_i} \cdots \to GL^1 \xrightarrow{G\partial_1} GL^0 \to k[h] \to \{0\}$$

is a resolution of the $A_0[h]$-module $k[h]$. Consider the complex

$$M^\bullet = (\mathrm{Hom}_{A_h}(L^\bullet, A_h),\ {}^t\partial_\bullet).$$

Recall that there is a natural filtration on $\mathrm{Hom}_{A_h}(L^i, A_h)$ defined by

$$F_t \mathrm{Hom}_{A_h}(L^i, A_h) = \{\lambda \in \mathrm{Hom}_{A_h}(L^i, A_h) \mid \lambda(F_p L^i) \subset F_{t+p} A_h\}.$$

One has an isomorphism of right FA-modules $F\,\mathrm{Hom}_{A_h}(L^i, A_h) = \bigoplus_{j=1}^{d_i} FA(m_{j,i})$. Hence,

$$GF\,\mathrm{Hom}_{A_h}(L^i, A_h) \simeq \underline{\mathrm{Hom}}_{GA_h}(GL^i, GA_h),$$

and the complex $\underline{\mathrm{Hom}}_{GA_h}(GL^i, GA_h)$ computes $\underline{\mathrm{Ext}}^i_{GA_h}(k[h], GA_h)$. We have the following isomorphisms of right $A_0[h]$-modules.

$$\underline{\mathrm{Ext}}^i_{GA_h}(k[h], GA_h) \simeq \underline{\mathrm{Ext}}^i_{A_0[h]}(k[h], A_0[h]) \simeq \mathrm{Ext}^i_{A_0}(k, A_0)[h].$$

If $i \neq d$, then $\underline{\mathrm{Ext}}^i_{GA_h}(k[h], GA_h) = \{0\}$. This means that the sequence

$$\underline{\mathrm{Hom}}_{GA}(GL_{i-1}, GA_h) \xrightarrow{{}^tG\partial_i} \underline{\mathrm{Hom}}_{GA}(GL_i, GA_h) \xrightarrow{{}^tG\partial_{i+1}} \underline{\mathrm{Hom}}_{GA}(GL_{i+1}, GA_h)$$

is an exact sequence of GA_h-modules. Applying Proposition 2.2, the sequence

$$F\,\mathrm{Hom}_{FA}(FL_{i-1}, FN) \xrightarrow{{}^t\partial_i} F\,\mathrm{Hom}_{FA}(FL_i, FN) \xrightarrow{{}^t\partial_{i+1}} F\,\mathrm{Hom}_{FA}(FL_{i+1}, FN)$$

is strict exact. As FL_i is finite free, the underlying module of $F\operatorname{Hom}_{FA}(FL_i, FN)$ is $\operatorname{Hom}_A(L_i, N)$. Hence, we have proved that $\operatorname{Ext}^i_{A_h}(K, A_h) = \{0\}$ if $i \neq d$.

We have also proved that all the maps ${}^t\partial_i$ are strict. Hence, by Proposition 2.4, we have

$$G\operatorname{Ext}^i_{A_h}(k[\![h]\!], A_h) \simeq \underline{\operatorname{Ext}}^i_{GA_h}(k[h], A_0[h]) \simeq \operatorname{Ext}^i_{A_0}(k, A_0)[h],$$

for all integers i. The FA_h-modules $\operatorname{Ext}^i_{A_h}(K, A_h)$ are finite-type FA-modules. They are therefore Hausdorff, in fact, they are even complete (Proposition 2.9). As $\operatorname{Ext}^d_{A_h}(K, A_h)$ is Hausdorff and $G\operatorname{Ext}^d_{A_h}(k[\![h]\!], A_h) \simeq \operatorname{Ext}^d_{A_0}(k, A_0)[h]$, the $k[\![h]\!]$-module $\operatorname{Ext}^d_{A_h}(K, A_h)$ is one-dimensional. This finishes the proof. □

From now on, we assume that A_h is a topological Hopf algebra and that its action on K is given by the counit. The antipode of A_h will be denoted by S_h.

If V is a left A_h-module, we define the right A_h-module V^r by

$$v \cdot_{S_h} a = S_h(a) \cdot v \quad \text{for all } a \in A_h \text{ and } v \in V,$$

and the right A_h-module V^ρ by

$$v \cdot_{S_h^{-1}} a = S_h^{-1}(a) \cdot v \quad \text{for all } a \in A_h \text{ and } v \in V.$$

Similarly, if W is a right A_h-module, we define the left A_h-module W^l by

$$a \cdot_{S_h} w = w \cdot S_h(a) \quad \text{for all } a \in A_h \text{ and } w \in W,$$

and the left A_h-module W^λ by

$$a \cdot_{S_h^{-1}} w = w \cdot S_h^{-1}(a) \quad \text{for all } a \in A_h \text{ and } w \in W.$$

One has $(V^r)^\lambda = V$, $(V^\rho)^l = V$, $(W^l)^\rho = W$ and $(W^\lambda)^r = W$. Thus, we have defined two (in the case where $S_h^2 \neq \mathrm{id}$) equivalences of categories between the category of left A_h-modules and the category of right A_h-modules, that is, left A_h^{op}-modules.

Let $\operatorname{Mod}(A_h)$ be the abelian category of left A_h-modules and $D(\operatorname{Mod}(A_h))$ be the derived category of the abelian category $\operatorname{Mod}(A_h)$. We may consider A_h as an $A_h \otimes A_h^{op}$-module. Introduce a functor D_{A_h} from $D(\operatorname{Mod}(A_h))$ to $D(Mod(A_h^{op}))$ by setting

$$D_{A_h}(M^\bullet) = R\operatorname{Hom}_{A_h}(M^\bullet, A_h) \quad \text{for all } M^\bullet \in D(A_h).$$

If M is a finitely generated module, the canonical arrow $M \to D_{A_h^{op}} \circ D_{A_h}(M)$ is an isomorphism.

Let V be a left A_h-module. Then, by transposition, $V^* = \operatorname{Hom}_K(V, K)$ is naturally endowed with a right A_h-module structure. Using the antipode, we can also see it as a left module structure. Thus, one has

$$u \cdot f = f \cdot S_h(u) \quad \text{for all } u \in A_h \text{ and } f \in V^*.$$

We endow $\Omega_{A_h} \otimes V^*$ with the right A_h-module structure given by

$$(\omega \otimes f) \cdot u = \lim_{n \to +\infty} \sum_j \theta_{A_h}(u'_{j,n}) \omega \otimes f \cdot S_h^2(u''_{j,n})$$

and $\Delta(u) = \lim_{n\to+\infty} \sum_j u'_{j,n} \otimes u''_{j,n}$, for all $u \in A_h$, all $f \in V^*$, and all $\omega \in \Omega_{A_h}$.

Theorem 4.5. *Let V be an A_h-module free of finite type as a $k[\![h]\!]$-module. Then $D_{A_h}(V)$ and $\Omega_{A_h} \otimes V^*$ are isomorphic in $D\left(A_h^{op}\right)$.*

To prove the theorem, we need the following lemma [Duflo 1982; Chemla 1994]:

Lemma 4.6. *Let W be a left A_h-module. $A_h \widehat{\otimes} W$ is endowed with two different $(A_h \otimes A_h^{op})$-module structures, as follows. Set*

$$\Delta(a) = \lim_{n \to +\infty} \sum_i a'_{i,n} \otimes a''_{i,n} \quad \text{for } a \in A_h. \tag{4-1}$$

The first structure, denoted by $(A_h \widehat{\otimes} W)_1$, is given by

$$(u \otimes w) \cdot a = ua \otimes w \quad \text{and} \quad a \cdot (u \otimes w) = \lim_{n \to +\infty} \sum_i a'_{i,n} u \otimes a''_{i,n} \cdot w,$$

where $w \in W$ and $u, a \in A_h$. The second structure, denoted by $(A_h \widehat{\otimes} W)_2$, is given by

$$a \cdot (u \otimes w) = au \otimes w \quad \text{and} \quad (u \otimes w) \cdot a = \lim_{n \to +\infty} \sum_i u a'_{i,n} \otimes S_h(a''_{i,n}) \cdot w.$$

The $A_h \otimes A_h^{op}$-modules $(A_h \widehat{\otimes} W)_1$ and $(A_h \widehat{\otimes} W)_2$ are isomorphic.

Proof of Lemma 4.6. The map $\Psi : (A_h \widehat{\otimes} W)_2 \to (A_h \widehat{\otimes} W)_1$ given by

$$u \otimes w \mapsto \lim_{n \to +\infty} \sum_i u'_{i,n} \otimes u''_{i,n} \cdot w,$$

with Δ as in (4-1), is an isomorphism of $A_h \otimes A_h^{op}$-modules from $(A_h \widehat{\otimes} W)_2$ to $(A_h \widehat{\otimes} W)_1$. Moreover, $\Psi^{-1}(u \otimes w) = \sum u'_{i,n} \otimes S_h(u''_{i,n}) \cdot w$. ☐

Proof of Theorem 4.5. Let $L^\bullet$ be a resolution of K by free A_h-modules. We endow $L^i \otimes V$ with the following left A_h-module structure:

$$a \cdot (l \otimes v) = \lim_{n \to +\infty} \sum_i a'_{i,n} \cdot l \otimes a''_{i,n} \cdot v.$$

Then $L^\bullet \otimes V$ is a resolution of V by free A_h-modules. Using the relation

$$a \cdot l \otimes v = \lim_{n \to +\infty} \sum_i a'_{i,n} \left(l \otimes S_h(a''_{i,n}) \cdot v \right),$$

one shows the sequence of A_h-isomorphisms

$$\begin{aligned} D_{A_h}(V) &\simeq \operatorname{Hom}_{A_h}(L\otimes V, A_h) \simeq \operatorname{Hom}_{A_h}\big(L, (A_h\otimes V^*)_1\big) \\ &\simeq \operatorname{Hom}_{A_h}\big(L, (A_h\otimes V^*)_2\big) \simeq R\operatorname{Hom}_{A_h}(K, A_h)\otimes V^*. \qquad \square \end{aligned}$$

5. Link with quantum duality

Review of the quantum dual principle [Drinfeld 1987, Gavarini 2002]. There are two functors,

$$(\ )' : \mathrm{QUEA} \to \mathrm{QFSA} \quad \text{and} \quad (\ )^\vee : \mathrm{QFSA} \to \mathrm{QUEA},$$

which are inverse to each other. If $U_h(\mathfrak{g})$ is a quantization of $U(\mathfrak{g})$ and $F_h[\![\mathfrak{g}]\!]$ is a quantization of $F[\![\mathfrak{g}]\!] = U(\mathfrak{g})^*$, then $U_h(\mathfrak{g})'$ is a quantization of $F[\![\mathfrak{g}^*]\!]$ and $F_h[\![\mathfrak{g}]\!]^\vee$ is a quantization of $U(\mathfrak{g}^*)$. We recall the construction of the functor $(\ )^\vee$, which is the one we will need. Let $\mathfrak{g}$ be a Lie bialgebra and $F_h[\![\mathfrak{g}]\!]$ a quantization of $F[\![\mathfrak{g}]\!] = U(\mathfrak{g})^*$. For simplicity we will write F_h instead of $F_h[\![\mathfrak{g}]\!]$. If ϵ_h denotes the counit of F_h, set $I := \epsilon_h^{-1}(hk[\![h]\!])$ and $J = \operatorname{Ker}\epsilon_h$. Let

$$F_h^\times := \sum_{n\geq 0} h^{-n} I^n = \sum_{n\geq 0} (h^{-1}I)^n = \bigcup_{n\geq 0} (h^{-1}I)^n$$

be the $k[\![h]\!]$-subalgebra of $k((h))\otimes_{k[\![h]\!]} F_h$ generated by $h^{-1}I$. As $I = J + hF_h$, one has

$$F_h^\times = \sum_{n\geq 0} h^{-n} J^n.$$

Define $F_h^\vee$ to be the h-adic completion of the $k[\![h]\!]$-module $F_h^\times$. The Hopf algebra structure on F_h induces a Hopf algebra structure on $F_h^\vee$. A precise description of $F_h^\vee$ is given in [Gavarini 2002]. The algebras F_h/hF_h and $k[\![\bar{x}_1, \dots, \bar{x}_n]\!]$ are isomorphic. We denote $\pi : F_h \to F_h/hF_h$ be the natural projection. We may choose $x_j \in \pi^{-1}(\bar{x}_j)$ for any j, such that $\epsilon_h(x_j) = 0$. Then F_h and $k[\![x_1, \dots, x_n, h]\!]$ are isomorphic as $k[\![h]\!]$-topological modules and J is the set of formal series f whose degree in the x_j, $\partial_X(f)$ (that is, the degree of the lowest-degree monomials occurring in the series with nonzero coefficients) is strictly positive. As F_h/hF_h is commutative, one has $x_ix_j - x_jx_i = h\chi_{i,j}$ with $\chi_{i,j} \in F_h$. Since $\chi_{i,j}$ is in J, it can be written as

$$\chi_{i,j} = \sum_{a=1}^{n} c_a(h)x_a + f_{i,j}(x_1, \dots, x_n, h), \quad \text{with } \partial_X(f_{i,j}) > 1.$$

If $\check{x}_i = h^{-1}x_j$, then

$$F_h^\vee = \left\{ f = \sum_{r\in\mathbb{N}} P_r(\check{x}_1, \dots, \check{x}_n)h^r \;\middle|\; P_r(X_1, \dots, X_n) \in k[X_1, \dots, X_n] \right\}.$$

The topological $k[\![h]\!]$-modules $F_h^\vee$ and $k[\check{x}_1, \dots, \check{x}_n][\![h]\!]$ are isomorphic. One has

$$\check{x}_i\check{x}_j - \check{x}_j\check{x}_i = \sum_{a=1}^{n} c_a(h)\check{x}_a + h^{-1}\check{f}_{i,j}(\check{x}_1, \dots, \check{x}_n, h),$$

where $\check{f}_{i,j}(\check{x}_1, \dots, \check{x}_n, h)$ is obtained from $f_{i,j}(x_1, \dots, x_n)$ by writing $x_j = h\check{x}_j$. The element $h^{-1}\check{f}_{i,j}(\check{x}_1, \dots, \check{x}_n, h)$ is in $hk[\check{x}_1, \dots, \check{x}_n][\![h]\!]$ (as $\partial_X(f_{i,j}) > 1$). The k-span of the set of cosets $\{e_i = \check{x}_i \bmod hF_h^\vee\}$ is a Lie algebra isomorphic to $\mathfrak{g}^*$, and the map $\Psi : F_h^\vee \to U(\mathfrak{g}^*)[\![h]\!]$ defined by

$$\Psi\Bigl(\sum_{r\in\mathbb{N}} P_r(\check{x}_1, \dots, \check{x}_n)h^r\Bigr) = \sum_{r\in\mathbb{N}} P_r(e_1, \dots, e_n)h^r$$

is an isomorphism of topological $k[\![h]\!]$-modules. Denote by $\cdot_h$ multiplication on F_h and its transposition to $U(\mathfrak{g}^*)[\![h]\!]$ by Ψ. If u and v are in $U(\mathfrak{g}^*)$, one writes $u \cdot_h v = \sum_{r\in\mathbb{N}} h^r \mu_r(u, v)$. One knows that the first nonzero μ_r is a 1-cocycle of the Hochschild cohomology.

If P in $k[X_1, \dots, X_n]$ can be written $P = \sum_{i_1,\dots,i_n} a_{i_1,\dots,i_n} X_1^{i_1} \dots X_n^{i_n}$ and $g \in k[X_1, \dots, X_n][\![h]\!]$ can be written $g = \sum_{i=1}^{r} P_r(X_1, \dots, X_r)h^r$, then one sets

$$P^{\otimes}(e_1, \dots, e_n) = \sum_{i_1,\dots,i_n} a_{i_1,\dots,i_n} e_1^{\otimes i_1} \dots e_n^{\otimes i_n} \in T_k\Bigl(\bigoplus_{i=1}^{n} ke_i\Bigr),$$

$$g^{\otimes}(e_1, \dots, e_n) = \sum_{i=1}^{r} P_r^{\otimes}(e_1, \dots, e_r)h^r.$$

$(F_h)^\vee$ is isomorphic as an algebra to

$$U_h(\mathfrak{g}^*) \simeq \frac{T_{k[\![h]\!]}\Bigl(\bigoplus_{i=1}^{n} k[\![h]\!]e_i\Bigr)}{I},$$

where I is the closure (in the h-adic topology) of the two sided ideal generated by the relations

$$e_i \otimes e_j - e_j \otimes e_i = \sum_{k=1}^{n} c_k(h)e_k + h^{-1}\check{f}_{i,j}^{\otimes}(e_1, \dots, e_n, h).$$

Quantum duality and deformation of the Koszul complex. We may construct resolutions of the trivial $F_h[\mathfrak{g}]$ and $F_h[\mathfrak{g}]^\vee$-modules that respect the quantum duality.

Theorem 5.1. *Let $\mathfrak{g}$ be a Lie bialgebra, $F_h[\mathfrak{g}]$ a QFSHA such that $F_h[\mathfrak{g}]/(hF_h[\mathfrak{g}])$ is isomorphic to $F[\mathfrak{g}]$ as a topological Poisson Hopf algebra and $F_h[\mathfrak{g}]^\vee = U_h(\mathfrak{g}^*)$, the quantization of $U(\mathfrak{g}^*)$ constructed from $F_h[\mathfrak{g}]$ by the quantum duality principle. Let $\bar{x}_1, \dots, \bar{x}_n$ be elements of $F[\mathfrak{g}]$ such that $F[\mathfrak{g}] \simeq k[\![\bar{x}_1, \dots, \bar{x}_n]\!]$. Choose $x_1, \dots, x_n$, elements of $F_h[\mathfrak{g}]$, such that $x_i = \bar{x}_i \bmod h$ and $\epsilon_h(x_i) = 0$. Then*

$U_h(\mathfrak{g}^*) \simeq k[\check{x}_1, \dots, \check{x}_n]\llbracket h \rrbracket$ with $\check{x}_i = h^{-1}x_i$. Let $(\epsilon_1, \dots, \epsilon_n)$ be a basis of $\mathfrak{g}^*$ and $C^a_{i,j}$ the structural constants of $\mathfrak{g}^*$ with respect to this basis. We can construct a resolution of the trivial $F_h[\mathfrak{g}]$-module $K^h_\bullet = \big(F_h[\mathfrak{g}] \otimes \bigwedge \mathfrak{g}^*, \partial^h_q\big)$ of the form

$$\begin{aligned}
&\partial^h_q(1 \otimes \epsilon_{p_1} \wedge \cdots \wedge \epsilon_{p_q}) \\
&\quad = \sum_{i=1}^{q} (-1)^{i-1} x_i \otimes \epsilon_{p_1} \wedge \cdots \wedge \widehat{\epsilon_{p_i}} \wedge \cdots \wedge \epsilon_{p_q} \\
&\qquad + \sum_{r<s} \sum_{a} (-1)^{r+s} h C^a_{p_r,p_s} 1 \otimes \epsilon_a \wedge \epsilon_{p_1} \wedge \cdots \wedge \widehat{\epsilon_{p_r}} \wedge \cdots \wedge \widehat{\epsilon_{p_s}} \wedge \cdots \wedge \epsilon_{p_q} \\
&\qquad + \sum_{t_1,\dots,t_{q-1}} h\alpha^{t_1,\dots,t_{q-1}}_{p_1,\dots,p_q} \otimes \epsilon_{t_1} \wedge \cdots \wedge \epsilon_{t_{q-1}},
\end{aligned}$$

such that $\alpha^{t_1,\dots,t_{q-1}}_{p_1,\dots,p_q} \in I = \epsilon_h^{-1}(hk\llbracket h \rrbracket)$. Set

$$\check{\alpha}^{t_1,\dots,t_{q-1}}_{p_1,\dots,p_q}(\check{x}_1, \dots, \check{x}_n) = \alpha^{t_1,\dots,t_{q-1}}_{p_1,\dots,p_q}(x_1, \dots, x_n).$$

$\check{\alpha}^{t_1,\dots,t_{q-1}}_{p_1,\dots,p_q}$ is in $hk[\check{x}_1, \dots, \check{x}_n]\llbracket h \rrbracket$. Now define the morphism of $U_h(\mathfrak{g}^*)$-modules $\check{\partial}^h_q : U_h(\mathfrak{g}^*) \otimes \bigwedge^q(\mathfrak{g}^*) \to U_h(\mathfrak{g}^*) \otimes \bigwedge^{q-1}(\mathfrak{g}^*)$ by

$$\begin{aligned}
&\check{\partial}^h_q(1 \otimes \epsilon_{p_1} \wedge \cdots \wedge \epsilon_{p_q}) \\
&\quad = \sum_{i=1}^{n} (-1)^{i-1} \check{x}_i \otimes \epsilon_{p_1} \wedge \cdots \wedge \widehat{\epsilon_{p_i}} \wedge \cdots \wedge \epsilon_{p_q} \\
&\qquad + \sum_{r<s} \sum_{a} (-1)^{r+s} C^a_{p_r,p_s} 1 \otimes \epsilon_a \wedge \epsilon_{p_1} \wedge \cdots \wedge \widehat{\epsilon_{p_r}} \wedge \cdots \wedge \widehat{\epsilon_{p_s}} \wedge \cdots \wedge \epsilon_{p_q} \\
&\qquad + \sum_{t_1,\dots,t_{q-1}} \check{\alpha}^{t_1,\dots,t_{q-1}}_{p_1,\dots,p_q} \otimes \epsilon_{t_1} \wedge \cdots \wedge \epsilon_{t_{q-1}}.
\end{aligned}$$

Then $\check{K}^\bullet_h = (U_h(\mathfrak{g}^*) \otimes \bigwedge^\bullet \mathfrak{g}^*, \check{\partial}^h_q)$ is a resolution of the trivial $U_h(\mathfrak{g}^*)$-module $k\llbracket h \rrbracket$.

Proof of Theorem 5.1. One sets $x_i x_j - x_j x_i = \sum_{a=1}^n hC^a_{i,j} x_a + hu^a_{i,j} x_a$. We know that $u^a_{i,j}$ is in I. Take $\partial^h_0 = \epsilon_h$, $\partial^h_1(1 \otimes \epsilon_i) = x_i$. Set

$$\partial^h_2(1 \otimes \epsilon_i \wedge \epsilon_j) = x_i \otimes \epsilon_j - x_j \otimes \epsilon_i - \sum_a hC^a_{i,j} \otimes \epsilon_a - h \sum_a u^a_{i,j} \otimes \epsilon_a.$$

We have $\partial^h_1 \circ \partial^h_2 = 0$ and we may choose $\alpha^a_{i,j} = u^a_{i,j}$.

Assume that $\partial^h_0, \partial^h_1, \dots, \partial^h_q$ have been constructed such that

- $\partial^h_{r-1} \partial^h_r = 0$ for all $r \in [1, q]$;
- $\operatorname{Im} \partial^h_r = \operatorname{Ker} \partial^h_{r-1}$ for all $r \in [1, q]$ (and the required relations are satisfied);
- $\alpha^{q_1,\dots,q_{r-1}}_{p_1,p_2,\dots,p_r} \in I$.

Let us show that we can construct ∂^h_{q+1} satisfying these three conditions.

A computation [Knapp 1988, page 173] shows that

$$\begin{aligned}
&\partial^h_q\Bigl(\sum_{i=1}^{q+1}(-1)^{i-1}x_{p_i}\otimes\epsilon_{p_1}\wedge\cdots\wedge\widehat{\epsilon_{p_i}}\wedge\cdots\wedge\epsilon_{p_{q+1}}\Bigr)\\
&\quad+\partial^h_q\Bigl(\sum_{k<l}\sum_a(-1)^{k+l}hC^a_{p_k,p_l}1\otimes\epsilon_a\wedge\epsilon_{p_1}\wedge\cdots\wedge\widehat{\epsilon_{p_k}}\wedge\cdots\wedge\widehat{\epsilon_{p_l}}\wedge\cdots\wedge\epsilon_{p_{q+1}}\Bigr)\\
&=\sum_{j<i}(-1)^{i+j}\Bigl(x_{p_i}x_{p_j}-x_{p_j}x_{p_i}-\sum_a hC^a_{p_i,p_j}x_a\Bigr)\otimes\epsilon_1\wedge\cdots\wedge\widehat{\epsilon_{p_j}}\wedge\cdots\wedge\widehat{\epsilon_{p_i}}\wedge\cdots\wedge\epsilon_{p_{q+1}}\\
&\quad+\sum_i(-1)^{i-1}hx_{p_i}\alpha_{p_1,\dots,\widehat{p_i},\dots,p_{q+1}}+\sum_{r<s}(-1)^{r+s}h^2C^a_{p_r,p_s}\alpha_{a,p_1,\dots,\widehat{p_r},\dots,\widehat{p_l},\dots,p_{q+1}}.
\end{aligned}$$

Modulo h, this expression is zero. Since $\partial^h_{q-1}\partial^h_q$, vanishes, this same expression is in $h\,\mathrm{Ker}\,\partial^h_{q-1}=h\,\mathrm{Im}\,\partial^h_q$. Hence it equals $-\partial^h_q(h\alpha^{t_1,\dots,t_q}_{p_1,\dots,p_{q+1}})$, for of an appropriate choice of $\alpha^{t_1,\dots,t_q}_{p_1,\dots,p_{q+1}}$ in $F_h[\mathfrak{g}]$.

We prove that $\alpha^{t_1,\dots,t_q}_{p_1,\dots,p_{q+1}}$ is in I. Clearly, $-\partial^h_q(h\alpha^{t_1,\dots,t_q}_{p_1,\dots,p_{q+1}}\otimes\epsilon_{t_1}\wedge\cdots\wedge\epsilon_{t_q})$ is an element of $I^3\otimes\bigwedge^q\mathfrak{g}^*$. Note that ∂^h_q sends $I^r\otimes\bigwedge^q\mathfrak{g}^*$ to $I^{r+1}\otimes\bigwedge^q\mathfrak{g}^*$. Let us write

$$\alpha^{t_1,\dots,t_q}_{p_1,\dots,p_{q+1}}=\sum_{i_1,\dots,i_n}(\alpha^{t_1,\dots,t_q}_{p_1,\dots,p_{q+1}})_{i_1,\dots,i_n}x_1^{i_1}\dots x_n^{i_n},$$

with $(\alpha^{t_1,\dots,t_q}_{p_1,\dots,p_{q+1}})_{i_1,\dots,i_n}$ in $k[\![h]\!]$. From the remarks just made, we see that

$$\partial^h_q\Bigl(h\sum_{t_1,\dots,t_q}(\alpha^{t_1,\dots,t_q}_{p_1,\dots,p_{q+1}})_{0,\dots,0}\epsilon_{t_1}\wedge\cdots\wedge\epsilon_{t_q}\Bigr)\in I^3\otimes\textstyle\bigwedge^q\mathfrak{g}^*.$$

Hence, $(\alpha^{t_1,\dots,t_q}_{p_1,\dots,p_{q+1}})_{0,\dots,0}$ is in $hk[\![h]\!]$.

Since $\mathrm{Im}\,G\partial^h_{q+1}=\mathrm{Ker}\,G\partial^h_q$, one has $\mathrm{Im}\,\partial^h_{q+1}=\mathrm{Ker}\,\partial^h_q$.

Set $\check{\alpha}^{t_1,\dots,t_{q-1}}_{p_1,\dots,p_q}(\check{x}_1,\dots,\check{x}_n)=\alpha^{t_1,\dots,t_{q-1}}_{p_1,\dots,p_q}(x_1,\dots,x_n)$. Then $\check{\partial}_0=\epsilon$, $\check{\partial}_1(1\otimes\epsilon_i)=\check{x}_i$, $\check{\partial}_2(1\otimes\epsilon_i\wedge\epsilon_j)=\check{x}_i\otimes\epsilon_j-\check{x}_j\otimes\epsilon_j\sum_a C^a_{i,j}\otimes\epsilon_a-\sum_a\check{u}^a_{i,j}\otimes\epsilon_a$, and

$$\begin{aligned}
&\check{\partial}^h_{q+1}(1\otimes\epsilon_{p_1}\wedge\cdots\wedge\epsilon_{p_{q+1}})\\
&\quad=\sum_{i=1}^{q+1}(-1)^{i-1}\check{x}_i\otimes\epsilon_{p_1}\wedge\cdots\wedge\hat{\epsilon}_{p_i}\wedge\cdots\wedge\epsilon_{p_{q+1}}\\
&\qquad+\sum_{r<s}\sum_a(-1)^{r+s}C^a_{p_r,p_s}1\otimes\epsilon_a\wedge\epsilon_{p_1}\wedge\cdots\wedge\hat{\epsilon}_{p_r}\wedge\cdots\wedge\hat{\epsilon}_{p_s}\wedge\cdots\wedge\epsilon_{p_{q+1}}\\
&\qquad+\sum_{t_1,\dots,t_{q-1}}\check{\alpha}^{t_1,\dots,t_q}_{p_1,\dots,p_{q+1}}\otimes\epsilon_{t_1}\wedge\cdots\wedge\epsilon_{t_q}.
\end{aligned}$$

If P is in F_h, one has $\partial_q(P\otimes\epsilon_{p_1}\wedge\cdots\wedge\epsilon_{p_q})=h\check{\partial}(\check{P}\otimes\epsilon_{p_1}\wedge\cdots\wedge\epsilon_{p_q})$. The relation $\check{\partial}_q\check{\partial}_{q+1}=0$ is obtained by multiplying the relation $\partial^h_q\partial^h_{q+1}=0$ by h^{-2}. As $G\check{\partial}^h_q$ is the differential of the Koszul complex of the trivial $U(\mathfrak{g}^*)[h]$-module, the complex $\check{K}^\bullet_h=\big(U_h(\mathfrak{g}^*)\otimes\bigwedge^\bullet\mathfrak{g}^*,\,\check{\partial}^h_n\big)$ is a resolution of the trivial $U_h(\mathfrak{g}^*)$-module. □

A link between θ_{F_h} and $\theta_{F_h^\vee}$. The remainder of this section is devoted to the proof of this equality:

Theorem 5.2. $$\theta_{F_h}=h\theta_{F_h^\vee}.$$

Proof. We keep the notation of the previous proposition and we will use the proof of Theorem 4.1.

The complex $\big(\bigwedge^\bullet\mathfrak{g}\otimes F_h,{}^t\,\partial^h_n\big)$ computes the $k[\![h]\!]$-modules $\operatorname{Ext}^i_{F_h}(k[\![h]\!],F_h)$. The cohomology class $\operatorname{cl}(1\otimes\epsilon_1^*\wedge\cdots\wedge\epsilon_n^*)$ is a basis of

$$\underline{\operatorname{Ext}}^n_{F[\mathfrak{g}][h]}(k[h],F[\mathfrak{g}][h])\simeq G\operatorname{Ext}^n_{F_h}(k[\![h]\!],F_h).$$

Hence, there exists $\sigma=1+h\sigma_1+\cdots\in\operatorname{Ker}{}^t\,\partial^h_n$ such that $[\operatorname{cl}(\sigma\otimes\epsilon_1^*\wedge\cdots\wedge\epsilon_n^*)]$ is a basis of $G\operatorname{Ext}^n_{F_h}(k[\![h]\!],F_h)$. As the filtration on $\operatorname{Ext}^n_{F_h}(k[\![h]\!],F_h)$ is Hausdorff, the cohomology class $\operatorname{cl}(\sigma\otimes\epsilon_1^*\wedge\cdots\wedge\epsilon_n^*)$ is a basis of $\operatorname{Ext}^n_{F_h}(k[\![h]\!],F_h)$.

Define $\check{\sigma}$ by $\check{\sigma}(\check{x}_1,\dots,\check{x}_n)=\sigma(x_1,\dots,x_n)$. One has ${}^t\partial_n=h\,{}^t\check{\partial}_n$, and it is easy to check that $\check{\sigma}\otimes\epsilon_1^*\wedge\cdots\wedge\epsilon_n^*$ is in $\operatorname{Ker}{}^t\,\check{\partial}^h_{n-1}$. If we had

$$\check{\sigma}\otimes\epsilon_1^*\wedge\cdots\wedge\epsilon_n^*={}^t\,\check{\partial}^h_{n-1}\Bigg(\sum_{i=1}^n\check{\sigma}_i\otimes\epsilon_1^*\wedge\cdots\wedge\widehat{\epsilon_i^*}\wedge\cdots\wedge\epsilon_n^*\Bigg),$$

then, reducing modulo h, we would get

$$\bar{\check{\sigma}}\otimes\epsilon_1^*\wedge\cdots\wedge\epsilon_n^*=\overline{{}^t\check{\partial}^h_{n-1}}\Bigg(\sum_{i=1}^n\bar{\check{\sigma}}_i\otimes\epsilon_1^*\wedge\cdots\wedge\widehat{\epsilon_i^*}\wedge\cdots\wedge\epsilon_n^*\Bigg).$$

This would imply that $\operatorname{cl}(1\otimes\epsilon_1^*\wedge\cdots\wedge\epsilon_n^*)$ is 0 in $\operatorname{Ext}^n_{U(\mathfrak{g}^*)}(k,U(\mathfrak{g}^*))$, which is impossible because $\operatorname{cl}(1\otimes\epsilon_1^*\wedge\cdots\wedge\epsilon_n^*)$ is a basis of $\operatorname{Ext}^n_{U(\mathfrak{g}^*)}(k,U(\mathfrak{g}^*))$. Thus, $\operatorname{cl}(\check{\sigma}\otimes\epsilon_1^*\wedge\cdots\wedge\epsilon_n^*)$ is a nonzero element of $\operatorname{Ext}^{\dim\mathfrak{g}^*}_{U_h(\mathfrak{g}^*)}(k[\![h]\!],U_h(\mathfrak{g}^*))$. For all i in $[1,n]$, one has the relation

$$\sigma x_i\otimes\epsilon_1^*\wedge\cdots\wedge\epsilon_n^*=\theta_{F_h}(x_i)\sigma\otimes\epsilon_1^*\wedge\cdots\wedge\epsilon_n^*+{}^t\,\partial^h_n(\mu)$$

Let us write $\mu=\sum_i\mu_i\otimes\epsilon_1^*\wedge\cdots\wedge\widehat{\epsilon_i^*}\wedge\cdots\wedge\epsilon_n^*$ with $\mu_i\in F_h[\mathfrak{g}]$. We set

$$\check{\mu}_i(\check{x}_1,\dots,\check{x}_n)=\mu_i(x_1,\dots,x_n)\quad\text{and}\quad\check{\mu}=\sum_i\check{\mu}_i\otimes\epsilon_1^*\wedge\cdots\wedge\widehat{\epsilon_i^*}\wedge\cdots\wedge\epsilon_n^*.$$

Then we have $h\check{\sigma}\check{x}_i\otimes\epsilon_1^*\wedge\cdots\wedge\epsilon_n^*=\theta_{F_h}(x_i)\,\check{\sigma}\otimes\epsilon_1^*\wedge\cdots\wedge\epsilon_n^*+h\,{}^t\check{\partial}^h_n(\check{\mu})$. □

6. Study of an example

We will now explicitly study an example suggested by B. Enriquez. Chloup [1997] introduced the triangular Lie bialgebra

$$(\mathfrak{g} = kX_1 \oplus kX_2 \oplus kX_3 \oplus kX_4 \oplus kX_5, r = 4(X_2 \wedge X_3)),$$

where the nonzero brackets are given by $[X_1, X_2] = X_3$, $[X_1, X_3] = X_4$ and $[X_1, X_4] = X_5$, and the cobracket $\delta_{\mathfrak{g}}$ is the following:

$$\text{if } X \in \mathfrak{g}, \quad \text{then } \delta(X) = X \cdot 4(X_2 \wedge X_3).$$

The dual Lie bialgebra of $\mathfrak{g}$ will be denoted by $\big(\mathfrak{a} = \bigoplus_{i=1}^{5} ke_i, \delta\big)$. The only nonzero Lie bracket of $\mathfrak{a}$ is $[e_2, e_4] = 2e_1$ and its cobracket δ is nonzero on the basis vectors e_3, e_4, e_5:

$$\delta(e_3) = e_1 \otimes e_2 - e_2 \otimes e_1 = 2e_1 \wedge e_2, \quad \delta(e_4) = 2e_1 \wedge e_3, \quad \delta(e_5) = 2e_1 \wedge e_4.$$

We may twist the trivial deformation of $\big(U(\mathfrak{g})[\![h]\!], \mu_0, \Delta_0, \iota_0, \epsilon_0, S_0\big)$ by the invertible element

$$R = \exp\big(h(X_2 \otimes X_3 - X_3 \otimes X_2)\big)$$

of $U(\mathfrak{g})[\![h]\!] \widehat{\otimes} U(\mathfrak{g})[\![h]\!]$ (see [Chari and Pressley 1994, page 130]). The topological Hopf algebra obtained has the same multiplication, antipode, unit and counit. However, its coproduct is $\Delta^R = R^{-1}\Delta_0 R$. It is a quantization of $(\mathfrak{g}, r)$. We will denote it by $U_h(\mathfrak{g})$. The Hopf algebra $U_h(\mathfrak{g})^*$ is a QFSHA and $(U_h(\mathfrak{g})^*)^\vee$ is a quantization of $(\mathfrak{a}, \delta_{\mathfrak{a}})$. We will compute it explicitly.

Proposition 6.1. (a) $(U(\mathfrak{g})^*)^\vee$ *is isomorphic as a topological Hopf algebra to the topological k[[h]]-algebra*

$$\frac{T_{k[\![h]\!]}(k[\![h]\!]e_1 \oplus k[\![h]\!]e_2 \oplus k[\![h]\!]e_3 \oplus k[\![h]\!]e_4 \oplus k[\![h]\!]e_5)}{I},$$

where I is the closure of the two-sided ideal generated by

$$e_2 \otimes e_4 - e_4 \otimes e_2 - 2e_1,$$

$$e_3 \otimes e_5 - e_5 \otimes e_3 - \tfrac{2}{3}h^2 e_1 \otimes e_1 \otimes e_1,$$

$$e_4 \otimes e_5 - e_5 \otimes e_4 - \tfrac{1}{6}h^3 e_1 \otimes e_1 \otimes e_1 \otimes e_1,$$

$$e_2 \otimes e_5 - e_5 \otimes e_2 + he_1 \otimes e_1,$$

$$e_3 \otimes e_4 - e_4 \otimes e_3 + he_1 \otimes e_1,$$

$$e_i \otimes e_j - e_j \otimes e_i, \quad \text{if } \{i, j\} \neq \{2, 4\}, \{3, 5\}, \{4, 5\}, \{2, 5\}, \{3, 4\},$$

with the coproduct Δ_h, counit ϵ_h and antipode S defined as follows:

$$\begin{aligned}
&\Delta_h(e_1) = e_1 \otimes 1 + 1 \otimes e_1,\\
&\Delta_h(e_2) = e_2 \otimes 1 + 1 \otimes e_2,\\
&\Delta_h(e_3) = e_3 \otimes 1 + 1 \otimes e_3 - he_2 \otimes e_1,\\
&\Delta_h(e_4) = e_4 \otimes 1 + 1 \otimes e_4 - he_3 \otimes e_1 + \tfrac{1}{2}h^2 e_2 \otimes e_1^2,\\
&\Delta_h(e_5) = e_5 \otimes 1 + 1 \otimes e_5 - he_4 \otimes e_1 + \tfrac{1}{2}h^2 e_3 \otimes e_1^2 - \tfrac{1}{6}h^3 e_2 \otimes e_1^3,\\
&\epsilon_h(e_i) = 0 \quad \textit{and} \quad S(e_i) = -e_i \quad \textit{for } i \in [1,5].
\end{aligned}$$

(b) $(U(\mathfrak{g})^*)^\vee$ *is not isomorphic to the trivial deformation of* $U(\mathfrak{a})$ *as an algebra.*

Proof of Proposition 6.1. Let ξ_i be the element of $U(\mathfrak{g})^*$ defined by

$$\langle \xi_i, X_1^{a_1} X_2^{a_2} X_3^{a_3} X_4^{a_4} X_5^{a_5} \rangle = \delta_{a_1,0} \dots \delta_{a_i,1} \dots \delta_{a_5,0}.$$

The algebras $U(\mathfrak{g})^*$ and $k[\![\xi_1, \dots, \xi_n]\!]$ are isomorphic. The topological Hopf algebra $(U_h(\mathfrak{g})^*, {}^t\Delta_0^R = \cdot_h, {}^t\mu_0 = \Delta_h, {}^t\epsilon_0, {}^t\iota_0 = \epsilon_h, {}^tS_0)$ is a QFSHA. $U_h(\mathfrak{g})^*$ and $k[\![\xi_1, \dots, \xi_n, h]\!]$ are isomorphic as $k[\![h]\!]$-modules. The elements $\xi_1, \dots, \xi_n$ generate topologically the $k[\![h]\!]$- algebra $U_h(\mathfrak{g})^*$ and satisfy $\epsilon_h(\xi_i) = 0$,

$$\langle \xi_2 \otimes \xi_4 - \xi_4 \otimes \xi_2, \Delta^R(X_1^{a_1} \dots X_5^{a_5}) \rangle \neq 0 \iff (a_1, a_2, a_3, a_4, a_5) = (1,0,0,0,0)$$

and $\langle \xi_2 \otimes \xi_4 - \xi_4 \otimes \xi_2, \Delta^R(X_1) \rangle = 2h$. Hence, $\xi_2 \cdot_h \xi_4 - \xi_4 \cdot_h \xi_2 = 2h\xi_1$. The other relations are obtained similarly.

Let us now compute the coproduct Δ_h of $U_h(\mathfrak{g})^*$:

$$\langle \Delta_h(\xi_5), X_1^{a_1} X_2^{a_2} X_3^{a_3} X_4^{a_4} X_5^{a_5} \otimes X_1^{b_1} X_2^{b_2} X_3^{b_3} X_4^{b_4} X_5^{b_5} \rangle \neq 0 \iff$$

$$(a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5) = \begin{cases} (0,0,0,0,1,0,0,0,0,0) & \text{or} \\ (0,0,0,0,0,0,0,0,0,1) & \text{or} \\ (0,0,0,1,0,1,0,0,0,0) & \text{or} \\ (0,0,1,0,0,2,0,0,0,0) & \text{or} \\ (0,1,0,0,0,3,0,0,0,0). \end{cases}$$

Moreover,

$$\langle \Delta_h(\xi_5), X_4 \otimes X_1 \rangle = -1, \quad \langle \Delta_h(\xi_5), X_3 \otimes X_1^2 \rangle = 1, \quad \langle \Delta_h(\xi_5), X_2 \otimes X_1^3 \rangle = -1.$$

Hence,

$$\Delta_h(\xi_5) = \xi_5 \otimes 1 + 1 \otimes \xi_5 - \xi_4 \otimes \xi_1 + \tfrac{1}{2}\xi_3 \otimes \xi_1 \cdot_h \xi_1 - \tfrac{1}{6}\xi_2 \otimes \xi_1 \cdot_h \xi_1 \cdot_h \xi_1.$$

$\Delta_h(\xi_1)$, $\Delta_h(\xi_2)$, $\Delta_h(\xi_3)$ and $\Delta_h(\xi_4)$ are computed similarly.

We set $\check{\xi}_i = h^{-1}\xi_i$ and $e_i = \check{\xi}_i \bmod h \; (U(\mathfrak{g})^*)^\vee$. From what we have reviewed in the first paragraph of this section, the first part of this theorem is proved.

Then $\Psi : (U(\mathfrak{g})^*)^\vee \to U(\mathfrak{a})[[h]]$, defined by

$$\Psi\Bigl(\sum_{r\in\mathbb{N}} P_r(\check{\xi}_1, \dots, \check{\xi}_n)h^r\Bigr) = \sum_{r\in\mathbb{N}} P_r(e_1, \dots, e_n)h^r,$$

is an isomorphism of topological $k[[h]]$-modules. Let $\cdot_h$ be the transposition of the multiplication of F_h to $U(\mathfrak{a})[[h]]$. If u and v are in $U(\mathfrak{a})$, one sets

$$u \cdot_h v = uv + \sum_{r=1}^{\infty} h^r \mu_r(u, v).$$

One has $\mu_1(e_3, e_4) = 0$, $\mu_1(e_4, e_3) = e_1^2$ and $\mu_1(e_2, e_5) = 0$, $\mu_1(e_5, e_2) = e_1^2$. Let us show that μ_1 is a coboundary in the Hochschild cohomology. The Hochschild cohomology $HH^*(U(\mathfrak{a}), U(\mathfrak{a}))$ is computed by the complex

$$\bigl(\mathrm{Hom}(U(\mathfrak{a})^{\otimes *}, U(\mathfrak{a})), b\bigr),$$

where

$$\begin{aligned} b(f)(a_0, \dots, a_n) = a_0 f(a_1, \dots, a_n) + \sum_{i=1}^{n} (-1)^i f(a_0, \dots, a_{i-1}a_i, \dots a_n) \\ + f(a_0, \dots, a_{n-1})a_n(-1)^n \end{aligned}$$

if $f \in \mathrm{Hom}(U(\mathfrak{a})^{\otimes n+1}, U(\mathfrak{a}))$. Using the explicit isomorphism between the Hochschild cohomology $HH^*(U(\mathfrak{a}), U(\mathfrak{a}))$ and the Lie algebra cohomology of $\mathfrak{a}$ with coefficients in $U(\mathfrak{a})^{ad}$ (with the adjoint action) and $H^*(\mathfrak{a}, U(\mathfrak{a})^{ad})$ [Loday 1998, Lemma 3], one can show that $\mu_1 = b(\alpha)$. The map $\alpha \in \mathrm{Hom}(U(\mathfrak{a}), U(\mathfrak{a}))$ is determined by

$$\alpha_{|\mathfrak{a}} = -\tfrac{1}{2}e_1e_2 \otimes e_3^* - \tfrac{1}{2}e_1e_4 \otimes e_5^*$$

and

$$\mu_1(u, v) = u\alpha(v) - \alpha(uv) + u\alpha(v) \quad \text{for all } (u, v) \in U(\mathfrak{a}).$$

We set $\beta_h = \mathrm{id} - h\alpha$. Then $\beta_h^{-1} = \sum_{i=0}^{\infty} h^i\alpha^i$. If u and v are elements of $U(\mathfrak{a})$, we put $u \cdot_h' v = \beta_h^{-1}(\beta_h(u) \cdot_h \beta_h(v))$. If i and j are different from 3 and 5, then $e_i \cdot_h' e_j = e_i \cdot_h e_j$. Computations lead to the relations:

$$e_1 \cdot_h' e_5 = e_5 \cdot_h' e_1, \quad e_2 \cdot_h' e_3 = e_3 \cdot_h' e_2, \quad e_2 \cdot_h' e_5 = e_5 \cdot_h' e_2, \quad e_3 \cdot_h' e_4 = e_4 \cdot_h' e_3,$$
$$e_1 \cdot_h' e_3 = e_3 \cdot_h' e_1, \quad e_3 \cdot_h' e_5 - e_5 \cdot_h' e_3 = \tfrac{1}{6}h^2e_1^3, \quad e_4 \cdot_h' e_5 - e_5 \cdot_h' e_4 = \tfrac{1}{6}-h^2e_1^3.$$

The topological algebras $[U(\mathfrak{a})[[h]], \cdot_h]$ and $\bigl[U(\mathfrak{a})[[h]], \cdot_h'\bigr]$ are isomorphic, hence, their centers are isomorphic. Using the commutation relations, one can compute the center $Z\bigl[U(\mathfrak{a})[[h]], \cdot_h'\bigr]$ of $\bigl[U(\mathfrak{a})[[h]], \cdot_h'\bigr]$:

$$Z\bigl[U(\mathfrak{a})[[h]], \cdot_h'\bigr] = \Bigl\{\sum_{n\geq 0} P_r(e_1)h^r \Bigm| P_r \in k[X_1]\Bigr\}.$$

But, the center of the trivial deformation of $U(\mathfrak{a})$ is

$$Z\big[U(\mathfrak{a})[\![h]\!], \mu_0\big] = \Big\{ \sum_{n\geq 0} P_r(e_1, e_3, e_5)h^r \;\Big|\; P_r \in k[X_1, X_3, X_5] \Big\}.$$

Hence, the algebras $[U(\mathfrak{a})[\![h]\!], \cdot'_h]$ and $[U(\mathfrak{a})[\![h]\!], \mu_0]$ are not isomorphic. □

Proposition 6.2. *We consider the quantized enveloping algebra of Proposition 6.1. We write the relations defining the ideal I as follows.*

$$e_i \otimes e_j - e_j \otimes e_i - \sum_a C^a_{i,j} e_a - P_{i,j}.$$

As all the $P_{i,j}$'s are monomials in e_1's, the notation $P_{i,j}/e_1$ makes sense. The complex

$$0 \to U_h(\mathfrak{a}) \otimes {\textstyle\bigwedge}^5 \mathfrak{a} \xrightarrow{\partial_5^h} U_h(\mathfrak{a}) \otimes {\textstyle\bigwedge}^4 \mathfrak{a} \xrightarrow{\partial_4^h} \cdots \xrightarrow{\partial_2^h} U_h(\mathfrak{a}) \otimes \mathfrak{a} \xrightarrow{\partial_1^h} U_h(\mathfrak{a}) \xrightarrow{\partial_0^h} k[\![h]\!] \to 0,$$

where the morphisms of $U_h(\mathfrak{a})$ and ∂_h^i are described below, is a resolution of the trivial $U_h(\mathfrak{a})$-module $k[\![h]\!]$. We set

$$\begin{aligned} &\partial_n(1 \otimes e_{p_1} \wedge \cdots \wedge e_{p_n}) \\ &\quad = \sum_{i=1}^{n} (-1)^{i-1} e_{p_i} \otimes e_{p_1} \wedge \cdots \wedge \widehat{e_{p_i}} \wedge \cdots \wedge e_{p_n} \\ &\qquad + \sum_{k<l} (-1)^{k+l} \sum_a C^a_{p_k,p_l} 1 \otimes e_a \wedge e_{p_1} \wedge \cdots \wedge \widehat{e_{p_k}} \wedge \cdots \wedge \widehat{e_{p_l}} \wedge \cdots \wedge e_{p_n}. \end{aligned}$$

Then,

$$\begin{aligned} &\partial_0^h = \epsilon_h, \\ &\partial_1^h(1 \otimes e_i) = e_i, \\ &\partial_2^h(1 \otimes e_i \wedge e_j) = \partial_2(1 \otimes e_i \wedge e_j) - \frac{P_{i,j}}{e_1} \otimes e_i, \\ &\partial_3^h(1 \otimes e_i \wedge e_j \wedge e_k) = \partial_3(1 \otimes e_i \wedge e_j \wedge e_k) - \frac{P_{i,j}}{e_1} \otimes e_1 \wedge e_k \\ &\qquad\qquad + \frac{P_{i,k}}{e_1} \otimes e_1 \wedge e_j - \frac{P_{j,k}}{e_1} \otimes e_1 \wedge e_i, \\ &\partial_4^h(1 \otimes e_1 \wedge e_i \wedge e_j \wedge e_k) = \partial_4(1 \otimes e_1 \wedge e_i \wedge e_j \wedge e_k), \\ &\partial_4^h(1 \otimes e_2 \wedge e_3 \wedge e_4 \wedge e_5) = \partial_4(1 \otimes e_2 \wedge e_3 \wedge e_4 \wedge e_5) + \frac{P_{3,5}}{e_1} \otimes e_1 \wedge e_2 \wedge e_4 \\ &\qquad\qquad - \frac{P_{3,4}}{e_1} \otimes e_1 \wedge e_2 \wedge e_5 - \frac{P_{4,5}}{e_1} \otimes e_1 \wedge e_2 \wedge e_3 - \frac{P_{2,5}}{e_1} \otimes e_1 \wedge e_3 \wedge e_4, \\ &\partial_5^h(1 \otimes e_1 \wedge e_2 \wedge e_3 \wedge e_4 \wedge e_5) = \partial_5(1 \otimes e_1 \wedge e_2 \wedge e_3 \wedge e_4 \wedge e_5). \end{aligned}$$

The character defined by right multiplication on $\mathrm{Ext}^5_{U_h(\mathfrak{a})}(k[\![h]\!], U_h(\mathfrak{a}))$ *of* $U_h(\mathfrak{a})$ *is zero.*

Proof of Proposition 6.2. The resolution of $k[\![h]\!]$ is obtained as in the proof of Theorem 5.1. The rest of the proposition follows by easy computations. □

7. Applications

Poincaré duality. Let M be an A_h^{op}-module and N an A_h-module. The right exact functor $M \underset{A_h}{\otimes} -$ has a left derived functor. We set

$$\mathrm{Tor}^i_{A_h}(M, N) = L^i(M \underset{A_h}{\otimes} -)(N).$$

Theorem 7.1. *Let A_h be a deformation algebra of A_0 satisfying the hypothesis of Theorem 4.1. Assume moreover that the A_h-module K is of finite projective dimension. Let M be an A_h-module. The K-modules* $\mathrm{Ext}^i_{A_h}(K, M)$ *and* $\mathrm{Tor}^{A_h}_{d_{A_h}-i}(\Omega_{A_h}, M)$ *are isomorphic.*

Remark. Theorem 7.1 generalizes classical Poincaré duality [Knapp 1988].

Proof of Theorem 7.1. As the A_h-module K admits a finite-length resolution by finitely generated projective A_h-modules, $P^\bullet \to K$, the canonical arrow

$$R\,\mathrm{Hom}_{A_h}(K, A_h) \otimes^L_{A_h} M \to R\,\mathrm{Hom}_{A_h}(K, M)$$

is an isomorphism in $D(\mathrm{Mod}\, A_h)$. □

Duality property for induced representations of quantum groups. From now on, we assume that A_h is a topological Hopf algebra.

In this section, we keep the notation of Theorem 4.5. Let V be a left A_h-module, then, by transposition, $V^* = \mathrm{Hom}_K(V, K)$ is naturally endowed with a right A_h-module structure. Using the antipode, we can also see V^* as a left module structure. Thus,

$$u \cdot f = f \cdot S(u) \quad \text{for all } u \in A_h \text{ and } f \in V^*.$$

We endow $\Omega_{A_h} \otimes V^*$ with the right A_h-module structure given by

$$(\omega \otimes f) \cdot u = \lim_{n \to +\infty} \sum_j \theta_{A_h}(u'_{j,n})\, \omega \otimes f \cdot S_h^2(u''_{j,n})$$

and $\Delta(u) = \lim_{n \to +\infty} \sum_j u'_{j,n} \otimes u''_{j,n}$, for all $u \in A_h$, all $f \in V^*$, and all $\omega \in \Omega_{A_h}$.

Let A_h be a topological Hopf deformation of A_0, and let B_h be a topological Hopf deformation of B_0. We assume, moreover, that there exists a morphism of Hopf algebras from B_h to A_h and that A_h is a flat B_h^{op}-module (by Proposition 3.6

this is verified if the induced B_0-module structure on A_0 is flat). If V is an A_h-module, we can define the induced representation from V as follows:

$$\mathrm{Ind}_{B_h}^{A_h}(V) = A_h \otimes_{B_h} V,$$

on which A_h acts by left multiplication.

Proposition 7.2. *Let A_h be a topological Hopf deformation of A_0 and let B_h be a topological deformation of B_0. We assume that there exists a morphism of Hopf algebras from B_h to A_h, such that A_h is a flat B_h^{op}-module. In addition, we assume that B_h satisfies the hypothesis of Theorem 4.1. Let V be a B_h-module which is a free finite-dimensional K-module. Then, $D_{B_h}(\mathrm{Ind}_{A_h}^{B_h}(V))$ is isomorphic to $(\Omega_{B_h} \otimes V^*)_{\otimes_{B_h}} A_h[-d_{B_h}]$ in $D(\mathrm{Mod}\, B_h^{op})$.*

Corollary 7.3. *Let A_h be a topological Hopf deformation of A_0 and let B_h be a topological deformation of B_0. We assume that there exists a morphism of Hopf algebras from B_h to A_h, such that A_h is a flat B_h^{op}-module. We also assume that B_h satisfies the condition of Theorem 4.1. Let V be a B_h-module which is a free finite-dimensional K-module. Then,*

(a) $\mathrm{Ext}^i_{A_h}(A_h \otimes_{B_h} V, A_h)$ *is reduced to* 0 *if i is different from d_{B_h}.*

(b) *The right A_h-module* $\mathrm{Ext}^{d_{B_h}}_{A_h}(A_h \otimes_{B_h} V, A_h)$ *is isomorphic to* $(\Omega_{B_h} \otimes V^*) \otimes_{B_h} A_h$.

Remark. Proposition 7.2 is already known in the case where $\mathfrak{g}$ is a Lie algebra, $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, and A and B are the corresponding enveloping algebras. In this case, one has $d_{B_h} = \dim \mathfrak{h}$ and $d_{C_h} = \dim \mathfrak{k}$. More precisely, It was proved by Brown and Levasseur [1985, page 410] and Kempf [1991] in the case where $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra, and $\mathrm{Ind}_{U(\mathfrak{h})}^{U(\mathfrak{g})}(V)$ is a Verma-module. In addition, Proposition 7.4 is proved in full generality for Lie superalgebras in [Chemla 1994].

Here are some examples of situations where we can apply Proposition 7.2.

Example. Let k be a field of characteristic 0. We set $K = k[[h]]$. Etingof and Kazhdan have constructed a functor Q from the category $LB(k)$ of Lie bialgebras over k to the category $HA(K)$ of topological Hopf algebras over K. If $(\mathfrak{g}, \delta)$ is a Lie bialgebra, its image by Q will be denoted by $U_h(\mathfrak{g})$.

Let $\mathfrak{g}$ be a Lie bialgebra and let $\mathfrak{h}$ be a Lie sub-bialgebra of $\mathfrak{g}$. The functoriality of the quantization implies the existence of an embedding of Hopf algebras from $U_h(\mathfrak{h})$ to $U_h(\mathfrak{g})$ which satisfies all our hypothesis.

Example. If $\mathfrak{g}$ is a Lie bialgebra, we will denote by $\mathscr{F}(\mathfrak{g})$ the formal group attached to it and by $\mathscr{F}_h(\mathfrak{g})$ its Etingof Kazhdan quantization. Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie algebras, and assume that there exists a surjective morphism of Lie bialgebras from $\mathfrak{g}$ to $\mathfrak{h}$. Then, $\mathscr{F}_h(\mathfrak{g})$ is a flat $\mathscr{F}_h(\mathfrak{h})$-module, and $A_h = \mathscr{F}_h(\mathfrak{g})$ and $B_h = \mathscr{F}_h(\mathfrak{h})$ satisfies the hypothesis of the theorem.

Example. If G is an affine algebraic Poisson group, we will denote by $\mathscr{F}(G)$ the algebra of regular functions on G and by $\mathscr{F}_h(G)$ its Etingof Kazhdan quantization. Let G and H be affine algebraic Poisson groups. Assume that there is a Poisson group map $G \to H$ such that $\mathscr{F}(G)$ is a flat $\mathscr{F}(H)^{op}$-module. By functoriality of Etingof Kazhdan quantization, $A_h = \mathscr{F}_h(G)$, and $B_h = \mathscr{F}_h(H)$ satisfies the hypothesis of the theorem.

The proof of Proposition 7.2 is analogous to that of [Chemla 2004, Proposition 3.2.4].

We now extend to Hopf algebras another duality property for induced representations of Lie algebras [Chemla 1994].

Proposition 7.4. *Let A_h be a Hopf deformation of A_0, B_h be a Hopf deformation of B_0 and C_h be a Hopf deformation of C_0. We assume that there exists a morphism of Hopf algebras from B_h to A_h and a morphism of Hopf algebras from C_h to A_h such that A_h is a flat B_h^{op}-module and a flat C_h^{op}-module. We also assume that B_h and C_h satisfy the hypothesis of Theorem 4.1. Let V* (*respectively W*) *be an B_h-module* (*respectively C_h-module*) *which is a free finite dimensional K-module. Then, for all integers n, one has an isomorphism*

$$\mathrm{Ext}^{n+d_{B_h}}_{A_h}\Big(A_h \underset{B_h}{\otimes} V,\, A_h \underset{C_h}{\otimes} W\Big) \simeq \mathrm{Ext}^{n+d_{C_h}}_{A_h^{op}}\Big((\Omega_{C_h} \otimes W^*) \underset{C_h}{\otimes} A_h,\, (\Omega_{B_h} \otimes V^*) \underset{C_h}{\otimes} A_h\Big).$$

Remark. Proposition 7.4 is already known in the case where $\mathfrak{g}$ is a Lie algebra, $\mathfrak{h}$ and $\mathfrak{k}$ are Lie subalgebras of $\mathfrak{g}$, and A, B and C are the corresponding enveloping algebras. In this case one has $d_{B_h} = \dim \mathfrak{h}$ and $d_{C_h} = \dim \mathfrak{k}$. More precisely, generalizing a result of G. Zuckerman [Boe and Collingwood 1985], A. Gyoja [2000] proved a part of this theorem (namely the case where $\mathfrak{h} = \mathfrak{g}$ and $n = \dim \mathfrak{h} = \dim \mathfrak{k}$) under the assumptions that $\mathfrak{g}$ is split semisimple and $\mathfrak{h}$ is a parabolic subalgebra of $\mathfrak{g}$. D. H. Collingwood and B. Shelton [1990] also proved a duality of this type (still under the semisimple hypothesis) but in a slightly different context.

M. Duflo [1987] proved Proposition 7.4 for a $\mathfrak{g}$ general Lie algebra, $\mathfrak{h} = \mathfrak{k}$, $V = W^*$ being one-dimensional representations.

Proposition 7.4 is proved in full generality in the context of Lie superalgebras in [Chemla 1994]. The proof in the present case is very similar to that of [Chemla 2004, Corollary 3.2.5].

Hochschild cohomology. In this subsection, A_h is a topological Hopf algebra. We set $A_h^e = A_h \otimes_{k[\![h]\!]} A_h^{op}$ and $\widehat{A_h^e} = A_h \widehat{\otimes_{k[\![h]\!]}} A_h^{op}$. If M is an $\widehat{A_h^e}$-module, we set

$$HH^i_{A_h}(M) = \mathrm{Ext}^i_{\widehat{A_h^e}}(A_h, M) \quad \text{and} \quad HH_i^{A_h}(M) = \mathrm{Tor}_i^{\widehat{A_h^e}}(A_h, M).$$

The next result was obtained in [Dolgushev and Etingof 2005] for a deformation of the algebra of regular functions on a smooth algebraic affine variety. Its proof in our setting is analogous to that of [Chemla 2004, Theorem 3.3.2].

Proposition 7.5. *Assume that A_0 satisfies the conditions of Theorem 4.1. Assume moreover that $A_0 \otimes A_0^{op}$ is noetherian. Consider $A_h \widehat{\otimes_{k[[h]]}} A_h$ with the $\widehat{A_h^e}$-module structure given by $\alpha \cdot (x \otimes y) \cdot \beta = \alpha x \otimes y\beta$. for $\alpha, \beta, x, y \in A_h$.*

(a) *$HH^i_{A_h}(A_h \widehat{\otimes_{k[[h]]}} A_h)$ is zero if $i \neq d_{A_h}$.*

(b) *The $\widehat{A_h^e}$-module $U = HH^{d_{A_h}}_{A_h}(A_h \widehat{\otimes_{k[[h]]}} A_h)$ is isomorphic to $\Omega_{A_h} \otimes A_h$ with the $\widehat{A_h^e}$-module structure given by*

$$\alpha \cdot (\omega \otimes x) \cdot \beta = \omega \theta_{A_h}(\beta_i') \otimes S(\beta_i'') x S^{-1}(\alpha)$$

for $\alpha, \beta, x \in A_h$, where $\beta = \sum_i \beta_i' \otimes \beta_i''$.

Proof. Using the antipode S_h of A_h, we have in $D(\mathrm{Mod}\,\widehat{A_h^e})$ the isomorphism

$$R\operatorname{Hom}_{\widehat{A_h^e}}(A_h, A_h \widehat{\otimes} A_h) \simeq R\operatorname{Hom}_{A_h \widehat{\otimes} A_h}\big((A_h)^\#, (A_h \widehat{\otimes} A_h)^\#\big),$$

where the structures on $(A_h)^\#$ and $(A_h \widehat{\otimes} A_h)^\#$ are given by $(\alpha \otimes \beta) \cdot u = \alpha u S_h(\beta)$, $(\alpha \otimes \beta) \cdot (u \otimes v) = \alpha u \otimes v S_h(\beta)$, and $(u \otimes v) \cdot \alpha \otimes \beta = u\alpha \otimes S_h(\beta) v$, for all $\alpha, \beta, u, v \in A_h$. Using the version of Lemma 4.6 for right modules [Chemla 2004, Lemma 1.1], one sees that $(A_h)^\#$ is isomorphic to $(A_h \widehat{\otimes} A_h) \otimes_{A_h} K$ as an $A_h \widehat{\otimes} A_h$-module. We get

$$\begin{aligned} R\operatorname{Hom}_{\widehat{A_h^e}}(A_h, A_h \widehat{\otimes} A_h) &\simeq R\operatorname{Hom}_{A_h \widehat{\otimes} A_h}(A_h \widehat{\otimes} A_h \otimes_{A_h} K, (A_h \widehat{\otimes} A_h)^\#) \\ &\simeq R\operatorname{Hom}_{A_h}(K, (A_h \widehat{\otimes} A_h)^\#) \\ &\simeq R\operatorname{Hom}_{A_h}(K, A_h) \otimes_{A_h} (A_h \widehat{\otimes} A_h)^\# \\ &\simeq \Omega_h \otimes_{A_h} (A_h \widehat{\otimes} A_h)^\#. \end{aligned}$$

Furthermore, the isomorphism $\mathrm{id} \otimes S_h^{-1}$ transforms $(A_h \widehat{\otimes} A_h)^\#$ into the natural $(A_h \widehat{\otimes} A_h) \otimes (A_h \widehat{\otimes} A_h)^{op}$-module $(A_h \widehat{\otimes} A_h)^{\mathrm{nat}}$, given by

$$(\alpha \otimes \beta) \cdot (u \otimes v) = \alpha u \otimes \beta v, \quad (u \otimes v) \cdot \alpha \otimes \beta = u\alpha \otimes v\beta$$

for all $(\alpha, \beta, u, v) \in A_h$.

Using Lemma 4.6, one sees that $\Omega_h \otimes_{A_h} (A_h \widehat{\otimes} A_h)^{\mathrm{nat}}$ is isomorphic to $\Omega_h \otimes A_h$ endowed with the $(A_h \widehat{\otimes} A_h)^{op}$-module structure given by

$$(u \otimes v) \cdot \alpha \otimes \beta = \sum_i u \theta_{A_h}(\alpha_i') \otimes S(\alpha_i'') v \beta \quad \text{for all } \alpha, \beta \in A_h.$$

This finishes the proof of the proposition. □

We are in the case where $\operatorname{Ext}^i_{\widehat{A^e_h}}(A_h, \widehat{A^e_h})$ is 0 except when $i = d_{A_h}$, so we get a duality between Hochschild homology and Hochschild cohomology [van den Bergh 1998].

Corollary 7.6. *Let A_0 be a k-algebra satisfying the hypothesis of Theorem 4.1. Assume moreover that $A_0^e = A_0 \otimes A_0^{op}$ is noetherian and that the $\widehat{A^e_h}$-module A_h is of finite projective dimension. Let M be an $\widehat{A^e_h}$-module. One has*

$$HH^i(M) \simeq HH_{d_{A_h}-i}(U \otimes_{A_h} M), \quad \text{where } U = \operatorname{Ext}^{d_{A_h}}_{\widehat{A^e_h}}(A_h, \widehat{A^e_h}).$$

Proof. The proof is similar to that of [van den Bergh 1998]. Assume first that M is a finite-type $\widehat{A^e_h}$-module. Let $P^\bullet \to A_h \to 0$ be a finite-length and finite-type projective resolution of the $\widehat{A^e_h}$-module A_h, and let $Q^\bullet \to M \to 0$ be a finite-type projective resolution of the $\widehat{A^e_h}$-module M. As Q^i and $U \otimes_{A_h} Q^i$ are complete, one has the following sequence of isomorphisms:

$$\begin{aligned} HH^i_{\widehat{A^e_h}}(M) &\simeq H^i\big(\operatorname{Hom}_{\widehat{A^e_h}}(P^\bullet, M)\big) \simeq H^i\big(\operatorname{Hom}_{\widehat{A^e_h}}(P^\bullet, \widehat{A^e_h}) \otimes_{\widehat{A^e_h}} M\big) \\ &\simeq H^i(U[-d] \otimes^L_{\widehat{A^e_h}} M) \simeq H^{i-d_{A_h}}\big(U \otimes_{\widehat{A^e_h}} Q^\bullet\big) \\ &\simeq H^{i-d_{A_h}}\big((A_h \otimes_{A_h} U) \otimes_{\widehat{A^e_h}} Q^\bullet\big) \\ &\simeq H^{i-d_{A_h}}\big(A_h \otimes_{\widehat{A^e_h}} (U \otimes_{A_h} Q^\bullet)\big) \simeq HH_{d_{A_h}-i}(U \otimes_{A_h} M). \end{aligned}$$

In the general case, when M is no longer a finite-type $\widehat{A^e_h}$-module. We have $M = \varinjlim M'$, where M' runs over all finitely generated $\widehat{A^e_h}$-submodules of M. This allows us to finish the proof. □

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REPRESENTATIONS OF THE CATEGORY OF MODULES OVER POINTED HOPF ALGEBRAS OVER $\mathbb{S}_3$ AND $\mathbb{S}_4$

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We classify exact indecomposable module categories over the representation category of all nontrivial Hopf algebras with coradical $\mathbb{S}_3$ and $\mathbb{S}_4$. As a byproduct, we compute all its Hopf–Galois extensions and we show that these Hopf algebras are cocycle deformations of their graded versions.

1. Introduction

Given a tensor category $\mathscr{C}$, an *exact module category* [Etingof and Ostrik 2004a] over $\mathscr{C}$ is an abelian category $\mathscr{M}$ equipped with a biexact functor $\otimes : \mathscr{C} \times \mathscr{M} \to \mathscr{M}$ subject to natural associativity and unit axioms, such that, for any projective object $P \in \mathscr{C}$ and any $M \in \mathscr{M}$, the object $P \otimes M$ is again projective.

Exact module categories, or *representations* of $\mathscr{C}$, are interesting objects to consider. They are implicitly present in many areas of mathematics and mathematical physics, such as subfactor theory [Böckenhauer et al. 2000], affine Hecke algebras [Bezrukavnikov and Ostrik 2004], extensions of vertex algebras [Kirillov and Ostrik 2002; Huang and Kong 2004], Calabi–Yau algebras [Ginzburg 2007], and conformal field theory, see for example [Barmeier et al. 2010; Fuchs and Schweigert 2003; Coquereaux and Schieber 2007; Coquereaux and Schieber 2008]. Module categories have been used in the study of fusion categories [Etingof et al. 2005], and in the theory of (weak) Hopf algebras [Ostrik 2003b; Mombelli 2010; Nikshych 2008].

The classification of exact module categories over a fixed finite tensor category $\mathscr{C}$ has been undertaken by several authors:

- when $\mathscr{C}$ is the semisimple quotient of $U_q(\mathfrak{sl}_2)$, by [Kirillov and Ostrik 2002; Etingof and Ostrik 2004b];
- over the tensor categories of representations of finite supergroups, by [Etingof and Ostrik 2004a];

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- over $\operatorname{Rep}(D(G))$, where $D(G)$ is the Drinfeld double of a finite group G, by [Ostrik 2003a];
- over the tensor category of representations of Lusztig's small quantum group $u_q(\mathfrak{sl}_2)$, by [Mombelli 2010];
- and more generally over $\operatorname{Rep}(H)$, where H is a lifting of a quantum linear space, by [Mombelli 2011].

The main goal of this paper is the classification of exact module categories over the representation category of any nontrivial (that is, different from the group algebra) finite-dimensional Hopf algebra with coradical $\Bbbk\mathbb{S}_3$ or $\Bbbk\mathbb{S}_4$.

Finite-dimensional Hopf algebras with coradical $\Bbbk\mathbb{S}_3$ or $\Bbbk\mathbb{S}_4$ were classified in [Andruskiewitsch et al. 2010] and [García and García Iglesias $\geq$ 2011], respectively. For all these Hopf algebras, the associated graded Hopf algebras $\operatorname{gr} H$ is of the form $\mathfrak{B}(X, q) \# \Bbbk\mathbb{S}_n$ for $n = 3$ or 4, where X is a finite set equipped with a map $\triangleright : X \times X \to X$ satisfying certain axioms that make it into a *rack*, and where $q : X \times X \to \Bbbk^\times$ is a 2-cocycle. We obtain the result:

Let $n = 3$ or 4, and let $\mathcal{M}$ be an exact indecomposable module category over $\operatorname{Rep}(\mathfrak{B}(X, q) \# \Bbbk\mathbb{S}_n)$. There exist

- a subgroup $F < \mathbb{S}_n$ and a 2-cocycle $\psi \in Z^2(F, \Bbbk^\times)$,
- a subset $Y \subseteq X$ invariant under the action of F,
- a family of scalars $\{\xi_C\}$ compatible (see Definition 7.1) with (F, ψ, Y),

such that $\mathcal{M} \simeq {}_{\mathcal{B}(Y,F,\psi,\xi)}\mathcal{M}$, where $\mathcal{B}(Y, F, \psi, \xi)$ is a left $\mathfrak{B}(X, q) \# \Bbbk\mathbb{S}_n$-comodule algebra constructed from the data (Y, F, ψ, ξ).

We also show that, if H is a finite-dimensional Hopf algebra with coradical $\Bbbk\mathbb{S}_3$ or $\Bbbk\mathbb{S}_4$, then H and $\operatorname{gr} H$ are cocycle deformations of each other. This implies that there is a bijective correspondence between module categories over $\operatorname{Rep}(H)$ and $\operatorname{Rep}(\operatorname{gr} H)$.

The content of the paper is as follows. In Section 3 we recall the basic results on module categories over finite-dimensional Hopf algebras. We recall the main result of [Mombelli 2011] that gives an isomorphism between Loewy-graded comodule algebras, and a semidirect product of a twisted group algebra and a homogeneous coideal subalgebra inside the Nichols algebra.

In Section 4 we show how to distinguish Morita equivariant classes of comodule algebras over pointed Hopf algebras.

In Section 5 we recall the definition of a rack X and a ql-datum $\mathcal{Q}$, and how to construct (quadratic approximations to) Nichols algebras $\widehat{\mathfrak{B}}_2(X, q)$ and pointed Hopf algebras $\mathcal{H}(\mathcal{Q})$ from them. In particular, we recall a presentation of all finite-dimensional Hopf algebras with coradical $\Bbbk\mathbb{S}_3$ or $\Bbbk\mathbb{S}_4$.

In Section 6, we give a classification of connected homogeneous left coideal subalgebras of $\widehat{\mathfrak{B}}_2(X, q)$ and also a presentation by generators and relations.

In Section 7 we introduce a family of comodule algebras large enough to classify module categories. We give an explicit Hopf–biGalois extension over $\widehat{\mathfrak{B}}_2(X, q) \# \Bbbk\mathbb{S}_n$, $n \in \mathbb{N}$, and a lifting $\mathcal{H}(\mathcal{Q})$, proving that there is a bijective correspondence between module categories over $\operatorname{Rep}(\widehat{\mathfrak{B}}_2(X, q) \# \Bbbk\mathbb{S}_n)$ and $\operatorname{Rep}(\mathcal{H}(\mathcal{Q}))$, $n = 3, 4$. In particular, we obtain that any pointed Hopf algebra over $\mathbb{S}_3$ or $\mathbb{S}_4$ is a cocycle deformation of its associated graded algebra, a result analogous to a theorem of Masuoka for abelian groups [Masuoka 2008]. Finally, the classification of module categories over $\operatorname{Rep}(\widehat{\mathfrak{B}}_2(X, q) \# \Bbbk\mathbb{S}_n)$ is presented in this section and, as a consequence, all Hopf–Galois objects over $\widehat{\mathfrak{B}}_2(X, q) \# \Bbbk\mathbb{S}_n$ are described.

2. Preliminaries and notation

We will denote by $\Bbbk$ an algebraically closed field of characteristic zero. The tensor product over the field $\Bbbk$ will be denoted by $\otimes$. All vector spaces, algebras and categories will be considered over $\Bbbk$. For any algebra A, ${}_A\mathcal{M}$ will denote the category of finite-dimensional left A-modules.

The symmetric group on n letters is denoted $\mathbb{S}_n$, and the conjugacy class of all j-cycles in $\mathbb{S}_n$ is denoted $\mathcal{O}^n_j$. For any group G, any 2-cocycle $\psi \in Z^2(G, \Bbbk^\times)$, and any $h \in G$, we will denote $\psi^h(x, y) = \psi(h^{-1}xh,\ h^{-1}yh)$ for all $x, y \in G$.

If H is a Hopf algebra, a 2-cocycle σ in H is a convolution-invertible linear map $\sigma : H \times H \to \Bbbk$ such that

$$\sigma(x_{(1)}, y_{(1)})\,\sigma(x_{(2)}\,y_{(2)}, z) = \sigma(y_{(1)}, z_{(1)})\,\sigma(x, y_{(2)}\,z_{(2)}) \tag{2-1}$$

and $\sigma(x, 1) = \sigma(1, x) = \varepsilon(x)$, for every $x, y, z \in H$. The set of 2-cocycles in H is denoted by $Z^2(H)$.

If A is an H-comodule algebra via $\lambda : A \to H \otimes A$, we will say that a (right) ideal J is H-costable if $\lambda(J) \subseteq H \otimes J$. We will say that A is (right) H-simple if there is no nontrivial (right) ideal H-costable in A.

If $H = \bigoplus H(i)$ is a coradically graded Hopf algebra, we will say that a left coideal subalgebra $K \subseteq H$ is *homogeneous* if $K = \bigoplus K(i)$ is graded as an algebra and, for any n, $K(n) \subseteq H(n)$ and $\Delta(K(n)) \subseteq \bigoplus_{i=0}^{n} H(i) \otimes K(n-i)$. K is said to be *connected* if $\mathcal{K} \cap H(0) = \Bbbk$.

If $H = \mathfrak{B}(V) \# \Bbbk G$, where V is a Yetter–Drinfeld module over G and $K \subseteq H$ is a coideal subalgebra, we will denote by $\operatorname{Stab} K$ the biggest subgroup of G such that the adjoint action of $\operatorname{Stab} K$ leaves K invariant.

If H is a finite-dimensional Hopf algebra, then $H_0 \subseteq H_1 \subseteq \cdots \subseteq H_m = H$ will denote the coradical filtration. When $H_0 \subseteq H$ is a Hopf subalgebra, the associated graded algebra $\operatorname{gr} H$ is a coradically graded Hopf algebra. If (A, λ) is a left

H-comodule algebra, the coradical filtration on H induces a filtration on A, given by $A_n = \lambda^{-1}(H_n \otimes A)$. This filtration is called the *Loewy series* on A.

The associated graded algebra gr A is a left gr H-comodule algebra. The algebra A is right H-simple if and only if gr A is right gr H-simple; see [Mombelli 2010, Section 4].

3. Representations of tensor categories

Given a tensor category $\mathscr{C} = (\mathscr{C}, \otimes, a, \mathbf{1})$, a *module category* over $\mathscr{C}$ (or a *representation* of $\mathscr{C}$) is an abelian category $\mathcal{M}$ equipped with an exact bifunctor

$$\overline{\otimes} : \mathscr{C} \times \mathcal{M} \to \mathcal{M}$$

and natural associativity and unit isomorphisms

$$m_{X,Y,M} : (X \otimes Y) \otimes M \to X \otimes (Y \otimes M)$$

and $\ell_M : \mathbf{1} \otimes M \to M$, satisfying natural associativity and unit axioms; see [Etingof and Ostrik 2004a; Ostrik 2003b]. We assume, as in the first of these papers, that all module categories have only finitely many isomorphism classes of simple objects.

A module category is *indecomposable* if it is not equivalent to a direct sum of two nontrivial module categories. A module category $\mathcal{M}$ over a finite tensor category $\mathscr{C}$ is *exact* [Etingof and Ostrik 2004a] if, for any projective $P \in \mathscr{C}$ and any $M \in \mathcal{M}$, the object $P \otimes M$ is again projective in $\mathcal{M}$.

If $\mathcal{M}$ is an exact module category over $\mathscr{C}$, then the dual category $\mathscr{C}^*_{\mathcal{M}}$ (see [Etingof and Ostrik 2004a]) is a finite tensor category. There is a bijective correspondence between the set of equivalence classes of exact module categories over $\mathscr{C}$ and over $\mathscr{C}^*_{\mathcal{M}}$; see [Etingof and Ostrik 2004a, Theorem 3.33]. This implies that, for any finite-dimensional Hopf algebra, there is a bijective correspondence between the set of equivalence classes of exact module categories over Rep(H) and Rep(H^*).

3A. ***Module categories over pointed Hopf algebras.*** We are interested in exact indecomposable module categories over the representation category of finite-dimensional Hopf algebras. If H is a Hopf algebra and $\lambda : \mathscr{A} \to H \otimes \mathscr{A}$ is a left H-comodule algebra, the category ${}^H\mathcal{M}_{\mathscr{A}}$ is the category of finite-dimensional right $\mathscr{A}$-modules left H-comodules, where the comodule structure is a $\mathscr{A}$-module morphism. If $\mathscr{A}'$ is another left H-comodule algebra the category ${}^H_{\mathscr{A}}\mathcal{M}_{\mathscr{A}'}$ is defined analogously.

The category of finite-dimensional left $\mathscr{A}$-modules ${}_{\mathscr{A}}\mathcal{M}$ is a representation of Rep(H). The action $\overline{\otimes}$: Rep$(H) \times {}_{\mathscr{A}}\mathcal{M} \to {}_{\mathscr{A}}\mathcal{M}$ is given by $V \,\overline{\otimes}\, M = V \otimes M$ for all $V \in$ Rep(H) and $M \in {}_{\mathscr{A}}\mathcal{M}$. The left $\mathscr{A}$-module structure on $V \otimes M$ is given by the coaction λ.

If $\mathcal{M}$ is an exact indecomposable module over $\operatorname{Rep}(H)$, then there exists a left H-comodule algebra $\mathcal{A}$ right H-simple with trivial coinvariants such that $\mathcal{M} \simeq {}_{\mathcal{A}}\mathcal{M}$ as modules over $\operatorname{Rep}(H)$; see [Andruskiewitsch and Mombelli 2007, Theorem 3.3].

If $\mathcal{A}$ and $\mathcal{A}'$ are two right H-simple left H-comodule algebras such that the categories ${}_{\mathcal{A}}\mathcal{M}$ and ${}_{\mathcal{A}'}\mathcal{M}$ are equivalent as representations over $\operatorname{Rep}(H)$, then there exists an equivariant Morita context (P, Q, f, g); that is, $P \in {}^{H}_{\mathcal{A}'}\mathcal{M}_{\mathcal{A}}$, $Q \in {}^{H}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}'}$, $f : P \otimes_{\mathcal{A}} Q \to \mathcal{A}'$ and $g : Q \otimes_{\mathcal{A}'} P \to \mathcal{A}$, such that the latter are bimodule isomorphisms. Moreover, it holds that $\mathcal{A}' \simeq \operatorname{End}_{\mathcal{A}}(P)$ as comodule algebras. The comodule structure on $\operatorname{End}_{\mathcal{A}}(P)$ is given by $\lambda(T) = T_{(-1)} \otimes T_{(0)}$, where

$$\langle \alpha, T_{(-1)} \rangle T_{(0)}(p) = \langle \alpha, T(p_{(0)})_{(-1)} \mathcal{S}^{-1}(p_{(-1)}) \rangle T(p_{(0)})_{(0)}, \tag{3-1}$$

for any $\alpha \in H^*$, $T \in \operatorname{End}_{\mathcal{A}}(P)$ and $p \in P$. See [Andruskiewitsch and Mombelli 2007] for more details.

From the previous paragraph, we can see that the categories ${}^{H}\mathcal{M}_{\mathcal{A}}$ play a central role in the theory. The following theorem will be of great use in the next section.

Theorem 3.1 [Skryabin 2007]. *Let H be a Hopf algebra and $\mathcal{A}$ a left H-comodule algebra, both finite-dimensional.*

(i) *If $\mathcal{A}$ is H-simple and $M \in {}^{H}\mathcal{M}_{\mathcal{A}}$, then there exists $t \in \mathbb{N}$ such that M^t, the direct sum of t copies of M, is a free $\mathcal{A}$-module.*

(ii) *$M \in {}^{H}\mathcal{M}_{\mathcal{A}}$ is free as an A-module if and only if there exists a maximal ideal $J \subset \mathcal{A}$ such that $M/M \cdot J$ is free as a $\mathcal{A}/J$-module.* □

Part (i) of this theorem is present in the proof of Theorem 3.5 of [Skryabin 2007]. Part (ii), which is Theorem 4.2 of the same paper, will be particularly useful when the ideal J is such that $\mathcal{A}/J = \Bbbk$, since in this case $M/M \cdot J$ is automatically free.

Theorem 3.2 [Mombelli 2011, Theorem 3.3]. *Let G be a finite group and let H be a finite-dimensional pointed Hopf algebra with coradical $\Bbbk G$. Assume there exists $V \in {}^{G}_{G}\mathcal{YD}$ such that* $\operatorname{gr} H = U = \mathfrak{B}(V) \# \Bbbk G$. *Let $\mathcal{A}$ be a left H-comodule algebra right H-simple with trivial coinvariants. There exist*

(1) *a subgroup $F \subseteq G$,*

(2) *a 2-cocycle $\psi \in Z^2(F, \Bbbk^\times)$,*

(3) *a homogeneous left coideal subalgebra $\mathcal{K} = \bigoplus_{i=0}^{m} \mathcal{K}(i) \subseteq \mathfrak{B}(V)$ where $\mathcal{K}(1) \subseteq V$ is a $\Bbbk G$-subcomodule invariant under the action of F,*

such that $\operatorname{gr} \mathcal{A} \simeq \mathcal{K} \# \Bbbk_\psi F$ *as left U-comodule algebras.* □

The algebra structure and the left U-comodule structure of $\mathcal{K} \# \Bbbk_\psi F$ is given as follows. If $x, y \in \mathcal{K}$, $f, g \in F$ then

$$(x \# g)(y \# f) = x(g \cdot y) \# \psi(g, f)\, gf,$$

$$\lambda(x \# g) = (x_{(1)} \# g) \otimes (x_{(2)} \# g),$$

where the action of F on $\mathcal{K}$ is the restriction of the action of G on $\mathfrak{B}(V)$ as an object in ${}^{G}_{G}\mathcal{Y}D$. Observe that F is necessarily a subgroup of $\operatorname{Stab}\mathcal{K}$.

4. Equivariant equivalence classes of comodule algebras

In this section we will show how to distinguish equivalence classes of some comodule algebras over pointed Hopf algebras, and then we will apply this result to our cases. Many of the ideas here are already contained in [Mombelli 2010; Mombelli 2011], although with less generality.

Let Γ be a finite group and H be a finite-dimensional pointed Hopf algebra with coradical $\Bbbk\Gamma$ and with coradical filtration $H_0 \subseteq H_1 \subseteq \cdots \subseteq H_m = H$. Assume there is $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ such that $\operatorname{gr} H = U = \mathfrak{B}(V) \# \Bbbk\Gamma$.

Lemma 4.1. *Take* Γ, U *as above, and let* $\sigma \in Z^2(\Gamma, \Bbbk^\times)$ *be a* 2*-cocycle. There exists a* 2*-cocycle* $\varsigma \in Z^2(U)$ *such that* $\varsigma|_{\Gamma\times\Gamma} = \sigma$.

Proof. Consider the linear map $\varsigma : U \times U \to \Bbbk$ defined on homogeneous elements $x, y \in U$ by

$$\varsigma(x, y) = \begin{cases} \sigma(x, y) & \text{if } x, y \in U(0); \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\varsigma(x, 1) = \varsigma(1, x) = \varepsilon(x)$ by definition. We have to check that (2-1) holds for $x \in U(m)$, $y \in U(n)$, $z \in U(k)$, and $m, n, k \in \mathbb{N}$. If $k > 0$, the left-hand side of (2-1) is zero. Set $\Delta(z) = \sum_{i=0}^{k} z^i \otimes z^{k-i}$, with $z^s \in U(s)$, $s = 0, \dots, k$. Analogously, set $\Delta(y) = \sum_{j=0}^{n} y^j \otimes y^{n-j}$, with $y^t \in U(t)$, $t = 0, \dots, n$. Then the right-hand side is

$$\sum_{i=0}^{k}\sum_{j=0}^{n} \varsigma(x, y^{n-j}z^{k-i})\, \varsigma(y^j, z^i) = \varsigma(x, y^n z^k) = 0,$$

and thus (2-1) holds. Both sides of this equation are similarly seen to be zero if $m > 0$ or $n > 0$, while the case $m = n = k = 0$ holds by definition of ς. This map is convolution invertible, and its inverse ς^{-1} is defined in an analogous manner, using σ^{-1}. □

Let $\mathcal{A}, \mathcal{A}'$ be two right H-simple left H-comodule algebras. Let $F, F' \subseteq \Gamma$ be subgroups and let $\psi \in Z^2(F, \Bbbk^\times)$, $\psi' \in Z^2(F', \Bbbk^\times)$ be two cocycles such that $\mathcal{A}_0 = \Bbbk_\psi F$ and $\mathcal{A}'_0 = \Bbbk_{\psi'} F'$. Let $K, K' \in \mathfrak{B}(V)$ be two homogeneous coideal subalgebras such that $\operatorname{gr}\mathcal{A} = K \# \Bbbk_\psi F$ and $\operatorname{gr}\mathcal{A}' = K' \# \Bbbk_{\psi'} F'$.

The main result of this section is this:

Theorem 4.2. *The categories* ${}_{\mathcal{A}}\mathcal{M}$ *and* ${}_{\mathcal{A}'}\mathcal{M}$ *are equivalent as modules over* $\operatorname{Rep}(H)$ *if and only if there exists an element* $g \in \Gamma$ *such that* $\mathcal{A}' \simeq g\mathcal{A}g^{-1}$ *as comodule algebras.*

Proof. Let us assume that ${}_{\mathcal{A}}\mathcal{M} \cong {}_{\mathcal{A}'}\mathcal{M}$ as $\operatorname{Rep}(H)$-modules. By [Andruskiewitsch and Mombelli 2007, Proposition 1.24] there exists an equivariant Morita context (P, Q, f, h); that is, $P \in {}^{H}_{\mathcal{A}'}\mathcal{M}_{\mathcal{A}}$, $Q \in {}^{H}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}'}$, $f : P \otimes_{\mathcal{A}} Q \to \mathcal{A}'$ and $h : Q \otimes_{\mathcal{A}'} P \to \mathcal{A}$, where the latter are bimodule isomorphisms, and $\mathcal{A}' \simeq \operatorname{End}_{\mathcal{A}}(P)$ as comodule algebras. The comodule structure on $\operatorname{End}_{\mathcal{A}}(P)$ is given by $\lambda : \operatorname{End}_{\mathcal{A}}(P) \to H \otimes \operatorname{End}_{\mathcal{A}}(P)$ with $\lambda(T) = T_{(-1)} \otimes T_{(0)}$, where

$$\langle \alpha, T_{(-1)} \rangle T_{(0)}(p) = \langle \alpha,\ T(p_{(0)})_{(-1)} \mathcal{S}^{-1}(p_{(-1)}) \rangle T(p_{(0)})_{(0)}, \tag{4-1}$$

for any $\alpha \in H^*$, $T \in \operatorname{End}_{\mathcal{A}}(P)$, and $p \in P$.

For any $i = 0, \dots, m$, define $P(i) = P_i / P_{i-1}$, where $P_{-1} = 0$. The graded vector space $\operatorname{gr} P = \bigoplus_{i=0}^{m} P(i)$ has an obvious structure that makes it into an object in the category ${}^{U}\mathcal{M}_{K \# \Bbbk_{\psi} F}$. Denote the coaction by $\overline{\delta} : \operatorname{gr} P \to U \otimes \operatorname{gr} P$. In particular, $\operatorname{gr} P \in {}^{U}\mathcal{M}_{K}$; thus, by Theorem 3.1(2), we have $\operatorname{gr} P \simeq M \otimes K$, where $M = \operatorname{gr} P / (\operatorname{gr} P \cdot K^{+})$ since $K / K^{+} = \Bbbk$.

We have $\overline{\delta}(\operatorname{gr} P \cdot K^{+}) \subset (U \otimes \operatorname{gr} P)(K^{+} \otimes 1 + U \otimes K^{+})$, since $K = \Bbbk \oplus K^{+}$ and thus the map $\overline{\delta}$ induces a new map $\widehat{\delta} : M \to U' \otimes M$, where $U' = U / U K^{+} U$. Notice that U' is a pointed Hopf algebra with coradical $\Bbbk\Gamma$, since U is coradically graded and the ideal $U K^{+} U$ is homogeneous and does not intersect U_0. M is also a $\Bbbk_{\psi} F$-module with $\overline{m} \cdot f = \overline{m \cdot f}$, for $f \in F$, $\overline{m} \in M$. This action is easily seen to be well defined and, moreover, $M \in {}^{U'}\mathcal{M}_{\Bbbk_{\psi} F}$.

Let $\Psi \in Z^2(\Gamma, \Bbbk^*)$ be a 2-cocycle such that $\Psi|_{F \times F} = \psi$; see [Brown 1982, Proposition III (9.5)]. Let $\zeta \in Z^2(U')$ be such that $\zeta|_{\Gamma \times \Gamma} = \Psi^{-1}$, as in Lemma 4.1. By [Mombelli 2010, Lemma 2.1], there exists and equivalence of categories

$$ {}^{U'^{\zeta}}\mathcal{M}_{\Bbbk F} \simeq {}^{U'}\mathcal{M}_{(\Bbbk F)_{\Psi}}. $$

By Theorem 3.1(2), any object in ${}^{U'^{\zeta}}\mathcal{M}_{\Bbbk F}$ is a free $\Bbbk F$-module. Thus, there is an object N in ${}^{U/U(\Bbbk F)^{+}}\mathcal{M}$ such that $\operatorname{gr} P \simeq N \otimes K \otimes \Bbbk_{\psi} F$. Therefore, $\dim P = (\dim N)(\dim \mathcal{A})$. Similarly, we can assume that there is an $s \in \mathbb{N}$ such that $\dim Q = s \dim \mathcal{A}'$.

Using Theorem 3.1(1), there exists $t \in \mathbb{N}$ such that P^t is a free right $\mathcal{A}$-module; that is, there is a vector space T such that $P^t \simeq T \otimes \mathcal{A}$, and hence

$$ t \dim N = \dim T. \tag{4-2}$$

Since $P \otimes_{\mathcal{A}} Q \simeq \mathcal{A}'$, we have $P^t \otimes_{\mathcal{A}} Q \simeq T \otimes Q \simeq \mathcal{A}'^{t}$ and so $s \dim T \dim \mathcal{A}' = t \dim \mathcal{A}'$. Using (4-2), we see that $s \dim N = 1$, so $\dim N = 1$, so $\dim P = \dim \mathcal{A}$.

Claim 4.1. *If $n \in P_0$, then $P = n \cdot \mathcal{A}$.*

Notice that $P_0 \neq 0$. In fact, if $P_0 = 0$ and $k \in \mathbb{N}$ is minimal with $P_k \neq 0$, then $\lambda(P_k) \subset \sum_{j=0}^{k} H_{k-j} \otimes P_j = H_0 \otimes P_k$, which is a contradiction. Let $g \in \Gamma$ be such that $\lambda(n) = g \otimes n$. Now, if $J = \{a \in \mathcal{A} : n \cdot a = 0\}$, then J is a right ideal of $\mathcal{A}$. We will prove that $J = 0$. Let $a \in J$ and write $\lambda(a) = \sum_{i=1}^{n} a^i \otimes a_i$ in such way that the set $\{a^i : i = 1, \dots, n\} \subset H$ is linearly independent. Now, $\{ga^i : i = 1, \dots, n\} \subset H$ is also linearly independent, and we have $0 = \lambda(n \cdot a) = \sum_{i=1}^{n} ga^i \otimes n \cdot a_i$. Thus, $n \cdot a_i = 0$ for all $i = 1, \dots, n$; that is, $\lambda(a) \in H \otimes J$ and J is H-costable. As $\mathcal{A}$ is right H-simple, we have $J = 0$. Therefore, the action $\cdot : N \otimes \mathcal{A} \to P$ is injective and, since $\dim P = \dim N \dim \mathcal{A}$, the claim follows.

It is not difficult to prove that the linear map $\phi : g\mathcal{A}g^{-1} \to \operatorname{End}_{\mathcal{A}}(P)$ given by $\phi(gag^{-1})(n \cdot b) = n \cdot ab$ is an isomorphism of H-comodule algebras.

Conversely, if $\mathcal{A}' \simeq g\mathcal{A}g^{-1}$ as comodule algebras and $M \in {}_{\mathcal{A}}\mathcal{M}$, then the set gMg^{-1} has a natural structure of $\mathcal{A}'$-module in such way that the functor $F : {}_{\mathcal{A}}\mathcal{M} \to {}_{\mathcal{A}'}\mathcal{M}$ with $M \mapsto gMg^{-1}$ is an equivalence of $\operatorname{Rep}(H)$-modules. □

5. Pointed Hopf algebras over $\mathbb{S}_3$ and $\mathbb{S}_4$

In this section we describe all pointed Hopf algebras whose coradical is the group algebra of the groups $\mathbb{S}_3$ and $\mathbb{S}_4$. These were classified in [Andruskiewitsch et al. 2010] and [García and García Iglesias ≥ 2011], respectively.

Recall that a *rack* is a pair $(X, \triangleright)$, where X is a nonempty set and $\triangleright : X \times X \to X$ is a function, such that, for all $i \in X$, $\phi_i = i \triangleright (\cdot) : X \to X$ is a bijection, and satisfies $i \triangleright (j \triangleright k) = (i \triangleright j) \triangleright (i \triangleright k)$ for all $i, j, k \in X$. See [Andruskiewitsch and Graña 2003a] for detailed information on racks.

Let $(X, \triangleright)$ be a rack. A 2-cocycle $q : X \times X \to \Bbbk^{\times}$, denoted by $(i, j) \mapsto q_{ij}$, is a function such that, for all $i, j, k \in X$,

$$q_{i, j \triangleright k}\, q_{j,k} = q_{i \triangleright j, i \triangleright k}\, q_{i,k}.$$

In this case, it is possible to generate a braiding c^q in the vector space $\Bbbk X$ with basis $\{x_i\}_{i \in X}$ by setting $c^q(x_i \otimes x_j) = q_{ij} x_{i \triangleright j} \otimes x_i$ for all $i, j \in X$. We denote by $\mathfrak{B}(X, q)$ the Nichols algebra of this braided vector space.

5A. *Quadratic approximations to Nichols algebras.*

Let

$$\mathcal{J} = \bigoplus_{r \geq 2} \mathcal{J}^r$$

be the defining ideal of the Nichols algebra $\mathfrak{B}(X, q)$. We give a description of the space $\mathcal{J}^2$ of quadratic relations.

Let $\mathcal{R}$ be the set of equivalence classes in $X \times X$ for the relation generated by $(i, j) \sim (i \triangleright j, i)$. Let $C \in \mathcal{R}$ and $(i, j) \in C$. Take $i_1 = j$, $i_2 = i$ and, recursively, $i_{h+2} = i_{h+1} \triangleright i_h$. Set $n(C) = \#C$ and

$$\mathcal{R}' = \left\{ C \in \mathcal{R} : \prod_{h=1}^{n(C)} q_{i_{h+1}, i_h} = (-1)^{n(C)} \right\}.$$

Let $\mathcal{T}$ be the free associative algebra in the variables $\{T_l\}_{l\in X}$. If $C \in \mathcal{R}'$, consider the quadratic polynomial

$$\phi_C = \sum_{h=1}^{n(C)} \eta_h(C)\, T_{i_{h+1}} T_{i_h} \in \mathcal{T}, \tag{5-1}$$

where $\eta_1(C) = 1$ and $\eta_h(C) = (-1)^{h+1} q_{i_2 i_1} q_{i_3 i_2} \dots q_{i_h i_{h-1}}$ with $h \geq 2$. Then, a basis of the space $\mathcal{J}^2$ is given by

$$\phi_C\big(\{x_i\}_{i\in X}\big) \quad \text{for } C \in \mathcal{R}'. \tag{5-2}$$

We denote by $\widehat{\mathfrak{B}}_2(X, q)$ the quadratic approximation of $\mathfrak{B}(X, q)$, that is, the algebra defined by relations $\langle \mathcal{J}^2 \rangle$. For more details, see [García and García Iglesias $\geq$ 2011, Lemma 2.2].

Let G be a finite group. A *principal YD-realization* of (X, q) over G [Andruskiewitsch and Graña 2003a, def 3.2] is a way to realize this braided vector space $(\Bbbk X, c^q)$ as a Yetter–Drinfeld module over G. Explicitly, it is a collection $(\,\cdot\,, g, (\chi_i)_{i\in X})$, where

- $\cdot$ is an action of G on X,
- $g : X \to G$ is a function such that $g_{h\cdot i} = h g_i h^{-1}$ and $g_i \cdot j = i \triangleright j$,
- the family $(\chi_i)_{i\in X}$, where $\chi_i : G \to \Bbbk^*$ is a 1-cocycle (that is,

  $$\chi_i(ht) = \chi_i(t)\, \chi_{t\cdot i}(h),$$

  for all $i \in X$ and $h, t \in G$) satisfies $\chi_i(g_j) = q_{ji}$.

If $(\,\cdot\,, g, (\chi_i)_{i\in X})$ is a principal YD-realization of (X, q) over G, then $\Bbbk X \in {}^{G}_{G}\mathcal{YD}$, as follows. The action and coaction of G are determined by

$$\delta(x_i) = g_i \otimes x_i, \quad h \cdot x_i = \chi_i(h)\, x_{h\cdot i} \quad \text{for } i \in X, h \in G.$$

Lemma 5.1. *If $(i \triangleright j) \triangleright i = j$ for any $i, j \in X$, then*

$$\chi_i(f)\, q_{f\cdot i \triangleright f\cdot j,\, f\cdot i} = \chi_j(f)\, q_{i\triangleright j,\, i} \quad \textit{for any } f \in G \textit{ and } i, j \in X. \tag{5-3}$$

5B. ***Nichols algebras over $\mathbb{S}_n$.*** Let X be $\mathcal{O}_2^n$ or $\mathcal{O}_4^4$, considered as racks with the map $\triangleright$ given by conjugation. Consider the maps

$$\operatorname{sgn} : \mathbb{S}_n \times X \to \Bbbk^*, \ (\sigma, i) \mapsto \operatorname{sgn}(\sigma),$$

$$\chi : \mathbb{S}_n \times \mathcal{O}_2^n \to \Bbbk^*, (\sigma, i) \mapsto \chi_i(\sigma) = \begin{cases} 1 & \text{if } i = (a, b) \text{ and } \sigma(a) < \sigma(b), \\ -1 & \text{if } i = (a, b) \text{ and } \sigma(a) > \sigma(b). \end{cases}$$

We will deal with the cocycles

$$\begin{aligned} -1 : X\times X &\to \Bbbk^*, & (j,i)&\mapsto \operatorname{sgn}(j)=-1, & i,j&\in X;\\ \chi : \mathcal{O}_2^n\times\mathcal{O}_2^n &\to \Bbbk^*, & (j,i)&\mapsto \chi_i(j) & i,j&\in\mathcal{O}_2^n. \end{aligned}$$

The quadratic approximations of the corresponding Nichols algebras are

$$\begin{aligned} \widehat{\mathfrak{B}}_2(\mathcal{O}_2^n,-1) = \Bbbk\big\langle x_{(lm)},\ 1\le l<m\le n \,\big|\, & x_{(ab)}^2,\ x_{(ab)}x_{(ef)}+x_{(ef)}x_{(ab)},\\ & x_{(ab)}x_{(bc)}+x_{(bc)}x_{(ac)}+x_{(ac)}x_{(ab)},\\ & 1\le a<b<c\le n,\ 1\le e<f\le n,\ \{a,b\}\cap\{e,f\}=\varnothing\big\rangle,\\ \widehat{\mathfrak{B}}_2(\mathcal{O}_2^n,\chi) = \Bbbk\big\langle x_{(lm)},\ 1\le l<m\le n \,\big|\, & x_{(ab)}^2,\ x_{(ab)}x_{(ef)}-x_{(ef)}x_{(ab)},\\ & x_{(ab)}x_{(bc)}-x_{(bc)}x_{(ac)}-x_{(ac)}x_{(ab)},\\ & x_{(bc)}x_{(ab)}-x_{(ac)}x_{(bc)}-x_{(ab)}x_{(ac)},\\ & 1\le a<b<c\le n,\ 1\le e<f\le n,\ \{a,b\}\cap\{e,f\}=\varnothing\big\rangle,\\ \widehat{\mathfrak{B}}_2(\mathcal{O}_4^4,-1) = \Bbbk\big\langle x_i,\ i\in\mathcal{O}_4^4 \,\big|\, & x_i^2,\ x_ix_{i^{-1}}+x_{i^{-1}}x_i,\\ & x_ix_j+x_kx_i+x_jx_k \text{ if } ij=ki \text{ and } j\ne i\ne k\in\mathcal{O}_4^4\big\rangle. \end{aligned}$$

Example 5.2. A principal YD-realization of $(\mathcal{O}_2^n,-1)$ or $(\mathcal{O}_2^n,\chi)$, respectively of $(X,q)=(\mathcal{O}_4^4,-1)$, over $\mathbb{S}_n$, respectively over $\mathbb{S}_4$, is given by the inclusion $X\hookrightarrow\mathbb{S}_n$, and the action $\cdot$ is the conjugation. The family $\{\chi_i\}$ is determined by the cocycle. In either case, g is injective. For $n=3,4,5$, this is in fact the only possible realization over $\mathbb{S}_n$.

Remark 5.3. Notice that all $(\mathcal{O}_2^n,-1)$ and $(\mathcal{O}_2^n,\chi)$ for any n, and $(\mathcal{O}_4^4,-1)$ satisfy that $\mathfrak{R}=\mathfrak{R}'$. When $n=3,4,5$, we have from [Andruskiewitsch and Graña 2003b; García and García Iglesias ≥ 2011]

$$\begin{gathered} \widehat{\mathfrak{B}}_2(\mathcal{O}_2^n,-1)=\mathfrak{B}(\mathcal{O}_2^n,-1),\\ \widehat{\mathfrak{B}}_2(\mathcal{O}_2^n,\chi)=\mathfrak{B}(\mathcal{O}_2^n,\chi),\\ \dim\mathfrak{B}(\mathcal{O}_2^n,-1),\ \dim\mathfrak{B}(\mathcal{O}_2^n,\chi)<\infty. \end{gathered}$$

5C. ***Pointed Hopf algebras constructed from racks.*** A *quadratic lifting datum* (or ql-datum) [García and García Iglesias ≥ 2011, definition 3.5] is a collection $\mathfrak{Q}=\big(X,q,G,\big(\cdot,g,(\chi_l)_{l\in X}\big),(\gamma_C)_{C\in\mathfrak{R}'}\big)$ consisting of a rack X, a 2-cocycle q, a finite group G, a principal YD-realization $\big(\cdot,g,(\chi_l)_{l\in X}\big)$ of (X,q) over G such that $g_i\ne g_jg_k$ for all $i,j,k\in X$, and a collection $(\gamma_C)_{C\in\mathfrak{R}'}\in\Bbbk$ satisfying that, for each $C=\{(i_2,i_1),\dots,(i_n,i_{n-1})\}\in\mathfrak{R}'$ and $k\in X$, we have

$$\gamma_C=0 \qquad \text{if } g_{i_2}g_{i_1}=1, \tag{5-4}$$

$$\gamma_C=q_{ki_2}q_{ki_1}\gamma_{k\triangleright C} \qquad \text{if } k\triangleright C=\{k\triangleright(i_2,i_1),\dots,k\triangleright(i_n,i_{n-1})\}. \tag{5-5}$$

To each ql-datum $\mathfrak{Q}$ is attached a pointed Hopf algebra $\mathcal{H}(\mathfrak{Q})$, generated as an algebra by $\{a_l, H_t : l \in X,\ t \in G\}$ subject to the relations

$$H_e = 1, \quad H_t H_s = H_{ts} \qquad \text{for } t, s \in G; \tag{5-6}$$

$$H_t\, a_l = \chi_l(t)\, a_{t\cdot l}\, H_t \qquad \text{for } t \in G,\ l \in X; \tag{5-7}$$

$$\phi_C(\{a_l\}_{l\in X}) = \gamma_C(1 - H_{g_i g_j}) \quad \text{for } C \in \mathcal{R}',\ (i, j) \in C. \tag{5-8}$$

Here, ϕ_C is as in (5-1) above. The algebra $\mathcal{H}(\mathfrak{Q})$ has a structure of pointed Hopf algebra by setting

$$\Delta(H_t) = H_t \otimes H_t, \quad \Delta(a_i) = g_i \otimes a_i + a_i \otimes 1 \qquad \text{for } t \in G, i \in X.$$

See [García and García Iglesias $\geq$ 2011] for further details.

5D. ***Pointed Hopf algebras over $\mathbb{S}_n$.*** The following ql-data provide examples of (possibly infinite-dimensional) pointed Hopf algebras over $\mathbb{S}_n$: for $\alpha, \beta, \lambda \in \Bbbk$ and $t = (\alpha, \beta)$,

(1) $\mathfrak{Q}_n^{-1}[t] = (\mathbb{S}_n, \mathcal{O}_2^n, -1,\ \cdot\ , \iota, \{0, \alpha, \beta\})$,
(2) $\mathfrak{Q}_n^{\chi}[\lambda] = (\mathbb{S}_n, \mathcal{O}_2^n, \chi,\ \cdot\ , \iota, \{0, 0, \alpha\})$, and
(3) $\mathfrak{D}[t] = (\mathbb{S}_4, \mathcal{O}_4^4, -1,\ \cdot\ , \iota, \{\alpha, 0, \beta\})$.

We will present explicitly the algebras $\mathcal{H}(\mathfrak{Q})$ associated to these data. It will follow that relations (5-8) for each $C \in \mathcal{R}'$ with the same cardinality are $\mathbb{S}_n$-conjugated. Thus, for each C with a given number of elements, it is enough to consider a single relation.

Example 5.4. $\mathcal{H}(\mathfrak{Q}_n^{-1}[t])$ is the algebra generated by $\{a_i, H_r : i \in \mathcal{O}_2^n, r \in \mathbb{S}_n\}$ with relations

$$\begin{aligned}
&H_e = 1, \quad H_r H_s = H_{rs} \quad \text{for } r, s \in \mathbb{S}_n;\\
&H_j a_i = -a_{jij} H_j \quad \text{for } i, j \in \mathcal{O}_2^n;\\
&a_{(12)}^2 = 0;\\
&a_{(12)}a_{(34)} + a_{(34)}a_{(12)} = \alpha(1 - H_{(12)}H_{(34)});\\
&a_{(12)}a_{(23)} + a_{(23)}a_{(13)} + a_{(13)}a_{(12)} = \beta(1 - H_{(12)}H_{(23)}).
\end{aligned}$$

Example 5.5. $\mathcal{H}(\mathfrak{Q}_n^{\chi}[\lambda])$ is the algebra generated by $\{a_i, H_r : i \in \mathcal{O}_2^n, r \in \mathbb{S}_n\}$ with relations

$$\begin{aligned}
&H_e = 1, \quad H_r H_s = H_{rs} \quad \text{for } r, s \in \mathbb{S}_n;\\
&H_j a_i = \chi_i(j)\, a_{jij} H_j \quad \text{for } i, j \in \mathcal{O}_2^n;\\
&a_{(12)}^2 = 0;\\
&a_{(12)}a_{(34)} - a_{(34)}a_{(12)} = 0;\\
&a_{(12)}a_{(23)} - a_{(23)}a_{(13)} - a_{(13)}a_{(12)} = \alpha(1 - H_{(12)}H_{(23)}).
\end{aligned}$$

Example 5.6. $\mathcal{H}(\mathfrak{D}[t])$ is the algebra generated by $\{a_i, H_r : i \in \mathcal{O}_4^4, r \in \mathbb{S}_4\}$ with relations

$$\begin{aligned}
&H_e = 1, \quad H_r H_s = H_{rs} \quad \text{for } r, s \in \mathbb{S}_n;\\
&H_j a_i = -a_{jij} H_j \quad \text{for } i \in \mathcal{O}_4^4,\ j \in \mathcal{O}_2^4;\\
&a_{(1234)}^2 = \alpha(1 - H_{(13)} H_{(24)});\\
&a_{(1234)} a_{(1432)} + a_{(1432)} a_{(1234)} = 0;\\
&a_{(1234)} a_{(1243)} + a_{(1243)} a_{(1423)} + a_{(1423)} a_{(1234)} = \beta(1 - H_{(12)} H_{(13)}).
\end{aligned}$$

These Hopf algebras have been defined in [Andruskiewitsch and Graña 2003b, def 3.7], [García and García Iglesias $\geq$ 2011, def 3.9] and [García and García Iglesias $\geq$ 2011, def 3.10], respectively. Each of these $\mathcal{H}(\mathfrak{Q})$ satisfies $\operatorname{gr} \mathcal{H}(\mathfrak{Q}) = \widehat{\mathfrak{B}}_2(X, q) \# \Bbbk G$ for $G = \mathbb{S}_n$, with n as appropriate [García and García Iglesias $\geq$ 2011, propositions 5.4, 5.5, 5.6].

Remark 5.7. We have the following classification results:

(1) $\mathcal{H}(\mathfrak{Q}_3^{-1}[t])$, with $t = (0, 0)$ or $t = (0, 1)$, are all the nontrivial finite-dimensional pointed Hopf algebras over $\mathbb{S}_3$ [Andruskiewitsch et al. 2010].

(2) $\mathcal{H}(\mathfrak{Q}_4^{-1}[t])$, $\mathcal{H}(\mathfrak{Q}_4^{\chi}[\zeta])$ and $\mathcal{H}(\mathfrak{D}[t])$, with $t \in \mathbb{P}^1_{\Bbbk} \cup \{(0, 0)\}$ and $\zeta \in \{0, 1\}$, is a complete list of the nontrivial finite-dimensional pointed Hopf algebras over $\mathbb{S}_4$ [García and García Iglesias $\geq$ 2011].

We will classify module categories over the category of representations of any pointed Hopf algebra over $\mathbb{S}_3$ or $\mathbb{S}_4$, that is, of the algebras listed in Remark 5.7.

6. Coideal subalgebras of quadratic Nichols algebras

A fundamental piece of information to determine simple comodule algebras is the computation of homogeneous coideal subalgebras inside the Nichols algebra. This is part of Theorem 3.2. The study of coideal subalgebras is an active field of research in the theory of Hopf algebras and quantum groups, see for example [Heckenberger and Kolb 2011; Heckenberger and Schneider $\geq$ 2011; Kharchenko $\geq$ 2011; Kharchenko and Sagahon 2008].

In this section we present a description of all homogeneous left coideal subalgebras in the quadratic approximations of the Nichols algebras constructed from racks.

Fix $n \in \mathbb{N}$, let $X = \{i_1, \dots, i_n\}$ be a rack of n elements and $q : X \times X \to \Bbbk^*$ a 2-cocycle. Let $\mathcal{R}$ be as in Section 5A. Assume that, for any equivalence class C in $\mathcal{R}$ and $i, j, k \in X$, we have

$$(i, j), (i, k) \in C \Rightarrow j = k \quad \text{and} \quad (i, j), (k, i) \in C \Rightarrow k = i \triangleright j. \tag{6-1}$$

Let G be a finite group and let $(\,\cdot\,, g, (\chi_i)_{i\in X})$ be a principal YD-realization of (X, q) over G. We further assume that

$$g \text{ is injective} \quad \text{and} \quad \mathcal{R} = \mathcal{R}'. \tag{6-2}$$

For each subset $Y \subseteq X$, with $Y = \{i_{j_1}, \dots, i_{j_r}\} \subseteq X$, denote by $\mathcal{K}_Y$ the subalgebra of $\widehat{\mathfrak{B}}_2(X, q) \,\#\, \Bbbk 1$ generated by $x_{j_1}, \dots, x_{j_r}$. Set $\mathcal{H} = \widehat{\mathfrak{B}}_2(X, q) \,\#\, \Bbbk G$.

Proposition 6.1. *For each set $Y = \{i_{j_1}, \dots, i_{j_r}\} \subseteq X$ the algebra $\mathcal{K}_Y$ is an homogeneous coideal subalgebra of $\mathcal{H}$. For each such selection, if $S = \{g_{i_{j_1}}, \dots, g_{i_{j_r}}\}$, then*

$$\operatorname{Stab} \mathcal{K}_Y = S_Y^G = \{h \in G : h S_Y h^{-1} = S_Y\}.$$

Moreover, if $\mathcal{K}$ is a homogeneous coideal subalgebra of $\mathcal{H}$ generated in degree one, then there exists a unique $Y \subseteq X$ such that

$$\mathcal{K} = \mathcal{K}_Y.$$

In particular, the set of homogeneous coideal subalgebras of $\mathcal{H}$ generated in degree one inside $\widehat{\mathfrak{B}}_2(X, q) \,\#\, \Bbbk 1$ is in bijective correspondence with the set 2^X of parts of X.

Proof. It is clear that $\mathcal{K} = \mathcal{K}_Y$ is a homogeneous coideal subalgebra. To describe $\operatorname{Stab} \mathcal{K}$ it is enough to compute the stabilizer of the vector space $\Bbbk\{x_{j_1}, \dots, x_{j_r}\}$. But $h \cdot x_{j_k} = \chi_{j_k}(h)\, x_{h\cdot j_k}$, $k = 1, \dots, r$, and $x_{h\cdot j_k} \in \{x_{j_1}, \dots, x_{j_r}\}$ if and only if $h \cdot j_k \in \{j_1, \dots, j_r\}$, if and only if $g_{h\cdot j_k} = g_{j_l}$ for some $l = 1, \dots, r$. And the first part of the proposition follows since $g_{h\cdot j_k} = h g_{j_k} h^{-1}$.

Let $\mathcal{K}$ be a homogeneous coideal subalgebra of $\mathcal{H}$ generated in degree one. If $\mathcal{K} = \Bbbk$ the result is trivial, so assume that $\mathcal{K} \neq \Bbbk$. Since $\mathcal{K}$ is homogeneous, $\mathcal{K}(1) \neq 0$. Let $0 \neq y = \sum_i \lambda_i x_i \in \mathcal{K}(1)$; then

$$\Delta(y) = y \otimes 1 + \sum_i \lambda_i H_{g_i} \otimes x_i \quad \text{which implies} \quad \sum_i \lambda_i H_{g_i} \otimes x_i \in \mathcal{H}_0 \otimes \mathcal{K}(1).$$

Let $\sum_i \lambda_i H_{g_i} \otimes x_i = \sum_{t\in G} H_t \otimes \kappa_t$, where $\kappa_t = \sum_{j\in X} \eta_{tj} x_j \in \mathcal{K}(1)$ with $\eta_{tj} \in \Bbbk$, for all t, j.

From the assumption (6-2) we know that $H_t = H_{g_j}$ if and only if $t = g_j$, and $g_i = g_j$ if and only if $i = j$, where $i, j \in X$ and $t \in G$. Hence $\eta_{tk} = 0$ if $t \neq g_k$ for some $k \in X$. Set $\eta_{ij} = \eta_{g_i j}$. Thus,

$$\sum_i \lambda_i H_{g_i} \otimes x_i = \sum_{i,j} \eta_{ij} H_{g_i} \otimes x_j.$$

Therefore, $\lambda_i \neq 0$ implies $\eta_{ij} = \delta_{i,j} \lambda_i$, and so $\kappa_i = x_i$. Thus, $\{x_i \mid \lambda_i \neq 0\} \subset \mathcal{K}$ and $\mathcal{K}(1) = \bigoplus_{x_i \in \mathcal{K}(1)} \Bbbk\{x_i\}$. Therefore, if $Y = \{i \in X : x_i \in \mathcal{K}(1)\}$, then $\mathcal{K} = \mathcal{K}_Y$. Finally, if $Y \neq Y'$, then it follows from the injectivity of g that $\mathcal{K}_Y \ncong \mathcal{K}_{Y'}$ as coideal subalgebras. $\square$

The next general lemma will be useful in Section 6A to prove that certain subalgebras are generated in degree one. Given a rack X, let us recall the notion of derivations δ_i associated to every element of the canonical basis $\{e_i\}_{i\in X}$. If $\{e^i\}_{i\in X}$ denotes the dual basis of $\{e_i\}_{i\in X}$, then $\delta_i = (\mathrm{id}\otimes e^i)\Delta$. For $i \in X$, we denote by X_i the set $X\setminus\{i\}$, and thus $\Bbbk X_i = \Bbbk\{x_j \mid j \in X_i\}$. Let us assume, furthermore, that

$$q_{ii} = -1 \quad \text{for all } i \in X. \tag{6-3}$$

By (6-2), this condition is satisfied if, for example, $\dim \widehat{\mathfrak{B}}_2(X, q) < \infty$ or X is such that $i \triangleright i = i$ for all i.

Lemma 6.2. *Let $\mathcal{K} \subset \widehat{\mathfrak{B}}_2(X, q) \# \Bbbk 1$ be a homogeneous coideal subalgebra of $\mathcal{H}$, and let $i \in X$. If there is an $\omega \in \mathcal{K}$ such that $\delta_i(\omega) \neq 0$, then $x_i \in \mathcal{K}(1)$.*

Proof. Let $\mathcal{K} = \bigoplus_s \mathcal{K}(s)$, $\omega \in T(\Bbbk X)$, and $i \in X$. In $\mathcal{H}$,

$$\omega = \alpha_i(\omega) + \beta_i(\omega)x_i \quad \text{with } \alpha_i(\omega), \beta_i(\omega) \in \mathcal{H}_{X_i}.$$

It suffices to see this for a homogeneous monomial ω. We see it by induction on $\ell = \ell(\omega) \in \mathbb{N}$ such that $\omega \in T^\ell(\Bbbk X)$. If $\ell = 0$ or $\ell = 1$, this is clear. Let us assume it holds for $\ell = n-1$, for some $n \in \mathbb{N}$. If $\ell(\omega) = n$ and $\omega = x_{j_1}\dots x_{j_n}$, two possibilities hold: $j_1 \neq i$ or $j_1 = i$. In the first case, let $\omega' = x_{j_2}\dots x_{j_n}$. Then, $\ell(\omega') \leq n-1$ and therefore there exist $\alpha_i(\omega'), \beta_i(\omega') \in \mathcal{H}_{X_i}$ such that $\omega' = \alpha_i(\omega') + \beta_i(\omega')x_i$. As $x_{j_1}\alpha_i(\omega'),\ x_{j_1}\beta_i(\omega') \in \mathcal{H}_{X_i}$, in this case the claim follows.

In the second case, let $j = j_2$ and let us note that $j \neq i$, by (6-3). By (6-2), we can consider the relation

$$x_i x_j = q_{ij} x_{i\triangleright j} x_i - q_{ij} q_{i\triangleright j\, i} x_j x_{i\triangleright j}.$$

Thus, if $\omega'' = x_{j_3}\dots x_{j_n}$, then $\omega = q_{ij}x_{i\triangleright j}x_i\omega'' - q_{ij}q_{i\triangleright j\, i}x_j x_{i\triangleright j}\omega''$ and both members of this sum belong to $\mathcal{H}_{X_i} + \mathcal{H}_{X_i}x_i$ because of the previous case; the claim follows.

Let $\pi : \bigoplus_{s=0}^m \mathcal{H}(s)\otimes\mathcal{H}(m-s) \to \mathcal{H}(m-1)\otimes\mathcal{H}(1)$ be the canonical linear projection. Let $\omega \in T(\Bbbk X)$, $i \in X$ and $\alpha_i(\omega)$, $\beta_i(\omega)$ as above. Then,

$$\pi\Delta(\omega) \in \beta_i(\omega)\otimes x_i + \bigoplus_{j\neq i}\mathcal{H}\otimes x_j.$$

Notice that $\delta_i(\omega) = \beta_i(\omega)$, and therefore, if $\delta_i(\omega) \neq 0$, it follows that $x_i \in \mathcal{K}(1)$, by using (6-2) as in the proof of Proposition 6.1. □

In this part we will assume that X is one of the racks $\mathcal{O}_2^n$, $n \in \mathbb{N}$, or $\mathcal{O}_4^4$, with q one of the cocycles in Section 5B. Notice that (6-1) is satisfied in these cases. Using the previous results, we will describe explicitly all connected homogeneous coideal subalgebras of the bosonization of the quadratic approximations to Nichols algebras described in Section 5B.

We first introduce some notation. Let $Y \subset X$ be a subset, and define

$$\begin{aligned} \mathcal{R}_1^Y &= \{C \in \mathcal{R} : C \subseteq Y \times Y\}, \\ \mathcal{R}_2^Y &= \{C \in \mathcal{R} : |C \cap Y \times Y| = 1\}, \\ \mathcal{R}_3^Y &= \{C \in \mathcal{R} : C \cap Y \times Y = \varnothing\}. \end{aligned}$$

Remark 6.3. For the ql-data in Section 5D, we have $\mathcal{R} = \mathcal{R}_1^Y \cup \mathcal{R}_2^Y \cup \mathcal{R}_3^Y$ for any subset Y. If $f \in \operatorname{Stab} \mathcal{H}_Y$ then $f \cdot \mathcal{R}_s^Y \subseteq \mathcal{R}_s^Y$ for any $s = 1, 2, 3$. Also, (6-3) holds.

Definition 6.4. Take the free associative algebra $\mathcal{T}$ in the variables $\{T_l\}_{l \in Y}$. According to this, we set $\vartheta_{C,Y}(\{T_l\}_{l\in Y})$ in $\mathcal{T}$ as

$$\vartheta_{C,Y}(\{T_l\}_{l\in Y}) = \begin{cases} \phi_C(\{T_l\}_{l\in X}) & \text{if } C \in \mathcal{R}_1^Y; \\ T_i T_j T_i - q_{i\triangleright j,i}\, T_j T_i T_j & \text{if } C \in \mathcal{R}_2^Y,\ (i,j) \in C \cap Y \times Y; \\ 0 & \text{if } C \in \mathcal{R}_3^Y. \end{cases} \tag{6-4}$$

We define the algebra $\mathcal{L}_Y$ as

$$\mathcal{L}_Y = \Bbbk\langle\{y_i\}_{i\in Y}\rangle \,/\, \langle \vartheta_{C,Y}(\{y_l\}_{l\in Y}) : C \in \mathcal{R}\rangle. \tag{6-5}$$

If $Y = X$, then $\mathcal{L}_X \cong \mathfrak{B}(X, q)$. For simplicity, we sometimes write ϑ_C for $\vartheta_{C,Y}$.

Take $\mathfrak{B}$ to be one of the quadratic (Nichols) algebras $\widehat{\mathfrak{B}}_2(\mathbb{O}_2^n, -1)$, $\widehat{\mathfrak{B}}_2(\mathbb{O}_2^n, \chi)$, or $\mathfrak{B}(\mathbb{O}_4^4, -1)$. Accordingly, set $X = \mathbb{O}_2^n$, $q = -1$, χ or $(X, q) = (\mathbb{O}_4^4, -1)$. Consider a YD-realization for (X, q) such that (6-2) is satisfied (for instance, one from Example 5.2). Set $\mathcal{H} = \mathfrak{B} \# \Bbbk G$.

Theorem 6.5. *Let $Y \subset X$. $\mathcal{L}_Y$ is an $\mathcal{H}$-comodule algebra with coaction*

$$\delta(y_i) = g_i \otimes y_i + x_i \otimes 1, \quad i \in Y.$$

The map $y_i \mapsto x_i$, $i \in Y$, defines an epimorphism of $\mathcal{H}$-comodule algebras $\mathcal{L}_Y \twoheadrightarrow \mathcal{H}_Y$. Moreover, if $n = 3$, it is an isomorphism and $\mathcal{L}_Y \cong \mathcal{H}_Y$.

Proof. The relations that define $\mathcal{L}_Y$ are satisfied in $\mathfrak{B}$. In fact, it suffices to check this in the case $C \in \mathcal{R}_2^Y$, since in the other ones we have $\vartheta_C = 0$ or $\vartheta_C = \phi_C$, and $\phi_C = 0$ in $\mathfrak{B}$; see (5-2). Now, if $C \in \mathcal{R}_2^Y$ and $(i, j) \in C \cap Y \times Y$, let $k = i \triangleright j$. By the definition of $\mathcal{R}_2^Y$, we necessarily have $k \neq i, j$. Then, if we multiply the relation $x_i x_j - q_{ij} x_{i\triangleright j} x_i + q_{ij} q_{i\triangleright j\, i} x_j x_{i\triangleright j} = 0$ by x_i on the right, and apply these relations to the outcome, we get

$$\begin{aligned} 0 &= x_i x_j x_i + q_{ij} q_{i\triangleright j\, i}\, x_j x_{i\triangleright j} x_i = x_i x_j x_i + q_{i\triangleright j\, i} x_j (x_i x_j + q_{ij} q_{i\triangleright j\, i}\, x_j x_{i\triangleright j}) \\ &= x_i x_j x_i + q_{i\triangleright j\, i}\, x_j x_i x_j. \end{aligned}$$

Thus, we have an algebra projection $\pi : \mathcal{L}_Y \twoheadrightarrow \mathcal{H}_Y$. It is straightforward to see that

$$\delta(\vartheta_{C,Y}(\{y_l\}_{l\in Y})) = \vartheta_{C,Y}(\{x_l\}_{l\in Y}) \otimes 1 + g_{C,Y} \otimes \vartheta_{C,Y}(\{y_l\}_{l\in Y}) \quad \text{for every } C \in \mathcal{R},$$

where

$$g_{C,Y} = \begin{cases} g_i g_j & \text{if } C \in \mathfrak{R}_1^Y,\ (i,j) \in C, \\ g_i g_j g_i & \text{if } C \in \mathfrak{R}_2^Y,\ (i,j) \in C \cap Y \times Y, \\ 0, & \text{if } C \in \mathfrak{R}_3^Y. \end{cases}$$

Therefore, δ provides $\mathcal{L}_Y$ with a structure of $\mathcal{H}$-comodule in such a way that π becomes a homomorphism.

We analyze now the particular case $n = 3$. If $|Y| = 1$, the result is clear. Let us suppose then that $Y = \{i, j\} \subset \mathbb{O}_2^3$. Notice that the map π is homogeneous. If $\gamma \in \ker(\pi)$, then $\pi(\gamma) = 0$ in $\mathfrak{B}(\mathbb{O}_2^3, -1)$. By (5-2), we necessarily have $\deg \gamma \geq 3$. Now, if $\deg \gamma = 3$, then

$$\gamma = \alpha y_i y_j y_i + \beta y_j y_i y_j = (\alpha + \beta) y_j y_i y_j.$$

for $\alpha, \beta \in \Bbbk$. Then, $\pi(\gamma) = 0$ implies that $\alpha = -\beta$ and $\gamma = 0$. Finally, we can see that there are no elements $\gamma \in \mathcal{L}_Y$ with $\deg \gamma \geq 4$. In fact, an element γ with $\deg \gamma = 4$ would be of the form

$$\gamma = \alpha y_i y_j y_i y_j + \beta y_j y_i y_j y_i = \alpha y_i y_i y_j y_i + \beta y_j y_j y_i y_j = 0.$$

This also shows there are no elements of greater degree. Therefore, $\mathcal{L}_Y = \mathcal{K}_Y$. □

Remark 6.6. If $n \neq 3$, then in general $\mathcal{L}_Y \neq \mathcal{K}_Y$. In fact, when $n = 4$, $q = -1$ and we take $Y = \{(13), (23), (34)\} \subseteq \mathbb{O}_2^4$, we have

$$\mathcal{L}_Y \cong \Bbbk\langle x, y, z : x^2, y^2, z^2, xyx - yxy, yzy - zyz, xzx - zxz\rangle.$$

Now, in the subalgebra of $\mathfrak{B}(\mathbb{O}_2^4, -1)$ generated by $x = x_{(23)}$, $y = x_{(34)}$ and $z = x_{(13)}$, we have the relation

$$\begin{aligned}(xyz)^2 &= x_{(23)}x_{(34)}x_{(13)}x_{(23)}x_{(34)}x_{(13)} \\ &= -x_{(23)}x_{(34)}\big(x_{(23)}x_{(12)} + x_{(12)}x_{(13)}\big)x_{(34)}x_{(13)} \\ &= x_{(23)}x_{(34)}x_{(23)}x_{(34)}x_{(12)}x_{(13)} + x_{(23)}x_{(12)}x_{(34)}x_{(13)}x_{(34)}x_{(13)} \\ &= x_{(23)}x_{(23)}x_{(34)}x_{(23)}x_{(12)}x_{(13)} + x_{(23)}x_{(12)}x_{(34)}x_{(34)}x_{(13)}x_{(34)} \\ &= 0.\end{aligned}$$

But $(xyz)^2 \neq 0$ in $\mathcal{L}_Y$. We will prove this using GAP [2008] with the package GBNP [Cohen and Gijsbers $\geq$ 2011]. See Proposition 6.9(6) for a description of $\mathcal{K}_Y$ in this case.

6A. *Coideal subalgebras of Hopf algebras over* $\mathbb{S}_n$.

Set $n = 3$ or 4, let $\mathfrak{B}$ be a finite-dimensional Nichols algebra over $\mathbb{S}_n$, and $\mathcal{H} = \mathfrak{B} \# \Bbbk\mathbb{S}_n$. Recall that these Nichols algebras coincide with their quadratic approximations. We will describe all the coideal subalgebras of $\mathcal{H}$. We will also calculate their stabilizer subgroups.

We start out by proving that in this case these coideal subalgebras are generated in degree one.

Theorem 6.7. *If $\mathcal{K}$ is a homogeneous left coideal subalgebra of $\mathcal{H}$, then $\mathcal{K}$ is generated in degree one. In particular, $\mathcal{K} = \mathcal{K}_Y$ for a unique $Y \subseteq X$.*

Proof. We will see that, given $\omega \in \mathcal{K}$, we have $\omega \in \langle x_i : \delta_i \omega \neq 0\rangle$. Then, by Lemma 6.2, it will follow that $\omega \in \langle \mathcal{K}(1)\rangle$. Let $I = \{i \in X : \delta_i \omega = 0\}$ and let us assume I has m elements. We will proceed case by case, for $m = 0, \dots, 6$.

The cases when $m = 0$ (that is, $x_i \in \mathcal{K}(1)$ for all $i \in X$), $m = 6$, and in general $m = n$ (since then $\omega = 0$, see [Andruskiewitsch and Graña 2003b, Section 6]) are clear. The case $m = 1$ is Lemma 6.2, which also holds for any $n \in \mathbb{N}$.

Consider the case $m = 2$, for any $n \in \mathbb{N}$. Let $I = \{i, p\}$. We know that there is an expression of ω without, say, x_i. Let us see that we can write ω without x_i nor x_p. Let $j \in X$ such that $p \triangleright j = i$. Using relations as in Lemma 6.2, and using that $x_l x_r x_l = -q_{l \triangleright r l} x_r x_l x_r$ and $x_r x_l x_r x_l = 0$ for all $l, r \in X$, we can assume that ω can be written as

$$\omega = \gamma^0 + \gamma^1 x_p + \gamma^2 x_p x_j$$

with γ^0, γ^1, γ^2 not containing x_i- or x_p-factors in their expressions.

In more detail, we can assume that $\omega \in T^\ell(\Bbbk X)$ is a homogeneous monomial. For each appearance of a factor $x_p x_l$ with $l \neq j$, we replace it by $q_{pl} x_l x_{p \triangleright l} + q_{pl} q_{p \triangleright l p} x_{p \triangleright l} x_p$. That is, we replace by an expression in which x_p is located more to the right, and an expression that does not contain x_i or x_p (in the position where we had an x_p). If we have a factor of the form $x_p x_j$, we move it to the right until we get to $x_p x_j x_p$, but we can replace this expression by $-q_{p \triangleright j p} x_j x_p x_j$.

Now, $0 = \delta_p \omega = \gamma^1 g_p + \gamma^2 g_p x_j = (\gamma^1 + q_{pj} \gamma^2 x_i) g_p$, and therefore we have

$$\omega = \gamma^0 + \gamma^1 x_p + q_{pj} \gamma^2 x_i x_p + q_{pj} q_{ip} \gamma^2 x_j x_i = \gamma^0 + q_{pj} q_{ip} \gamma^2 x_j x_i.$$

But then, $0 = \delta_i \omega = q_{pj} q_{ip} \gamma^2 x_j g_i$, and therefore $\omega = \gamma^0$ can be written without x_i or x_p.

This finishes the case $n = 3$, since in this case $|X| = 3$. We now set $n = 4$, and deal with the cases $m = 3, 4, 5$.

Consider the case $m = 3$. Fix $I = \{i_1, i_2, p\}$. There are three possibilities:

$$I = \{i, j, i \triangleright j\}; \tag{6-6}$$

$$I = \{i, j, k\} \quad \text{such that } i \triangleright k = k \text{ or } j \triangleright k = k; \tag{6-7}$$

$$I = \{i, j, l\} \quad \text{(the remaining case)}. \tag{6-8}$$

Let $j_1, j_2 \in X$ be such that $p \triangleright j_s = i_s$ for $s = 1, 2$. We can assume that ω is written without x_{i_s} for $s = 1, 2$. Notice that j_1, j_2 do not always exist. For instance, in (6-6) there are no j_1 or j_2, and in (6-7) j_1 or j_2 do not exist. We analyze the three cases separately.

In (6-6), as there are no j_1, j_2, we can write ω in the form $\omega = \gamma^0 + \gamma^1 x_p$, with γ^0, γ^1 without factors x_j, $j \in I$. But from $\delta_p \omega = 0$ it follows that $\omega = \gamma^0$ and, therefore, we can write ω without factors x_j, $j \in I$.

Case (6-7) is similar. Assume, for example, that $i_2 \triangleright p = p$. Then, we have no j_2. Accordingly, we can assume that ω is of the form

$$\omega = \gamma^0 + \gamma^1 x_p + \gamma^2 x_p x_{j_1} = \gamma^0 + \gamma^1 x_p + q_{pj_1} \gamma^2 x_{i_1} x_p + q_{pj_1} q_{j_1 i_1} \gamma^2 x_{j_1} x_{i_1}$$

with $\gamma^1, \gamma^2, \gamma^3$ without factors x_j, $j \in I$. Now, $0 = \delta_p \omega = (\gamma^1 p + q_{pj_1} \gamma^2 x_{i_1}) g_p$ and thus $\omega = \gamma^0 + q_{pj_1} q_{j_1 i_1} \gamma^2 x_{j_1} x_{i_1}$ but, as $\delta_{i_1} \omega = 0$, it follows $\omega = \gamma^0$ and therefore ω is written without factors x_j, $j \in I$.

It remains to see (6-8). The existence of j_1, j_2 makes this case more subtle than the previous ones. Let us analyze the set $I = \{i_1, i_2, p\}$. We have $k = i_1 \triangleright i_2 = i_2 \triangleright i_1 \notin I$ but, moreover, we have $X = \{i_1, i_2, p, k, j_2, j_1\}$. In fact, we can have neither $i_1 \triangleright i_2 = j_1$ (since this implies $i_2 = p$) nor $i_1 \triangleright i_2 = j_2$ (since this implies $i_1 = p$). More, we have $i_2 \triangleright j_1 = j_1$, and therefore $x_{i_2} x_{j_1} = q_{i_2 j_1} x_{j_1} x_{i_2}$. Set

$$\begin{aligned} a &= x_p, & b &= x_{j_1}, & c &= x_{j_2}, \\ d &= x_{i_1}, & e &= x_{i_2}, & f &= x_k. \end{aligned}$$

We analyze which are the longest words that we can write with the "conflicting" factors a, b and c, starting with a. Recall that $aba = \pm bab$ and $abb = 0$. Starting with ab, we can preliminary form the words $abca$ and $abcb$. Now, $abcac = \pm babca$, and thus we discard it. Consider $abcb$. Since $abcabc = 0$, we are left with $abcaba$. As $abcabab = 0$, we reach $abcabac$. As $abcabaca = abcabacb = 0$, we keep this word. In the case of $abcb$, arguing similarly, we reach $abcbacb$. If we start with acb, as $acbc = \pm abcb$, we consider those words starting with $acba$. The longest one is $acbacab$, but this is $\pm abcbacb$. So the longest word we can form that was not considered before is $acbac$.

In consequence, we can assume there exist $\gamma^i \in \mathcal{K}$, $i = 0, \dots, 15$, without factors x_j, $j \in I$, such that ω is of the form

$$\begin{aligned} \omega = \gamma^0 + \gamma^1 a + \gamma^2 ab + \gamma^3 abc + \gamma^4 abca + \gamma^5 abcab + \gamma^6 abcaba \\ + \gamma^7 abcabac + \gamma^8 abcb + \gamma^9 abcba + \gamma^{10} abcbac + \gamma^{11} abcbacb \\ + \gamma^{12} ac + \gamma^{13} acb + \gamma^{14} acba + \gamma^{15} acbac. \end{aligned}$$

Using the relations and the fact that $\delta_s \omega = 0$ for $s = p, i_1, i_2$, we will show that we can write ω without factors x_s, $s = p, i_1, i_2$. When using the relations, by abuse of notation, we will omit the scalars $q_{..}$ that may appear, including those in the (new) factors γ^i. When needed, we will denote by $\gamma^{i'}, \gamma^{i''}, \gamma^{i'''} \in \mathcal{K}$ some of these scalar multiples of the γ^i.

As $\delta_p\omega = 0$, we can rewrite ω as

$$\begin{aligned}\omega = \gamma^0 + \gamma^2 bd + \gamma^3 bdc + \gamma^{3'} dce + \gamma^5 bdcbd + \gamma^{5'} dcebd + \gamma^7 abcabce \\ + \gamma^8 bdcb + \gamma^{8'} dceb + \gamma^{8''} debd + \gamma^{10} abcbce + \gamma^{11} abcbebd \\ + \gamma^{11'} abcbceb + \gamma^{12} ce + \gamma^{13} ebd + \gamma^{13'} ceb + \gamma^{15} acbea + \gamma^{15'} acbce.\end{aligned}$$

Using that $\delta_{i_1}\omega = 0$ together with the relations $dc = \pm cd$, $be = \pm eb$, $bcb = \pm cbc$ and $abcabc = bcbc = 0$, we see that

$$\begin{aligned}\omega = \gamma^0 + \gamma^2 bd + \gamma^3 bcd + \gamma^{3'} cde + \gamma^5 bdcbd + \gamma^{5'} dcebd \\ + \gamma^7 abcdeae + \gamma^8 bdcb + \gamma^{8'} dcbe + \gamma^{8''} dbed + \gamma^{11} abcbeda \\ + \gamma^{12} ce + \gamma^{13} bed + \gamma^{13'} cbe + \gamma^{15} acbce + \gamma^{15'} edaea.\end{aligned}$$

Using that $\delta_{i_2}\omega = 0$ together with the relations, we get to

$$\begin{aligned}\omega = \gamma^0 + \gamma^2 bd + \gamma^3 bcd + \gamma^5 bcbad + \gamma^{5'} cbafe + \gamma^{5''} cbaed \\ + \gamma^8 bcad + \gamma^{8'} bcba + \gamma^{8''} baed + \gamma^{8'''} bafe + \gamma^{11} abcbebd \\ + \gamma^{11'} abcbeab + \gamma^{11''} abcbfea + \gamma^{11'''} abcbfac + \gamma^{13} bed + \gamma^{13'} bfe.\end{aligned}$$

Using now that $\delta_{i_1}\omega = 0$,

$$\begin{aligned}\omega = \gamma^0 + \gamma^5 cbafe + \gamma^8 bcba + \gamma^{8'} bafe + \gamma^{11} abcbacb \\ + \gamma^{11'} abcbfce + \gamma^{11''} abcbfac + \gamma^{13} bfe.\end{aligned}$$

Using again that $\delta_{i_2}\omega = 0$,

$$\begin{aligned}\omega &= \gamma^0 + \gamma^8 bcba + \gamma^{11} abcbacb + \gamma^{11'} abcbafc \\ &= \beta^0 + \beta^1 a + \beta^2 abcbacb + \beta^3 abcbabf\end{aligned}$$

for $\beta^i \in \mathcal{K}$, $i = 0, \ldots, 3$, without factors x_j, $j \in I$. Using that $\delta_p\omega = 0$,

$$\omega = \beta^0 + \beta^2 dedaeda + \beta^3 dedadaf = \beta^0,$$

since $edaeda = dada = 0$. That is, we can write ω without any factors x_j, $j \in I$.

In the case $m = 4$, we look at the different subsets of three elements of I. If we have a subset of three elements that corresponds to the case (6-8), it follows that ω can be written without the factors x_j with j in that subset, and then ω is in an algebra isomorphic to $\mathfrak{B}(\mathbb{O}_2^3, -1)$, for which we have already proved the result. If we have a subset as in the case (6-6), when we add to this subset a fourth element we obtain another subset as in the case (6-8). If our subset corresponds to the case (6-7), in order to get to a case different from (6-8), we necessarily have to add a fourth element such that I is

$$I = \{i, j, k, l\} \quad \text{with } i \triangleright k = k \text{ and } j \triangleright l = l.$$

We analyze this case. If $p \in I$ is fixed and ω is written without factors x_j with $j \in I \setminus \{p\} = \{i_1, i_2, i_3\}$, notice that if $p \triangleright i_3 = i_3$ there is no other j_3 such that $p \triangleright j_3 = i_3$ and, moreover, if j_1, j_2 are such that $p \triangleright j_s = i_s$ for $s = 1, 2$, then $x_{j_1} x_{j_2} = \pm x_{j_2} x_{j_1}$. Therefore, we can assume that there are γ^i for $i = 0, \dots, 4$, such that they do not contain factors x_j for $j \in I$, and such that ω can be written as

$$\begin{aligned}
\omega &= \gamma^0 + \gamma^1 x_p + \gamma^2 x_p x_{j_1} + \gamma^3 x_p x_{j_1} x_{j_2} + \gamma^4 x_p x_{j_1} x_{j_2} x_p \\
&= \gamma^0 + \gamma^2 x_{j_1} x_{i_1} + \gamma^3 x_{j_1} x_{i_1} x_{j_2} + \gamma^{3'} x_{i_1} x_{j_2} x_{i_2} + \gamma^4 x_{i_1} x_{i_2} x_p && (\text{since } \delta_p \omega = 0) \\
&= \gamma^0 + \gamma^2 x_{j_1} x_{i_1} + \gamma^3 x_{j_1} x_{i_1} x_{j_2} + \gamma^{3'} x_{i_1} x_{j_2} x_{i_2} && (\text{since } \delta_p \omega = 0) \\
&= \gamma^0 + \gamma^2 x_{j_1} x_{i_1} + \gamma^3 x_{j_1} x_{i_1} x_{j_2} && (\text{since } \delta_{i_2} \omega = 0) \\
&= \gamma^0 + \gamma^3 x_{j_1} x_{j_2} x_{i_2} && (\text{since } \delta_{i_1} \omega = 0) \\
&= \gamma^0 && (\text{since } \delta_{i_2} \omega = 0).
\end{aligned}$$

Then, we can write ω without x_j for $j \in I$. In the case when $m = 5$, ω necessarily belongs to an algebra isomorphic to $\mathfrak{B}(\mathbb{O}_2^3, -1)$. ☐

Now, we apply Theorems 6.5 and 6.7 to calculate the coideal subalgebras and stabilizer subgroups of $\mathcal{H} = \mathfrak{B}(\mathbb{O}_2^3) \# \Bbbk \mathbb{S}_3$.

Corollary 6.8. *The following are all the proper homogeneous left coideal subalgebras of* $\mathfrak{B}(\mathbb{O}_2^3, -1) \# \Bbbk \mathbb{S}_3$:

(1) $\mathcal{K}_i = \langle x_i \rangle \cong \Bbbk[x]/\langle x^2 \rangle$ *for* $i \in \mathbb{O}_2^3$,

(2) $\mathcal{K}_{i,j} = \langle x_i, x_j \rangle \cong \Bbbk\langle x, y \rangle / \langle x^2, y^2, xyx - yxy \rangle$ *for* $i, j \in \mathbb{O}_2^3$.

The nontrivial stabilizer subgroups of $\mathbb{S}_3$ *are*, *respectively*, *case*

(1) $\operatorname{Stab} \mathcal{K}_i = \mathbb{Z}_2 \cong \langle i \rangle \subset \mathbb{S}_3$,

(2) $\operatorname{Stab} \mathcal{K}_{i,j} = \mathbb{Z}_2 \cong \langle k \rangle \subset \mathbb{S}_3$ *for* $k \neq i, j$. ☐

Next, we use the computer program [GAP 2008], together with the package [Cohen and Gijsbers ≥ 2011], to compute the coideal subalgebras of the finite-dimensional Nichols algebras over $\mathbb{S}_4$ associated to the rack of transpositions $\mathbb{O}_2^4$. In the same way can be computed the coideal subalgebras of the Nichols algebra $\mathfrak{B}(\mathbb{O}_4^4, -1)$ associated to the rack of 4-cycles. The presentation of these algebras may not be minimal, in the sense that there may be redundant relations. Moreover, in the general case, non-redundant relations in a coideal subalgebra $\mathcal{K}$ may become redundant when computing the bosonization with a subgroup $F \leq \operatorname{Stab} \mathcal{K}$.

First, we need to establish some notation and conventions. Let $\Bbbk\langle x, y, z \rangle$ be the free algebra in the variables x, y, z. We set the ideals

$$R^{\pm}(x, y, z) = \langle x^2, y^2, z^2, xy + yz \pm zx \rangle \subset \Bbbk\langle x, y, z \rangle.$$

Set $\mathfrak{B}_4^+ = \mathfrak{B}(\mathbb{O}_2^4, -1)$ and $\mathfrak{B}_4^- = \mathfrak{B}(\mathbb{O}_2^4, \chi)$. Recall that Y stands for a subset of $\mathbb{O}_2^4$.

Proposition 6.9. *Let* $\varepsilon = \pm$. *Any homogeneous proper coideal subalgebra* $\mathcal{H}^\varepsilon$ *of* $\mathfrak{B}_4^\varepsilon \# \Bbbk 1$ *is isomorphic to one of the algebras in the following list*:

- $\dim \mathcal{H}^\varepsilon(1) = 1$:
  - (1) $Y = \{i\}$, $\mathcal{H}^\varepsilon = \Bbbk[x]/\langle x^2\rangle$, *and* $\dim \mathcal{H}^\varepsilon = 2$.
- $\dim \mathcal{H}^\varepsilon(1) = 2$:
  - (2) $Y = \{i, j\}$, $i \triangleright j = j$, $\mathcal{H}^\varepsilon = \Bbbk\langle x, z\rangle/\langle x^2, z^2, xz + \varepsilon zx\rangle$, *and* $\dim \mathcal{H}^\varepsilon = 4$.
  - (3) $Y = \{i, j\}$, $i \triangleright j \neq j$, $\mathcal{H}^\varepsilon = \Bbbk\langle x, y\rangle/\langle x^2, y^2, xyx - \varepsilon yxy\rangle$, *and* $\dim \mathcal{H}^\varepsilon = 6$.
- $\dim \mathcal{H}^\varepsilon(1) = 3$:
  - (4) $Y = \{i, j, k\}$, $i \triangleright j = k$, $\mathcal{H}^\varepsilon = \Bbbk\langle x, y, z\rangle/\langle R^\varepsilon(x, y, z)\rangle$, *and* $\dim \mathcal{H}^\varepsilon = 12$.
  - (5) $Y = \{i, j, k\}$, $i \triangleright j \neq j, k$, $i \triangleright k = k$,
    $\mathcal{H}^\varepsilon_{i,j,k} := \Bbbk\langle x, y, z\rangle/\langle x^2, y^2, z^2, xyx - \varepsilon yxy, zyz - \varepsilon yzy, xz + \varepsilon zx\rangle$,
    and $\dim \mathcal{H}^\varepsilon = 24$.
  - (6) $Y = \{i, j, k\}$, $i \triangleright j$, $j \triangleright k$, $i \triangleright k \notin \{i, j, k\}$,
    $$\begin{aligned}\mathcal{H}^\varepsilon_Y = \Bbbk\langle x, y, z \;:\; & x^2, y^2, z^2,\\ & yxy - \varepsilon xyx,\ zxz - \varepsilon xzx,\ zyz - \varepsilon yzy,\\ & zxyz + yzxy + xyzx,\ zyxz + yxzy + xzyx\\ & zxyxzx + \varepsilon yzxyxz, zxyxzy + \varepsilon xzxyxz\rangle,\end{aligned}$$
    and $\dim \mathcal{H}^\varepsilon = 48$.
- $\dim \mathcal{H}^\varepsilon(1) = 4$:
  - (7) $Y = \{i, j, k, l\}, i \triangleright j = k, i \triangleright l = l$,
    $$\begin{aligned}\mathcal{H}^\varepsilon_Y = \Bbbk\langle x, y, z, w \;:\; & x^2,\ y^2,\ z^2,\ w^2,\\ & zx + \varepsilon yz + \varepsilon xy,\ zy + yx + \varepsilon xz,\ wz + \varepsilon zw,\\ & yxy - \varepsilon xyx,\ wxw - \varepsilon xwx,\ wyw - \varepsilon ywy,\\ & wyx + \varepsilon wxz - \varepsilon zwy,\ wyz + wxy - zwx\\ & wxyz - zwxz,\ wxzw + xwxz,\\ & wxyw + ywxy + xywx,\ wxyxz - \varepsilon zwxyx,\\ & wxyxwx + \varepsilon ywxyxw,\ wxyxwy + \varepsilon xwxyxw\rangle,\end{aligned}$$
    and $\dim \mathcal{H}^\varepsilon = 96$.
  - (8) $Y = \{i, j, k, l\}$, $i \triangleright j \neq j, k$, $i \triangleright k = k$, $j \triangleright l = l$,

$$\begin{aligned}\mathcal{K}^{\varepsilon}_{Y} = \Bbbk\langle x, y, z, w \,:\, & x^2,\ y^2,\ z^2,\ w^2,\ zy+\varepsilon yz,\ wx+\varepsilon xw, \\ & yxy-\varepsilon xyx,\ zxz-\varepsilon xzx,\ wyw-\varepsilon ywy, \\ & wzw-\varepsilon zwz,\ zxyx+yzxy,\ zxyz+\varepsilon xzxy, \\ & wyx-\varepsilon zwy-yxz+\varepsilon xzw, \\ & wzx-\varepsilon zxy-ywz+\varepsilon xyw, \\ & wyzxy-\varepsilon ywyzx-xyzwy+xyxzw, \\ & wyzxw+zxywz-yxzwy-xwyzx, \\ & wyzw-\varepsilon zxwz-yzxw+yxwy+\varepsilon xwyz-\varepsilon xyzx\rangle,\end{aligned}$$

and $\dim \mathcal{K}^{\varepsilon} = 144$.

- $\dim \mathcal{K}^{\varepsilon}(1) = 5$:

  (9) $Y = \{i, j, k, l, m\}$,

  $i \triangleright j = k,\ i \triangleright l = m,\ j \triangleright l \neq l,\ k \triangleright m \neq m,\ j \triangleright m = m,\ k \triangleright l = l,$

$$\begin{aligned}\mathcal{K}^{\varepsilon} = \Bbbk\langle x, y, z, w, u \,:\, & x^2,\ y^2,\ z^2,\ w^2,\ u^2,\ wz+\varepsilon zw,\ uy+\varepsilon yu, \\ & zx+\varepsilon yz+\varepsilon xy,\ zy+yx+\varepsilon xz, \\ & ux+\varepsilon wu+\varepsilon xw,\ uw+wx+\varepsilon xu, \\ & yxy-\varepsilon xyx,\ wxw-\varepsilon xwx, \\ & wyw-\varepsilon ywy,\ uzu-\varepsilon zuz, \\ & wyx+\varepsilon wxz-\varepsilon zwy,\ wyz+wxy-zwx, \\ & uzw-\varepsilon wxz-xuz,\ wxyz-zwxz, \\ & wxyw+ywxy+xywx, \\ & wxyxz-\varepsilon zwxyx,\ wxzw+xwxz, \\ & wxyxwx+\varepsilon ywxyxw,\ wxyxwy+\varepsilon xwxyxw\rangle,\end{aligned}$$

  and $\dim \mathcal{K}^{\varepsilon} = 288$.

The stabilizers subgroups of $\mathbb{S}_4$ *are*, *respectively*,

(1) $\mathbb{Z}_2 \times \mathbb{Z}_2 \cong \langle g_i, g_j\rangle \subset \mathbb{S}_4$ *with* $i \triangleright j = j$;

(2) $D_4 \cong \langle g_i, \sigma\rangle \subset \mathbb{S}_4$ (*if*, *for example*, $g_i = (12)$ *and* $\sigma = (1324)$);

(3) $\mathbb{Z}_2 \cong \langle g_k\rangle \subset \mathbb{S}_4,\ k = i \triangleright j$.

(4) $\mathbb{S}_3 \cong \langle g_i, g_j, g_k\rangle \subset \mathbb{S}_4,\ i \triangleright j = k$;

(5) $\mathbb{Z}_2 \cong \langle g_j g_l\rangle,\ j \neq l,\ j \triangleright l = l$;

(6) $\mathbb{S}_3 \cong \langle g_{i\triangleright j}, g_{j\triangleright k}, g_{k\triangleright i}\rangle \subset \mathbb{S}_4$;

(7) *If* $\mathcal{K}^{\varepsilon}$ *belongs to items* (7) *or* (8), *then* $\operatorname{Stab} \mathcal{K}^{\varepsilon} = 1$. ☐

Examples 6.10. We give, as an illustration, an example of a subset $Y \subseteq \mathcal{O}_2^4$ for each case in the previous proposition. Note that, for any comodule algebra $\mathcal{K}_{Y'}$, if Y' is not on the following list, then $\mathcal{K}_{Y'}$ is $\mathbb{S}_4$-conjugated to another algebra $\mathcal{K}_Y$ with Y on the list.

(1) $Y = \{(12)\}$,

(2) $Y = \{(12), (34)\}$,

(3) $Y = \{(12), (13)\}$,

(4) $Y = \{(12), (13), (23)\}$,

(5) $Y = \{(12), (13), (34)\}$,

(6) $Y = \{(12), (13), (14)\}$,

(7) $Y = \{(12), (13), (23), (14)\}$,

(8) $Y = \{(12), (13), (24), (34)\}$,

(9) $Y = \{(12), (13), (23), (14), (24)\}$. □

Remark 6.11. Let $Y \subset \mathcal{O}_2^4$ and let $Z \subset \mathcal{O}_2^4$ be such that $\mathcal{O}_2^4 = Y \sqcup Z$, as sets. Denote by Y_j one of the subsets of item (j) of Proposition 6.9, and by Z_j the corresponding complement. Notice that we have the following bijections

$$Z_1 \cong Y_9, \quad Z_2 \cong Y_8, \quad Z_3 \cong Y_7, \quad Z_4 \cong Y_6, \quad Z_5 \cong Y_5.$$

Therefore, $\dim \mathcal{K}_Y \dim \mathcal{K}_Z = \dim \mathfrak{B}^{\varepsilon}$ for every Y. An analogous statement holds when $X = \mathcal{O}_4^4$.

7. Representations of $\mathrm{Rep}(\widehat{\mathfrak{B}}_2(X, q) \# \Bbbk G)$

In this section, we take $\mathfrak{Q} = \big(X, q, G, \big(\cdot, g, (\chi_l)_{l\in X}\big), (\lambda_C)_{C\in\mathfrak{R}'}\big)$ as one of the ql-data from Section 5D. Note that in this case the set $C_i = \{(i, i)\}$ belongs to $\mathfrak{R} = \mathfrak{R}'$ and $(i \triangleright j) \triangleright i = j$ for any $i, j \in X$. Let $\mathcal{H}(\mathfrak{Q})$ be the corresponding Hopf algebra defined in Section 5C, and set $\mathcal{H} = \widehat{\mathfrak{B}}_2(X, q) \# \Bbbk G$. We will assume that $\dim \widehat{\mathfrak{B}}_2(X, q) < \infty$ (and thus $\dim \mathcal{H}(\mathfrak{Q}) < \infty$; see [García and García Iglesias $\geq$ 2011, Proposition 4.2]). In particular, this holds for $n = 3, 4, 5$.

7A. $\widehat{\mathfrak{B}}_2(X, q) \# \Bbbk G$-comodule algebras.

We will construct families of comodule algebras over quadratic approximations of Nichols algebras. These families are large enough to classify module categories in all of our examples.

Definition 7.1. Let $F < G$ be a subgroup and $\psi \in Z^2(F, \Bbbk^{\times})$. If $Y \subseteq X$ is a subset such that $F \cdot Y \subseteq Y$, that is, $F < \mathrm{Stab}\, \mathcal{K}_Y$, then we will say that a family of scalars $\xi = \{\xi_C\}_{C\in\mathfrak{R}}$ with $\xi_C \in \Bbbk$ is *compatible* with the triple (Y, F, ψ) if, for any

$f \in \operatorname{Stab}\mathcal{K}_Y$, we have

$$\xi_{f\cdot C}\,\chi_i(f)\chi_j(f) = \xi_C\,\psi(f, g_ig_j)\,\psi(fg_ig_j, f^{-1}) \quad \text{if } C \in \mathcal{R}_1^Y, (i, j) \in C;$$
$$\xi_{f\cdot C}\,\chi_i^2(f)\chi_j(f) = \xi_C\,\psi(f, g_ig_jg_i)\,\psi(fg_ig_jg_i, f^{-1}) \quad \text{if } C \in \mathcal{R}_2^Y, (i, j) \in C;$$
$$\xi_{C_i} = \xi_{C_j} = 0 \quad \text{if } C \in \mathcal{R}_2^Y, (i, j) \in C.$$

We will assume that the family ξ is normalized by $\xi_C = 0$ if either $C \in \mathcal{R}_1^Y$, $(i, j) \in C$, and $g_ig_j \notin F$, or if $C \in \mathcal{R}_2^Y$, $(i, j) \in C$, and $g_ig_jg_i \notin F$.

We now introduce the comodule algebras we will work with.

Definition 7.2. Let $F < G$ be a subgroup, $\psi \in Z^2(F, \Bbbk^\times)$, and $Y \subseteq X$ a subset such that $F \cdot Y \subseteq Y$. Let $\xi = \{\xi_C\}_{C\in\mathcal{R}'}$ be compatible with (Y, F, ψ). Define $\mathcal{A}(Y, F, \psi, \xi)$ to be the algebra generated by $\{y_l, e_f : l \in Y, f \in F\}$ and relations

$$e_1 = 1 \quad\text{and}\quad e_re_s = \psi(r, s)\,e_{rs} \quad \text{for } r, s \in F, \tag{7-1}$$
$$e_f\,y_l = \chi_l(f)\,y_{f\cdot l}\,e_f \quad \text{for } f \in F,\ l \in Y, \tag{7-2}$$
$$\vartheta_{C,Y}(\{y_l\}_{l\in X}) = \begin{cases} \xi_C\,e_C & \text{if } e_C \in F \\ 0 & \text{if } e_C \notin F \end{cases} \qquad \text{for } C \in \mathcal{R}. \tag{7-3}$$

Here, $\vartheta_{C,Y}$ was defined in (6-4) while the element e_C is defined by

$$e_C = \begin{cases} e_{g_ig_j} & \text{if } C \in \mathcal{R}_1^Y \text{ and } (i, j) \in C, \\ e_{g_ig_jg_i} & \text{if } C \in \mathcal{R}_2^Y \text{ and } (i, j) \in C \cap Y \times Y, \\ 0, & \text{if } C \in \mathcal{R}_3^Y. \end{cases} \tag{7-4}$$

If $Z \subseteq X$ is a subset invariant under the action of F, we define $\mathcal{B}(Z, F, \psi, \xi)$ as the subalgebra of $\mathcal{A}(X, F, \psi, \xi)$ generated by the elements $\{y_l, e_f : l \in Z, f \in F\}$.

Remark 7.3. (a) Applying $\operatorname{ad}(f)$, with $f \in \operatorname{Stab}\mathcal{K}_Y$, to Equation (7-3) and using (5-3) one can deduce the equations in Definition 7.1.

(b) It may happen that $\mathcal{B}(Z, F, \psi, \xi) \neq \mathcal{A}(Z, F, \psi, \xi)$.

Let $\lambda : \mathcal{A}(Y, F, \psi, \xi) \to \mathcal{H} \otimes \mathcal{A}(Y, F, \psi, \xi)$ be the map defined by

$$\lambda(e_f) = f \otimes e_f, \qquad \lambda(y_l) = x_l \otimes 1 + g_l \otimes y_l, \tag{7-5}$$

for all $f \in F, l \in Y$.

Lemma 7.4. *$\mathcal{A}(Y, F, \psi, \xi)$ is a left $\mathcal{H}$-comodule algebra with coaction λ as in (7-5) and $\mathcal{B}(Z, F, \psi, \xi)$ is a subcomodule algebra of $\mathcal{A}(X, F, \psi, \xi)$.*

Proof. We first prove that the map λ is well defined. It is easy to see that $\lambda(e_fy_l) = \chi_l(f)\,\lambda(y_{f\cdot l}\,e_g)$ for any $f \in F, l \in X$.

Let $C \in \mathcal{R}_1^Y$ and $(i, j) \in C$. In this case, $\vartheta_C = \phi_C$. We will prove that $\lambda(\phi_C(\{y_l\}_{l\in X})) = \lambda(\xi_C\, e_{g_i g_j})$. Using the definition of the polynomial ϕ_C, we obtain that

$$\begin{aligned}\lambda(\phi_C(\{y_l\}_{l\in X})) &= \sum_{h=1}^{n(C)} \eta_h(C)\, x_{i_{h+1}} x_{i_h} \otimes 1 + x_{i_{h+1}} g_{i_h} \otimes y_{i_h}\\ &\qquad\qquad + g_{i_{h+1}} x_{i_h} \otimes y_{i_{h+1}} + g_{i_{h+1}} g_{i_h} \otimes y_{i_{h+1}} y_{i_h}\\ &= \phi_C(\{x_l\}_{l\in X}) \otimes 1 + g_i g_j \otimes \phi_C(\{y_l\}_{l\in X})\\ &= \xi_C\, g_i g_j \otimes e_{g_i g_j} = \lambda(\xi_C\, e_{g_i g_j}).\end{aligned}$$

The second equality follows since $i_{n(C)+1} = i_1$,

$$g_{i_{h+1}} x_{i_h} = q_{i_{h+1} i_h}\, x_{i_{h+2}} g_{i_{h+1}} \quad \text{and} \quad \eta_h(C) q_{i_{h+1} i_h} = -\eta_{h+1}(C).$$

Now, let $C \in \mathcal{R}_2^Y$, $(i, j) \in C$ and $i \triangleright j \notin Y$. In this case relation (7-3) is

$$y_i y_j y_i + q_{i\triangleright j i}\, y_j y_i y_j = \xi_C\, e_{g_i g_j g_i}.$$

Note that assumption $\xi_{C_i} = \xi_{C_j} = 0$ implies that $y_i^2 = 0 = y_j^2$. The proof that $\lambda(y_i y_j y_i + q_{i\triangleright j i}\, y_j y_i y_j) = \xi_C \lambda(e_{g_i g_j g_i})$ is a straightforward computation. □

Theorem 7.5. *Let $Y \subseteq X$ be an F-invariant subset. If $\mathcal{A}(X, F, \psi, \xi) \neq 0$, then the following statements hold*:

(1) *The algebras $\mathcal{A}(X, G, \psi, \xi)$ are left $\mathcal{H}$-Galois extensions.*

(2) *If ξ satisfies*

$$\xi_C = \begin{cases} -\lambda_C & \text{if } \lambda_C \neq 0,\\ 0 & \text{if } \lambda_C = 0 \text{ and } g_j g_i \neq 1,\\ \text{arbitrary} & \text{if } \lambda_C = 0 \text{ and } g_j g_i = 1,\end{cases} \tag{7-6}$$

then $\mathcal{A}(X, G, 1, \xi)$ is a $(\mathcal{H}, \mathcal{H}(\mathcal{Q}))$-biGalois object.

(3) *$\mathcal{B}(Y, F, \psi, \xi)_0 = \Bbbk_\psi F$, and thus $\mathcal{B}(Y, F, \psi, \xi)$ is a right $\mathcal{H}$-simple left $\mathcal{H}$-comodule algebra.*

(4) *There is an isomorphism of comodule algebras* $\operatorname{gr}\mathcal{B}(Y, F, \psi, \xi) \simeq \mathcal{H}_Y \# \Bbbk_\psi F$.

(5) *There is an isomorphism $\mathcal{B}(Y, F, \psi, \xi) \simeq \mathcal{B}(Y', F', \psi', \xi')$ of comodule algebras if and only if $Y = Y'$, $F = F'$, $\psi = \psi'$ and $\xi = \xi'$.*

Proof. Step 1: To prove that $\mathcal{A}(X, G, \psi, \xi)$ is a Galois extension, observe that the canonical map

$$\begin{aligned}&\operatorname{can} : \mathcal{A}(X, G, \psi, \xi) \otimes \mathcal{A}(X, G, \psi, \xi) \to \mathcal{H} \otimes \mathcal{A}(X, G, \psi, \xi),\\ &\operatorname{can}(x \otimes y) = x_{(-1)} \otimes x_{(0)} y,\end{aligned}$$

is surjective. Indeed, for any $f \in G$, $l \in X$, we have $\operatorname{can}(e_f \otimes e_{f^{-1}}) = f \otimes 1$ and

$$\operatorname{can}(y_l \otimes 1 - e_{g_l} \otimes e_{g_l^{-1}} y_l) = x_l \otimes 1.$$

Step 2: Define the map $\rho : \mathcal{A}(X, G, 1, \xi) \to \mathcal{A}(X, G, 1, \xi) \otimes \mathcal{H}(\mathcal{Q})$ by

$$\rho(e_f) = e_f \otimes H_f \quad \text{and} \quad \rho(y_l) = y_l \otimes 1 + e_{g_l} \otimes a_l \quad \text{for } l \in X,\ f \in G.$$

The map ρ is well defined. Indeed, if $C \in \mathcal{R}$ and $(i, j) \in C$, then

$$\begin{aligned}\rho(\phi_C(\{y_l\}_{l\in X})) &= \phi_C(\{y_l\}_{l\in X}) \otimes 1 + e_{g_i g_j} \otimes \phi_C(\{a_l\}_{l\in X})\\ &= \xi_C\, e_{g_i g_j} \otimes 1 + \lambda_C\, e_{g_i g_j} \otimes (1 - H_{g_i g_j}).\end{aligned}$$

Clearly, if ξ satisfies (7-6), then $\rho(\phi_C(\{y_l\}_{l\in X})) = \xi_C\, \rho(e_{g_i g_j})$. The proof that $\mathcal{A}(X, G, 1, \xi)$ is a $(\mathcal{H}, \mathcal{H}(\mathcal{Q}))$-bicomodule and a right $\mathcal{H}(\mathcal{Q})$-Galois object is done by a straightforward computation.

Step 3: If $\mathcal{A}(X, F, \psi, \xi) \neq 0$, then there is a group $\overline{F}$ with a projection $F \twoheadrightarrow \overline{F}$ such that $\mathcal{A}(Y, F, \psi, \xi)_0 = \Bbbk_{\overline{\psi}} \overline{F}$. The map $\mathcal{A}(Y, F, \psi, \xi)_0 \otimes \mathcal{A}(Y, F, \psi, \xi)_0 \to \Bbbk F \otimes \mathcal{A}(Y, F, \psi, \xi)_0$, defined by $e_f \otimes e_g \mapsto f \otimes \psi(f, g)\, e_{fg}$, is surjective. Hence, $F = \overline{F}$. This implies that $\mathcal{B}(Z, F, \psi, \xi)_0 = \Bbbk_\psi F$ and, by [Mombelli 2010, Prop. 4.4], it follows that $\mathcal{B}(Z, F, \psi, \xi)$ is a right $\mathcal{H}$-simple left $\mathcal{H}$-comodule algebra.

Step 4: It follows from Theorem 3.2(3) that $\operatorname{gr}\mathcal{B}(Y, F, \psi, \xi) \simeq \mathcal{K} \# \Bbbk_\psi F$ for some homogeneous left coideal subalgebra $\mathcal{K} \subseteq \widehat{\mathfrak{B}}_2(X, q)$. Recall that $\mathcal{K}$ is identified with the subalgebra of $\operatorname{gr}\mathcal{B}(Y, F, \psi, \xi)$ given by

$$\big\{a \in \operatorname{gr}\mathcal{A}(Y, F, \psi, \xi) : (\mathrm{id} \otimes \pi)\lambda(a) \in \mathcal{H} \otimes 1\big\};$$

see [Mombelli 2010, Proposition 7.3 (3)]. There, it is also proved that the composition

$$\operatorname{gr}\mathcal{B}(Y, F, \psi, \xi) \xrightarrow{(\vartheta\otimes\pi)\lambda} \mathcal{K} \# \Bbbk_\psi F \xrightarrow{\mu} \operatorname{gr}\mathcal{B}(Y, F, \psi, \xi),$$

is the identity map, where $\vartheta : \mathcal{H} \to \widehat{\mathfrak{B}}_2(X, q)$ and $\pi : \operatorname{gr}\mathcal{B}(Y, F, \psi, \xi) \to \Bbbk_\psi F$ are the canonical projections, and μ is the multiplication map. Both maps are bijections and, since for any $l \in Y$ we have$(\vartheta \otimes \pi)\lambda(y_l) = x_l$, it follows that $\mathcal{K} = \mathcal{K}_Y$.

Step 5: Let $\beta : \mathcal{B}(Y, F, \psi, \xi) \to \mathcal{B}(Y', F', \psi', \xi')$ be a comodule algebra isomorphism. The restriction of β to $\mathcal{B}(Y, F, \psi, \xi)_0$ induces an isomorphism between $\Bbbk_\psi F$ and $\Bbbk_{\psi'} F'$, and thus $F = F'$ and $\psi = \psi'$. Since β is a comodule morphism, it is clear that $Y = Y'$ and $\xi_C = \xi'_C$ for any $C \in \mathcal{R}$. □

Corollary 7.6. *If* $\mathcal{A}(X, G, 1, \xi) \neq 0$ *for some* ξ *satisfying* (7-6), *then*

1. *the Hopf algebras* $\mathcal{H} = \widehat{\mathfrak{B}}_2(X, q) \# \Bbbk G$ *and* $\mathcal{H}(\mathcal{Q})$ *are cocycle deformations of each other*;
2. *there is a bijective correspondence between equivalence classes of exact module categories over* $\operatorname{Rep}(\mathcal{H})$ *and* $\operatorname{Rep}(\mathcal{H}(\mathcal{Q}))$.

Remark 7.7. Under the assumptions in Corollary 7.8, we obtain in particular that $\operatorname{gr}\mathcal{H}(\mathcal{Q}) = \widehat{\mathfrak{B}}_2(X, q) \# \Bbbk G$, since the latter is a quotient of the first.

The following corollary uses Propositions A.14 and A.18, where certain algebras are shown to be not null. These propositions will be proven in the Appendix, and their proofs are independent of the other results in the article.

Corollary 7.8. *If H is a nontrivial pointed Hopf algebra over $\mathbb{S}_3$ or $\mathbb{S}_4$, then H is a cocycle deformation of* $\operatorname{gr} H$.

Proof. Finite-dimensional Nichols algebras over $\mathbb{S}_3$ and $\mathbb{S}_4$ coincide with their quadratic approximations. That is, if H is a finite-dimensional pointed Hopf algebra over $\mathbb{S}_n$ with $n = 3, 4$, then $\operatorname{gr} H \cong \widehat{\mathfrak{B}}_2(X, q) \# \Bbbk\mathbb{S}_n$. By Main Theorem of [García and García Iglesias $\geq$ 2011] we know that $H \cong \mathcal{H}(\mathcal{Q})$. Therefore, the theorem follows from Corollary 7.6, since in Propositions A.14 and A.18 we show the existence of nonzero $(\operatorname{gr} \mathcal{H}(\mathcal{Q}), \mathcal{H}(\mathcal{Q}))$-biGalois objects in these cases.

When dealing with either $\mathcal{Q}_4^{-1}[t]$ or $\mathcal{D}[t]$, notice that the condition $\xi_2 = 2\xi_1$ in Proposition A.18 does not interfere with the proof, since, by (7-6), ξ_1 and respectively ξ_2 can be chosen arbitrarily. □

Remark 7.9. In [2008, Theorem A1], Masuoka proved that the Hopf algebras $u(\mathcal{D}, \lambda, \mu)$ associated to a datum of finite Cartan type $\mathcal{D}$ appearing in the classification of [Andruskiewitsch and Schneider 2010] are cocycle deformations of the associated graded Hopf algebras $u(\mathcal{D}, 0, 0)$.

Corollaries 7.6(1) and 7.8 provide a similar result for some families of Hopf algebras constructed from Nichols algebras not of diagonal type. It would be interesting to generalize this kind of result for larger classes of Nichols algebras.

7B. ***Module categories over* Rep($\mathcal{H}(\mathcal{Q})$).** Let A be a $\mathcal{H}$-comodule algebra with $\operatorname{gr} A = \mathcal{K}_Y \# \Bbbk_\psi F$ for $F \leq \operatorname{Stab} \mathcal{K}_Y$ and $\psi \in Z^2(F, \Bbbk^*)$. Let Z be such that, as sets, $X = Y \sqcup Z$. Notice that $F \leq \operatorname{Stab} \mathcal{K}_Z$.

Lemma 7.10. *Under these assumptions, there exists a family of scalars ξ compatible with (X, F, ψ) such that $A \simeq \mathcal{B}(Y, F, \psi, \xi)$ as comodule algebras.*

Proof. The canonical projection $\pi : A_1 \to A_1/A_0 \simeq \mathcal{K}_Y(1) = \Bbbk Y$ is a morphism of $\mathcal{A}_0$-bimodules. Let $\iota : \Bbbk Y \to A_1$ be a section of $\mathcal{A}_0$-bimodules of π. Since the elements $\{x_l : l \in Y\}$ are in the image of π, we can choose elements $\{y_l : l \in Y\}$ in A_1 such that $\iota(x_l) = y_l$ for any $l \in Y$. It is straightforward to verify that $\lambda(y_l) = x_l \otimes 1 + g_l \otimes y_l$ and $e_f\, y_l = \chi_l(f)\, y_{f \cdot l}\, e_f$ for $f \in F$, $l \in Y$. Since $\operatorname{gr} A$ is generated by the elements $\{x_l, e_f : l \in Y, f \in F\}$, it follows that A is generated as an algebra by the elements $\{y_l, e_f : l \in Y, f \in F\}$.

Now, let $B = A \otimes \mathcal{K}_Z$. Then, B has an comodule algebra structure for which the canonical inclusion $A \hookrightarrow A \otimes 1 \subset B$ is a homomorphism. The algebra structure is given as follows:

For $i \in Y$, $j \in Z$, $f \in F$, set

$$\begin{aligned}
&(e_f \otimes 1)(1 \otimes y_j) = e_f \otimes y_j,\\
&(1 \otimes y_j)(e_f \otimes 1) = \chi_j^{-1}(f) e_f \otimes y_{f^{-1}\cdot j},\\
&(y_i \otimes 1)(1 \otimes y_j) = (y_i \otimes y_j),\\
&(1 \otimes y_j)(y_i \otimes 1) =
\end{aligned}$$

$$\begin{cases} q_{ji} y_i \otimes y_j + \xi_C e_C \otimes 1 & \text{if } i \triangleright j = j,\\ q_{ji} y_{j\triangleright i} \otimes y_j - q_{ji} q_{j\triangleright i\, j} y_i y_{j\triangleright i} \otimes 1 + \xi_C e_C \otimes 1 & \text{if } i \triangleright j \neq j,\ i \triangleright j \in Y,\\ q_{ji} 1 \otimes y_{j\triangleright i} y_j - q_{ji} q_{j\triangleright i\, j} y_i \otimes y_{j\triangleright i} + \xi_C e_C \otimes 1 & \text{if } i \triangleright j \neq j,\ i \triangleright j \notin Y. \end{cases}$$

Here, C stands for the class $C \in \mathcal{R}'$ such that $(j, i) \in C$. Recall that, by definition, $\xi_C = 0$ if $g_C \notin F$. Then, the map

$$m : B \to \mathcal{A}(X, F, \psi, \xi), \quad a \otimes x \mapsto ax, \tag{7-7}$$

is an algebra epimorphism. Now, if

$$A \ni a \longmapsto a_{(-1)} \otimes a_{(0)} \in \mathcal{H} \otimes A \quad \text{and} \quad \mathcal{K}_Z \ni x \longmapsto x_{(-1)} \otimes x_{(0)} \in \mathcal{H} \otimes \mathcal{K}_Z$$

denote the corresponding coactions, define $\lambda : B \to \mathcal{H} \otimes B$ by $\lambda(a \otimes x) = a_{(-1)} x_{(-1)} \otimes a_{(0)} \otimes x_{(0)}$. It is straightforward to check that λ is well defined. We check this case by case in the above definition of the multiplication of B. For instance, if $i \triangleright j \neq j$ and $i \triangleright j \in Y$, then we have

$$\begin{aligned}
&\lambda(1 \otimes y_j)\lambda(y_i \otimes 1)\\
&\quad = (g_j \otimes (1 \otimes y_j) + x_j \otimes (1 \otimes 1))(g_i \otimes (y_i \otimes 1) + x_i \otimes (1 \otimes 1))\\
&\quad = (g_j \otimes (1 \otimes y_j))(g_i \otimes (y_i \otimes 1)) + (x_j \otimes (1 \otimes 1))(g_i \otimes (y_i \otimes 1))\\
&\qquad\qquad + (g_j \otimes (1 \otimes y_j))(x_i \otimes (1 \otimes 1)) + (x_j \otimes (1 \otimes 1))(x_i \otimes (1 \otimes 1))\\
&\quad = g_j g_i \otimes (1 \otimes y_j)(y_i \otimes 1) + x_j g_i \otimes (y_i \otimes 1)\\
&\qquad\qquad + q_{ji} x_{j\triangleright i} g_j \otimes (1 \otimes y_j) + x_j x_i \otimes (1 \otimes 1)\\
&\quad = g_j g_i \otimes (q_{ji} y_{j\triangleright i} \otimes y_j - q_{ji} q_{j\triangleright i\, j} y_i y_{j\triangleright i} \otimes 1 + \xi_C g_C \otimes 1)\\
&\qquad\qquad + x_j g_i \otimes (y_i \otimes 1) + q_{ji} x_{j\triangleright i} g_j \otimes (1 \otimes y_j)\\
&\qquad\qquad + (q_{ji} x_{j\triangleright i} x_j - q_{ji} q_{j\triangleright i\, j} x_i x_{j\triangleright i} \otimes 1) \otimes (1 \otimes 1),
\end{aligned}$$

which coincides with $\lambda(q_{ji} y_{j\triangleright i} \otimes y_j - q_{ji} q_{j\triangleright i\, j} y_i y_{j\triangleright i} \otimes 1 + \xi_C g_C \otimes 1)$.

Thus, B is an $\mathcal{H}$-comodule algebra, with

$$\dim B = \dim A \dim \mathcal{K}_Z = \dim \mathcal{K}_Y \dim \mathcal{K}_Z |F| = \dim \mathcal{A}(X, F, \psi, \xi)$$

by Remark 6.11. Then, the map m from (7-7) is an isomorphism. □

We can now formulate the main result of the paper. For any $h \in G$, we write $\xi_C^h = \xi_{h^{-1} \cdot C}$. Recall that we denote by $\mathcal{B}(Y, F, \psi, \xi)$ the sub-comodule algebra of $\mathcal{A}(X, F, \psi, \xi)$ generated by $\{y_i\}_{i \in Y}$.

Theorem 7.11. (1) *Let $\mathcal{M}$ be an exact indecomposable module category over* $\operatorname{Rep}(\mathcal{H}(\mathcal{Q}))$. *There exist*

(i) *a subgroup $F < G$ and a 2-cocycle $\psi \in Z^2(F, \Bbbk^\times)$,*
(ii) *a subset $Y \subset X$ with $F \cdot Y \subset Y$, and*
(iii) *a family of scalars $\{\xi_C\}_{C \in \mathcal{R}'}$ compatible with (X, F, ψ),*

such that there is a module equivalence $\mathcal{M} \simeq {}_{\mathcal{B}(Y,F,\psi,\xi)}\mathcal{M}$.

(2) *Let (Y, F, ψ, ξ) and (Y', F', ψ', ξ') be two families as before. There is an equivalence of module categories ${}_{\mathcal{B}(Y,F,\psi,\xi)}\mathcal{M} \simeq {}_{\mathcal{B}(Y',F',\psi',\xi')}\mathcal{M}$ if and only if there exists an element $h \in G$ such that $F' = hFh^{-1}$, $\psi' = \psi^h$, $Y' = h \cdot Y$ and $\xi' = \xi^h$.*

Proof. Step 1: By Corollary 7.8, we can assume that $\mathcal{M}$ is an exact indecomposable module category over $\operatorname{gr}\mathcal{H}(\mathcal{Q}) = \mathcal{H}$. It follows from [Andruskiewitsch and Mombelli 2007, Theorem 3.3] that there is a right $\mathcal{H}$-simple left $\mathcal{H}$-comodule algebra $\mathcal{A}$ such that $\mathcal{M} \simeq {}_{\mathcal{A}}\mathcal{M}$. Theorem 3.2 implies that there is a subgroup $F < G$, a 2-cocycle $\psi \in Z^2(F, \Bbbk^\times)$ and a subset $Y \subset X$ with $F \cdot Y \subset Y$, such that $\operatorname{gr}\mathcal{A} = \mathcal{K}_Y \# \Bbbk_\psi F$. Here, $\mathcal{A}_0 = \Bbbk_\psi F$. The result then follows from Lemma 7.10.

Step 2: If the module categories ${}_{\mathcal{B}(Y,F,\psi,\xi)}\mathcal{M}$ and ${}_{\mathcal{B}(Y',F',\psi',\xi')}\mathcal{M}$ are equivalent, then Theorem 4.2 implies that there exists an element $h \in G$ such that $\mathcal{B}(Y', F', \psi', \xi') \simeq h\mathcal{B}(Y, F, \psi, \xi)h^{-1}$ as H-comodule algebras.

The algebra map $\alpha : h\mathcal{B}(Y, F, \psi, \xi)h^{-1} \to \mathcal{B}(h \cdot Y, hFh^{-1}, \psi^h, \xi^h)$, defined by $\alpha(he_f h^{-1}) = e_{hfh^{-1}}$ and $\alpha(hy_l h^{-1}) = \chi_l(h)\, y_{h \cdot l}$ for all $f \in F$ and $l \in Y$, is a well-defined comodule algebra isomorphism. It follows that $\mathcal{B}(Y', F', \psi', \xi') \simeq \mathcal{B}(h \cdot Y, hFh^{-1}, \psi^h, \xi^h)$ and, by using Theorem 7.5(3), we get the result. □

As a consequence of Theorem 7.11 we have:

Corollary 7.12. *Any $\mathcal{H}$-Galois object is of the form $\mathcal{A}(X, G, \psi, \xi)$.*

Proof. Let A be a $\mathcal{H}$-Galois object. Then, ${}_A\mathcal{M}$ is an exact module category over $\operatorname{Rep}\mathcal{H}$. Moreover, ${}_A\mathcal{M}$ is indecomposable; otherwise, by [Andruskiewitsch and Mombelli 2007, Proposition 1.18], there would exist a proper bilateral ideal $J \subset A$ $\mathcal{H}$-stable. Thus, $\operatorname{can}(A \otimes J) = \operatorname{can}(J \otimes A)$, which contradicts the bijectivity of can. By Theorem 7.11, there exists (X, G, ψ, ξ) such that $A \cong \mathcal{A}(X, G, \psi, \xi)$. □

7C. *Modules categories over $\mathfrak{B}(\mathbb{O}_2^3, -1) \# \Bbbk\mathbb{S}_3$.* We apply Theorem 7.11 to exhibit explicitly all module categories in this particular case. In this case the rack is

$$\mathbb{O}_2^3 = \{(12), (13), (23)\}.$$

For each $i \in \mathcal{O}_2^3$, we denote by g_i the element i when thought of as an element of the group $\mathbb{S}_3$. We will show in the Appendix that the algebras in the following result are not null; then, the next corollary will follow from Theorem 7.11.

Corollary 7.13. *Let $\mathcal{M}$ be an indecomposable exact module category over*

$$\operatorname{Rep}(\mathfrak{B}(\mathcal{O}_2^3, -1) \# \Bbbk\mathbb{S}_3).$$

There is a module equivalence $\mathcal{M} \simeq {}_{\mathcal{A}}\mathcal{M}$ where $\mathcal{A}$ is one (and only one) of the comodule algebras in following list, where $i, j, k \in \mathcal{O}_2^3$ and $\xi, \mu, \eta \in \Bbbk$.

(1) *For any subgroup $F \subseteq \mathbb{S}_3$, $\psi \in Z^2(F, \Bbbk^\times)$, the twisted group algebra $\Bbbk_\psi F$.*

(2) *The algebra $\mathcal{A}(\{i\}, \xi, 1) = \langle y_i : y_i^2 = \xi 1\rangle$ with coaction $\lambda(y_i) = x_i \otimes 1 + g_i \otimes y_i$.*

(3) *The algebra $\mathcal{A}(\{i\}, \xi, \mathbb{Z}_2) = \langle y_i, h : y_i^2 = \xi 1,\ h^2 = 1,\ hy_i = -y_i h\rangle$ with coaction $\lambda(y_i) = x_i \otimes 1 + g_i \otimes y_i$ and $\lambda(h) = g_i \otimes h$.*

(4) *The algebra $\mathcal{A}(\{i, j\}, 1) = \langle y_i, y_j : y_i^2 = y_j^2 = 0,\ y_i y_j y_i = y_j y_i y_j\rangle$ with coaction $\lambda(y_i) = x_i \otimes 1 + g_i \otimes y_i$ and $\lambda(y_j) = x_j \otimes 1 + g_j \otimes y_j$.*

(5) *The algebra*

$$\mathcal{A}(\{i, j\}, \mathbb{Z}_2) = \langle y_i, y_j, h : y_i^2 = y_j^2 = 0, h^2 = 1, hy_i = -y_j h, y_i y_j y_i = y_j y_i y_j\rangle$$

with coaction determined by $\lambda(y_i) = x_i \otimes 1 + g_i \otimes y_i$, $\lambda(y_j) = x_j \otimes 1 + g_j \otimes y_j$ and $\lambda(h) = g_k \otimes h$, where $k \neq i, j$.

(6) *The algebra $\mathcal{A}(\mathcal{O}_2^3, \xi, 1)$ generated by $\{y_{(12)}, y_{(13)}, y_{(23)}\}$ with relations*

$$y_{(12)}^2 = y_{(13)}^2 = y_{(23)}^2 = \xi 1,$$
$$y_{(12)} y_{(13)} + y_{(13)} y_{(23)} + y_{(23)} y_{(12)} = 0,$$
$$y_{(13)} y_{(12)} + y_{(23)} y_{(13)} + y_{(12)} y_{(23)} = 0.$$

The coaction is determined by $\lambda(y_s) = x_s \otimes 1 + g_s \otimes y_s$ for any $s \in \mathcal{O}_2^3$.

(7) *The algebra $\mathcal{A}(\mathcal{O}_2^3, \xi, \mathbb{Z}_2)$ generated by $\{y_{(12)}, y_{(13)}, y_{(23)}, h\}$ with relations*

$$y_{(12)}^2 = y_{(13)}^2 = y_{(23)}^2 = \xi 1, \quad h^2 = 1, \quad hy_{(12)} = -y_{(12)} h, \quad hy_{(13)} = -y_{(23)} h,$$
$$y_{(12)} y_{(13)} + y_{(13)} y_{(23)} + y_{(23)} y_{(12)} = 0.$$

The coaction is determined by $\lambda(h) = g_{(12)} \otimes h$ and $\lambda(y_s) = x_s \otimes 1 + g_s \otimes y_s$ for any $s \in \mathcal{O}_2^3$.

(8) *The algebra $\mathcal{A}(\mathcal{O}_2^3, \xi, \mu, \eta, \mathbb{Z}_3)$ generated by $\{y_{(12)}, y_{(13)}, y_{(23)}, h\}$ with relations*

$$y_{(12)}^2 = y_{(13)}^2 = y_{(23)}^2 = \xi 1, \quad h^3 = 1,$$
$$hy_{(12)} = y_{(13)} h, \quad hy_{(13)} = y_{(23)} h, \quad hy_{(23)} = y_{(12)} h,$$
$$y_{(12)} y_{(13)} + y_{(13)} y_{(23)} + y_{(23)} y_{(12)} = \mu\, h,$$

$$y_{(13)}y_{(12)} + y_{(23)}y_{(13)} + y_{(12)}y_{(23)} = \eta\, h^2.$$

The coaction is determined by $\lambda(h) = g_{(132)} \otimes h$ *and* $\lambda(y_s) = x_s \otimes 1 + g_s \otimes y_s$ *for any* $s \in \mathbb{O}_2^3$.

(9) *For each* $\psi \in Z^2(\mathbb{S}_3, \Bbbk^\times)$, *the algebra* $\mathcal{A}(\mathbb{O}_2^3, \xi, \mu, \mathbb{S}_3, \psi)$ *generated by* $\{y_{(12)}, y_{(13)}, y_{(23)}, e_h : h \in \mathbb{S}_3\}$ *with relations*

$$e_h e_t = \psi(h,t)\, e_{ht} \quad \text{and} \quad e_h y_s = -y_{h\cdot s} e_h \quad \text{for } h, t \in \mathbb{S}_3,\ s \in \mathbb{O}_2^3,$$
$$y_{(12)}^2 = y_{(13)}^2 = y_{(23)}^2 = \xi 1, \quad y_{(12)}y_{(13)} + y_{(13)}y_{(23)} + y_{(23)}y_{(12)} = \mu\, e_{(123)}.$$

The coaction is determined by $\lambda(e_h) = h \otimes e_h$ *and* $\lambda(y_s) = x_s \otimes 1 + g_s \otimes y_s$ *for any* $s \in \mathbb{O}_2^3$.

Appendix: $\mathcal{A}(Y, F, \psi, \xi) \neq 0$

We will complete the proofs of Corollaries 7.8 and 7.13, by showing that the algebras in their statements are not null.

Proposition A.14. *If* $\mathcal{A}(Y, F, \psi, \xi)$ *is one of the algebras in Corollary 7.13, then* $\mathcal{A}(Y, F, \psi, \xi) \neq 0$.

Proof. The case $Y \neq \mathbb{O}_2^3$ is clear. Set $Y = \mathbb{O}_2^3$. Note that each one of these algebras is naturally a right $\Bbbk F$-module via $a \leftharpoonup t = a e_t$ for $a \in \mathcal{A}(Y, F, \psi, \xi)$, $t \in F$. Thus, we can consider the induced representation $W = \mathcal{A}(Y, F, \psi, \xi) \otimes_{\Bbbk F} W_\varepsilon$, where $W_\varepsilon = \Bbbk\{z\}$ is the trivial $\Bbbk F$-module. Let

$$\begin{aligned} B = \{1,\ & y_{(12)},\ y_{(13)},\ y_{(23)},\ y_{(13)}y_{(12)},\ y_{(12)}y_{(13)},\ y_{(12)}y_{(23)},\ y_{(13)}y_{(23)}, \\ & y_{(12)}y_{(13)}y_{(23)},\ y_{(13)}y_{(12)}y_{(23)},\ y_{(12)}y_{(13)}y_{(12)},\ y_{(12)}y_{(13)}y_{(12)}y_{(23)}\} \end{aligned}$$

and consider the linear subspace V of W generated by $B \otimes z$. We show that this is a nontrivial submodule in the four cases left, namely $F = 1$, $\mathbb{Z}_2$, $\mathbb{Z}_3$ or $\mathbb{S}_3$. In all these cases, the action of $y_{(12)}$ is determined by the matrix

$$y_{(12)} = \begin{bmatrix} 0 & \xi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \xi & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \xi & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Now, take $F = \mathbb{S}_3$, $\psi \equiv 1$. The action of $e_{(12)}$ and $e_{(13)}$ is determined, respectively, by the matrices

$$\begin{bmatrix}
1 & 0 & 0 & 0 & \mu & 0 & 0 & \mu & 0 & 0 & 0 & 0\\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & -\mu & 0\\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \mu & \xi & 0\\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \xi & -\mu & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & -\mu\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & \mu\\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

and

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & \mu & \mu & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & \mu & 0 & \xi & 0\\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & -\mu & 0\\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & \xi & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & -\mu\\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & \mu\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.$$

The action of $e_{(23)}$ is given by $e_{(12)}e_{(13)}e_{(12)}$. Finally, we use Mathematica to check that these matrices satisfy the relations defining the algebra in each case.

We deal now with a generic 2-cocycle $\psi \in Z^2(\mathbb{S}_3, \Bbbk^\times)$. Fix $\mathscr{A} = \mathscr{A}(Y, F, 1, \xi)$ and $\mathscr{A}' = \mathscr{A}(Y, F, \psi, \xi)$. Also, set $U = \mathcal{H}_Y \# \Bbbk F$ and $U' = \mathcal{H}_Y \# \Bbbk_\psi F$. If $\overline{\psi} \in Z^2(U)$ is the 2-cocycle such that $\overline{\psi}_{F\times F} = \psi$ (see Lemma 4.1), it follows that $U' = U^{\overline{\psi}}$. Now, as $\mathscr{A}$ is an U-comodule algebra which is isomorphic to U as U-comodules, it follows that there exists a 2-cocycle $\gamma \in Z^2(U)$ such that $\mathscr{A} \cong {}_\gamma U$ (see [Montgomery 1993, Sec. 7 & 8]). It is easy to check then that $\mathscr{A}' = {}_\gamma U'$ by computing the multiplication on the generators. Thus, $\mathscr{A}' \neq 0$. □

To finish the proof of Corollary 7.8, we present three families of nontrivial algebras $\mathscr{A}(X, G, 1, \xi)$ for $X = \mathbb{O}_2^4$, $G = \mathbb{S}_4$, and certain collections of scalars $\{\xi_C\}_{C\in\mathscr{R}'}$ satisfying (7-6). In Proposition A.18, we will show that $\mathscr{A}(X, G, 1, \xi) \neq 0$.

Definition A.15. Let $\psi \in Z^2(\mathbb{S}_4, \Bbbk^\times)$ and $\alpha, \beta \in \Bbbk$.

(1) $\mathcal{A}^{-1}_{\psi}(\alpha,\beta)$ is the algebra generated by $\{y_i, e_g : i \in \mathcal{O}^4_2, g \in \mathbb{S}_4\}$ with relations

$$\begin{gathered} e_1 = 1, \quad e_r e_s = \psi(r,s)\, e_{rs} \ \text{ for } r,s \in \mathbb{S}_4, \\ e_g y_l = \operatorname{sgn}(g)\, y_{g\cdot l}\, e_g \ \text{ for } g \in \mathbb{S}_4,\ l \in \mathcal{O}^4_2, \\ y^2_{(12)} = \alpha\, 1, \quad y_{(12)}y_{(34)} + y_{(34)}y_{(12)} = 2\alpha\, e_{(12)(34)}, \\ y_{(12)}y_{(23)} + y_{(23)}y_{(13)} + y_{(13)}y_{(12)} = \beta\, e_{(132)}. \end{gathered}$$

(2) $\mathcal{A}^{4}_{\psi}(\alpha,\beta)$ is the algebra generated by $\{y_i, e_g : i \in \mathcal{O}^4_4, g \in \mathbb{S}_4\}$ with relations

$$\begin{gathered} e_1 = 1, \quad e_r e_s = \psi(r,s)\, e_{rs} \ \text{ for } r,s \in \mathbb{S}_4, \\ e_g y_l = \operatorname{sgn}(g)\, y_{g\cdot l} e_g \ \text{ for } g \in \mathbb{S}_4,\ l \in \mathcal{O}^4_4, \\ y^2_{(1234)} = \alpha\, e_{(13)(24)}, \quad y_{(1234)}y_{(1432)} + y_{(1432)}y_{(1234)} = 2\alpha 1, \\ y_{(1234)}y_{(1243)} + y_{(1243)}y_{(1423)} + y_{(1423)}y_{(1234)} = \beta\, e_{(132)}. \end{gathered}$$

(3) $\mathcal{A}^{\chi}_{\psi}(\alpha,\beta)$ is the algebra generated by $\{y_i, e_g : i \in \mathcal{O}^4_2, g \in \mathbb{S}_4\}$ with relations

$$\begin{gathered} e_1 = 1, \quad e_r e_s = \psi(r,s)\, e_{rs} \ \text{ for } r,s \in \mathbb{S}_4, \\ e_g\, y_l = \chi_l(g)\, y_{g\cdot l}\, e_g \ \text{ for } g \in \mathbb{S}_4,\ l \in \mathcal{O}^4_2, \\ y^2_{(12)} = \alpha\, 1, \quad y_{(12)}y_{(34)} - y_{(34)}y_{(12)} = 0, \\ y_{(12)}y_{(23)} - y_{(23)}y_{(13)} - y_{(13)}y_{(12)} = \beta\, e_{(132)}. \end{gathered}$$

Remark A.16. Let $\mathfrak{Q} = \mathfrak{Q}^{-1}[t]$. It is clear $\mathcal{A}^{-1}_{\psi}(\alpha,\beta) \cong \mathcal{A}(\mathcal{O}^4_2, \mathbb{S}_4, \psi, \xi)$ for the family $\xi = \{\xi_C\}_{C\in\mathfrak{R}}$ where, for $i = 1, 2, 3$, $\xi_C = \xi_i$ is constant in the classes C with the same cardinality $|C| = i$ and where, in this case, $\xi_1 = \alpha$, $\xi_2 = 2\alpha$, $\xi_3 = \beta$.

Analogously, if $\mathfrak{Q} = \mathfrak{Q}^{\chi}[t]$, then $\mathcal{A}^{\chi}_{\psi}(\alpha,\beta)$ is the algebra $\mathcal{A}(\mathcal{O}^4_2, \mathbb{S}_4, \psi, \xi)$ for a certain family ξ subject to similar conditions as in the previous paragraph. The same holds for $\mathfrak{Q} = \mathfrak{D}[t]$, $\mathcal{A}^{4}_{\psi}(\alpha,\beta)$ and $\mathcal{A}(\mathcal{O}^4_4, \mathbb{S}_4, \psi, \xi)$.

Recall that there is a group epimorphism $\pi : \mathbb{S}_4 \to \mathbb{S}_3$ with kernel $H = \langle (12)(34), (13)(24), (23)(14)\rangle$. Moreover, $\pi(\mathcal{O}^4_2) = \mathcal{O}^3_2$. Let $\mathfrak{Q}$ be one of the ql-data from Section 5D for $n = 4$.

Lemma A.17. *Let $\mathfrak{Q}$ be as above, and take $\gamma = 0$ if $\mathfrak{Q} = \mathfrak{Q}^{-1}_4$. There is an epimorphism of algebras $\mathcal{H}(\mathfrak{Q}) \twoheadrightarrow \mathcal{H}(\mathfrak{Q}^{-1}_3[\lambda])$.*

Proof. Consider the ideal I in $\mathcal{H}(\mathfrak{Q})$ generated by the element $H_{(12)}H_{(34)} - 1$, and let $\mathcal{L} = \mathcal{H}(\mathfrak{Q})/I$. We have

$$\begin{aligned} H_{(14)}H_{(23)} &= \operatorname{ad}(H_{(24)})(H_{(12)}H_{(34)}), &&\text{so } H_{(14)}H_{(23)} = 1 &&\text{in } \mathcal{L}, \\ a_{(34)} &= \operatorname{ad}(H_{(14)}H_{(23)})(a_{12}), &&\text{so } a_{(34)} = a_{(12)} &&\text{in } \mathcal{L}. \end{aligned}$$

Analogously, $H_{(13)} = H_{(24)}$, $a_{(14)} = a_{(23)}$ and $a_{(24)} = a_{(13)}$ in $\mathscr{L}$. Since, for this ql-data, the action $\cdot : \mathbb{S}_4 \times X \to X$ is given by conjugation, and $g : X \to \mathbb{S}_4$ is the inclusion, the relations (5-6) and (5-7) in the definition of $\mathscr{H}(\mathfrak{Q})$ are satisfied in the quotient. It is now easy to check that the quadratic relations (5-8) defining $\mathscr{H}(\mathfrak{Q})$ become in the quotient the corresponding ones defining the algebra $\mathscr{H}(\mathfrak{Q}_3^{-1}[\lambda])$. □

Proposition A.18. *Assume that* (Y, F, ψ, ξ) *satisfies* $\xi_{C_i} = \xi_{C_j}$ *for all* $i, j \in Y$. *If* $\mathfrak{Q} \neq \mathfrak{Q}_4^{\chi}(\lambda)$, *assume in addition that*

$$i, j \in Y,\ i \triangleright j = j\ (i, j) \in C \ \Rightarrow\ \xi_C = 2\xi_i.$$

Then, the algebra $\mathscr{A}(Y, F, \psi, \xi)$ *is not null.*

Proof. Assume first that $\psi \equiv 1$. Now, given a datum (Y, F, ψ, ξ), we have $\pi(F) < \mathbb{S}_3$ and it is easy to see that $\pi(Y)$ is a subrack of $\mathcal{O}_2^3$. Moreover, it follows that ξ is compatible with the triple $(\pi(Y), \pi(F), \psi)$. Then, we have the algebra $\mathscr{A}(\pi(Y), \pi(F), \psi, \xi)$. As in Lemma A.17, it is easy to see that, if we quotient out by the ideal generated by $\langle e_f e_g : fg^{-1} \in N\rangle$, then we have an algebra epimorphism $\mathscr{A}(Y, F, \psi, \xi) \twoheadrightarrow \mathscr{A}(\pi(Y), \pi(F), \psi, \xi)$. As these algebras are nonzero by Proposition A.14, so is $\mathscr{A}(Y, F, \psi, \xi)$.

Notice that, in the case in which (Y, F, ψ, ξ) is associated with the ql-datum $\mathfrak{Q}_4^{\chi}(\lambda)$, assumption (ii) is not needed, since the first equation in Definition 7.1 implies that, if $i, j \in Y$ are such that $i \triangleright j = i$ and $C \in \mathscr{R}'$ is the corresponding class, then $\xi_C = 0$ and this relation is contained in the ideal by which we quotient.

The case $\psi \neq 1$ follows now as in the proof of Proposition A.14. □

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(p, p)-GALOIS REPRESENTATIONS ATTACHED TO AUTOMORPHIC FORMS ON GL_n

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We study the local reducibility at p of the p-adic Galois representation attached to a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$. In the case that the underlying Weil–Deligne representation is Frobenius semisimple and indecomposable, we analyze the reducibility completely. We use methods from p-adic Hodge theory, and work under a transversality assumption on the Hodge and Newton filtrations in the corresponding filtered module.

1. Introduction

Let $f = \sum_{n=1}^{\infty} a_n(f)q^n$ be a primitive elliptic modular cusp form of weight $k \geq 2$, level $N \geq 1$, and nebentypus $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$. Let K_f denote the number field generated by the Fourier coefficients of f. Fix an embedding of $\bar{\mathbb{Q}}$ into $\bar{\mathbb{Q}}_p$, and let $\wp$ be the prime of $\bar{\mathbb{Q}}$ determined by this embedding. Let $\wp$ also denote the induced prime of K_f, and let $K_{f,\wp}$ be the completion of K_f at $\wp$. For a global or local field F of characteristic 0, let G_F denote the absolute Galois group of F. There is a global Galois representation

$$\rho_{f,\wp} : G_{\mathbb{Q}} \to \mathrm{GL}_2(K_{f,\wp})$$

associated to f (and $\wp$) by Deligne which has the property that for all primes $\ell \nmid Np$,

$$\mathrm{trace}(\rho_{f,\wp}(\mathrm{Frob}_\ell)) = a_\ell(f) \quad \text{and} \quad \det(\rho_{f,\wp}(\mathrm{Frob}_\ell)) = \chi(\ell)\ell^{k-1}.$$

Thus $\det(\rho_{f,\wp}) = \chi\chi_{\mathrm{cyc},p}^{k-1}$, where $\chi_{\mathrm{cyc},p}$ is the p-adic cyclotomic character.

It is a well-known result of Ribet that the global representation $\rho_{f,\wp}$ is irreducible. However, if f is ordinary at $\wp$, that is, if $a_p(f)$ is a $\wp$-adic unit, then an important theorem of Wiles, valid more generally for Hilbert modular forms, says that the corresponding local representation is reducible.

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Theorem 1.1 [Wiles 1988]. *Let f be a $\wp$-ordinary primitive form as above. Then the restriction of $\rho_{f,\wp}$ to the decomposition subgroup $G_{\mathbb{Q}_p}$ is reducible. More precisely, there exists a basis in which*

$$\rho_{f,\wp}|_{G_{\mathbb{Q}_p}} \sim \begin{pmatrix} \chi_p \cdot \lambda(\beta/p^{k-1}) \cdot \chi_{\mathrm{cyc},p}^{k-1} & u \\ 0 & \lambda(\alpha) \end{pmatrix},$$

where $\chi = \chi_p \chi'$ is the decomposition of χ into its p and prime-to-p parts,

$$\lambda(x) : G_{\mathbb{Q}_p} \to K_{f,\wp}^{\times}$$

is the unramified character taking arithmetic Frobenius to x, and $u : G_{\mathbb{Q}_p} \to K_{f,\wp}$ is a continuous function; moreover, α is

(i) *the unit root of $X^2 - a_p(f)X + p^{k-1}\chi(p)$ if $p \nmid N$,*

(ii) *the unit $a_p(f)$ if $p \parallel N$, $p \nmid \operatorname{cond}\chi$, and $k = 2$,*

(iii) *the unit $a_p(f)$ if $p \mid N$ and $v_p(N) = v_p(\operatorname{cond}\chi)$,*

and β is given by $\alpha\beta = \chi'(p)p^{k-1}$.

Moreover, in case (ii), $a_p(f)$ is a unit if and only if $k = 2$, and one can show that $\rho_{f,\wp}|_{G_{\mathbb{Q}_p}}$ is irreducible when $k > 2$.

Urban has generalized Theorem 1.1 to the case of primitive Siegel modular cusp forms of genus 2. We briefly recall this result here. Let π be a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$ whose Archimedean component π_∞ belongs to the discrete series, with cohomological weights $(a, b; a+b)$ with $a \geq b \geq 0$. For each prime p, Laumon, Taylor and Weissauer have defined a four-dimensional Galois representation

$$\rho_{\pi,p} : G_{\mathbb{Q}} \to \mathrm{GL}_4(\bar{\mathbb{Q}}_p)$$

with standard properties. Let p be an unramified prime for π. Tilouine and Urban have generalized the notion of ordinariness for such primes p in three ways to what they call Borel ordinary, Siegel ordinary, and Klingen ordinary (these terms come from the underlying parabolic subgroups of $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$). In the Borel case, the p-ordinariness of π implies that the Hecke polynomial of π_p, namely

$$(X-\alpha)(X-\beta)(X-\gamma)(X-\delta),$$

has the property that the p-adic valuations of α, β, γ, and δ are 0, $b+1$, $a+2$, and $a+b+3$, respectively.

Theorem 1.2 [Urban 2005; Tilouine and Urban 1999]. *Say π is a Borel p-ordinary cuspidal automorphic representation of* $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$ *that is stable at ∞ with cohomological weights $(a, b; a+b)$. Then the restriction of $\rho_{\pi,p}$ to the decomposition subgroup $G_{\mathbb{Q}_p}$ is upper-triangular. More precisely, there is a basis in which*

$$\rho_{\pi,p}|_{G_{\mathbb{Q}_p}} \sim \begin{pmatrix} \lambda(\delta/p^{a+b+3})\cdot\chi_{\mathrm{cyc},p}^{a+b+3} & * & * & * \\ 0 & \lambda(\gamma/p^{a+2})\cdot\chi_{\mathrm{cyc},p}^{a+2} & * & * \\ 0 & 0 & \lambda(\beta/p^{b+1})\cdot\chi_{\mathrm{cyc},p}^{b+1} & * \\ 0 & 0 & 0 & \lambda(\alpha) \end{pmatrix},$$

where $\lambda(x)$ *is the unramified character that takes arithmetic Frobenius to* x.

We remark that $\rho_{\pi,p}$ above is the contragredient of the one used in [Urban 2005] (we also use the arithmetic Frobenius in defining our unramified characters), so the theorem matches exactly with Corollary 1(iii) of that work. Similar results in the Siegel and Klingen cases can be found in [Urban 2005].

The local Galois representations appearing in Theorems 1.1 and 1.2 are sometimes referred to as (p, p)-Galois representations. The goal of this paper is to prove structure theorems for the local (p, p)-Galois representations attached to automorphic representations of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$ for any $n \geq 1$.

Let now π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$. We assume that the global p-adic Galois representation $\rho_{\pi,p}$ attached to π exists, and that it satisfies several natural properties; for example, it lives in a strictly compatible system of Galois representations, and satisfies local-global compatibility. Recently, much progress has been made on this front: such Galois representations have been attached to what are referred to as RAESDC (regular, algebraic, essentially self-dual, cuspidal) automorphic representations of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$ by Clozel, Harris, Kottwitz and Taylor, and for conjugate self-dual automorphic representations over CM fields these representations were shown by Taylor and Yoshida to satisfy local-global compatibility away from p.

The assumptions above allow us to specify the Weil–Deligne parameter at p. We study the (p, p)-Galois representation $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ attached to π, given this parameter. In fact, as the expert reader will note, since our methods are local, our results could equally well have been phrased purely in terms of this parameter.

A key tool in our analysis is the celebrated result of Colmez and Fontaine establishing an equivalence of categories between potentially semistable representations and filtered (φ, N)-modules with coefficients and descent data. Under some standard hypotheses, such as Assumption 3.6 that the Hodge and Newton filtrations are in general position in the corresponding crystal, we show that in several cases the corresponding local (p, p)-representation $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ has an upper-triangular form, and completely determines the diagonal characters. In other cases, and perhaps more interestingly, we give conditions under which this local representation is irreducible. For instance, we directly generalize the comment about irreducibility made just after Theorem 1.1. As a sample of our results, we state the following theorem, which is a collation of Theorems 5.7, 5.8, and 6.10.

Theorem 1.3 (indecomposable case). *Say π is a cuspidal automorphic representation of $\mathrm{GL}_{mn}(\mathbb{A}_{\mathbb{Q}})$ with infinitesimal character given by integers $-\beta_1 > \cdots > -\beta_{mn}$. Suppose the Weil–Deligne representation attached to π_p is Frobenius semisimple and indecomposable, that is,*

$$\mathrm{WD}(\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}) \sim \tau_m \otimes \mathrm{Sp}(n),$$

where τ_m is an irreducible representation of $W_{\mathbb{Q}_p}$ of dimension $m \geq 1$, and $\mathrm{Sp}(n)$ *is the special representation for $n \geq 1$. Let Assumption 3.6 hold.*

(i) *Suppose $m = 1$ and $\tau_1 = \chi_0 \cdot \chi'$ is a character, where χ_0 is the ramified part, and χ' is an unramified character mapping arithmetic Frobenius to α.*

(a) *If π is ordinary at p (i.e., $v_p(\alpha) = -\beta_1$), then the β_i are necessarily consecutive integers, and*

$$\rho_{\pi,p}|_{G_{\mathbb{Q}_p}} \sim$$

$$\begin{pmatrix} \chi_0\cdot\lambda\Bigl(\dfrac{\alpha}{p^{v_p(\alpha)}}\Bigr)\cdot\chi_{\mathrm{cyc},p}^{-\beta_1} & * & * \\ 0 & \chi_0\cdot\lambda\Bigl(\dfrac{\alpha}{p^{v_p(\alpha)}}\Bigr)\cdot\chi_{\mathrm{cyc},p}^{-\beta_1-1} & * \\ & & \ddots & \\ 0 & 0 & \chi_0\cdot\lambda\Bigl(\dfrac{\alpha}{p^{v_p(\alpha)}}\Bigr)\cdot\chi_{\mathrm{cyc},p}^{-\beta_1-(n-1)} \end{pmatrix},$$

where $\lambda(x)$ is the unramified character taking arithmetic Frobenius to x.

(b) *If π is not p-ordinary, $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ is irreducible.*

(ii) *Suppose $m \geq 2$. Then $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ is irreducible.*

The theorem gives complete information about the reducibility of the (p, p)-Galois representation in the indecomposable case (under Assumption 3.6). In particular, the image of the (p, p)-representation tends to be either in a minimal parabolic subgroup or a maximal parabolic subgroup of GL_n. While this is forced in the GL_2 setting, it is somewhat surprising that the image does not lie in any "intermediate" parabolic subgroups even in the GL_n setting. We also point out that parts (i)(b) and (ii) of the theorem imply that the global representation $\rho_{\pi,p}$ is irreducible (see also [Taylor and Yoshida 2007, Corollary B] for the case of conjugate self-dual representations over CM fields).

The theorem is proved in Sections 5 and 6 using methods from p-adic Hodge theory. Recall that the category of Weil–Deligne representations is equivalent to the category of (φ, N)-modules. In Section 6, we classify the (φ, N)-submodules of the (φ, N)-module associated with the indecomposable Weil–Deligne representation in the theorem. This classification plays a key role in analyzing the (p, p)-representation once the Hodge filtration is introduced. Along the way, we take a slight detour to write down explicitly the filtered (φ, N)-module attached to an m-dimensional "unramified supercuspidal" representation, since this might be a useful addition to the literature (see [Ghate and Mézard 2009] for the two-dimensional case).

The term "indecomposable case" in the discussion above refers to the standard fact that every Frobenius semisimple indecomposable Weil–Deligne representation has the form stated in the theorem. Some partial results in the decomposable case, where the Weil–Deligne representation is a direct sum of indecomposables, can be found in Section 8 of [Ghate and Kumar 2010]. The principal series case is treated completely in Section 4 of the present paper. In the spherical case our results overlap with those in D. Geraghty's thesis [2010], and we thank T. Gee for pointing this out to us.

We also refer to [Ghate and Kumar 2010, §3] for another proof of Theorem 1.1 along the lines of this paper. The original proof used Dieudonné theory only in weight 2 and then Hida theory [1986] (see also [Banerjee et al. 2010]) to extend to higher weights. Of the remaining sections, Section 2 recalls some useful facts from p-adic Hodge theory, whereas Section 3 recalls some general facts and conjectures about Galois representations associated with automorphic representations of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$.

2. *p*-adic Hodge theory

We start by recalling some results we need from p-adic Hodge theory. For the basic definitions in the subject, e.g., of Fontaine's ring $\mathbf{B}_{\mathrm{st}}$, filtered (φ, N)-modules with coefficients and descent data, and Newton and Hodge numbers, see [Fontaine 1994; Savitt 2005; Fontaine and Ouyang; Ghate and Mézard 2009, §2]. Also, see [Ghate and Kumar 2010, §2] for proofs.

Newton and Hodge numbers. We start by stating some facts about Newton and Hodge numbers, which do not seem to be in the literature when the coefficients are not necessarily $\mathbb{Q}_p$.

Let F and E be two finite field extensions of $\mathbb{Q}_p$, and assume that all the conjugates of F are contained in E.

Lemma 2.1 (Newton number). *Suppose D is a filtered (φ, N, F, E)-module of rank n such that the action of φ is E-semisimple, that is, there exists a basis $\{e_1, \dots, e_n\}$ of D such that $\varphi(e_i) = \alpha_i e_i$, for some $\alpha_i \in E^\times$. Then*

$$t_N(D) = [E : \mathbb{Q}_p] \cdot \sum_{i=1}^{n} v_p(\alpha_i).$$

Lemma 2.2 (Hodge number). *Suppose D is a filtered (φ, N, F, E)-module of rank n. Then*

$$t_H(D) = [E : \mathbb{Q}_p] \cdot \sum_{i \in \mathbb{Z}} i \cdot \mathrm{rank}_{F \otimes_{\mathbb{Q}_p} E} \ \mathrm{gr}^i D_F.$$

Remark. By the last two lemmas, one can drop the common factor of $[E : \mathbb{Q}_p]$ when checking the admissibility of a filtered (φ, N, F, E)-module.

Lemma 2.3. *Let D_1 and D_2 be filtered (φ, N, F, E)-modules of rank r_1 and r_2, respectively. Assume that the action of φ on D_1 and D_2 is semisimple. Then*

$$t_N(D_1 \otimes D_2) = \operatorname{rank}(D_1)\, t_N(D_2) + \operatorname{rank}(D_2)\, t_N(D_1),$$
$$t_H(D_1 \otimes D_2) = \operatorname{rank}(D_1)\, t_H(D_2) + \operatorname{rank}(D_2)\, t_H(D_1).$$

Remark. The formulas above are well-known if $E = \mathbb{Q}_p$.

Potentially semistable representations. Let E and F be two finite extensions of $\mathbb{Q}_p$, and let V be a finite dimensional vector space over E.

Definition. A representation $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}(V)$ is said to be *semistable* over F or *F-semistable*, if

$$\dim_{F_0} D_{\mathrm{st},F}(V) = \dim_{F_0}(\mathbf{B}_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_F} = \dim_{\mathbb{Q}_p} V,$$

where $F_0 = \mathbf{B}_{\mathrm{st}}^{G_F}$. If such an F exists, ρ is said to be a potentially semistable representation. If $F = \mathbb{Q}_p$, we say that ρ is semistable.

Remark. If ρ is F-semistable, ρ is F'-semistable for any finite extension of F'/F. Hence we may and do assume that F is Galois over $\mathbb{Q}_p$.

The following fundamental theorem plays a key role in subsequent arguments.

Theorem 2.4 [Colmez and Fontaine 2000]. *There is an equivalence of categories between F-semistable representations $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(E)$ with Hodge–Tate weights $-\beta_n \leq \cdots \leq -\beta_1$ and admissible filtered (φ, N, F, E)-modules D of rank n over $F_0 \otimes_{\mathbb{Q}_p} E$ such that the jumps in the Hodge filtration $\mathrm{Fil}^i D_F$ on $D_F := F \otimes_{F_0} D$ are at $\beta_1 \leq \cdots \leq \beta_n$.*

The equivalence of categories in the theorem is induced by Fontaine's functor $D_{\mathrm{st},F}$. The Frobenius φ, monodromy N, and filtration on $\mathbf{B}_{\mathrm{st}}$ induce the corresponding structures on $D_{\mathrm{st},F}(V)$. There is also an induced action of $\mathrm{Gal}(F/\mathbb{Q}_p)$ on $D_{\mathrm{st},F}(V)$.

As an illustration of the power of the theorem we recall a useful and well-known fact:

Corollary 2.5. *Every potentially semistable character $\chi : G_{\mathbb{Q}_p} \to E^\times$ is of the form $\chi = \chi_0 \cdot \lambda(a_0) \cdot \chi_{\mathrm{cyc},p}^i$, where χ_0 is a finite order character of $\mathrm{Gal}(F/\mathbb{Q}_p)$ for a cyclotomic extension F of $\mathbb{Q}_p$, $-i \in \mathbb{Z}$ is the Newton number of $D_{st,F}(\chi)$, and $\lambda(a_0)$ is the unramified character that takes arithmetic Frobenius to the unit $a_0 = p^{-i}/a \in \mathcal{O}_E^\times$, where $a = \varphi(v)/v$ for any nonzero vector v in $D_{st,F}(\chi)$.*

Weil–Deligne representations. We now recall the definition of the Weil–Deligne representation associated with an F-semistable representation $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(E)$, due to Fontaine. We assume that $F/\mathbb{Q}_p$ is Galois and $F \subseteq E$. Let W_F denote the Weil group of F. For any (φ, N, F, E)-module D, we have the decomposition

$$D \simeq \prod_{i=1}^{[F_0:\mathbb{Q}_p]} D_i, \tag{2-1}$$

where $D_i = D \otimes_{(F_0 \otimes_{\mathbb{Q}_p} E, \sigma^i)} E$, and σ is the arithmetic Frobenius of $F_0/\mathbb{Q}_p$.

Definition 2.6 (Weil–Deligne representation). Let $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(E)$ be an F-semistable representation. Let D be the corresponding filtered module. Noting $W_{\mathbb{Q}_p}/W_F = \mathrm{Gal}(F/\mathbb{Q}_p)$, we let

$$g \in W_{\mathbb{Q}_p} \text{ act on } D \text{ by } (g \bmod W_F) \circ \varphi^{-\alpha(g)},$$

where the image of g in $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ is the $\alpha(g)$-th power of the arithmetic Frobenius at p. We also have an action of N via the monodromy operator on D. These actions induce a Weil–Deligne action on each D_i in (2-1), and the resulting Weil–Deligne representations are all isomorphic. This isomorphism class is defined to be the Weil–Deligne representation $\mathrm{WD}(\rho)$ associated with ρ.

Remark. If $F/\mathbb{Q}_p$ is totally ramified and $\mathrm{Frob}_p \in W_{\mathbb{Q}_p}$ is the arithmetic Frobenius, then observe that $\mathrm{WD}(\rho)(\mathrm{Frob}_p)$ acts by φ^{-1}.

Lemma 2.7. *Let $\rho : \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}_n(E)$ be a potentially semistable representation. If $\mathrm{WD}(\rho)$ is irreducible, so is ρ.*

Compatible systems. We recall the notion of a strictly compatible system of Galois representations following [Khare and Wintenberger 2009, §5], where it was used to great effect in the two-dimensional case. Let F be a number field, ℓ a prime, and $\rho : G_F \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$ a continuous global Galois representation.

Definition. Say that ρ is *geometric* if it is unramified outside a finite set of primes of F and its restrictions to the decomposition groups at primes above ℓ are potentially semistable.

A geometric representation defines, for every prime q of F, a representation of the Weil–Deligne group at q, denoted by WD_q, with values in $\mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$, well-defined up to conjugacy. For q of characteristic not ℓ, the definition is classical, and comes from the theory of Deligne–Grothendieck, and for q of characteristic ℓ, the definition comes from Fontaine theory (Definition 2.6).

Definition. Let L be a number field. An L-rational, n-dimensional *strictly compatible system* of geometric representations (ρ_ℓ) of G_F is the collection of data consisting of:

(1) For each prime ℓ and each embedding $i : L \hookrightarrow \bar{\mathbb{Q}}_\ell$, a continuous, semisimple representation $\rho_\ell : G_F \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$ that is geometric.

(2) For each prime q of F, an F-semisimple (Frobenius semisimple) representation r_q of the Weil–Deligne group WD_q with values in $\mathrm{GL}_n(L)$ such that
   - r_q is unramified for all q outside a finite set;
   - for each ℓ and each $i : L \hookrightarrow \bar{\mathbb{Q}}_\ell$, the Frobenius semisimple Weil–Deligne representation $\mathrm{WD}_q \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$ associated with $\rho_\ell|_{D_q}$ is conjugate to r_q (via the embedding $i : L \hookrightarrow \bar{\mathbb{Q}}_\ell$); and
   - there are n distinct integers $\beta_1 < \cdots < \beta_n$ such that ρ_ℓ has Hodge–Tate weights $\{-\beta_1, \ldots, -\beta_n\}$. (The minus signs arise since the weights are the negatives of the jumps in the Hodge filtration on the associated filtered module.)

The existence of strictly compatible systems attached to classical cusp forms is well-known [Katz and Messing 1974; Saito 1997]. For general cuspidal automorphic representations, we will not use the full strength of this definition. In fact we only use it to obtain information about the Weil–Deligne parameter at p. Our results could equally well be stated using this parameter as the *starting point* of our analysis.

3. The case of GL_n

The goal of this paper is to prove various generalizations of Theorem 1.1 for the local (p, p)-Galois representations attached to automorphic representations of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$. In this section we collect together some facts about such automorphic representations and their Galois representations needed for the proof. The main results we need are the local Langlands correspondence [Henniart 2000; Harris and Taylor 2001] and the existence of strictly compatible systems of Galois representations attached to cuspidal automorphic representations of GL_n (much progress has been made on this by Clozel, Harris, Kottwitz, and Taylor [Clozel et al. 2008]).

Local Langlands correspondence. We will need a few results concerning the local Langlands correspondence. We follow [Kudla 1994] in our exposition, noting that that article follows [Rodier 1982], which in turn is based on the original work of Bernstein and Zelevinsky.

Let F be a complete non-Archimedean local field of residue characteristic p, let $n \geq 1$, and let $G = \mathrm{GL}_n(F)$. For a partition $n = n_1 + n_2 + \cdots + n_r$ of n, let P be the corresponding parabolic subgroup of G, let M be the Levi subgroup of P, and N the unipotent radical of P. Let δ_P denote the modulus character of the adjoint action of M on N. If $\sigma = \sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_r$ is a smooth representation of M on V, we let

$$I_P^G(\sigma) = \{f : G \to V \mid f \text{ smooth on } G \text{ and } f(nmg) = \delta_P^{1/2}(m)(\sigma(m)(f(g))\}$$

for $n \in N$, $m \in M$, and $g \in G$. The group G acts on functions in $I_P^G(\sigma)$ by right translation and $I_P^G(\sigma)$ is the usual induced representation of σ. It is an admissible representation of finite length.

A result of Bernstein and Zelevinsky says that if all the σ_i are supercuspidal and σ is irreducible, smooth and admissible, then $I_P^G(\sigma)$ is reducible if and only if $n_i = n_j$ and $\sigma_i = \sigma_j(1)$ for some $i \neq j$. For the partition $n = m + m + \cdots + m$ (r times), and for a supercuspidal representation σ of $\mathrm{GL}_m(F)$, call the data

$$(\sigma, \sigma(1), \cdots, \sigma(r-1)) = [\sigma, \sigma(r-1)] = \Delta$$

a segment. Clearly $I_P^G(\Delta)$ is reducible. By [Kudla 1994, Theorem 1.2.2], the induced representation $I_P^G(\Delta)$ has a unique irreducible quotient $Q(\Delta)$ that is essentially square-integrable.

Two segments

$$\Delta_1 = [\sigma_1, \sigma_1(r_1 - 1)] \quad \text{and} \quad \Delta_2 = [\sigma_2, \sigma_2(r_2 - 1)]$$

are said to be linked if $\Delta_1 \not\subseteq \Delta_2$, $\Delta_2 \not\subseteq \Delta_1$, and $\Delta_1 \cup \Delta_2$ is a segment. We say that Δ_1 precedes Δ_2 if Δ_1 and Δ_2 are linked and if $\sigma_2 = \sigma_1(k)$ for some $k \in \mathbb{N}$.

Theorem 3.1 (Langlands classification). *Let $\Delta_1, \dots, \Delta_r$ be segments such that if $i < j$ then Δ_i does not precede Δ_j.*

(1) *The induced representation $I_P^G(Q(\Delta_1) \otimes \cdots \otimes Q(\Delta_r))$ admits a unique irreducible quotient $Q(\Delta_1, \cdots, \Delta_r)$, called the Langlands quotient. Moreover, r and the segments Δ_i up to permutation are uniquely determined by the Langlands quotient.*

(2) *Every irreducible admissible representation of* $\mathrm{GL}_n(F)$ *is isomorphic to some $Q(\Delta_1, \cdots, \Delta_r)$.*

(3) *The induced representation $I_P^G(Q(\Delta_1) \otimes \cdots \otimes Q(\Delta_r))$ is irreducible if and only if no two of the segments Δ_i and Δ_j are linked.*

So much for the automorphic side. We now turn to the Galois side. Recall that a representation of W_F is said to be Frobenius semisimple if arithmetic Frobenius acts semisimply. An admissible representation of the Weil–Deligne group of F is one for which the action of W_F is Frobenius semisimple. Let $\mathrm{Sp}(r)$ denote the Weil–Deligne representation of order r with the usual definition. When $F = \mathbb{Q}_p$, there is a basis $\{f_i\}$ of $\mathrm{Sp}(r)$ for which $\varphi f_i = p^{i-1} f_i$, and $N f_i = f_{i-1}$ for $i > 1$ and $N f_1 = 0$. It is well-known that every indecomposable admissible representation of W_F is of the form $\tau \otimes \mathrm{Sp}(r)$, where τ is an irreducible admissible representation of W_F and $r \geq 1$. Moreover (cf. [Rohrlich 1994, §5, Corollary 2]), every admissible representation of W_F is of the form

$$\bigoplus_i \tau_i \otimes \mathrm{Sp}(r_i),$$

where the τ_i are irreducible admissible representations of W_F and the $r_i \in \mathbb{N}$.

Theorem 3.2 (local Langlands correspondence: [Harris and Taylor 2001, VII.2.20; Henniart 2000; Kutzko 1980]). *There is a bijection between isomorphism classes of irreducible admissible representations of* $\mathrm{GL}_n(F)$ *and isomorphism classes of admissible n-dimensional representations of* W_F.

The correspondence is given as follows. The key point is to construct a bijection $\Phi_F : \sigma \mapsto \tau = \Phi_F(\sigma)$ between the set of isomorphism classes of supercuspidal representations of $\mathrm{GL}_n(F)$ and the set of isomorphism classes of irreducible admissible representations of W_F. This was done in [Henniart 2000] and [Harris and Taylor 2001]. Then, to $Q(\Delta)$, for the segment $\Delta = [\sigma, \sigma(r-1)]$, one associates the indecomposable admissible representation $\Phi_F(\sigma) \otimes \mathrm{Sp}(r)$ of the Weil–Deligne group of F. More generally, to the Langlands quotient $Q(\Delta_1, \cdots, \Delta_r)$, where $\Delta_i = [\sigma_i, \sigma_i(r_i - 1)]$, for $i = 1$ to r, one associates the admissible representation $\bigoplus_i \Phi_F(\sigma_i) \otimes \mathrm{Sp}(r_i)$ of the Weil–Deligne group of F.

***Automorphic forms on* $\mathbf{GL}_n$.** The Harish-Chandra isomorphism identifies the center $\mathfrak{z}_n$ of the universal enveloping algebra of the complexified Lie algebra $\mathfrak{gl}_n$ of GL_n, with the algebra $\mathbb{C}[X_1, X_2, \cdots, X_n]^{S_n}$, where the symmetric group S_n acts by permuting the X_i. Given a multiset $H = \{x_1, x_2, \ldots, x_n\}$ of n complex numbers one obtains an infinitesimal character of $\mathfrak{z}_n$ given by $\chi_H : X_i \mapsto x_i$.

Cuspidal automorphic forms with infinitesimal character χ_H (or more simply just H) are smooth functions $f : \mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A}_\mathbb{Q}) \to \mathbb{C}$ satisfying the usual finiteness condition under a maximal compact subgroup, a cuspidality condition, and a growth condition, for which we refer the reader to [Taylor 2004]. In addition, if $z \in \mathfrak{z}_n$, then $z \cdot f = \chi_H(z) f$. The space of such functions is denoted by

$$\mathcal{A}_H^\circ(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})).$$

This space is a direct sum of irreducible admissible $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q}^{(\infty)}) \times (\mathfrak{gl}_n, O(n))$-modules each occurring with multiplicity one, and these irreducible constituents are referred to as cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$ with infinitesimal character χ_H. Let π be such an automorphic representation. By a result of Flath, π is a restricted tensor product $\pi = \bigotimes_p' \pi_p$ [Bump 1997, Theorem 3.3.3] of local automorphic representations.

Galois representations. Let π be an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$ with infinitesimal character χ_H, where H is a multiset of integers. The following very strong, but natural, conjecture seems to be part of the folklore.

Conjecture 3.3. *Let H consist of n distinct integers. There is a strictly compatible system of Galois representations $(\rho_{\pi,\ell})$ associated with π, with Hodge–Tate weights H, such that local-global compatibility holds.*

Here local-global compatibility means that the underlying semisimplified Weil–Deligne representation at p in the compatible system (which is independent of the residue characteristic ℓ of the coefficients by hypothesis) corresponds to π_p via the local Langlands correspondence. Considerable evidence towards this conjecture is available for self-dual representations, thanks to the work of Clozel, Kottwitz, Harris, and Taylor. We quote the following theorem from [Taylor 2004], referring to that paper for the original references (e.g., [Clozel 1991]).

Theorem 3.4 [Taylor 2004, Theorem 3.6]. *Let H consist of n distinct integers. Suppose that the contragredient representation $\pi^\vee = \pi \otimes \psi$ for some character $\psi : \mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}^\times \to \mathbb{C}^\times$, and suppose that for some prime q, the representation π_q is square-integrable. Then there is a continuous representation*

$$\rho_{\pi,\ell} : G_{\mathbb{Q}} \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$$

such that $\rho_{\pi,\ell}|_{G_{\mathbb{Q}_\ell}}$ is potentially semistable with Hodge–Tate weights given by H, and such that for any prime $p \neq \ell$, the semisimplification of the Weil–Deligne representation attached to $\rho_{\pi,\ell}|_{G_{\mathbb{Q}_p}}$ is the same as the Weil–Deligne representation associated by the local Langlands correspondence with π_p, except possibly for the monodromy operator.

Subsequent work of Taylor and Yoshida [2007] shows that the two Weil–Deligne representations in the theorem above are in fact the same (i.e., the monodromy operators also match).

In any case, for the rest of this paper we shall *assume* that Conjecture 3.3 holds. In particular, we assume that the Weil–Deligne representation at p associated with a p-adic member of the compatible system of Galois representations attached to π using Fontaine theory is the same as the Weil–Deligne representation at p attached to an ℓ-adic member of the family, for $\ell \neq p$.

A variant. A variant of the result above can be found in [Clozel et al. 2008]. We state it now, using the notation and terminology from §4.3 of that reference.

Say π is an RAESDC (regular, algebraic, essentially self-dual, cuspidal) automorphic representation if π is a cuspidal automorphic representation such that

- $\pi^\vee = \pi \otimes \chi$ for some character $\chi : \mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}^\times \to \mathbb{C}^\times$, and
- π_∞ has the same infinitesimal character as some irreducible algebraic representation of GL_n.

Let $a \in \mathbb{Z}^n$ satisfy

$$a_1 \geq \cdots \geq a_n. \tag{3-1}$$

Let Ξ_a denote the irreducible algebraic representation of GL_n with highest weight a. We say that an RAESDC automorphic representation π has weight a if π_∞ has the same infinitesimal character as $\Xi_a^\vee$; in this case there is an integer w_a such that $a_i + a_{n+1-i} = w_a$ for all i.

Let S be a finite set of primes of $\mathbb{Q}$. For $v \in S$ let ρ_v be an irreducible square-integrable representation of $\mathrm{GL}_n(\mathbb{Q}_v)$. Say that an RAESDC representation π has type $\{\rho_v\}_{v \in S}$ if for each $v \in S$, π_v is an unramified twist of $\rho_v^\vee$.

With this setup, Clozel, Harris, and Taylor attached a Galois representation to an RAESDC π.

Theorem 3.5 [Clozel et al. 2008, Proposition 4.3.1]. *Let $\iota : \bar{\mathbb{Q}}_\ell \simeq \mathbb{C}$. Let π be an* RAESDC *automorphic representation as above of weight a and type $\{\rho_v\}_{v \in S}$. There is a continuous semisimple Galois representation $r_{\ell,\iota}(\pi) : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$ such that*:

(1) $r_{\ell,\iota}(\pi)|^{\mathrm{ss}}_{G_{\mathbb{Q}_p}} = (r_\ell(\iota^{-1}\pi_p)^\vee)(1-n)^{\mathrm{ss}}$ *for every prime $p \nmid \ell$.* (Here r_ℓ is the reciprocity map defined in [Harris and Taylor 2001].)

(2) *If $\ell = p$, then the restriction $r_{\ell,\iota}(\pi)|_{G_{\mathbb{Q}_p}}$ is potentially semistable and if π_p is unramified, it is crystalline, with Hodge–Tate weights $-(a_j + n - j)$ for $j = 1, \ldots, n$.*

The Newton and Hodge filtrations. Let $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ be the (p, p)-representation attached to an automorphic representation π, and let D be the corresponding filtered (φ, N, F, E)-module (for suitable choices of F and E).

There are two natural filtrations on D_F, the Hodge filtration $\mathrm{Fil}^i D_F$ and the Newton filtration defined by ordering the slopes of the crystalline Frobenius (the valuations of the roots of φ). To keep the analysis of the structure of the (p, p)-representation $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ within reasonable limits, we make this assumption:

Assumption 3.6. The Newton filtration on D_F is in general position with respect to the Hodge filtration $\mathrm{Fil}^i D_F$.

Here, if V is a space and $\mathrm{Fil}_1^i V$ and $\mathrm{Fil}_2^j V$ are two filtrations on V, we say they are in general position if each $\mathrm{Fil}_1^i V$ is as transverse as possible to each $\mathrm{Fil}_2^j V$. We remark that the condition above is in some sense generic since two random filtrations on a space tend to be in general position.

(Quasi)ordinary representations. As mentioned earlier, our goal is to prove that the (p, p)-Galois representation attached to π is upper triangular in several cases. To this end it is convenient to recall some terminology (see, e.g., [Greenberg 1994, p. 152] or [Ochiai 2001, Definition 3.1]).

Definition. Let F be a number field. A p-adic representation V of G_F is called *ordinary* (respectively *quasiordinary*) if the following conditions are satisfied:

(1) For each place v of F over p, there is a decreasing filtration of G_{F_v}-modules $\cdots \operatorname{Fil}_v^i V \supseteq \operatorname{Fil}_v^{i+1} V \supseteq \cdots$ such that $\operatorname{Fil}_v^i V = V$ for $i \ll 0$ and $\operatorname{Fil}_v^i V = 0$ for $i \gg 0$.

(2) For each v and i, I_v acts on $\operatorname{Fil}_v^i V / \operatorname{Fil}_v^{i+1} V$ via the character $\chi_{\mathrm{cyc},p}^i$, where $\chi_{\mathrm{cyc},p}$ is the p-adic cyclotomic character (respectively, there exists an open subgroup of I_v acting on $\operatorname{Fil}_v^i V / \operatorname{Fil}_v^{i+1} V$ via $\chi_{\mathrm{cyc},p}^i$).

4. Principal series

Let π be an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$ with infinitesimal character H, for a set of distinct integers H. Let π_p denote the local automorphic representation of $\mathrm{GL}_n(\mathbb{Q}_p)$. In this section we study the behavior of the (p, p)-Galois representation assuming that π_p is in the principal series.

Spherical case. Assume that π_p is an unramified principal series representation. Since π_p is a spherical representation of $\mathrm{GL}_n(\mathbb{Q}_p)$, there exist unramified characters $\chi_1, \ldots, \chi_n$ of $\mathbb{Q}_p^\times$ such that π_p is the Langlands quotient $Q(\chi_1, \ldots, \chi_n)$. We can parametrize the isomorphism class of this representation by the Satake parameters $\alpha_1, \ldots, \alpha_n$ for $\alpha_i = \chi_i(\omega)$, where ω is a uniformizer for $\mathbb{Q}_p$.

Note that $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ is crystalline with Hodge–Tate weights H. Let D be the corresponding filtered φ-module, having a filtration with jumps $\beta_1 < \beta_2 < \cdots < \beta_n$ (so that the Hodge–Tate weights H are $-\beta_1 > \cdots > -\beta_n$).

Definition 4.1. An automorphic representation π is *p-ordinary* if $\beta_i + v_p(\alpha_i) = 0$ for all $i = 1, \ldots, n$. (In particular, the $v_p(\alpha_i)$ are integers.)

Theorem 4.2 (spherical case). *Suppose that π is an automorphic representation of* $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$ *with infinitesimal character given by the integers $-\beta_1 > \cdots > -\beta_n$ and such that π_p is in the unramified principal series with Satake parameters $\alpha_1, \ldots, \alpha_n$. If π is p-ordinary, then*

$$\rho_{\pi,p}|_{G_{\mathbb{Q}_p}} \sim$$

$$\begin{pmatrix} \lambda\Big(\frac{\alpha_1}{p^{v_p(\alpha_1)}}\Big)\cdot\chi_{\mathrm{cyc},p}^{-\beta_1} & * & * & * \\ 0 & \lambda\Big(\frac{\alpha_2}{p^{v_p(\alpha_2)}}\Big)\cdot\chi_{\mathrm{cyc},p}^{-\beta_2} & * & * \\ & & \ddots & \\ 0 & 0 & \lambda\Big(\frac{\alpha_{n-1}}{p^{v_p(\alpha_{n-1})}}\Big)\cdot\chi_{\mathrm{cyc},p}^{-\beta_{n-1}} & * \\ 0 & 0 & 0 & \lambda\Big(\frac{\alpha_n}{p^{v_p(\alpha_n)}}\Big)\cdot\chi_{\mathrm{cyc},p}^{-\beta_n} \end{pmatrix}.$$

In particular, $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ *is ordinary.*

Proof. Since π_p is p-ordinary, we have $v_p(\alpha_n) < v_p(\alpha_{n-1}) < \cdots < v_p(\alpha_1)$. By strict compatibility, the characteristic polynomial of the inverse of crystalline Frobenius of D_n is equal to $\prod_i (X - \alpha_i)$.

Since the $v_p(\alpha_i)$ are distinct, there exists a basis of eigenvectors of D_n for the operator φ, say $\{e_i\}$, with corresponding eigenvalues $\{\alpha_i^{-1}\}$. For $1 \le i \le n$, let D_i be the φ-submodule generated by $\{e_1, \dots, e_i\}$. Since D_n is admissible we know that $t_H(D_i) \le t_N(D_i)$ for all $i = 1, \dots, n$.

The filtration on D_n is

$$\cdots \subseteq 0 \subsetneq \mathrm{Fil}^{\beta_n}(D_n) \subseteq \cdots \subsetneq \mathrm{Fil}^{\beta_1}(D_n) = D_n \subseteq \cdots .$$

Since D_n is admissible, we have

$$\sum_{i=1}^{n} \beta_i = -\sum_{i=1}^{n} v_p(\alpha_i). \tag{4-1}$$

By Assumption 3.6, the jumps in the induced filtration on D_{n-1} are $\beta_1, \dots, \beta_{n-1}$. By (4-1), we have

$$t_H(D_{n-1}) = \sum_{i=1}^{n-1} \beta_i = -\sum_{i=1}^{n-1} v_p(\alpha_i) = t_N(D_{n-1}),$$

since $\beta_n = -v_p(\alpha_n)$. This implies that D_{n-1} is admissible. Moreover, D_n/D_{n-1} is also admissible since $t_H(D_n/D_{n-1}) = \beta_n$ and $t_N(D_n/D_{n-1}) = -v_p(\alpha_n)$ since φ acts on D_n/D_{n-1} by α_n^{-1}. Therefore, the Galois representation $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ looks like

$$\rho \sim \begin{pmatrix} \rho_{n-1} & * \\ 0 & \lambda\left(\dfrac{\alpha_n}{p^{v_p(\alpha_n)}}\right) \cdot \chi_{\mathrm{cyc},p}^{-\beta_n} \end{pmatrix},$$

where ρ_{n-1} is the $(n-1)$-dimensional representation of $G_{\mathbb{Q}_p}$ corresponding to D_{n-1}.

Successive application of this argument to $D_{n-1}, D_{n-2}, \dots, D_1$ yields the result. □

Variant, following [Clozel et al. 2008]. Let π now be an RAESDC representation of weight a as in Theorem 3.5, and let π_p denote the local p-adic automorphic representation associated with π. For any $i = 1, \dots, n$, set $\beta'_{n+1-i} := a_i + n - i$, where the a_i's are as in (3-1). We have $\beta'_n > \beta'_{n-1} > \cdots > \beta'_1$, and the Hodge–Tate weights are $-\beta'_n < -\beta'_{n-1} < \cdots < -\beta'_1$.

Assume that π_p is in the unramified principal series, so $\pi_p = Q(\chi_1, \chi_2, \dots, \chi_n)$, where the χ_i are unramified characters of $\mathbb{Q}_p^\times$. Set $\alpha'_i = \chi_i(\omega) p^{(n-1)/2}$. Let $t_p^{(j)}$

be the eigenvalue of $T_p^{(j)}$ on $\pi_p^{\mathrm{GL}_n(\mathbb{Z}_p)}$, where $T_p^{(j)}$ is the j-th Hecke operator as in [Clozel et al. 2008], and $\pi_p^{\mathrm{GL}_n(\mathbb{Z}_p)}$ is spanned by a $\mathrm{GL}_n(\mathbb{Z}_p)$-fixed vector, unique up to a constant. We would like to compute the right-hand side of the equality in Theorem 3.5(1). By [Clozel et al. 2008, Corollary 3.1.2], in the spherical case, one has

$$\begin{aligned}&(r_\ell(\iota^{-1}\pi_p)^\vee)(1-n)(\mathrm{Frob}_p^{-1})\\ &\quad=\prod_i (X-\alpha_i')\\ &\quad= X^n - t_p^{(1)}X^{n-1} + \cdots + (-1)^j p^{j(j-1)/2} t_p^{(j)} X^{n-j} + \cdots + (-1)^n p^{n(n-1)/2} t_p^{(n)},\end{aligned}$$

where Frob_p^{-1} is geometric Frobenius. Let s_j denote the j-th elementary symmetric polynomial. Then, by the equation above, for any $j = 1, \dots, n$, we have

$$p^{j(j-1)/2} t_p^{(j)} = s_j(\alpha_i') = p^{j(n-1)/2} s_j(\chi_i(p)),$$

and hence $t_p^{(j)} = s_j(\chi_i(p))p^{j(n-j)/2}$. In this setting, we have:

Definition 4.3. An automorphic representation π is *p-ordinary* if $\beta_i' + v_p(\alpha_i') = 0$ for all $i = 1, \dots, n$.

Again, if π is p-ordinary, the $v_p(\alpha_i')$ are integers.

By strict compatibility, crystalline Frobenius has its characteristic polynomial exactly as above. The next theorem is proved like Theorem 4.2.

Theorem 4.4 (spherical case, variant). *Let π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$ of weight a, as in Theorem 3.5. Let $r_{p,\iota}(\pi)$ be the corresponding p-adic Galois representation, with Hodge–Tate weights $-\beta_{n+1-i}' := a_i + n - i$, for $i = 1, \dots, n$. Suppose π_p is in the principal series with Satake parameters $\alpha_1, \dots, \alpha_n$, and set $\alpha_i' = \alpha_i p^{(n-1)/2}$. If π is p-ordinary, then*

$$r_{p,\iota}(\pi)|_{G_{\mathbb{Q}_p}} \sim \begin{pmatrix} \lambda\Big(\frac{\alpha_1'}{p^{v_p(\alpha_1')}}\Big)\cdot\chi_{\mathrm{cyc},p}^{-\beta_1'} & * & * & * \\ 0 & \lambda\Big(\frac{\alpha_2'}{p^{v_p(\alpha_2')}}\Big)\cdot\chi_{\mathrm{cyc},p}^{-\beta_2'} & * & * \\ & & \ddots & \\ 0 & 0 & \lambda\Big(\frac{\alpha_{n-1}'}{p^{v_p(\alpha_{n-1}')}}\Big)\cdot\chi_{\mathrm{cyc},p}^{-\beta_{n-1}'} & * \\ 0 & 0 & 0 & \lambda\Big(\frac{\alpha_n'}{p^{v_p(\alpha_n')}}\Big)\cdot\chi_{\mathrm{cyc},p}^{-\beta_n'} \end{pmatrix}.$$

In particular, $r_{p,\iota}(\pi)|_{G_{\mathbb{Q}_p}}$ is ordinary.

Theorem 4.4 was also obtained by D. Geraghty in the course of proving modularity lifting theorems for GL_n (see [Geraghty 2010, Lemma 2.7.7 and Corollary 2.7.8]). We thank T. Gee for pointing this out to us.

Ramified principal series case. Returning to the case where π is an automorphic representation with infinitesimal character H, we assume now that the automorphic representation $\pi_p = Q(\chi_1, \dots, \chi_n)$, where the χ_i are possibly ramified characters of $\mathbb{Q}_p^\times$.

By the local Langlands correspondence, we think of the χ_i as characters of the Weil group $W_{\mathbb{Q}_p}$. In particular, the restriction of the χ_i to the inertia group have finite image. By strict compatibility,

$$\mathrm{WD}(\rho)|_{I_p} \simeq \bigoplus_i \chi_i|_{I_p}.$$

The characters $\chi_i|_{I_p}$ factor through $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{nr}}(\zeta_{p^m})/\mathbb{Q}_p^{\mathrm{nr}}) \simeq \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^m})/\mathbb{Q}_p)$ for some $m \geq 1$. Denote $\mathbb{Q}_p(\zeta_{p^m})$ by F. Observe that F is a finite abelian totally ramified extension of $\mathbb{Q}_p$. Let $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}} : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(E)$ be the corresponding (p, p)-representation. Note that $\rho_{\pi,p}|_{G_F}$ is crystalline.

Let D_n be the corresponding filtered module. Then $D_n = Ee_1 + \cdots + Ee_n$, where $g \in \mathrm{Gal}(F/\mathbb{Q}_p)$ acts by χ_i on e_i. A short computation shows that $\varphi(e_i) = \alpha_i^{-1} e_i$, where $\alpha_i = \chi_i(\omega_F)$ for ω_F a uniformizer of F.

Using Corollary 2.5, and following the proof of Theorem 4.2, we obtain:

Theorem 4.5 (ramified principal series). *Say $\pi_p = Q(\chi_1, \dots, \chi_n)$ is in the ramified principal series. If π is p-ordinary,*

$$\rho_{\pi,p}|_{G_{\mathbb{Q}_p}} \sim \begin{pmatrix} \chi_1 \cdot \lambda\left(\frac{\alpha_1}{p^{v_p(\alpha_1)}}\right) \cdot \chi_{\mathrm{cyc},p}^{-\beta_1} & * & & * \\ 0 & \chi_2 \cdot \lambda\left(\frac{\alpha_2}{p^{v_p(\alpha_2)}}\right) \cdot \chi_{\mathrm{cyc},p}^{-\beta_2} & & * \\ 0 & 0 & \ddots & * \\ & & & \chi_n \cdot \lambda\left(\frac{\alpha_n}{p^{v_p(\alpha_n)}}\right) \cdot \chi_{\mathrm{cyc},p}^{-\beta_n} \end{pmatrix}.$$

In particular, $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ is quasiordinary.

5. The Steinberg case

In this section we treat the case where the Weil–Deligne representation attached to π_p is a twist of the special representation $\mathrm{Sp}(n)$. The case of unramified twists occupies most of the section; ramified twists are the subject of Theorem 5.8 at the end.

We start with the case where the Weil–Deligne representation attached to π_p is of the form $\chi \otimes \mathrm{Sp}(n)$, where χ is an unramified character.

Let D be the filtered $(\varphi, N, \mathbb{Q}_p, E)$-module attached to $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$. Thus D is a vector space over E. Note $N^n = 0$ and $N^{n-1} \neq 0$ so that there is a basis $\{f_n, f_{n-1}, \dots, f_1\}$ of D with $f_{i-1} := Nf_i$ for $1 < i \leq n$ and $Nf_1 = 0$, i.e.,

$$f_n \overset{N}{\mapsto} f_{n-1} \overset{N}{\mapsto} \cdots \overset{N}{\mapsto} f_1 \overset{N}{\mapsto} 0.$$

Say χ takes arithmetic Frobenius to α. Since $N\varphi = p\varphi N$, we may assume that $\varphi(f_i) = \alpha_i^{-1} f_i$ for all $i = 1, \dots, n$, where $\alpha_i^{-1} = p^{i-1}/\alpha$. When $\alpha = 1$, D reduces to the Weil–Deligne representation $\mathrm{Sp}(n)$ mentioned after Theorem 3.1.

For each $1 \leq i \leq n$, let D_i denote the subspace $\langle f_i, \dots, f_1\rangle$. Clearly, $\dim(D_i) = i$ and $D_1 \subsetneq D_2 \subsetneq \cdots \subsetneq D_n$. One can easily prove:

Lemma 5.1. *For every integer $1 \leq r \leq n$, there is a unique N-submodule of D, of rank r, namely D_r.*

Let $\beta_n > \cdots > \beta_1$ be the jumps in the Hodge filtration on D. We assume that the Hodge filtration is in general position with respect to the Newton filtration given by the D_i (Assumption 3.6). An example of such a filtration is

$$\langle f_n\rangle \subsetneq \langle f_n, f_{n-1}\rangle \subsetneq \cdots \subsetneq \langle f_n, f_{n-1}, \dots, f_2\rangle \subsetneq \langle f_n, f_{n-1}, \dots, f_1\rangle.$$

The following elementary lemma plays an important role in later proofs.

Lemma 5.2. *Let m be a natural number. Let $\{a_i\}_{i=1}^n$ be an increasing sequence of integers such that $|a_{i+1} - a_i| = m$. Let $\{b_i\}_{i=1}^n$ be another increasing sequence of integers, such that $|b_{i+1} - b_i| \geq m$. Assume that $\sum_i a_i = \sum_i b_i$. If $a_n = b_n$ or $a_1 = b_1$, then $a_i = b_i$ for all i.*

The same holds with "decreasing" instead of "increasing".

Proof. Let us prove the lemma when $a_n = b_n$ and the a_i are increasing. The proof in the other cases is similar. We have

$$m(n-1+n-2+\cdots+1) \leq \sum_{i=1}^{n}(b_n - b_i) = \sum_{i=1}^{n}(a_n - a_i) = m(n-1+n-2+\cdots+1).$$

The first equality follows from $a_n = b_n$. From the equation above, we see that $b_n - b_i = a_n - a_i$ for every $1 \leq i \leq n$. Since $a_n = b_n$, we have $a_i = b_i$ for every $1 \leq i \leq n$. ☐

By Lemma 5.1, the D_i are the only (φ, N)-submodules of D. The following proposition shows that if two consecutive submodules D_i and D_{i+1} are admissible, all the D_i are admissible.

Proposition 5.3. *Suppose there exists an integer* $1 \leq i \leq n$ *such that both* D_i *and* D_{i+1} *are admissible. Then each* D_r, *for* $1 \leq r \leq n$, *is admissible. Moreover, the* β_i *are consecutive integers.*

Proof. Since D_i and D_{i+1} are admissible, we have the equalities

$$\beta_1+\beta_2+\cdots+\beta_i = -\sum_{r=1}^{i} v_p(\alpha_r) \quad\text{and}\quad \beta_1+\beta_2+\cdots+\beta_{i+1} = -\sum_{r=1}^{i+1} v_p(\alpha_r), \tag{5-1}$$

whose difference gives

$$-v_p(\alpha_{i+1}) = \beta_{i+1}. \tag{5-2}$$

Define $a_r = -v_p(\alpha_r)$ and $b_r = \beta_r$ for $1 \leq r \leq n$. Hence,

$$\begin{aligned} a_n > \cdots > a_{i+2} > a_{i+1} > a_i > \cdots > a_1, \\ b_n > \cdots > b_{i+2} > b_{i+1} > b_i > \cdots > b_1. \end{aligned} \tag{5-3}$$

By (5-2), $a_{i+1} = b_{i+1}$. By Lemma 5.2 and (5-1), we have $a_r = b_r$ for all $1 \leq r \leq i+1$.

Since D_n is admissible,

$$t_H(D_n) = \sum_{r=1}^{n} \beta_r = -\sum_{r=1}^{n} v_p(\alpha_r) = t_N(D_n). \tag{5-4}$$

From (5-1) and (5-4), we have

$$\sum_{r=i+1}^{n} \beta_r = -\sum_{r=i+1}^{n} v_p(\alpha_r).$$

Again, by (5-3) and Lemma 5.2, we have $a_r = b_r$ for all $i+1 \leq r \leq n$. Hence $\beta_r = -v_p(\alpha_r)$ for all $1 \leq r \leq n$. This shows that all the other D_i's are admissible. Also, the β_i are consecutive integers since the $v_p(\alpha_i)$ are consecutive integers. ☐

Corollary 5.4. *Keeping the notation as above, the admissibility of* D_1 *or* D_{n-1} *implies the admissibility of all other* D_i.

Theorem 5.5. *Assume that the Hodge filtration on* D *is in general position with respect to the* D_i (*Assumption 3.6*). *Then the crystal* D *is either irreducible or reducible, in which case each* D_i, *for* $1 \leq i \leq n$, *is admissible.*

Proof. If D is irreducible, we are done. If not, there exists an i, such that D_i is admissible. If D_{i-1} or D_{i+1} is admissible, then by Proposition 5.3, all the D_r are admissible. So, it is enough to consider the case where neither D_{i-1} nor D_{i+1} is admissible (and D_i is admissible). We have

$$\beta_1 + \beta_2 + \cdots + \beta_{i-1} < -\sum_{r=1}^{r=i-1} v_p(\alpha_r), \tag{5-5a}$$

$$\beta_1 + \beta_2 + \cdots + \beta_i = -\sum_{r=1}^{r=i} v_p(\alpha_r), \tag{5-5b}$$

$$\beta_1 + \beta_2 + \cdots + \beta_{i+1} < -\sum_{r=1}^{r=i+1} v_p(\alpha_r). \tag{5-5c}$$

Subtracting (5-5b) from (5-5a), we get $-\beta_i < v_p(\alpha_i)$. Subtracting (5-5b) from (5-5c), we get $\beta_{i+1} < -v_p(\alpha_{i+1}) = -v_p(\alpha_i) + 1$. Adding these inequalities, we obtain $\beta_{i+1} - \beta_i < 1$. But this is a contradiction, since $\beta_{i+1} > \beta_i$. This proves the theorem. □

Definition 5.6. Say π is *p-ordinary* if $\beta_1 + v_p(\alpha) = 0$.

If π is p-ordinary, D_1 is admissible, so the flag $D_1 \subset D_2 \subset \cdots \subset D_n$ is an admissible flag by Theorem 5.5 (an easy check shows that if π is p-ordinary, Assumption 3.6 holds automatically).

Applying the discussion above to the local Galois representation $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$, we obtain:

Theorem 5.7 (unramified twist of Steinberg representation). *Say π is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$ with infinitesimal character given by the integers $-\beta_1 > \cdots > -\beta_n$. Suppose that π_p is an unramified twist of the Steinberg representation, that is, $\mathrm{WD}(\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}) \sim \chi \otimes \mathrm{Sp}(n)$, where χ is the unramified character mapping arithmetic Frobenius to α. If π is ordinary at p (that is, $v_p(\alpha) = -\beta_1$), then the β_i are necessarily consecutive integers and*

$$\rho_{\pi,p}|_{G_{\mathbb{Q}_p}} \sim \begin{pmatrix} \lambda\Big(\frac{\alpha}{p^{v_p(\alpha)}}\Big)\cdot\chi_{\mathrm{cyc},p}^{-\beta_1} & * & * \\ 0 & \lambda\Big(\frac{\alpha}{p^{v_p(\alpha)}}\Big)\cdot\chi_{\mathrm{cyc},p}^{-\beta_1-1} & * \\ & & \ddots & \\ 0 & 0 & \lambda\Big(\frac{\alpha}{p^{v_p(\alpha)}}\Big)\cdot\chi_{\mathrm{cyc},p}^{-\beta_1-(n-1)} \end{pmatrix},$$

where $\lambda(\alpha/p^{v_p(\alpha)})$ is an unramified character that takes arithmetic Frobenius to $\alpha/p^{v_p(\alpha)}$, and in particular, $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ is ordinary. If π is not p-ordinary and Assumption 3.6 holds, then $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ is irreducible.

Proof. By strict compatibility, D is the filtered $(\varphi, N, \mathbb{Q}_p, E)$-module attached to $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$. If π is p-ordinary, we are done and the characters on the diagonal are determined by Corollary 2.5.

If π is not p-ordinary, D is irreducible. Indeed, if D is reducible, then by Theorem 5.5, all D_i, and in particular D_1, are admissible, so π is p-ordinary. $\square$

Theorem 5.8 (ramified twist of Steinberg representation). *Let the notation and hypotheses be as in Theorem 5.7, except that this time assume that*

$$\mathrm{WD}(\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}) \sim \chi \otimes \mathrm{Sp}(n),$$

where χ *is an arbitrary, possibly ramified, character. Write* $\chi = \chi_0 \cdot \chi'$ *where* χ_0 *is the ramified part of* χ, *and* χ' *is an unramified character taking arithmetic Frobenius to* α. *If* π *is* p*-ordinary* ($\beta_1 = -v_p(\alpha)$), *then the* β_i *are consecutive integers and*

$$\rho_{\pi,p}|_{G_{\mathbb{Q}_p}} \sim$$

$$\begin{pmatrix} \chi_0 \cdot \lambda\Big(\frac{\alpha}{p^{v_p(\alpha)}}\Big) \cdot \chi_{\mathrm{cyc},p}^{-\beta_1} & * & * \\ 0 & \chi_0 \cdot \lambda\Big(\frac{\alpha}{p^{v_p(\alpha)}}\Big) \cdot \chi_{\mathrm{cyc},p}^{-\beta_1-1} & * \\ & & \ddots & \\ 0 & 0 & \chi_0 \cdot \lambda\Big(\frac{\alpha}{p^{v_p(\alpha)}}\Big) \cdot \chi_{\mathrm{cyc},p}^{-\beta_1-(n-1)} \end{pmatrix}.$$

If π *is not* p*-ordinary and Assumption 3.6 holds, then* $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ *is irreducible.*

Proof. Let F be a totally ramified abelian (cyclotomic) extension of $\mathbb{Q}_p$ such that $\chi_0|_{I_F} = 1$. Then the reducibility of $\rho_{\pi,p}|_{G_F}$ over F can be shown exactly as in Theorem 5.7, and the theorem over $\mathbb{Q}_p$ follows using the descent data of the underlying filtered module. If π is not p-ordinary, then by arguments similar to those used in proving Theorem 5.7, $\rho_{\pi,p}|_{G_F}$ is irreducible, so that $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ is also irreducible. $\square$

6. Supercuspidal ⊗ Steinberg

We now turn to the case where the Weil–Deligne representation attached to π_p is indecomposable. Thus we assume that $\mathrm{WD}(\rho_{\pi,p}|_{G_{\mathbb{Q}_p}})$ is Frobenius semisimple and is of the form $\tau \otimes \mathrm{Sp}(n)$, where τ is an irreducible m-dimensional representation corresponding to a supercuspidal representation of GL_m for $m \geq 1$, and $\mathrm{Sp}(n)$ for $n \geq 1$ denotes the usual special representation.

(φ, N)-submodules. We start by classifying the (φ, N, F, E)-submodules of D, the crystal attached to the local representation $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$, when $\mathrm{WD}(\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}) = \tau \otimes \mathrm{Sp}(n)$ for $m \geq 1$ and $n \geq 1$. This will play a key role in the study of the structure of $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$, when taking the Hodge filtration on D into account.

Recall from [Breuil and Schneider 2007, Proposition 4.1] that there is an equivalence of categories between (φ, N)-modules with coefficients and descent data, and

Weil–Deligne representations. We write D_τ for the (φ, N)-modules corresponding to τ, and likewise $D_{\mathrm{Sp}(n)}$ for those corresponding to $\mathrm{Sp}(n)$, and so on.

Theorem 6.1. *All the (φ, N, F, E)-submodules of $D = D_\tau \otimes D_{\mathrm{Sp}(n)}$ are of the form $D_\tau \otimes D_{\mathrm{Sp}(r)}$ for some $1 \le r \le n$.*

The case that $m = 1$ was treated in the previous section (twist of Steinberg), and the case $n = 1$ is vacuously true.

Assume first that τ is an induced representation of dimension m of the form $\mathrm{Ind}_{W_K}^{W_p} \chi$, where K is a p-adic field such that $[K : \mathbb{Q}_p] = m$, and χ is a character of W_K. This is known to always hold if $(p, m) = 1$ or $p > m$. For simplicity, we shall assume that K is the unique unramified extension of $\mathbb{Q}_p$, namely $\mathbb{Q}_{p^m}$, and refer to τ in this case as an unramified supercuspidal representation. Let σ be the generator of $\mathrm{Gal}(\mathbb{Q}_{p^m}/\mathbb{Q}_p)$, and let I_{p^m} denote the inertia subgroup of $\mathbb{Q}_{p^m}$. Then

$$\tau|_{I_p = I_{p^m}} \simeq (\mathrm{Ind}_{W_{p^m}}^{W_p} \chi)|_{I_{p^m}} \simeq \bigoplus_{i=1}^{m} \chi^{\sigma^i}|_{I_{p^m}}.$$

Since τ is irreducible, by Mackey's criterion, we have $\chi \neq \chi^{\sigma^i}$ for all i, on W_{p^m} and also on I_{p^m}. Moreover, $\chi^{\sigma^j} \neq \chi^{\sigma^i}$ for any $i \neq j$.

Following the methods of [Ghate and Mézard 2009], it is possible to explicitly write down the crystal D_τ whose underlying Weil–Deligne representation is the unramified supercuspidal representation τ above. For the details we refer the reader to [Ghate and Kumar 2010, §7.2]. In particular, one may write down appropriate finite extensions F_0 and F of $\mathbb{Q}_p$, so that D_τ is a free rank m module over $F_0 \otimes E$ with basis e_i, $i = 1, \dots, m$, such that D_τ is given by

$$\begin{aligned} &\varphi(e_i) = t_m^{-1/m} e_i, \quad N(e_i) = 0, \quad \sigma(e_i) = e_{i+1}, \\ &g(e_i) = (1 \otimes \chi^{\sigma^{i-1}}(g))(e_i), \quad g \in I(F/K) \end{aligned} \tag{6-1}$$

for all $1 \le i \le m$ and some constant $t_m \in \mathbb{O}_E$. When $m = 2$, this (φ, N)-module is exactly the one given in [Ghate and Mézard 2009, §3.3], though the e_i used here differ by a scalar from the e_i used there.

Recall that the module $D_{\mathrm{Sp}(n)}$ has a basis $\{f_n, f_{n-1}, \dots, f_1\}$, with properties as in Section 5. Using (6-1) and a basis of $D_\tau \otimes D_{\mathrm{Sp}(r)}$ of the form $e_i \otimes f_j$, it is possible to give an explicit proof of Theorem 6.1, when τ is an unramified supercuspidal representation of dimension m [Ghate and Kumar 2010, §7.2.5]. However, it is also possible to prove the theorem for general τ, independently of any explicit formulas. We give this proof now. The following general lemma is useful:

Lemma 6.2. *The theory of Jordan canonical forms can be extended to nilpotent operators on free finite-rank $(F_0 \otimes E)$-modules. We call the number of blocks in the Jordan decomposition of the monodromy operator N as the index of N.*

Proof. One simply extends the usual theory of Jordan canonical forms on each projection under (2-1) to modules over $F_0 \otimes E$-modules. □

Returning to our situation:

Lemma 6.3. *There are no rank r (φ, N, F, E)-submodules of $D = D_\tau \otimes D_{\mathrm{Sp}(n)}$ on which N acts trivially for $1 \le r \le m-1$.*

Proof. Suppose there exists such a module, say $\tilde{D}$, of rank $r < m$. Since N acts trivially on $\tilde{D}$, we have $\tilde{D} \subseteq D_\tau \otimes \langle f_1 \rangle = D_\tau \otimes D_{\mathrm{Sp}(1)} \simeq D_\tau$. But τ is irreducible, so D_τ is irreducible by Lemma 2.7, a contradiction. □

Corollary 6.4. *The index of N on a (φ, N, F, E)-submodule of D is m.*

Proof of Theorem 6.1. Let D' be a (φ, N, F, E)-submodule of $D = D_\tau \otimes D_{\mathrm{Sp}(n)}$. By the corollary above, there are m blocks in the Jordan canonical form of N on D'. Without loss of generality, assume that the blocks have sizes $r_1 \le r_2 \le \cdots \le r_m$ with $\sum_{i=1}^m r_i = \mathrm{rank}\ D'$. Suppose $w_1, \ldots, w_m$ are the corresponding basis vectors in D' such that the order of nilpotency of N on w_i is r_i, so that the $N^j(w_i)$ form a basis of D'. If all the r_i are equal to say r, an easy argument shows $D' = D_\tau \otimes D_{\mathrm{Sp}(r)}$. We show that this is indeed the case.

Suppose towards a contradiction that $r_i \ne r_{i+1}$ for some $1 \le i < m$. For $1 \le i \le n$, let D_i denote the submodule $D_\tau \otimes \mathrm{Ker}(N^i) = D_\tau \otimes D_{\mathrm{Sp}(i)}$. Now, arrange the basis vectors $N^j w_k$ of D' as follows:

$$\begin{array}{rrr}
 & & w_1,\ Nw_1,\ \ldots,\ N^{r_1-1}w_1, \\
 & & \vdots \\
 & & w_k,\ Nw_k,\ \ldots,\ N^{r_k-1}w_k, \\
w_{k+1},\ Nw_{k+1},\ \ldots, & N^{r_{k+1}-r_1-1}w_{k+1}, & N^{r_{k+1}-r_1}w_{k+1},\ \ldots,\ N^{r_{k+1}-1}w_{k+1}, \\
 & & \vdots \\
w_m,\ Nw_m,\ N^2w_m,\ \ldots, & N^{r_m-r_1-1}w_m, & N^{r_m-r_1}w_m,\ \ldots,\ N^{r_m-1}w_m.
\end{array}$$

With respect to this arrangement, denote the span of the vectors in the last i columns by A_i. Since $r_i \ne r_{i+1}$, the rank of the space A_{r_i+1}/A_{r_i} is less than m. Moreover, A_{r_i+1}/A_{r_i} is a subspace of D_{r_i+1}/D_{r_i}; that is, there is an inclusion of (φ, N, F, E)-modules $A_{r_i+1}/A_{r_i} \hookrightarrow D_{r_i+1}/D_{r_i}$. Now

$$\begin{aligned}
D_{r_i+1}/D_{r_i} &= (D_\tau \otimes D_{\mathrm{Sp}(r_i+1)})/(D_\tau \otimes D_{\mathrm{Sp}(r_i)}) \\
&\simeq D_\tau \otimes (D_{\mathrm{Sp}(r_i+1)}/D_{\mathrm{Sp}(r_i)}) \simeq D_\tau \otimes D_{\mathrm{Sp}(1)} \simeq D_\tau.
\end{aligned}$$

All the isomorphisms above are isomorphisms of (φ, N, F, E)-modules over $F_0 \otimes E$. By Lemma 6.3, the inclusion above is not possible! Hence all the r_i are indeed equal. This finishes the proof of Theorem 6.1. □

Filtration on $D = D_\tau \otimes D_{\mathrm{Sp}(n)}$. We can now apply the discussion above to write down the structure of the (p, p)-Galois representation attached to a cuspidal automorphic representation of $\mathrm{GL}_{mn}(\mathbb{A}_{\mathbb{Q}})$.

We start with some remarks. Suppose D_1 and D_2 are two admissible filtered modules. It is well-known (see [Totaro 1996]) that the tensor product $D_1 \otimes D_2$ is also admissible. The difficulty in proving this lies in the fact that one does not have much information about the structure of the (φ, N)-submodules of the tensor product. If they are of the form $D' \otimes D''$, where D' and D'' are admissible (φ, N)-submodules of D_1 and D_2 respectively, then $D' \otimes D''$ is also admissible by Lemma 2.3. But not all the submodules of $D_1 \otimes D_2$ are of this form.

However, we saw in Theorem 6.1 that for $D = D_\tau \otimes D_{\mathrm{Sp}(n)}$, all the (φ, N, F, E)-submodules of D are of the form $D_\tau \otimes D_{\mathrm{Sp}(r)}$ for some $1 \leq r \leq n$. This fact allows us to study the crystal D and its submodules, once we introduce the Hodge filtration.

Filtration in general position. Assume that the Hodge filtration on D is in general position with respect to the Newton filtration (Assumption 3.6). Let m be the rank of D_τ. Let $\{\beta_{i,j}\}_{i=1,j=1}^{i=n,j=m}$ be the jumps in the Hodge filtration with $\beta_{i_1,j_1} > \beta_{i_2,j_2}$, if $i_1 > i_2$, or if $i_1 = i_2$ and $j_1 > j_2$. Thus

$$\beta_{n,m} > \beta_{n,m-1} > \cdots > \beta_{n,1} > \beta_{n-1,m} > \cdots > \beta_{1,m} > \cdots > \beta_{1,1}.$$

Define, for every $1 \leq k \leq n$,

$$b_k = \sum_{j=1}^{j=m} \beta_{k,j},$$

and

$$a_k = t_N(D_\tau \otimes D_{\mathrm{Sp}(k)}) - t_N(D_\tau \otimes D_{\mathrm{Sp}(k-1)}) = t_N(D_\tau) + m(k-1),$$

where the last equality follows from Lemma 2.3. Clearly,

$$b_n > b_{n-1} > \cdots > b_2 > b_1,$$
$$a_n > a_{n-1} > \cdots > a_2 > a_1.$$

Observe that $b_{i+1} - b_i \geq m^2$ and $a_{i+1} - a_i = m$ for every $1 \leq i \leq n$. Since D is admissible, the submodule $D_\tau \otimes D_{\mathrm{Sp}(i)}$ of D is admissible if and only if $\sum_{k=1}^{i} b_k = \sum_{k=1}^{i} a_k$.

The arguments below are similar to the ones used when analyzing the Steinberg case. We start with an analog of Lemma 5.2.

Lemma 6.5. *Let $\{a_i\}_{i=1}^n$ be an increasing sequence of integers such that $a_{i+1} - a_i = m$ for every i and for some fixed natural number m. Let $\{b_i\}_{i=1}^n$ be an increasing sequence of integers such that $b_{i+1} - b_i \geq m^2$ for every i. Suppose that $\sum_i a_i = \sum_i b_i$. If $a_n = b_n$ or $a_1 = b_1$, then $m = 1$ and hence $a_i = b_i$ for all i.*

Proof. We prove the lemma when $a_n = b_n$; the case of $a_1 = b_1$ is similar. Write

$$m^2(n-1+n-2+\cdots+1) \le \sum_{i=1}^{n}(b_n-b_i) = \sum_{i=1}^{n}(a_n-a_i) = m(n-1+n-2+\cdots+1),$$

where the first equality follows from $a_n = b_n$. From the inequality we see that $m = 1$. Now, the rest of the proof follows from Lemma 5.2. □

Theorem 6.6. *If $D_\tau \otimes D_{\mathrm{Sp}(i)}$ and $D_\tau \otimes D_{\mathrm{Sp}(i+1)}$ are admissible submodules of D, then $m = 1$, in which case all the $D_\tau \otimes D_{\mathrm{Sp}(i)}$, for $1 \le i \le n$, are admissible.*

Proof. Since $D_\tau \otimes D_{\mathrm{Sp}(i)}$ and $D_\tau \otimes D_{\mathrm{Sp}(i+1)}$ are admissible, we have

$$b_1 + b_2 + \cdots + b_i = \sum_{r=1}^{i} a_r \quad \text{and} \quad b_1 + b_2 + \cdots + b_{i+1} = \sum_{r=1}^{i+1} a_r. \tag{6-2}$$

From these expressions, $b_{i+1} = a_{i+1}$. As recalled above:

$$b_n > \cdots > b_{i+2} > b_{i+1} > b_i > \cdots > b_1,$$
$$a_n > \cdots > a_{i+2} > a_{i+1} > a_i > \cdots > a_1.$$

Since $a_{i+1} = b_{i+1}$ and (6-2) holds, by Lemma 6.5 we have $m = 1$ and $a_i = b_i$ for all $1 \le i \le n$. This shows that all the $D_\tau \otimes D_{\mathrm{Sp}(i)}$ are admissible. □

Theorem 6.7. *Let $D = D_\tau \otimes D_{\mathrm{Sp}(n)}$ and assume that the Hodge filtration on D is in general position (Assumption 3.6). Then either D is irreducible or D is reducible, in which case $m = 1$ and the (φ, N, F, E)-submodules $D_\tau \otimes D_{\mathrm{Sp}(i)}$, for $1 \le i \le n$, are all admissible.*

Proof. Let $D_i = D_\tau \otimes D_{\mathrm{Sp}(i)}$ for $1 \le i \le n$. If D is irreducible, we are done. If not, by Theorem 6.1, there exists an $1 \le i \le n$ such that D_i is admissible. If D_{i-1} or D_{i+1} is also admissible, then by the theorem above, $m = 1$ and hence all the (φ, N, F, E)-submodules of D are admissible. So, assume D_{i-1} and D_{i+1} are not admissible, but D_i is admissible. We shall show that this is not possible. Indeed, we have

$$b_1 + b_2 + \cdots + b_{i-1} < \sum_{r=1}^{r=i-1} a_r, \tag{6-3a}$$

$$b_1 + b_2 + \cdots + b_i = \sum_{r=1}^{r=i} a_r, \tag{6-3b}$$

$$b_1 + b_2 + \cdots + b_{i+1} < \sum_{r=1}^{r=i+1} a_r. \tag{6-3c}$$

Subtracting (6-3b) from (6-3a), we get $-b_i < -a_i$. Subtracting (6-3b) from (6-3c), we get $b_{i+1} < a_{i+1}$. Adding these two inequalities, we get $b_{i+1} - b_i < a_{i+1} - a_i = m$. But this is a contradiction, since $b_{i+1} - b_i \geq m$. □

For emphasis we state separately:

Corollary 6.8. *With assumptions as above, the crystal* $D = D_\tau \otimes D_{\mathrm{Sp}(n)}$ *is irreducible if* $m \geq 2$.

Definition 6.9. Say π is *ordinary* at p if $a_1 = b_1$, that is, $t_N(D_\tau) = \sum_{j=1}^{m} \beta_{1,j}$.

This condition implies $m = 1$; the definition then coincides with Definition 5.6.

Applying the discussion above to the local (p, p)-Galois representation in a strictly compatible system, we obtain:

Theorem 6.10 (indecomposable case). *Say* π *is a cuspidal automorphic representation with infinitesimal character consisting of distinct integers. Suppose that*

$$\mathrm{WD}(\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}) \sim \tau_m \otimes \mathrm{Sp}(n),$$

where τ_m *is an irreducible representation of* $W_{\mathbb{Q}_p}$ *of dimension* $m \geq 1$, *and* $n \geq 1$. *Assume that Assumption 3.6 holds.*

- *If* π *is ordinary at* p, *then* $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ *is reducible, in which case* $m = 1$, τ_1 *is a character, and* $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ *is (quasi)ordinary as in Theorems 5.7 and 5.8.*
- *If* π *is not ordinary at* p, *then* $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ *is irreducible.*

Tensor product filtration. One might wonder what happens if the filtration on D is not necessarily in general position. As an example, we consider here just one case arising from the so-called tensor product filtration.

Assume that D_τ and $D_{\mathrm{Sp}(n)}$ are the usual filtered (φ, N, F, E)-modules, and equip $D_\tau \otimes D_{\mathrm{Sp}(n)}$ with the tensor product filtration. By the formulas in Lemma 2.3, one can prove:

Lemma 6.11. *Suppose that* $D = D_\tau \otimes D_{\mathrm{Sp}(n)}$ *has the tensor product filtration. Fix* $1 \leq r \leq n$. *Then* $D_\tau \otimes D_{\mathrm{Sp}(r)}$ *is an admissible submodule of* D *if and only if* $D_{\mathrm{Sp}(r)}$ *is an admissible submodule of* $D_{\mathrm{Sp}(n)}$.

We recall that if the filtration on $D_{\mathrm{Sp}(n)}$ is in general position (as in Assumption 3.6), then we have shown that furthermore $D_{\mathrm{Sp}(r)}$ is an admissible submodule of $D_{\mathrm{Sp}(n)}$ if and only if $D_{\mathrm{Sp}(1)}$ is an admissible submodule.

The lemma can be used to give an example where the tensor product filtration on D is not in general position (i.e., does not satisfy Assumption 3.6). Suppose that τ is an irreducible representation of dimension $m = 2$ and $D_{\mathrm{Sp}(2)}$ has weight 2 (as in [Breuil 2001] or [Ghate and Mézard 2009, §3.1]). Note that $D_{\mathrm{Sp}(1)}$ is an admissible submodule of $D_{\mathrm{Sp}(2)}$. Hence, by the lemma, $D_\tau \otimes D_{\mathrm{Sp}(1)}$ is an admissible submodule of $D_\tau \otimes D_{\mathrm{Sp}(2)}$. If the tensor product filtration satisfies Assumption 3.6, then the admissibility of $D_\tau \otimes D_{\mathrm{Sp}(1)}$ would contradict Theorem 6.7, since $m = 2$.

In any case, we have the following application to local Galois representations.

Proposition 6.12. *Suppose that* $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}} \sim \rho_\tau \otimes \rho_{\mathrm{Sp}(n)}$ *is a tensor product of two* (p, p)*-Galois representations, with underlying Weil–Deligne representations* τ *and* $\mathrm{Sp}(n)$*, respectively. If* $\rho_{\mathrm{Sp}(n)}$ *is irreducible, so is* $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$.

Errata

We end this paper by correcting some errors in [Ghate and Mézard 2009]:

- p. 2254, lines 6 and 7: $\mathbf{Q}$ should be $\mathbf{Q}_p$.
- p. 2257, first two lines should be $\varphi(e_1) = (1/\sqrt{t})\, e_1$ and $\varphi(e_2) = (1/\sqrt{t})\, e_2$.
- p. 2260: first three lines in the middle display should be $\varphi(e_1) = (1/\sqrt{t})\, e_1$, $\varphi(e_2) = (1/\sqrt{t})\, e_2$, and $t \in \mathbb{O}_E$, $\mathrm{val}_p(t) = k - 1$. Moreover, t is to be chosen in §3.4.3 satisfying $t^2 = 1/c$ (we may take $c = d$, since ι commutes with φ, and s is no longer required).

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ON INTRINSICALLY KNOTTED OR COMPLETELY 3-LINKED GRAPHS

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We say that a graph is intrinsically knotted or completely 3-linked if every embedding of the graph into the 3-sphere contains a nontrivial knot or a 3-component link each of whose 2-component sublinks is nonsplittable. We show that a graph obtained from the complete graph on seven vertices by a finite sequence of △Y-exchanges and Y△-exchanges is a minor-minimal intrinsically knotted or completely 3-linked graph.

1. Introduction

Throughout this paper we work in the piecewise linear category. Let f be an embedding of a finite graph G into the 3-sphere. Then f is called a *spatial embedding* of G and $f(G)$ is called a *spatial graph*. We denote the set of all spatial embeddings of G by $\mathrm{SE}(G)$. We call a subgraph γ of G that is homeomorphic to the circle a *cycle* of G. For a positive integer n, let $\Gamma^{(n)}(G)$ denote the set of all cycles of G if $n = 1$ and the set of all unions of n mutually disjoint cycles of G if $n \geq 2$. For simplicity, we also write $\Gamma(G)$ for $\Gamma^{(1)}(G)$. For an element λ in $\Gamma^{(n)}(G)$ and a spatial embedding f of G, $f(\lambda)$ is a knot if $n = 1$ and an n-component link if $n \geq 2$.

A graph G is said to be *intrinsically linked* (IL) if for every spatial embedding f of G, $f(G)$ contains a nonsplittable 2-component link. Conway and Gordon [1983] and Sachs [1984] showed that K_6 is IL, where K_m denotes the *complete graph* on m vertices. Also, IL graphs have been completely characterized as follows. For a graph G and an edge e of G, we denote the subgraph $G \setminus \operatorname{int} e$ by $G - e$. Let $e = \overline{uv}$ be an edge of G that is not a loop. We call the graph obtained from $G - e$ by identifying the end vertices u and v the *edge contraction of G along e*, and denote it by G/e. A graph H is called a *minor* of a graph G if there exists a subgraph G' of G and edges $e_1, e_2, \ldots, e_m$ of G' such that H is obtained from G' by a

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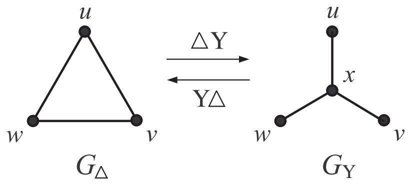
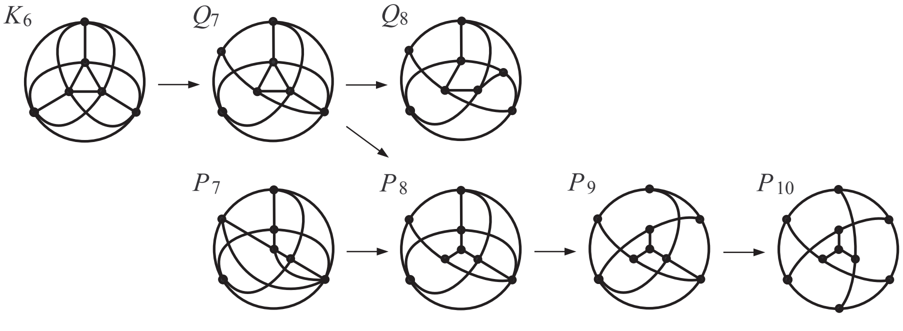
sequence of edge contractions along $e_1, e_2, \dots, e_m$. A minor H of G is called a *proper minor* if H does not equal G. Let $\mathcal{P}$ be a property for graphs that is *closed* under minor reductions; that is, for any graph G that does not have $\mathcal{P}$, all minors of G also do not have $\mathcal{P}$. A graph G is said to be *minor-minimal* with respect to $\mathcal{P}$ if G has $\mathcal{P}$ but all proper minors of G do not have $\mathcal{P}$. Note that G has $\mathcal{P}$ if and only if G has a minor-minimal graph with respect to $\mathcal{P}$ as a minor. By the famous theorem of Robertson and Seymour [2004], there are finitely many minor-minimal graphs with respect to $\mathcal{P}$. Nešetřil and Thomas [1985] showed that IL is closed under minor reductions, and Robertson, Seymour and Thomas [Robertson et al. 1995] showed that the set of all minor-minimal graphs with respect to IL equals the *Petersen family*, which is the set of all graphs obtained from K_6 by a finite sequence of $\triangle$Y*-exchanges* and Y$\triangle$*-exchanges*. A $\triangle$Y-exchange is the left-to-right operation shown here:



That is, a graph $G_\triangle$ containing a three-edge cycle $\triangle$ is changed into a new graph G_Y by removing the edges of the cycle and adding a new vertex x connected to each of the vertices of the deleted cycle, thus forming a Y. A Y$\triangle$-exchange is the reverse of this operation. $\triangle$Y- and Y$\triangle$-exchanges preserve IL: if $G_\triangle$ is IL, so is G_Y [Motwani et al. 1988], and if G_Y is IL, so is $G_\triangle$ [Robertson et al. 1995].

The Petersen family contains seven graphs, including the *Petersen graph* P_{10}:



(An arrow between two graphs indicates the application of a single $\triangle$Y-exchange.)

A graph G is said to be *intrinsically knotted* (IK) if for every spatial embedding f of G, $f(G)$ contains a nontrivial knot. Conway and Gordon [1983] showed that K_7 is IK. Fellows and Langston [1988] showed that IK is closed under minor reductions. Motwani, Raghunathan, and Saran [Motwani et al. 1988] showed that K_7 is a minor-minimal IK graph, and additional minor-minimal IK graphs were given in [Kohara and Suzuki 1992] and [Foisy 2002; 2003].

IK graphs have not been completely characterized yet. If $G_\triangle$ is IK then G_{Y} is also IK [Motwani et al. 1988], but if G_{Y} is IK then $G_\triangle$ may not always be IK. That is, the Y△-exchange does not preserve IK in general. Flapan and Naimi [2008] showed that there exists a graph G_{FN} obtained from K_7 by five △Y-exchanges and two Y△-exchanges that is not IK. We call the set of all graphs obtained from K_7 by a finite sequence of △Y and Y△-exchanges the *Heawood family*.[1] This family contains exactly twenty graphs, as illustrated in Figure 1; of these, C_{14} is the *Heawood graph* (Remark 4.7).

Kohara and Suzuki [1992] showed that a graph G in the Heawood family is a minor-minimal IK graph if G is obtained from K_7 by a finite sequence of △Y-exchanges, that is, if G is one of fourteen graphs K_7, H_8, $H_9, \dots, H_{12}$, F_9, F_{10}, E_{10}, E_{11} and $C_{11}, C_{12}, \dots, C_{14}$.[2] On the other hand, N'_{10} is isomorphic to G_{FN}, that is, N'_{10} is not IK. Our first purpose in this paper is to determine completely when a graph in the Heawood family is IK.

Theorem 1.1. *For a graph G in the Heawood family, the following are equivalent:*

1. *G is IK.*
2. *G is obtained from K_7 by a finite sequence of △Y-exchanges.*
3. *$\Gamma^{(3)}(G)$ is the empty set.*

Hence the members N_9, N_{10}, N_{11}, N'_{10}, N'_{11} and N'_{12} of the Heawood family are not IK, and only they contain a union of three mutually disjoint cycles.

Our second purpose is to show that any of the graphs in the Heawood family is a minor-minimal graph with respect to a certain kind of intrinsic nontriviality even if it is not IK. We say that a graph G is *intrinsically knotted or completely 3-linked*—I(K or C3L) for short—if for every spatial embedding f of G, $f(G)$ contains a nontrivial knot or a 3-component link all of whose 2-component sublinks are nonsplittable. An IK graph is I(K or C3L). As we show in Proposition 2.2, I(K or C3L) is closed under minor reductions.

Theorem 1.2. *All graphs in the Heawood family are minor-minimal I(K or C3L) graphs.*

As we have seen, N_9, N_{10}, N_{11}, N'_{10}, N'_{11} and N'_{12} are not IK, but they are but I(K or C3L) and are minor-minimal with respect to I(K or C3L).

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[1] Van der Holst [2006] calls the set of all graphs obtained from K_7 or $K_{3,3,1,1}$ by a finite sequence of △Y-exchanges and Y△-exchanges the Heawood family, where $K_{3,3,1,1}$ is the complete 4-partite graph on $3+3+1+1$ vertices.

[2] One edge of F_{10} in [Kohara and Suzuki 1992, Figure 5] is wanting.

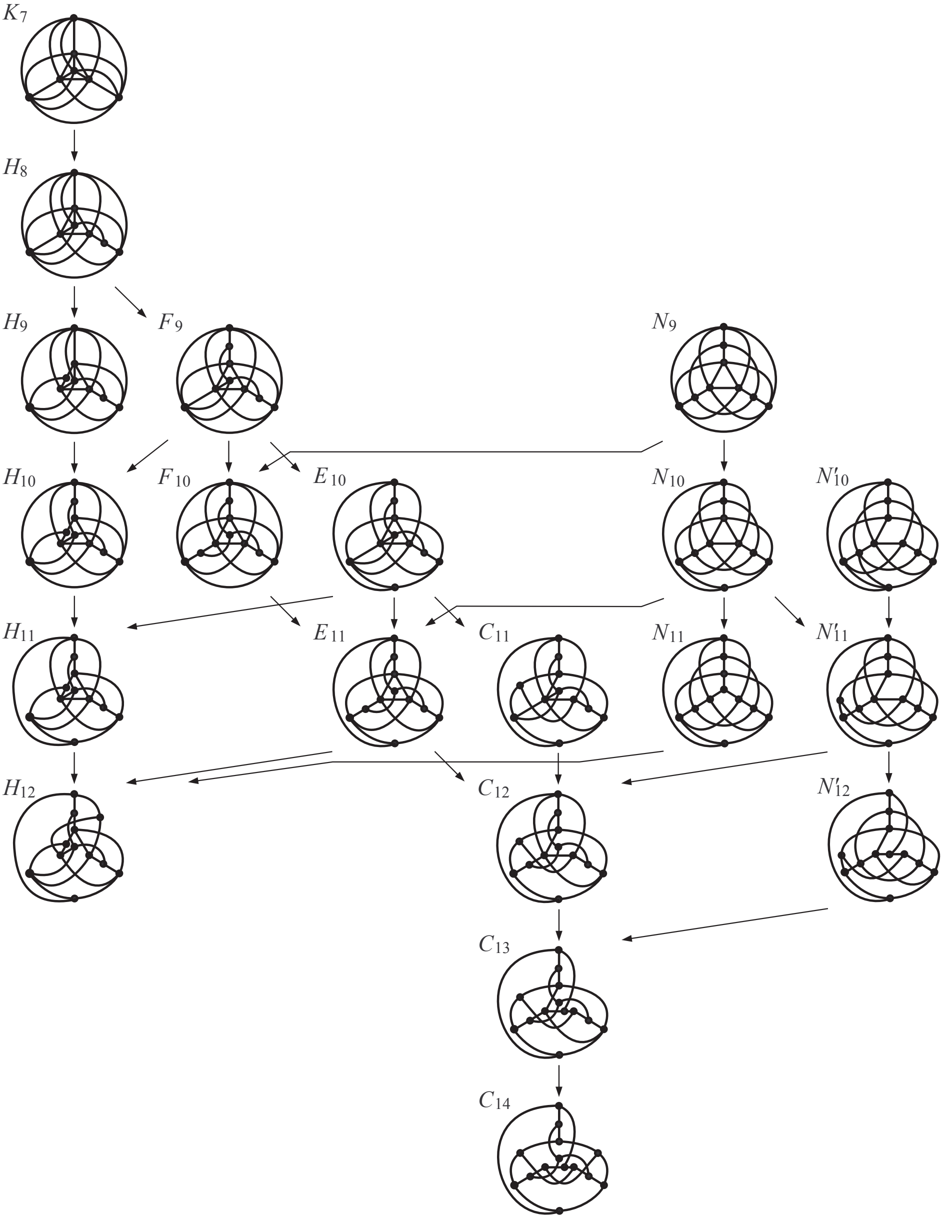


Figure 1. The Heawood family. An arrow between two graphs indicates the application of a single $\triangle$Y-exchange.

Remark 1.3. A graph G is said to be *intrinsically n-linked* (InL) if for every spatial embedding f of G, $f(G)$ contains a nonsplittable n-component link [Flapan et al. 2001a; 2001b]. I2L coincides with IL. Let G be a graph in the Heawood family that is not IK. Then we show in Example 4.6 that there exists a spatial embedding f of G such that $f(G)$ does not contain a nonsplittable 3-component link. That is, G is neither IK nor I3L.

Remark 1.4. A graph G is called *intrinsically knotted or* 3*-linked* — I(K or 3L) for short — if for every spatial embedding f of G, $f(G)$ contains a nontrivial knot or a nonsplittable 3-component link. Clearly I(K or C3L) implies I(K or 3L), but the converse is not true: [Foisy 2006] exhibits an I(K or 3L) graph G and a spatial embedding f of G such that $f(G)$ contains no nontrivial knot and all nonsplittable 3-component links contained in $f(G)$ have split 2-component sublinks.

The rest of this paper is organized as follows. Section 2 contains general results about graph minors, $\triangle$Y-exchanges and spatial graphs. We prove Theorem 1.1 in Section 3 and Theorem 1.2 in Section 4.

2. Graph minors, ΔY-exchanges and spatial graphs

Let H be a minor of a graph G. Then there exists a natural injection

$$\Psi^{(n)} = \Psi^{(n)}_{H,G} : \Gamma^{(n)}(H) \longrightarrow \Gamma^{(n)}(G)$$

for any positive integer n. We write Ψ for $\Psi^{(1)}$. Let f be a spatial embedding of G and e an edge of G that is not a loop. Then by contracting $f(e)$ into one point, we obtain a spatial embedding $\psi(f)$ of G/e. Similarly, we can also obtain a spatial embedding $\psi(f)$ of H from f. Thus we obtain a map

$$\psi = \psi_{G,H} : \mathrm{SE}(G) \longrightarrow \mathrm{SE}(H).$$

Then we immediately have:

Proposition 2.1. *For a spatial embedding f of G and an element λ in $\Gamma^{(n)}(H)$, $\psi(f)(\lambda)$ is ambient isotopic to $f(\Psi^{(n)}(\lambda))$.* □

Proposition 2.2. *I(K or C3L) is closed under minor reductions.*

Proof. Let G be a graph that is not I(K or C3L), and H be a minor of G. Let f be a spatial embedding of G that contains neither a nontrivial knot nor a 3-component link all of whose 2-component sublinks are nonsplittable. Then by Proposition 2.1, $\psi(f)$ has the same property. This implies that H is not I(K or C3L). □

Remark 2.3. Proposition 2.1 also implies that IK, InL and I(K or 3L) are closed under minor reductions.

Let $G_\triangle$ and G_{Y} be two graphs such that G_{Y} is obtained from $G_\triangle$ by a single $\triangle$Y-exchange, as in the previous section. Let λ be an element in $\Gamma^{(n)}(G_\triangle)$ that does not contain $\triangle$. Then there exists an element $\Phi^{(n)}(\lambda)$ in $\Gamma^{(n)}(G_{\mathrm{Y}})$ such that

$\lambda \setminus \triangle = \Phi^{(n)}(\lambda) \setminus \mathrm{Y}$. Thus we obtain a map

$$\Phi^{(n)} = \Phi^{(n)}_{G_\triangle, G_\mathrm{Y}} : \{\lambda \in \Gamma^{(n)}(G_\triangle) \mid \lambda \not\supset \triangle\} \longrightarrow \Gamma^{(n)}(G_\mathrm{Y}),$$

for any positive integer n. We denote $\Phi^{(1)}$ by Φ. Note that $\Phi^{(n)}$ is surjective and the inverse image of λ by $\Phi^{(n)}$ contains at most two elements in $\Gamma^{(n)}(G_\triangle)$ for any element λ in $\Gamma^{(n)}(G_\mathrm{Y})$. The surjectivity of $\Phi^{(n)}$ implies Proposition 2.4.

Proposition 2.4. *For $n \geq 2$, if $\Gamma^{(n)}(G_\triangle) = \varnothing$, then $\Gamma^{(n)}(G_\mathrm{Y}) = \varnothing$.* □

Let f be a spatial embedding of G_Y, and let D be a 2-disk in the 3-sphere such that $D \cap f(G_\mathrm{Y}) = f(\mathrm{Y})$ and $\partial D \cap f(G_\mathrm{Y}) = \{f(u), f(v), f(w)\}$. (Throughout the paper we use u, v, w, x for the vertices of the Y of interest, as in the first figure on page 408), Let $\varphi(f)$ be a spatial embedding of $G_\triangle$ such that $\varphi(f)(x) = f(x)$ for $x \in G_\mathrm{Y} \setminus \mathrm{Y}$ and $\varphi(f)(G_\triangle) = (f(G_\mathrm{Y}) \setminus f(\mathrm{Y})) \cup \partial D$. Then we obtain a map

$$\varphi = \varphi_{G_\mathrm{Y}, G_\triangle} : \mathrm{SE}(G_\mathrm{Y}) \longrightarrow \mathrm{SE}(G_\triangle),$$

and we immediately have Proposition 2.5.

Proposition 2.5. *For a spatial embedding f of G_Y and an element λ in $\Gamma^{(n)}(G_\mathrm{Y})$, $f(\lambda)$ is ambient isotopic to $\varphi(f)(\lambda')$ for each element λ' in the inverse image of λ by $\Phi^{(n)}$.* □

Lemma 2.6. *If $G_\triangle$ is I(K or C3L), then G_Y is also I(K or C3L).*

Proof. Assume that G_Y is not I(K or C3L), that is, that there exists a spatial embedding f of G_Y that contains neither a nontrivial knot nor a 3-component link all of whose 2-component sublinks are nonsplittable. We show that $\varphi(f)(G_\triangle)$ also has the same property.

Let γ be an element in $\Gamma(G_\triangle)$. If γ is not $\triangle$, then $\varphi(f)(\gamma)$ is ambient isotopic to $f(\Phi(\gamma))$ by Proposition 2.5, and $f(\Phi(\gamma))$ is a trivial knot by the assumption. Since $\varphi(f)(\triangle)$ is also a trivial knot, it follows that $\varphi(f)(G_\triangle)$ does not contain a nontrivial knot. Let λ be an element in $\Gamma^{(3)}(G_\triangle)$. If λ does not contain $\triangle$, then $\varphi(f)(\lambda)$ is ambient isotopic to $f(\Phi^{(3)}(\lambda))$ by Proposition 2.5, and $f(\Phi^{(3)}(\lambda))$ is a 3-component link that contains a split 2-component sublink by the assumption. If λ contains $\triangle$, then $\varphi(f)(\lambda)$ is a split 3-component link. Thus we see that $\varphi(f)(G_\triangle)$ does not contain a 3-component link with a nonsplittable 2-component sublink. □

Lemma 2.7. *If G_Y is minor-minimal for I(K or C3L), then $G_\triangle$ is also minor-minimal for I(K or C3L).*

Proof. (This lemma has already been proven in more general form [Ozawa and Tsutsumi 2007, Lemma 3.1, Exercise 3.2], but we prove it here for convenience.) We show that for any edge e of $G_\triangle$ that is not a loop, there exist a spatial embedding f of $G_\triangle - e$ and a spatial embedding g of $G_\triangle / e$ such that each of $f(G_\triangle - e)$ and

$g(G_\triangle/e)$ contains neither a nontrivial knot nor a 3-component link all of whose 2-component sublink are nonsplittable. If e is not one of the edges $\overline{uv}$, $\overline{vw}$ or $\overline{wu}$ of the $\triangle$ then there exist a spatial embedding f' of $G_Y - e$ and a spatial embedding g' of G_Y/e such that both $f'(G_Y - e)$ and $g'(G_Y/e)$ contain neither a nontrivial knot nor a 3-component link all of whose 2-component sublinks are nonsplittable. The graph $G_Y - e$ is obtained from $G_\triangle - e$, and likewise G_Y/e from $G_\triangle/e$, by a single $\triangle$Y-exchange at the same $\triangle$. Then we see that each of $\varphi(f')(G_\triangle - e)$ and $\varphi(g')(G_\triangle/e)$ contains neither a nontrivial knot nor a 3-component link having only nonsplittable 2-component sublinks, in a way similar to the proof of Lemma 2.6. If e is one of $\overline{uv}$, $\overline{vw}$ and $\overline{wu}$, we may assume that $e = \overline{uv}$ without loss of generality. Now there exists a spatial embedding f' of $G_Y/\overline{xw}$ such that $f'(G_Y/\overline{xw})$ contains neither a nontrivial knot nor a 3-component link having only nonsplittable 2-component sublinks. Then we can see that $G_\triangle - \overline{uv} = G_Y/\overline{xw}$. On the other hand, there exists a spatial embedding g' of $G_Y/\overline{xv}/\overline{xu}$ such that $g'(G_Y/\overline{xv}/\overline{xu})$ contains neither a nontrivial knot nor a 3-component link having only nonsplittable 2-component sublink. Take a 2-disk D' in the 3-sphere such that $D' \cap g'(G_Y/\overline{xv}/\overline{xu}) = g'(\overline{uw})$ and $\partial D' \cap g'(G_Y/\overline{xv}/\overline{xu}) = \{g'(u), g'(w)\}$. Then $(g'(G_Y/\overline{xv}/\overline{xu}) \setminus \operatorname{int} g'(\overline{uw})) \cup \partial D'$ may be regarded as the image of a spatial embedding of $G_\triangle/\overline{uv}$, denoted by g. Clearly $g(G_\triangle/\overline{uv})$ contains neither a nontrivial knot nor a 3-component link having only nonsplittable 2-component sublink. □

3. Proof of Theorem 1.1

Lemma 3.1. *Each of the graphs N_9, N_{10}, N_{11}, N'_{10}, N'_{11} and N'_{12} in the Heawood family is not IK.*

Proof. For N'_{10}, see [Flapan and Naimi 2008]. We show that N_9, N_{10}, N_{11}, N'_{11} and N'_{12} are not IK. Let f_9 be the spatial embedding of N_9 illustrated in Figure 2. It can be checked directly that $f_9(N_9)$ does not contain a nontrivial knot. Thus N_9 is

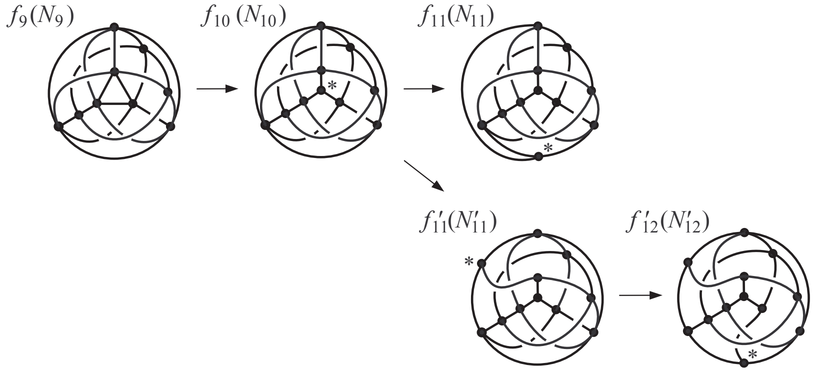
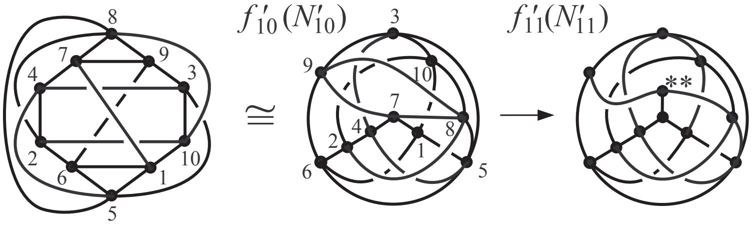


Figure 2

not IK. Let f_{10} be the spatial embedding of N_{10} illustrated in Figure 2. Let φ_{N_{10},N_9} be the map from $\mathrm{SE}(N_{10})$ to $\mathrm{SE}(N_9)$ induced by the Y$\triangle$-exchange from N_{10} to N_9 at the Y-fork marked $*$ in Figure 2. Then clearly $\varphi(f_{10}) = f_9$. Since $f_9(N_9)$ does not contain a nontrivial knot, by Proposition 2.5 it follows that $f_{10}(N_{10})$ also does not contain a nontrivial knot. Thus, N_{10} is not IK. By repeating this argument, we can see that each of the graphs N_{11}, N'_{11} and N'_{12} is also not IK; see Figure 2. □

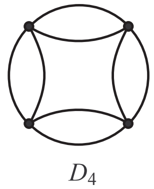
Proof of Theorem 1.1. First we show that (1) and (2) are equivalent. Since we already know that (2) implies (1), we show that (1) implies (2). If G is IK, then by Lemma 3.1 we see that G is not one of N_9, N_{10}, N_{11}, N'_{10}, N'_{11} or N'_{12}. Thus G is obtained from K_7 by a finite sequence of $\triangle$Y-exchanges. Next we show that (2) and (3) are equivalent. Assume that G is obtained from K_7 by a finite sequence of $\triangle$Y-exchanges. $\Gamma^{(3)}(K_7)$ is the empty set. Thus, by Proposition 2.4, we see that $\Gamma^{(3)}(G)$ is the empty set. Conversely, if G is one of N_9, N_{10}, N_{11}, N'_{10}, N'_{11}, and N'_{12}, then $\Gamma^{(3)}(G)$ is not the empty set. This completes the proof. □

Remark 3.2. Let f'_{11} be the spatial embedding of N'_{11} illustrated in Figure 2, and let f'_{10} be the spatial embedding of N'_{10} illustrated in the figure below. Let $\varphi_{N'_{11},N'_{10}}$ be the map from $\mathrm{SE}(N'_{11})$ to $\mathrm{SE}(N'_{10})$ induced by the Y$\triangle$-exchange from N'_{11} to N'_{10} at the Y-fork marked $**$. Then clearly $\varphi(f'_{11}) = f'_{10}$. Also, we can see that f'_{10} coincides with Flapan and Naimi's example [2008] of a spatial embedding of N'_{10} whose image does not contain a nontrivial knot, as illustrated in the leftmost diagram:



4. Proof of Theorem 1.2

Lemma 4.1 [Conway and Gordon 1983; Taniyama and Yasuhara 2001]. *Let G be a graph in the Petersen family and f a spatial embedding of G. Then there exists an element λ in $\Gamma^{(2)}(G)$ such that $\mathrm{lk}(f(\lambda))$ is odd, where* lk *denotes the linking number in the* 3*-sphere.*



D_4

Let D_4 be the graph illustrated on the right. We denote the set of all cycles of D_4 with exactly four edges by $\Gamma_4(D_4)$. For a spatial embedding f of D_4, we define

$$\alpha(f) \equiv \sum_{\gamma\in\Gamma_4(D_4)} a_2(f(\gamma)) \pmod 2,$$

where a_2 denotes the second coefficient of the *Conway polynomial*. Note that $a_2(K)$ of a knot K is congruent to the *Arf invariant* modulo 2 [Kauffman 1983].

Lemma 4.2 [Taniyama and Yasuhara 2001]. *Let f be a spatial embedding of D_4 and λ, λ' all elements in $\Gamma^{(2)}(D_4)$. If both $\mathrm{lk}(f(\lambda))$ and $\mathrm{lk}(f(\lambda'))$ are odd, then $\alpha(f) = 1$.*

Let G be a graph that contains D_4 as a minor and f a spatial embedding of G. Then we define

$$\alpha(f) \equiv \sum_{\gamma \in \Gamma_4(D_4)} a_2\bigl(f(\Psi_{D_4,G}(\gamma))\bigr) \pmod 2.$$

Lemma 4.3. *Let G be a graph that contains D_4 as a minor and let f be a spatial embedding of G. For two elements μ and μ' in $\Psi^{(2)}_{D_4,G}(\Gamma^{(2)}(D_4))$, if both $\mathrm{lk}(f(\mu))$ and $\mathrm{lk}(f(\mu'))$ are odd, then $\alpha(f) = 1$.*

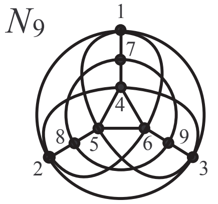
Proof. For two elements λ and λ' in $\Gamma^{(2)}(D_4)$, we see that both $\mathrm{lk}\bigl(f(\Psi^{(2)}_{D_4,G}(\lambda))\bigr)$ and $\mathrm{lk}\bigl(f(\Psi^{(2)}_{D_4,G}(\lambda'))\bigr)$ are odd by the assumption. Then by Proposition 2.1, it follows that $\mathrm{lk}(\psi_{G,D_4}(f)(\lambda))$ and $\mathrm{lk}(\psi_{G,D_4}(f)(\lambda'))$ are also odd. Therefore, by Lemma 4.2, we have that

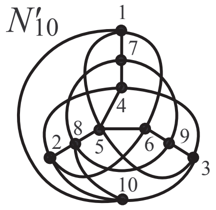
$$\alpha(f) \equiv \sum_{\gamma \in \Gamma_4(D_4)} a_2\bigl(f(\Psi_{D_4,G}(\gamma))\bigr) = \sum_{\gamma \in \Gamma_4(D_4)} a_2(\psi_{G,D_4}(f)(\gamma)) \equiv 1 \pmod 2. \qquad \square$$

The next theorem is the most important part of the proof of Theorem 1.2.

Theorem 4.4. *Let G be N_9 or N'_{10}. For every spatial embedding f of G, there exists an element γ in $\Gamma(G)$ such that $a_2(f(\gamma))$ is odd, or there exists an element λ in $\Gamma^{(3)}(G)$ such that each 2-component sublink of $f(\lambda)$ has an odd linking number.*

Proof. We will denote by $[i_1\, i_2\, \dots\, i_k]$ the cycle $\overline{i_1 i_2} \cup \overline{i_2 i_3} \cup \cdots \cup \overline{i_{k-1} i_k} \cup \overline{i_k i_1}$ of G. We label each vertex of G as follows:





First we show the case of $G = N_9$. Let f be a spatial embedding of N_9. Note that N_9 contains K_6 as the proper minor

$$(((N_9 - \overline{7\,8}) - \overline{8\,9}) - \overline{9\,7})/\overline{4\,7}/\overline{5\,8}/\overline{6\,9}.$$

By Lemma 4.1, there is thus an element ν in $\Gamma^{(2)}(K_6)$ such that $\mathrm{lk}(\psi_{N_9,K_6}(f)(\nu))$ is odd. Hence, by Proposition 2.1, there exists an element μ in $\Psi^{(2)}_{K_6,N_9}(\Gamma^{(2)}(K_6))$ such that $\mathrm{lk}(f(\mu))$ is odd. $\Psi^{(2)}_{K_6,N_9}(\Gamma^{(2)}(K_6))$ consists of ten elements, and by the symmetry of N_9, we may assume that $\mu = [1\,7\,4\,3] \cup [2\,6\,5\,8]$ or $[1\,2\,3] \cup [4\,5\,6]$ without loss of generality.

Case 1. Let $\mu = [1\,7\,4\,3] \cup [2\,6\,5\,8]$. Note that N_9 contains P_7 as the proper minor

$$(((((N_9 - \overline{6\,1}) - \overline{6\,2}) - \overline{6\,4}) - \overline{6\,5}) - \overline{6\,9})/\overline{3\,9}.$$

Thus, by Lemma 4.1, there is an element ν' in $\Gamma^{(2)}(P_7)$ such that $\mathrm{lk}(\psi_{N_9,P_7}(f)(\nu'))$ is odd. Hence, by Proposition 2.1, there exists an element μ' in $\Psi^{(2)}_{P_7,N_9}(\Gamma^{(2)}(P_7))$ such that $\mathrm{lk}(f(\mu'))$ is odd. $\Psi^{(2)}_{P_7,N_9}(\Gamma^{(2)}(P_7))$ consists of the nine elements

$$\begin{aligned} &\mu'_1 = [3\,4\,5] \cup [1\,2\,8\,7], && \mu'_2 = [1\,5\,4\,7] \cup [2\,3\,9\,8], && \mu'_3 = [2\,8\,5\,4] \cup [3\,1\,7\,9], \\ &\mu'_4 = [1\,2\,4\,7] \cup [3\,5\,8\,9], && \mu'_5 = [1\,2\,3] \cup [4\,7\,8\,5], && \mu'_6 = [1\,2\,8\,5] \cup [3\,4\,7\,9], \\ &\mu'_7 = [2\,3\,4] \cup [1\,5\,8\,7], && \mu'_8 = [7\,8\,9] \cup [1\,2\,4\,5], && \mu'_9 = [1\,5\,3] \cup [2\,8\,7\,4]. \end{aligned}$$

For $i = 1, 2, \ldots, 9$, let J^i be the subgraph of N_9 that is $\mu \cup \mu'_i \cup \overline{6\,9}$ if $i = 3, 6$ and $\mu \cup \mu'_i$ if $i \neq 3, 6$. Assume that $\mathrm{lk}(f(\mu'_i))$ is odd for some $i \neq 8$. Then it can be easily seen that J^i contains a graph D^i as a minor, such that D^i is isomorphic to D_4 and $\{\mu, \mu'_i\} = \Psi^{(2)}_{D^i,J^i}(\Gamma^{(2)}(D^i))$. Since both $\mathrm{lk}(f(\mu))$ and $\mathrm{lk}(f(\mu'_i))$ are odd, by Lemma 4.3 there exists an element γ in $\Gamma(J^i)$ such that $a_2(f(\gamma))$ is odd. Next assume that $\mathrm{lk}(f(\mu'_8))$ is odd. We denote two elements $[7\,8\,9] \cup [1\,2\,6\,5]$ and $[7\,8\,9] \cup [4\,2\,6\,5]$ in $\Gamma^{(2)}(J^8)$ by $\mu'_{8,1}$ and $\mu'_{8,2}$, respectively. We denote the subgraph $\mu \cup \mu'_{8,j}$ of J^8 by $J^{8,j}$ $(j = 1, 2)$. Then it can be easily seen that $J^{8,j}$ contains a graph $D^{8,j}$ as a minor, such that $D^{8,j}$ is isomorphic to D_4 and $\{\mu, \mu'_{8,j}\} = \Psi^{(2)}_{D^{8,j},J^{8,j}}(\Gamma^{(2)}(D^{8,j}))$ $(j = 1, 2)$. Note that

$$[1\,2\,4\,5] = [1\,2\,6\,5] + [4\,2\,6\,5]$$

in $H_1(J^8; \mathbb{Z}_2)$, where $H_*(\cdot\,; \mathbb{Z}_2)$ denotes the homology group with $\mathbb{Z}_2$-coefficients. Then, by the homological property of the linking number, we have that

$$1 \equiv \mathrm{lk}(f(\mu'_8)) \equiv \mathrm{lk}(f(\mu'_{8,1})) + \mathrm{lk}(f(\mu'_{8,2})) \pmod 2.$$

Thus we see that $\mathrm{lk}(f(\mu'_{8,1}))$ is odd or $\mathrm{lk}(f(\mu'_{8,2}))$ is odd. In either case, by Lemma 4.3 there exists an element γ in $\Gamma(J^{8,j})$ such that $a_2(f(\gamma))$ is odd.

Case 2. Let $\mu = [1\,2\,3] \cup [4\,5\,6]$. Note that N_9 contains P_9 as the proper minor

$$(((((N_9 - \overline{1\,2}) - \overline{2\,3}) - \overline{3\,1}) - \overline{4\,5}) - \overline{5\,6}) - \overline{6\,4}.$$

Thus, by Lemma 4.1, there is an element ν' in $\Gamma^{(2)}(P_9)$ such that $\mathrm{lk}(\psi_{N_9,P_9}(f)(\nu'))$ is odd. Hence by Proposition 2.1, there exists an element μ' in $\Psi^{(2)}_{P_9,N_9}(\Gamma^{(2)}(P_9))$ such that $\mathrm{lk}(f(\mu'))$ is odd. $\Psi^{(2)}_{P_9,N_9}(\Gamma^{(2)}(P_9))$ consists of seven elements, and by the symmetry of N_9, we may assume, without loss of generality, that $\mu' = [1\,5\,8\,7] \cup [2\,6\,9\,3\,4]$ or $[7\,8\,9] \cup [1\,5\,3\,4\,2\,6]$. Denote by J the subgraph $\mu \cup \mu'$ of N_9. Assume that $\mu' = [1\,5\,8\,7] \cup [2\,6\,9\,3\,4]$. We denote the two elements $[1\,5\,8\,7] \cup [4\,3\,2]$ and $[1\,5\,8\,7] \cup [6\,9\,3\,2]$ in $\Gamma^{(2)}(J)$ by μ'_1 and μ'_2, respectively. We denote the subgraph $\mu \cup \mu'_i$ of J by J^i $(i = 1, 2)$. Then J^i contains a graph D^i as a minor such that D^i is isomorphic to D_4 and

$$\{\mu, \mu'_i\} = \Psi^{(2)}_{D^i, J^i}(\Gamma^{(2)}(D^i)) \quad (i = 1, 2).$$

Since $[2\,6\,9\,3\,4] = [4\,3\,2] + [6\,9\,3\,2]$ in $H_1(J; \mathbb{Z}_2)$, it follows that

$$1 \equiv \mathrm{lk}(f(\mu')) \equiv \mathrm{lk}(f(\mu'_1)) + \mathrm{lk}(f(\mu'_2)) \pmod 2.$$

This implies that $\mathrm{lk}(f(\mu'_1))$ is odd or $\mathrm{lk}(f(\mu'_2))$ is odd. In both cases, by Lemma 4.3 there exists an element γ in $\Gamma(J^i)$ such that $a_2(f(\gamma))$ is odd. Next assume that $\mu' = [7\,8\,9] \cup [1\,5\,3\,4\,2\,6]$. We denote four elements $[7\,8\,9] \cup [3\,4\,5]$, $[7\,8\,9] \cup [4\,5\,6]$, $[7\,8\,9] \cup [1\,5\,6]$ and $[7\,8\,9] \cup [2\,4\,6]$ in $\Gamma^{(2)}(J)$ by μ'_1, μ'_2, μ'_3 and μ'_4, respectively. Since $[1\,5\,3\,4\,2\,6] = [3\,4\,5] + [4\,5\,6] + [1\,5\,6] + [2\,4\,6]$ in $H_1(J; \mathbb{Z}_2)$, we get

$$1 \equiv \mathrm{lk}(\mu') \equiv \mathrm{lk}(\mu'_1) + \mathrm{lk}(\mu'_2) + \mathrm{lk}(\mu'_3) + \mathrm{lk}(\mu'_4) \pmod 2.$$

This implies that $\mathrm{lk}(\mu'_i)$ is odd for some $i = 1, 2, 3$ or 4. Moreover, by the symmetry of J, we may assume that $\mathrm{lk}(\mu'_1)$ is odd or $\mathrm{lk}(\mu'_2)$ is odd without loss of generality. Assume that $\mathrm{lk}(\mu'_1)$ is odd. We denote the subgraph $\mu \cup \mu'_1 \cup \overline{1\,7} \cup \overline{6\,9}$ of N_9 by J^1. Then J^1 contains a graph D^1 as a minor such that D^1 is isomorphic to D_4 and $\{\mu, \mu'_1\} = \Psi^{(2)}_{D^1, J^1}(\Gamma^{(2)}(D^1))$. Since both $\mathrm{lk}(f(\mu))$ and $\mathrm{lk}(f(\mu'_1))$ are odd, by Lemma 4.3 there exists an element γ in $\Gamma(J^1)$ such that $a_2(f(\gamma))$ is odd. Next assume that $\mathrm{lk}(\mu'_2)$ is odd. We denote four elements $[7\,8\,9] \cup [1\,2\,6]$, $[7\,8\,9] \cup [1\,2\,3]$, $[7\,8\,9] \cup [2\,3\,4]$ and $[7\,8\,9] \cup [1\,3\,5]$ in $\Gamma^{(2)}(J)$ by μ'_5, μ'_6, μ'_7 and μ'_8, respectively. Since $[1\,5\,3\,4\,2\,6] = [1\,2\,6] + [1\,2\,3] + [2\,3\,4] + [1\,3\,5]$ in $H_1(J; \mathbb{Z}_2)$, we have

$$1 \equiv \mathrm{lk}(\mu') \equiv \mathrm{lk}(\mu'_5) + \mathrm{lk}(\mu'_6) + \mathrm{lk}(\mu'_7) + \mathrm{lk}(\mu'_8) \pmod 2.$$

Thus we see that $\mathrm{lk}(\mu'_i)$ is odd for some $i = 5, 6, 7$ or 8. Moreover, by the symmetry of J, we may assume that $\mathrm{lk}(\mu'_5)$ is odd or $\mathrm{lk}(\mu'_6)$ is odd without loss of generality. Assume that $\mathrm{lk}(\mu'_5)$ is odd. We denote the subgraph $\mu \cup \mu'_5 \cup \overline{4\,7} \cup \overline{3\,9}$ of N_9 by J^5. Then J^5 contains a graph D^5 as a minor such that D^5 is isomorphic to D_4 and $\{\mu, \mu'_5\} = \Psi^{(2)}_{D^5, J^5}(\Gamma^{(2)}(D^5))$. Since both $\mathrm{lk}(f(\mu))$ and $\mathrm{lk}(f(\mu'_5))$ are odd, by Lemma 4.3 there exists an element γ in $\Gamma(J^5)$ such that $a_2(f(\gamma))$ is odd. Finally, assume that $\mathrm{lk}(\mu'_6)$ is odd. Let us consider the 3-component link $L = f([1\,2\,3] \cup [4\,5\,6] \cup [7\,8\,9])$. Since all 2-component sublinks of L are $f(\mu)$, $f(\mu'_2)$ and $f(\mu'_6)$, each of the 2-component sublinks of L has an odd linking number.

Now we show the case of $G = N'_{10}$. Let f be a spatial embedding of N'_{10}. Note that N'_{10} contains P_7 as the proper minor

$$(((N'_{10} - \overline{7\,8}) - \overline{8\,9}) - \overline{9\,7})/\overline{4\,7}/\overline{5\,8}/\overline{6\,9}.$$

Thus by Lemma 4.1, there is an element ν in $\Gamma^{(2)}(P_7)$ such that $\mathrm{lk}(\psi_{N'_{10},P_7}(f)(\nu))$ is odd. Hence by Proposition 2.1, there exists an element μ in $\Psi^{(2)}_{P_7,N'_{10}}(\Gamma^{(2)}(P_7))$ such that $\mathrm{lk}(f(\mu))$ is odd. $\Psi^{(2)}_{P_7,N'_{10}}(\Gamma^{(2)}(P_7))$ consists of nine elements, and by the symmetry of N'_{10}, we may assume that $\mu = [1\,7\,4\,5] \cup [2\,10\,3\,9\,6]$, $[2\,4\,5\,8] \cup [1\,10\,3\,9\,6]$, $[3\,10\,8\,5] \cup [1\,6\,2\,4\,7]$, $[3\,4\,5] \cup [1\,10\,2\,6]$ or $[2\,8\,10] \cup [1\,6\,9\,3\,4\,7]$ without loss of generality.

Case 1. Let $\mu = [1\,7\,4\,5] \cup [2\,10\,3\,9\,6]$. Note that N'_{10} contains P_9 as the proper minor

$$(((((N'_{10} - \overline{5\,1}) - \overline{5\,3}) - \overline{5\,4}) - \overline{5\,6}) - \overline{5\,8}) - \overline{7\,9}.$$

Thus by Lemma 4.1, there is an element ν' in $\Gamma^{(2)}(P_9)$ such that $\mathrm{lk}(\psi_{N'_{10},P_9}(f)(\nu'))$ is odd. Hence by Proposition 2.1, there exists an element μ' in $\Psi^{(2)}_{P_9,N'_{10}}(\Gamma^{(2)}(P_9))$ such that $\mathrm{lk}(f(\mu'))$ is odd. $\Psi^{(2)}_{P_9,N'_{10}}(\Gamma^{(2)}(P_9))$ consists of seven elements

$$\begin{aligned}
\mu'_1 &= [3\,10\,8\,9] \cup [1\,6\,2\,4\,7], & \mu'_2 &= [1\,7\,8\,10] \cup [2\,4\,3\,9\,6],\\
\mu'_3 &= [1\,10\,2\,6] \cup [3\,4\,7\,8\,9], & \mu'_4 &= [2\,4\,3\,10] \cup [1\,7\,8\,9\,6],\\
\mu'_5 &= [2\,4\,7\,8] \cup [1\,10\,3\,9\,6], & \mu'_6 &= [2\,8\,9\,6] \cup [1\,10\,3\,4\,7],\\
\mu'_7 &= [2\,8\,10] \cup [1\,6\,9\,3\,4\,7]. & &
\end{aligned}$$

For $i = 1, 2, \ldots, 7$, let J^i be the subgraph of N'_{10} that is $\mu \cup \mu'_i \cup \overline{5\,8}$ if $i = 1, 6, 7$ and $\mu \cup \mu'_i$ if $i = 2, 3, 4, 5$. Assume that $\mathrm{lk}(f(\mu'_i))$ is odd for some i. Then J^i contains a graph D^i as a minor such that D^i is isomorphic to D_4 and $\{\mu, \mu'_i\} = \Psi^{(2)}_{D^i,J^i}(\Gamma^{(2)}(D^i))$. Because both $\mathrm{lk}(f(\mu))$ and $\mathrm{lk}(f(\mu'_i))$ are odd, by Lemma 4.3 there exists an element γ in $\Gamma(J^i)$ such that $a_2(f(\gamma))$ is odd.

Case 2. Let $\mu = [2\,4\,5\,8] \cup [1\,10\,3\,9\,6]$. Note that N'_{10} contains another P_9 as the proper minor

$$(((((N'_{10} - \overline{8\,2}) - \overline{8\,5}) - \overline{8\,7}) - \overline{8\,9}) - \overline{8\,10}) - \overline{3\,4}.$$

Thus by Lemma 4.1, there is an element ν' in $\Gamma^{(2)}(P_9)$ such that $\mathrm{lk}(\psi_{N'_{10},P_9}(f)(\nu'))$ is odd. Hence by Proposition 2.1, there exists an element μ' in $\Psi^{(2)}_{P_9,N'_{10}}(\Gamma^{(2)}(P_9))$ such that $\mathrm{lk}(f(\mu'))$ is odd. $\Psi^{(2)}_{P_9,N'_{10}}(\Gamma^{(2)}(P_9))$ consists of the seven elements

$$\begin{aligned}
\mu'_1 &= [1\,6\,9\,7] \cup [2\,4\,5\,3\,10], & \mu'_2 &= [1\,7\,4\,5] \cup [2\,10\,3\,9\,6],\\
\mu'_3 &= [3\,5\,6\,9] \cup [1\,10\,2\,4\,7], & \mu'_4 &= [1\,5\,3\,10] \cup [2\,4\,7\,9\,6],\\
\mu'_5 &= [1\,10\,2\,6] \cup [3\,9\,7\,4\,5], & \mu'_6 &= [1\,5\,6] \cup [2\,4\,7\,9\,3\,10],\\
\mu'_7 &= [2\,4\,5\,6] \cup [1\,10\,3\,9\,7]. & &
\end{aligned}$$

For $i = 1, 2, \ldots, 7$, let J^i be the subgraph of N'_{10} that is $\mu \cup \mu'_i \cup \overline{7\,8}$ if $i = 1, 7$ and $\mu \cup \mu'_i$ if $i \neq 1, 7$. Assume that $\mathrm{lk}(f(\mu'_i))$ is odd for some i. Then J^i contains a graph D^i as a minor such that D^i is isomorphic to D_4 and

$$\{\mu, \mu'_i\} = \Psi^{(2)}_{D^i,J^i}(\Gamma^{(2)}(D^i)).$$

Since both $\mathrm{lk}(f(\mu))$ and $\mathrm{lk}(f(\mu'_i))$ are odd, by Lemma 4.3 there exists an element γ in $\Gamma(J^i)$ such that $a_2(f(\gamma))$ is odd.

Case 3. Let $\mu = [3\,10\,8\,5] \cup [1\,6\,2\,4\,7]$. Let P_9 be the proper minor of N'_{10} and μ'_i the element in

$$\Psi^{(2)}_{P_9,N'_{10}}(\Gamma^{(2)}(P_9)) \quad (i = 1, 2, \dots, 7)$$

as in Case 2. For $i = 1, 2, \dots, 7$, let J^i be the subgraph of N'_{10} that is $\mu \cup \mu'_i \cup \overline{8\,9}$ if $i = 1, 4$ and $\mu \cup \mu'_i$ if $i \neq 1, 4$. Assume that $\mathrm{lk}(f(\mu'_i))$ is odd for some i. Then J^i contains a graph D^i as a minor such that D^i is isomorphic to D_4 and $\{\mu, \mu'_i\} = \Psi^{(2)}_{D^i,J^i}(\Gamma^{(2)}(D^i))$. Because both $\mathrm{lk}(f(\mu))$ and $\mathrm{lk}(f(\mu'_i))$ are odd, by Lemma 4.3 there exists an element γ in $\Gamma(J^i)$ such that $a_2(f(\gamma))$ is odd.

Case 4. Let $\mu = [3\,4\,5] \cup [1\,10\,2\,6]$. Note that N'_{10} contains another P_7 as the proper minor

$$(((N'_{10} - \overline{3\,4}) - \overline{4\,5}) - \overline{5\,3})/\overline{3\,9}/\overline{4\,7}/\overline{5\,8}.$$

Thus by Lemma 4.1, there is an element ν' in $\Gamma^{(2)}(P_7)$ such that $\mathrm{lk}(\psi_{N'_{10},P_7}(f)(\nu'))$ is odd. Hence by Proposition 2.1, there exists an element μ' in $\Psi^{(2)}_{P_7,N'_{10}}(\Gamma^{(2)}(P_7))$ such that $\mathrm{lk}(f(\mu'))$ is odd. $\Psi^{(2)}_{P_7,N'_{10}}(\Gamma^{(2)}(P_7))$ consists of the nine elements

$$\begin{aligned}
&\mu'_1 = [5\,6\,9\,8] \cup [1\,10\,2\,4\,7], && \mu'_2 = [3\,10\,8\,9] \cup [1\,6\,2\,4\,7],\\
&\mu'_3 = [1\,5\,8\,10] \cup [2\,4\,7\,9\,6], && \mu'_4 = [7\,8\,9] \cup [1\,10\,2\,6],\\
&\mu'_5 = [2\,8\,10] \cup [1\,6\,9\,7], && \mu'_6 = [2\,8\,5\,6] \cup [1\,10\,3\,9\,7],\\
&\mu'_7 = [1\,7\,8\,5] \cup [2\,10\,3\,9\,6], && \mu'_8 = [1\,5\,6] \cup [2\,4\,7\,9\,3\,10],\\
&\mu'_9 = [2\,4\,7\,8] \cup [1\,10\,3\,9\,6].
\end{aligned}$$

For $i = 1, 2, \dots, 9$, let J^i be the subgraph of N'_{10} that is $\mu \cup \mu'_5 \cup \overline{4\,7} \cup \overline{5\,8}$ if $i = 5$ and $\mu \cup \mu'_i$ if $i \neq 5$. Assume that $\mathrm{lk}(f(\mu'_i))$ is odd for some $i \neq 4, 8$. Then J^i contains a graph D^i as a minor such that D^i is isomorphic to D_4 and $\{\mu, \mu'_i\} = \Psi^{(2)}_{D^i,J^i}(\Gamma^{(2)}(D^i))$. Since both $\mathrm{lk}(f(\mu))$ and $\mathrm{lk}(f(\mu'_i))$ are odd, by Lemma 4.3 there exists an element γ in $\Gamma(J^i)$ such that $a_2(f(\gamma))$ is odd. Next assume that $\mathrm{lk}(f(\mu'_8))$ is odd. We denote two elements $[1\,5\,6] \cup [2\,4\,3\,10]$ and $[1\,5\,6] \cup [3\,4\,7\,9]$ in $\Gamma^{(2)}(J^8)$ by $\mu'_{8,1}$ and $\mu'_{8,2}$, respectively. We denote the subgraph $\mu \cup \mu'_{8,1}$ of J^8 by $J^{8,1}$ and the subgraph $\mu \cup \mu'_{8,2} \cup \overline{8\,9} \cup \overline{8\,10}$ of N'_{10} by $J^{8,2}$. Then $J^{8,j}$ contains a graph $D^{8,j}$ as a minor such that $D^{8,j}$ is isomorphic to D_4 and $\{\mu, \mu'_{8,j}\} = \Psi^{(2)}_{D^{8,j},J^{8,j}}(\Gamma^{(2)}(D^{8,j}))$ $(j = 1, 2)$. Since $[2\,4\,7\,9\,3\,10] = [2\,4\,3\,10] + [3\,4\,7\,9]$ in $H_1(J^8; \mathbb{Z}_2)$, it follows that

$$1 \equiv \mathrm{lk}(f(\mu'_8)) \equiv \mathrm{lk}(f(\mu'_{8,1})) + \mathrm{lk}(f(\mu'_{8,2})) \pmod 2.$$

This implies that $\mathrm{lk}(f(\mu'_{8,1}))$ is odd or $\mathrm{lk}(f(\mu'_{8,2}))$ is odd. In either case, by Lemma 4.3 there exists an element γ in $\Gamma(J^{8,j})$ such that $a_2(f(\gamma))$ is odd. Finally assume that $\mathrm{lk}(f(\mu'_4))$ is odd. Note that N'_{10} contains another P_9 as the proper minor

$$(((((N'_{10}-\overline{2\,4})-\overline{2\,6})-\overline{2\,8})-\overline{2\,10})-\overline{5\,1})-\overline{5\,3}.$$

Thus by Lemma 4.1, there is an element ν'' in $\Gamma^{(2)}(P_9)$ such that $\mathrm{lk}(\psi_{N'_{10},P_9}(f)(\nu''))$ is odd. Hence by Proposition 2.1, there exists an element μ'' in $\Psi^{(2)}_{P_9,N'_{10}}(\Gamma^{(2)}(P_9))$ such that $\mathrm{lk}(f(\mu''))$ is odd. $\Psi^{(2)}_{P_9,N'_{10}}(\Gamma^{(2)}(P_9))$ consists of the seven elements

$$\mu''_1=[5\,6\,9\,8]\cup[1\,10\,3\,4\,7],\quad \mu''_2=[4\,5\,8\,7]\cup[1\,10\,3\,9\,6],$$
$$\mu''_3=[1\,7\,8\,10]\cup[3\,4\,5\,6\,9],\quad \mu''_4=[3\,10\,8\,9]\cup[1\,7\,4\,5\,6],$$
$$\mu''_5=[1\,6\,9\,7]\cup[3\,4\,5\,8\,10],\quad \mu''_6=[3\,9\,7\,4]\cup[1\,10\,8\,5\,6],$$
$$\mu''_7=[7\,8\,9]\cup[1\,10\,3\,4\,5\,6].$$

For $j=1,2,\dots,7$, let $J^{4,j}$ be the subgraph of N'_{10} which is $\mu'_4\cup\mu''_j\cup\overline{2\,4}$ if $j=2,6$ and $\mu'_4\cup\mu''_j$ if $j\neq 2,6$. Assume that $\mathrm{lk}(f(\mu''_j))$ is odd for some $j\neq 7$. Then $J^{4,j}$ contains a graph $D^{4,j}$ as a minor such that $D^{4,j}$ is isomorphic to D_4 and $\{\mu'_4,\mu''_i\}=\Psi^{(2)}_{D^{4,j},J^{4,j}}(\Gamma^{(2)}(D^{4,j}))$. Since both $\mathrm{lk}(f(\mu'_4))$ and $\mathrm{lk}(f(\mu''_j))$ are odd, by Lemma 4.3 there exists an element γ in $\Gamma(J^{4,j})$ such that $a_2(f(\gamma))$ is odd. Next assume that $\mathrm{lk}(f(\mu''_7))$ is odd. We denote three elements $[7\,8\,9]\cup[1\,5\,3\,10]$, $[7\,8\,9]\cup[1\,5\,6]$ and $[7\,8\,9]\cup[3\,4\,5]$ in $\Gamma^{(2)}(N'_{10})$ by $\mu''_{7,1}$, $\mu''_{7,2}$ and $\mu''_{7,3}$. We denote the subgraph $\mu\cup\mu''_{7,k}\cup\overline{4\,7}\cup\overline{2\,8}$ of N'_{10} by $J^{4,7,k}$ $(k=1,2)$. Then $J^{4,7,k}$ contains a graph $D^{4,7,k}$ as a minor such that $D^{4,7,k}$ is isomorphic to D_4 and $\{\mu,\mu''_{7,k}\}=\Psi^{(2)}_{D^{4,7,k},J^{4,7,k}}(\Gamma^{(2)}(D^{4,7,k}))$ $(k=1,2)$. Since $[1\,10\,3\,4\,5\,6]=[1\,5\,3\,10]+[1\,5\,6]+[3\,4\,5]$ in $H_1(N'_{10};\mathbb{Z}_2)$, it follows that

$$1\equiv\mathrm{lk}(f(\mu'_7))\equiv\mathrm{lk}(f(\mu''_{7,1}))+\mathrm{lk}(f(\mu''_{7,2}))+\mathrm{lk}(f(\mu''_{7,3}))\pmod 2.$$

This implies that $\mathrm{lk}(f(\mu''_{7,k}))$ is odd for some k. If $\mathrm{lk}(f(\mu''_{7,1}))$ is odd or $\mathrm{lk}(f(\mu''_{7,2}))$ is odd, then by Lemma 4.3 there exists an element γ in $\Gamma(J^{4,7,k})$ such that $a_2(f(\gamma))$ is odd. If $\mathrm{lk}(f(\mu''_{7,3}))$ is odd, let us consider the 3-component link

$$L=f([3\,4\,5]\cup[7\,8\,9]\cup[1\,10\,2\,6]).$$

Since all 2-component sublinks of L are $f(\mu)$, $f(\mu'_4)$ and $f(\mu''_{7,3})$, each of the 2-component sublinks of L has an odd linking number.

Case 5. Let $\mu=[2\,8\,10]\cup[1\,6\,9\,3\,4\,7]$. We denote two elements $[2\,8\,10]\cup[1\,6\,9\,7]$ and $[2\,8\,10]\cup[3\,9\,7\,4]$ in $\Gamma^{(2)}(N'_{10})$ by μ_1 and μ_2, respectively. Since $[1\,6\,9\,3\,4\,7]=[1\,6\,9\,7]+[3\,9\,7\,4]$ in $H_1(N'_{10};\mathbb{Z}_2)$, it follows that

$$1\equiv\mathrm{lk}(f(\mu))\equiv\mathrm{lk}(f(\mu_1))+\mathrm{lk}(f(\mu_2))\pmod 2.$$

This implies that $\operatorname{lk}(f(\mu_1))$ is odd or $\operatorname{lk}(f(\mu_2))$ is odd. By the symmetry of N'_{10}, we may assume that $\operatorname{lk}(f(\mu_1))$ is odd. Note that N'_{10} contains another P_7 as the proper minor

$$(((N'_{10}-\overline{2\,8})-\overline{8\,10})-\overline{10\,2})/\overline{2\,6}/\overline{3\,10}/\overline{5\,8}.$$

Thus by Lemma 4.1, there is an element ν' in $\Gamma^{(2)}(P_7)$ such that $\operatorname{lk}(\psi_{N'_{10},P_7}(f)(\nu'))$ is odd. Hence by Proposition 2.1, there exists an element μ' in $\Psi^{(2)}_{P_7,N'_{10}}(\Gamma^{(2)}(P_7))$ such that $\operatorname{lk}(f(\mu'))$ is odd. $\Psi^{(2)}_{P_7,N'_{10}}(\Gamma^{(2)}(P_7))$ consists of the nine elements

$$\begin{aligned}
\mu'_1 &= [3\,5\,8\,9]\cup[1\,6\,2\,4\,7], & \mu'_2 &= [1\,7\,8\,5]\cup[2\,4\,3\,9\,6],\\
\mu'_3 &= [1\,5\,6]\cup[3\,9\,7\,4], & \mu'_4 &= [3\,4\,5]\cup[1\,6\,9\,7],\\
\mu'_5 &= [5\,6\,9\,8]\cup[1\,10\,3\,4\,7], & \mu'_6 &= [4\,5\,8\,7]\cup[1\,10\,3\,9\,6],\\
\mu'_7 &= [1\,5\,3\,10]\cup[2\,4\,7\,9\,6], & \mu'_8 &= [2\,4\,5\,6]\cup[1\,10\,3\,9\,7],\\
\mu'_9 &= [7\,8\,9]\cup[1\,10\,3\,4\,2\,6].
\end{aligned}$$

For $i=1,2,\dots,9$, let J^i be the subgraph of N'_{10} that is $\mu_1\cup\mu'_3\cup\overline{3\,10}\cup\overline{5\,8}$ if $i=3$ and $\mu_1\cup\mu'_i$ if $i\neq 3$. Assume that $\operatorname{lk}(f(\mu'_i))$ is odd for some $i\neq 4,9$. Then J^i contains a graph D^i as a minor such that D^i is isomorphic to D_4 and $\{\mu_1,\mu'_i\}=\Psi^{(2)}_{D^i,J^i}(\Gamma^{(2)}(D^i))$. Since both $\operatorname{lk}(f(\mu_1))$ and $\operatorname{lk}(f(\mu'_i))$ are odd, by Lemma 4.3 there exists an element γ in $\Gamma(J^i)$ such that $a_2(f(\gamma))$ is odd. Next assume that $\operatorname{lk}(f(\mu'_9))$ is odd. We denote two elements $[7\,8\,9]\cup[1\,6\,2\,10]$ and $[7\,8\,9]\cup[2\,4\,3\,10]$ in $\Gamma^{(2)}(J^9)$ by $\mu'_{9,1}$ and $\mu'_{9,2}$, respectively. We denote the subgraph $\mu_1\cup\mu'_{8,1}$ of J^9 by $J^{9,1}$ and the subgraph $\mu_1\cup\mu'_{9,2}\cup\overline{5\,3}\cup\overline{5\,1}$ of N'_{10} by $J^{9,2}$. Then $J^{9,j}$ contains a graph $D^{9,j}$ as a minor such that $D^{9,j}$ is isomorphic to D_4 and

$$\{\mu_1,\mu'_{9,j}\}=\Psi^{(2)}_{D^{9,j},J^{9,j}}(\Gamma^{(2)}(D^{9,j}))\quad(j=1,2).$$

Since $[1\,10\,3\,4\,2\,6]=[1\,6\,2\,10]+[2\,4\,3\,10]$ in $H_1(J^9;\mathbb{Z}_2)$, it follows that

$$1\equiv\operatorname{lk}(f(\mu'_9))\equiv\operatorname{lk}(f(\mu'_{9,1}))+\operatorname{lk}(f(\mu'_{9,2}))\ (\operatorname{mod} 2).$$

This implies that $\operatorname{lk}(f(\mu'_{9,1}))$ is odd or $\operatorname{lk}(f(\mu'_{9,2}))$ is odd. In either case, by Lemma 4.3 there exists an element γ in $\Gamma(J^{9,j})$ such that $a_2(f(\gamma))$ is odd. Finally assume that $\operatorname{lk}(f(\mu'_4))$ is odd. N'_{10} contains another P_9 as the proper minor

$$(((((N'_{10}-\overline{6\,1})-\overline{6\,2})-\overline{6\,5})-\overline{6\,9})-\overline{8\,7})-\overline{8\,10}.$$

Thus, by Lemma 4.1, there is $\nu''\in\Gamma^{(2)}(P_9)$ such that $\operatorname{lk}(\psi_{N'_{10},P_9}(f)(\nu''))$ is odd. Hence by Proposition 2.1, there exists $\mu''\in\Psi^{(2)}_{P_9,N'_{10}}(\Gamma^{(2)}(P_9))$ such that $\operatorname{lk}(f(\mu''))$

is odd. The set $\Psi^{(2)}_{P_9,N'_{10}}(\Gamma^{(2)}(P_9))$ consists of the seven elements

$$\begin{aligned}
\mu''_1 &= [3\,5\,8\,9]\cup[1\,10\,2\,4\,7], & \mu''_2 &= [3\,9\,7\,4]\cup[1\,5\,8\,2\,10],\\
\mu''_3 &= [1\,7\,4\,5]\cup[2\,8\,9\,3\,10], & \mu''_4 &= [2\,4\,5\,8]\cup[1\,10\,3\,9\,7],\\
\mu''_5 &= [2\,4\,3\,10]\cup[1\,5\,8\,9\,7], & \mu''_6 &= [1\,5\,3\,10]\cup[2\,4\,7\,9\,8],\\
\mu''_7 &= [3\,4\,5]\cup[1\,10\,2\,8\,9\,7].
\end{aligned}$$

For $j=1,2,\dots,7$, let $J^{4,j}$ be the subgraph of N'_{10} that is $\mu'_4\cup\mu''_j\cup\overline{2\,6}$ if $j=4,5$ and $\mu'_4\cup\mu''_j$ if $j\neq 4,5$. Assume that $\operatorname{lk}(f(\mu''_j))$ is odd for some $j\neq 7$. Then $J^{4,j}$ contains a graph $D^{4,j}$ as a minor such that $D^{4,j}$ is isomorphic to D_4 and $\{\mu'_4,\mu''_i\}=\Psi^{(2)}_{D^{4,j},J^{4,j}}(\Gamma^{(2)}(D^{4,j}))$. Since both $\operatorname{lk}(f(\mu'_4))$ and $\operatorname{lk}(f(\mu''_j))$ are odd, by Lemma 4.3 there exists an element γ in $\Gamma(J^{4,j})$ such that $a_2(f(\gamma))$ is odd. Next assume that $\operatorname{lk}(f(\mu''_7))$ is odd. We denote two elements $[3\,4\,5]\cup[1\,10\,8\,9\,7]$ and $[3\,4\,5]\cup[2\,8\,10]$ in $\Gamma^{(2)}(N'_{10})$ by $\mu''_{7,1}$ and $\mu''_{7,2}$, respectively. We denote the subgraph $\mu_1\cup\mu''_{7,1}\cup\overline{2\,4}\cup\overline{5\,6}$ of N'_{10} by $J^{4,7}$. Then $J^{4,7}$ contains a graph $D^{4,7}$ as a minor such that $D^{4,7}$ is isomorphic to D_4 and

$$\{\mu_1,\mu''_{7,1}\}=\Psi^{(2)}_{D^{4,7},J^{4,7}}(\Gamma^{(2)}(D^{4,7})).$$

Since $[1\,10\,2\,8\,9\,7]=[1\,10\,8\,9\,7]+[2\,8\,10]$ in $H_1(N'_{10};\mathbb{Z}_2)$, it follows that

$$1\equiv\operatorname{lk}(f(\mu''_7))\equiv\operatorname{lk}(f(\mu''_{7,1}))+\operatorname{lk}(f(\mu''_{7,2}))\pmod 2.$$

This implies that $\operatorname{lk}(f(\mu''_{7,1}))$ is odd or $\operatorname{lk}(f(\mu''_{7,2}))$ is odd. If $\operatorname{lk}(f(\mu''_{7,1}))$ is odd, then by Lemma 4.3 there exists an element γ in $\Gamma(J^{4,7})$ such that $a_2(f(\gamma))$ is odd. If $\operatorname{lk}(f(\mu''_{7,2}))$ is odd, let us consider the 3-component link

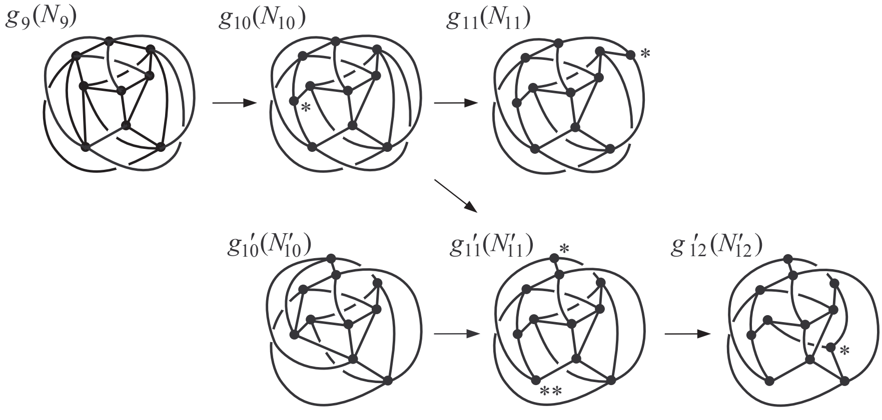
$$L=f([3\,4\,5]\cup[2\,8\,10]\cup[1\,6\,9\,7]).$$

Since all 2-component sublinks of L are $f(\mu_1)$, $f(\mu'_4)$ and $f(\mu''_{7,2})$, each of the 2-component sublinks of L has an odd linking number. This completes the proof. □

Proof of Theorem 1.2. A graph in the Heawood family is obtained from one of K_7, N_9 and N'_{10} by a finite sequence of $\triangle$Y-exchanges. Thus by Lemma 2.6, Theorem 4.4, and the fact that K_7 is IK — and thus I(K or C3L) — it follows that every graph in the Heawood family is I(K or C3L). On the other hand, a graph in the Heawood family is obtained from one of H_{12} and C_{14} by a finite sequence of Y$\triangle$-exchanges. Since each of H_{12} and C_{14} is a minor-minimal IK graph and $\Gamma^{(3)}(H_{12})$ and $\Gamma^{(3)}(C_{14})$ are the empty sets, it follows that H_{12} and C_{14} are minor-minimal I(K or C3L) graphs. By Lemma 2.7, we have the desired conclusion. □

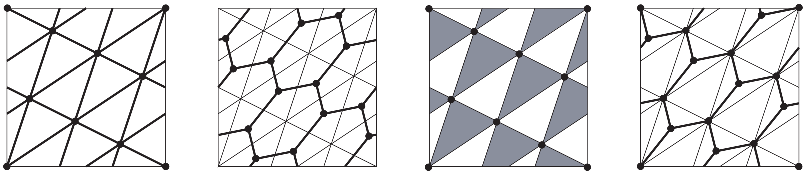
Remark 4.5. A graph is said to be 2-*apex* if it can be embedded in the 2-sphere after the deletion of at most two vertices and all of their incidental edges. It is not hard to see that any 2-apex graph may have a spatial embedding whose image contains neither a nontrivial knot nor a 3-component link all of whose 2-component sublinks are nonsplittable. Thus any 2-apex graph is not I(K or C3L). It is known that every graph of at most twenty edges is 2-apex [Mattman 2011] (see also [Johnson et al. 2010]). Since the number of all edges of every graph in the Heawood family is twenty-one, we see that any proper minor of a graph in the Heawood family is 2-apex, and thus not I(K or C3L). This also implies that any graph in the Heawood family is minor-minimal for I(K or C3L).

Example 4.6. Let g_9 be the spatial embedding of N_9 and g'_{10} the spatial embedding of N'_{10} illustrated here:



Then it can be checked directly that both $g_9(N_9)$ and $g'_{10}(N'_{10})$ do not contain a nonsplittable 3-component link. Thus neither N_9 nor N'_{10} is I3L. Also, we can see that N_{10}, N_{11}, N'_{11} and N'_{12} are not I3L in a similar way as the proof of Lemma 3.1 (see figure above).

Remark 4.7. The Heawood graph is IK. The Heawood graph H is the dual graph of K_7, which is embedded in a torus. It is known that there exists a unique graph C_{14} obtained from K_7 by seven applications of $\triangle Y$-exchanges [Kohara and Suzuki 1992]. The seven triangles correspond to the black triangles of a black-and-white coloring of the torus by K_7. Then C_{14} and H are mapped to each other by a translation of the torus:



Thus they are isomorphic. Since C_{14} is IK, we have the result.

Remark 4.8. It is known that all twenty-six graphs obtained from the complete four-partite graph $K_{3,3,1,1}$ by a finite sequence of $\triangle$Y-exchanges are minor-minimal IK graphs [Kohara and Suzuki 1992; Foisy 2002]. There exist thirty-two graphs that are obtained from $K_{3,3,1,1}$ by a finite sequence of $\triangle$Y-exchanges and Y$\triangle$-exchanges but that cannot be obtained from $K_{3,3,1,1}$ by a finite sequence of $\triangle$Y-exchanges. Recently, Goldberg, Mattman, and Naimi [2011] announced that these thirty-two graphs are also minor-minimal IK graphs.

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CONNECTION RELATIONS AND EXPANSIONS

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We give new proofs of the evaluation of the connection relation for the Askey–Wilson polynomials and for expressing the Askey–Wilson basis in those polynomials using q-Taylor series. This led to some inverse relations. We also evaluate the coefficients in the expansions of $(x+b)^n$ in various q-orthogonal polynomials, including the Askey–Wilson polynomials, which leads to explicit expressions for the moments of the Askey–Wilson weight function. We generalize the q-plane wave expansion by expanding $\mathcal{E}_q(x;\alpha)$ in Askey–Wilson polynomials. Further, we prove a bibasic extension of the Nassrallah–Rahman integral and establish a recently conjectured identity of George Andrews.

1. Introduction

Richard Askey and James Wilson introduced the polynomials that bear their names in their memoir [1985], where they derived, among other properties, the connection relation between Askey–Wilson polynomials with different parameters. One fundamental result of theirs is the evaluation of the Askey–Wilson q-beta integral,

$$\int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{\prod_{j=1}^4 (t_j e^{i\theta}, t_j e^{-i\theta}; q)_\infty}\, d\theta = \frac{2\pi (t_1t_2t_3t_4; q)_\infty}{(q;q)_\infty \prod_{1\le j<k\le 4}(t_jt_k;q)_\infty}. \tag{1-1}$$

All this work was done in the late 1970s and the results were made available to researchers in the area, but the writing took a long time. In the mean time, Nassrallah and Rahman [1985] generalized the Askey–Wilson integral to

$$\begin{aligned}\int_0^\pi & \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty\, (t_6e^{i\theta}, t_6e^{-i\theta}; q)_\infty}{\prod_{j=1}^5 (t_j e^{i\theta}, t_j e^{-i\theta}; q)_\infty}\, d\theta \\ &= \frac{2\pi (t_1t_2t_3t_4t_5/t_6; q)_\infty \prod_{j=1}^5 (t_jt_6;q)_\infty}{(q, t_6^2; q)_\infty \prod_{1\le j<k\le 5}(t_jt_k;q)_\infty} \\ &\quad \times {}_8W_7(t_6^2/q; t_6/t_1, t_6/t_2, t_6/t_3, t_6/t_4, t_6/t_5; q, t_1t_2t_3t_4t_5/t_6).\end{aligned} \tag{1-2}$$

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Remark. The preceding equation is (6.3.9) in [Gasper and Rahman 2004]. As in that reference and in [Ismail 2009], we follow the notation of [Andrews et al. 1999] for q-shifted factorials and basic hypergeometric series, and that of [Koekoek and Swarttouw 1998] for orthogonal polynomials.

The Askey–Wilson and Nassrallah–Rahman integrals play a fundamental role in the derivation of the results of this article, which is laid out as follows. Section 2 contains many of the formulas needed, other than (1-1) and (1-2). In particular, the Askey–Wilson polynomials are defined in (2-14).

In Section 3, we first solve the connection-coefficient problem of expanding an Askey–Wilson basis element

$$(ae^{i\theta}, ae^{-i\theta}; q)_n$$

in Askey–Wilson polynomials. The proof utilizes the q-integration by parts technique of [Brown et al. 1996]. One application of this expansion is to give a new derivation of a q-analogue of the plane wave expansion [Ismail 2009, (4.8.3)]

$$e^{ixy} = (2/y)^{\nu}\,\Gamma(\nu) \sum_{n=0}^{\infty} (n+\nu)\, i^n J_{n+\nu}(y)\, C_n^{\nu}(x), \tag{1-3}$$

a result first proved in [Ismail and Zhang 1994]. More importantly, we generalize the q-plane wave expansion to expand the Ismail–Zhang q-exponential function $\mathscr{E}_q(x;\alpha)$ in Askey–Wilson polynomials, which is a new result. The aforementioned connection-coefficient problem is also used to give a new proof of the connection relation of the Askey–Wilson polynomials. Each connection relation may be used to discover an inverse relation of the form $y_n = \sum_{k=0}^{n} Y_{n,k} x_k$ if and only if $x_n = \sum_{k=1}^{n} X_{n,k} y_k$. Inverse relations play a fundamental role in combinatorial-enumeration problems, as discussed in Riordan's classic [1968]. In the 1970s, interpretations of inverse relations involving q-shifted factorials and q-binomial coefficients were shown to be instances of Möbius inversion [Rota 1964] and of counting problems involving vector spaces over a finite field [Goldman and Rota 1970]. More recently, very general inverse relations were derived in [Krattenthaler 1989, 1996; Krattenthaler and Schlosser 1999].

Section 4 contains expansions of x^n and $(1 \pm x)^{\rho}$ in q-ultraspherical polynomials.

Section 5 contains the evaluation of two bibasic integrals which extend the Nassrallah–Rahman integral. They are stated as Theorems 5.1 and 5.2; the latter contains as a special case the evaluation of the moments of the Askey–Wilson weight function. [Corteel and Williams 2007] recently found a beautiful combinatorial expression for the n-th moment of the Askey–Wilson measure; this is also part of the results announced in [Corteel and Williams 2010]. Our analytic expression of the moments of the Askey–Wilson weight function is a double sum.

George Andrews [2011] studied identities involving the Catalan numbers he introduced in [Andrews 1987]. One of his identities was motivated by earlier work of L. Shapiro. Andrews' investigations led him to two summation theorems. One summation theorem is

$$(1\text{-}4)\qquad {}_4\phi_3\left(\begin{matrix} q^{-2n}, a, b, q^{1-2n}/ab \\ q^{2-2n}/a, q^{2-2n}/b, qab \end{matrix}\,\middle|\, q^2, q^2\right) = \frac{q^{-n}(a, b, -q; q)_n\,(ab; q^2)_n}{(ab; q)_n\,(a, b; q^2)_n},$$

which he proved. He conjectured the validity of the other summation theorem,

$$(1\text{-}5)\qquad {}_4\phi_3\left(\begin{matrix} q^{-2n}, a, b, q^{3-2n}/ab \\ q^{2-2n}/a, q^{2-2n}/b, qab \end{matrix}\,\middle|\, q^2, q^2\right)$$
$$= \frac{q^{-n}(a, b/q, -q; q)_n\,(q-ab)}{(1-b/q)\,(ab-q^2)\,(1-abq^{2n-1})}$$
$$\times \frac{(ab/q^2; q^2)_n}{(ab; q)_n\,(a, b/q^2; q^2)_n}\big(abq^{2n-2}(q^2-b) + abq^{n-1}(1-q) + b - q\big).$$

Andrews verified (1-5) for $1 \le n \le 6$. In Section 6, we give basic hypergeometric-series proofs of both (1-4) and the conjectured identity (1-5). We show that both (1-4) and (1-5) follow from a limiting case of the ${}_5\phi_4$ to ${}_{12}\phi_{11}$ transformation stated in [Gasper and Rahman 2004, (2.8.4)].

2. Preliminaries

The expansions of x^n and $(1-x)^\rho$ in ultraspherical polynomials are

$$(2\text{-}1)\qquad \frac{(2x)^n}{n!} = \sum_{k=0}^{\lfloor n/2\rfloor} \frac{\nu+n-2k}{k!\,(\nu)_{n+1-k}} C_{n-2k}^{\nu}(x)$$

[Rainville 1960, (36), p. 283], and

$$(2\text{-}2)\qquad (1-x)^\rho = \Gamma(\nu)\,\Gamma(\nu+\rho+1/2)\,\frac{2^{2\nu+\rho}}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(k+\nu)\,(-\rho)_k}{\Gamma(k+2\nu+\rho+1)}\, C_k^{\nu}(x),$$

valid for $-1 < x < 1$, $-\rho < \frac{1}{2}(\nu+1)$ if $\nu \ge 0$, and $-\rho < \nu + \frac{1}{2}$ if $-\frac{1}{2} < \nu \le 0$ [Erdélyi et al. 1953, (10.20.6)]. The Chebyshev polynomials are the special cases

$$(2\text{-}3)\qquad T_n(x) = \lim_{\nu\to 0} \frac{n+2\nu}{2\nu}\, C_n^{\nu}(x) \quad\text{and}\quad U_n(x) = C_n^1(x).$$

The Chebyshev polynomials are also special cases of the continuous q-ultraspherical polynomials, since

$$(2\text{-}4)\qquad T_n(x) = \lim_{\beta\to 1} \frac{1-\beta q^n}{1-\beta^2} C_n^{\nu}(x; \beta \mid q) \quad\text{and}\quad U_n(x) = C_n(x; q \mid q).$$

The Rogers connection relation for the q-ultraspherical polynomials is

$$C_n(x;\gamma\mid q)=\sum_{k=0}^{\lfloor n/2\rfloor}\frac{\beta^k(\gamma/\beta;q)_k(\gamma;q)_{n-k}}{(q;q)_k(q\beta;q)_{n-k}}\,\frac{1-\beta q^{n-2k}}{1-\beta}C_{n-2k}(x;\beta\mid q) \tag{2-5}$$

[Ismail 2009, (13.3.1)]. The Ismail–Zhang q-exponential function is

$$\begin{aligned}\mathscr{E}_q(\cos\theta;\alpha)\\ &=\frac{(\alpha^2;q^2)_\infty}{(q\alpha^2;q^2)_\infty}\sum_{n=0}^{\infty}(-ie^{i\theta}q^{(1-n)/2},-ie^{-i\theta}q^{(1-n)/2};q)_n\frac{(-i\alpha)^n}{(q;q)_n}q^{n^2/4}\end{aligned} \tag{2-6}$$

[Ismail 2009, §14.1].

We shall always use the notation

$$x=\cos\theta,\qquad z=e^{i\theta},\qquad f(x)=\breve{f}(z). \tag{2-7}$$

The set of polynomials $\{(ae^{i\theta},ae^{-i\theta};q)_n : n=0,1,\dots\}$ is a basis for the space of all polynomials, and is called the Askey–Wilson basis. The connection formula for the Askey–Wilson basis is

$$\frac{(be^{i\theta},be^{-i\theta};q)_n}{(q,ab;q)_n}=\sum_{k=0}^{n}\frac{(ae^{i\theta},ae^{-i\theta};q)_k}{(q,ab;q)_k}\,\frac{(b/a;q)_{n-k}}{(q;q)_{n-k}}\left(\frac{b}{a}\right)^k \tag{2-8}$$

[Ismail 1995]; see also the proof of Theorem 12.2.3 in [Ismail 2009].

We recall the definition of the Askey–Wilson operator,

$$(\mathscr{D}_q f)(x)=\frac{\breve{f}(q^{1/2}z)-\breve{f}(q^{-1/2}z)}{(q^{1/2}-q^{-1/2})(z-1/z)/2}. \tag{2-9}$$

It is easy to see that

$$\mathscr{D}_q(ae^{i\theta},ae^{-i\theta};q)_n=-\frac{2a(1-q^n)}{1-q}(aq^{1/2}e^{i\theta},aq^{1/2}e^{-i\theta};q)_{n-1} \tag{2-10}$$

[Ismail 2009, (12.2.2)]. We shall use the inner product

$$\langle f,g\rangle:=\int_{-1}^{1}f(x)\,\overline{g(x)}\,\frac{dx}{\sqrt{1-x^2}}. \tag{2-11}$$

Let

$$H_\nu:=\left\{f : f((z+1/z)/2)\text{ is analytic for } q^\nu\le|z|\le q^{-\nu}\right\}. \tag{2-12}$$

The following theorem — an analogue of integration by parts — is due to Brown, Evans and Ismail [Brown et al. 1996]; see also [Ismail 2009, §16.1].

Theorem 2.1. *The Askey–Wilson operator* $\mathcal{D}_q$ *satisfies, for* $f, g \in H_{1/2}$,

$$\langle \mathcal{D}_q f, g\rangle = \frac{\pi\sqrt{q}}{1-q}\left[f\Big(\frac{q^{1/2}+q^{-1/2}}{2}\Big)\overline{g(1)} - f\Big(-\frac{q^{1/2}+q^{-1/2}}{2}\Big)\overline{g(-1)}\right] - \Big\langle f, \sqrt{1-x^2}\,\mathcal{D}_q\big(g(x)(1-x^2)^{-1/2}\big)\Big\rangle. \tag{2-13}$$

The Askey–Wilson polynomials have the basic hypergeometric representation

$$\begin{aligned} &p_n(x;\mathbf{t}\mid q)\\ &\quad = t_1^{-n}(t_1t_2, t_1t_3, t_1t_4; q)_n\, {}_4\phi_3\left(\left.\begin{matrix} q^{-n}, t_1t_2t_3t_4q^{n-1}, t_1e^{i\theta}, t_1e^{-i\theta} \\ t_1t_2,\ t_1t_3,\ t_1t_4\end{matrix}\,\right|\, q, q\right), \end{aligned} \tag{2-14}$$

where $\mathbf{t}$ stands for the ordered quadruple (t_1, t_2, t_3, t_4). Their weight function is

$$w(x, \mathbf{t}\mid q) = \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{\prod_{j=1}^4 (t_je^{i\theta}, t_je^{-i\theta}; q)_\infty}\frac{1}{\sqrt{1-x^2}}, \qquad x=\cos\theta\in(-1,1), \tag{2-15}$$

The Askey–Wilson polynomials satisfy the orthogonality relation

$$\begin{aligned} \int_{-1}^1 p_m(x;\mathbf{t}\mid q)\, p_n(x;\mathbf{t}\mid q)\, w(x;\mathbf{t}\mid q)\, dx &= h_n(\mathbf{t})\,\delta_{m,n}\\ &= \frac{2\pi(t_1t_2t_3t_4q^{2n};q)_\infty\,(t_1t_2t_3t_4q^{n-1};q)_n}{(q^{n+1};q)_\infty \prod\limits_{1\le j<k\le 4}(t_jt_kq^n;q)_\infty}\,\delta_{m,n}, \end{aligned} \tag{2-16}$$

for $\max\{|t_1|, |t_2|, |t_3|, |t_4|\} < 1$. The Askey–Wilson polynomials also satisfy the Rodrigues-type formula

$$w(x;\mathbf{t}\mid q)\, p_n(x;\mathbf{t}\mid q) = \Big(\frac{q-1}{2}\Big)^n q^{n(n-1)/4}\,\mathcal{D}_q^n w(x; q^{n/2}\mathbf{t}\mid q). \tag{2-17}$$

The Chebyshev polynomials are also special Askey–Wilson polynomials; indeed,

$$\begin{aligned} &p_n(x; q, -q, \sqrt{q}, -\sqrt{q}\mid q) = (q^{n+2};q)_n U_n(x),\\ &p_0(x;\mathbf{t}\mid q) = T_0(x) = 1,\\ &p_n(x; 1, -1, \sqrt{q}, -\sqrt{q}\mid q) = 2(q^n;q)_n T_n(x) \qquad \text{for } n>0. \end{aligned} \tag{2-18}$$

We shall also use the q-Taylor expansion stated next.

Theorem 2.2 [Ismail 1995]. *Let*

$$x_n = (aq^{n/2} + q^{-n/2}/a)/2 \quad \textit{for } 0<q<1, 0<a<1, \tag{2-19}$$

If $f(x)$ is a polynomial, then

$$f(x)=\sum_{k=0}^{\infty} f_k\,(ae^{i\theta},ae^{-i\theta};q)_k,$$

with

$$f_k=\frac{(q-1)^k}{(2a)^k\,(q;q)_k}q^{-k(k-1)/4}\,(\mathcal{D}_q^k f)(x_k).$$

For a proof and details, see [Ismail 2009, Theorem 12.2.2].

3. Connection formulas and expansions

Lemma 3.1. *We have the integral evaluation*

$$\int_{-1}^{1}(ae^{i\theta},ae^{-i\theta};q)_n\,w(x;\mathbf{t}\,|\,q)\,dx= \tag{3-1}$$
$$=\frac{2\pi\,(t_1a,a/t_1;q)_n\,(t_1t_2t_3t_4;q)_\infty}{(q;q)_\infty\prod_{1\le j<m\le 4}(t_jt_m;q)_\infty}\,{}_4\phi_3\left(\begin{matrix}q^{-n},\ t_1t_2,\ t_1t_3,\ t_1t_4\\ t_1a,\,t_1t_2t_3t_4,\,q^{1-n}t_1/a\end{matrix}\,\middle|\,q,q\right).$$

This integral can be evaluated by writing

$$(ae^{i\theta},ae^{-i\theta};q)_n=\frac{(ae^{i\theta},ae^{-i\theta};q)_\infty}{(aq^ne^{i\theta},aq^ne^{-i\theta};q)_\infty},$$

then using the Nassrallah–Rahman integral (1-2) and the Watson transformation [Gasper and Rahman 2004, (III.18)]. It also follows by expanding $(ae^{i\theta},ae^{-i\theta};q)_n$ in $\{(t_1e^{i\theta},t_1e^{-i\theta};q)_k:0\le k\le n\}$ by using (2-8), and then applying the Askey–Wilson integral (1-1); see also [Ismail and Stanton 1998, Thm. 3].

Our first result is the next expansion of $(be^{i\theta},be^{-i\theta};q)_n$ in Askey–Wilson polynomials.

Theorem 3.2.

$$(be^{i\theta},be^{-i\theta};q)_n=\sum_{k=0}^{n} f_{n,k}(b,\mathbf{t})\,p_k(x;\mathbf{t}\,|\,q), \tag{3-2}$$

where

$$f_{n,k}(b,\mathbf{t})=\frac{(-b)^k q^{\binom{k}{2}}\,(q;q)_n\,(b/t_4,bt_4q^k;q)_{n-k}}{(q,t_1t_2t_3t_4q^{k-1};q)_k\,(q;q)_{n-k}} \tag{3-3}$$
$$\times{}_4\phi_3\left(\begin{matrix}q^{k-n},\ t_1t_4q^k,\ t_2t_4q^k,\ t_3t_4q^k\\ bt_4q^k,\,t_1t_2t_3t_4q^{2k},\,q^{1-n+k}t_4/b\end{matrix}\,\middle|\,q,q\right).$$

Proof. It is clear that

$$f_{n,k}h_k(\mathbf{t})=\left\langle p_k(x;\mathbf{t}\,|\,q)\,w(x;\mathbf{t}\,|\,q),\ \sqrt{1-x^2}(be^{i\theta},be^{-i\theta};q)_n\right\rangle$$

$$
\begin{aligned}
&= \Big(\frac{q-1}{2}\Big)^k q^{k(k-1)/4} \big\langle \mathcal{D}_q^k w(x; q^{k/2}\mathbf{t} \mid q),\ \sqrt{1-x^2}(be^{i\theta}, be^{-i\theta}; q)_n \big\rangle \\
&= \Big(\frac{1-q}{2}\Big)^k q^{k(k-1)/4} \int_{-1}^{1} w(x; q^{k/2}\mathbf{t} \mid q)\, \mathcal{D}_q^k (be^{i\theta}, be^{-i\theta}; q)_n \, dx \\
&= \frac{(-b)^k (q;q)_n}{(q;q)_{n-k}} q^{\binom{k}{2}} \int_{-1}^{1} (bq^{k/2}e^{i\theta}, bq^{k/2}e^{-i\theta}; q)_{n-k}\, w(x; q^{k/2}\mathbf{t} \mid q)\, dx.
\end{aligned}
$$

In these steps we used the Rodrigues formula (2-17), as well as (2-13) and (2-10). The result follows from a slight variation of Lemma 3.1. □

Our first application of Theorem 3.2 is the connection relation for the Askey–Wilson polynomials.

Corollary 3.3. *We have the connection relation*

$$
p_n(x; \mathbf{b}) = \sum_{k=o}^{n} c_{n,k}(\mathbf{b}, \mathbf{a})\, p_k(x; \mathbf{a}), \tag{3-4}
$$

where

$$
\begin{aligned}
c_{n,k}(\mathbf{b}, \mathbf{a}) = {} & \frac{b_4^{k-n} (b_1b_2b_3b_4q^{n-1}; q)_k\, (q, b_1b_4, b_2b_4, b_3b_4; q)_n}{(q;q)_{n-k}\, (q, a_1a_2a_3a_4q^{k-1}; q)_k\, (b_1b_4, b_2b_4, b_3b_4; q)_k} \\
& \times q^{k(k-n)} \sum_{j,l\ge 0} \frac{(q^{k-n}, b_1b_2b_3b_4q^{n+k-1}, a_4b_4q^k; q)_{j+l}\, q^{j+l}}{(b_1b_4q^k, b_2b_4q^k, b_3b_4q^k; q)_{j+l}\, (q;q)_j\, (q;q)_l} \\
& \qquad \times \frac{(a_1a_4q^k, a_2a_4q^k, a_3a_4q^k; q)_l\, (b_4/a_4; q)_j}{(a_4b_4q^k, a_1a_2a_3a_4q^{2k}; q)_l} \Big(\frac{b_4}{a_4}\Big)^l.
\end{aligned} \tag{3-5}
$$

Proof. The follows by expanding the left-hand side of (3-4) in the Askey–Wilson basis $\{(a_1e^{i\theta}, a_1e^{-i\theta}; q)_k\}$, then applying Theorem 3.2. □

Corollary 3.3 is Theorem 14.4.2 in [Ismail 2009]. When $a_4 = b_4$, the double series in (3-4) reduces to a ${}_5\phi_4$ and we get a result of [Askey and Wilson 1985]. See also [Gasper and Rahman 2004, (7.6.2)–(7.6.3)]. For another proof, see [Ismail and Zhang 2005], which also uses (2-13). Note that, in view of the orthogonality relation (2-16), Corollary 3.3 is equivalent to Theorem 3.2.

The special case $b = t_3$ of Theorem 3.2 is interesting. The result, after interchanging t_1 and t_3, is

$$
\begin{aligned}
(t_1e^{i\theta}, t_1e^{-i\theta}; q)_n = {} & \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-t_1)^k q^{\binom{k}{2}} \frac{(t_1t_2, t_1t_3, t_1t_4; q)_n}{(t_1t_2, t_1t_3, t_1t_4; q)_k} \frac{1 - t_1t_2t_3t_4q^{2k-1}}{1 - t_1t_2t_3t_4/q} \\
& \times \frac{(t_1t_2t_3t_4/q; q)_k}{(t_1t_2t_3t_4; q)_{n+k}}\, p_k(x; \mathbf{t} \mid q).
\end{aligned} \tag{3-6}
$$

Theorem 3.4. *The following relations are equivalent*:

$$B_n = \frac{(t_1t_2, t_1t_3, t_1t_4; q)_n}{t_1^n} \sum_{k=0}^{n} \frac{(q^{-n}, t_1t_2t_3t_4q^{n-1}; q)_k}{(q;q)_k \prod_{j=2}^{4}(t_1t_j;q)_k} q^k A_k \tag{3-7}$$

$$A_n = \sum_{k=0}^{n} \frac{t_1^k q^{\binom{k}{2}} (t_1/t_4, t_1t_2q^k, t_1t_3q^k, t_1t_4q^k; q)_{n-k}}{(q, t_1t_2t_3t_4q^{k-1}; q)_k \, (q, t_1t_2t_3t_4q^{2k}; q)_{n-k}} B_k \tag{3-8}$$

Proof. We set $b = t_1$ in (3-2) and take (2-14) into account. The ${}_4\phi_3$ in (3-3) becomes a ${}_3\phi_2$, and can be summed by the q-analogue of the Pfaff–Saalschütz theorem. □

Theorem 3.4 is known [Krattenthaler 1989; 1996]. An interesting question is to explore where such inverse pair lives from the point of view of the Möbius function on lattices [Rota 1964], because the lattices which will lead to such a deep result will be very interesting. It is also interesting to explore the concept of Bailey pairs [Andrews 1986] from the Möbius-inversion point of view.

The q-ultraspherical polynomials are special Askey–Wilson polynomials, since

$$p_n\big(x; \sqrt{\beta}, -\sqrt{\beta}, \sqrt{q\beta}, -\sqrt{q\beta} \mid q\big) = \frac{(q, \beta^2q^n; q)_n}{(\beta;q)_n} C_n(x; \beta \mid q). \tag{3-9}$$

The q-plane wave expansion in q-ultraspherical polynomials is

$$\begin{aligned} \mathcal{E}_q(x; i\alpha) = \frac{(\alpha)^{-\nu}(q;q)_\infty}{(-q\alpha^2; q^2)_\infty (q^{\nu+1}; q)_\infty} \sum_{n=0}^{\infty} \frac{(1-q^{n+\nu})}{(1-q^\nu)} q^{n^2/4} i^n \\ \times J_{\nu+n}^{(2)}(2\alpha; q)\, C_n(x; q^\nu \mid q); \end{aligned} \tag{3-10}$$

see [Ismail and Zhang 1994].

Another application of Theorem 3.2 is this generalization of (3-10):

Theorem 3.5. *We have the following generalization of the q-plane wave expansion function*:

$$\begin{aligned} \mathcal{E}_q(x; \alpha) = \frac{(\alpha^2; q^2)_\infty}{(q\alpha^2; q)_\infty} \sum_{n=0}^{\infty} \frac{\alpha^n q^{n^2/4} p_n(x; \mathbf{t})}{(q, t_1t_2t_3t_4q^{n-1}; q)_n} \\ \times \sum_{k=0}^{\infty} \frac{(-\alpha/t_4)^k}{(q;q)_k} (-q^{1+n-k}t_4^2; q^2)_k \, q^{k(k-2n)/4} \\ \times {}_4\phi_3\left(\begin{matrix} q^{-k}, \ t_1t_4q^n, \ t_2t_4q^n, \ t_3t_4q^n \\ -it_4q^{(1-k+n)/2}, \ it_4q^{(1-k+n)/2}, \ t_1t_2t_3t_4q^{2n} \end{matrix} \,\middle|\, q, q \right). \end{aligned} \tag{3-11}$$

Proof. Expand the $\mathcal{E}_q$ in the Askey–Wilson basis via (2-6), then apply (3-2). □

Another proof of Theorem 3.5. Since $\mathscr{E}_q(x;\alpha) \in L_2[-1,1,w(x;\mathbf{t})]$, we set

$$\mathscr{E}_q(x;\alpha) = \sum_{n=0}^{\infty} c_n p_n(x;\mathbf{t}).$$

Using (2-17), the divided-difference relation $\mathscr{D}_q\mathscr{E}_q(x;\alpha) = 2\alpha q^{1/4}/(1-q)\mathscr{E}_q(x;\alpha)$ and the q-integration by parts (2-13), we find that

$$\begin{aligned} c_n h_n(\mathbf{t}) &= \int_{-1}^{1} \mathscr{E}_q(x;\alpha)\, p_n(x;\mathbf{t})\, w(x;\mathbf{t})\, dx \\ &= \Big(\frac{q-1}{2}\Big)^n q^{\binom{n}{2}/2} \int_{-1}^{1} \mathscr{E}_q(x;\alpha)\, \mathscr{D}_q^n w(x; q^{n/2}\mathbf{t})\, dx \\ &= \alpha^n q^{n^2/4} \int_{-1}^{1} \mathscr{E}_q(x;\alpha)\, w(x; q^{n/2}\mathbf{t})\, dx \\ &= \frac{(\alpha^2;q^2)_\infty}{(q\alpha^2;q^2)_\infty} \alpha^n q^{n^2/4} \sum_{k=0}^{\infty} \frac{(-i\alpha)^k}{(q;q)_k} q^{k^2/4} \\ &\qquad \times \int_0^{\pi} w(\cos\theta; q^{n/2}\mathbf{t})\, (-iq^{(1-k)/2}e^{i\theta}, -iq^{(1-k)/2}e^{-i\theta}; q)_k \sin\theta\, d\theta. \end{aligned}$$

The integral above is

$$\frac{2\pi(-it_4q^{(1+n-k)/2}, -it_4q^{(1-n-k)/2}/t_4; q)_k\, (t_1t_2t_3t_4q^{2n};q)_\infty}{(q;q)_\infty \prod_{1\le j<m\le 4}(t_jt_mq^n;q)_\infty} \times {}_4\phi_3\left(\begin{matrix} q^{-k},\ t_1t_4q^n,\ t_2t_4q^n,\ t_3t_4q^n \\ -it_4q^{(1-k+n)/2}, it_4q^{(1-k+n)/2}, t_1t_2t_3t_4q^{2n} \end{matrix} \,\middle|\, q, q\right).$$

The result now follows from (2-16). ☐

In the case of q-ultraspherical polynomials, the ${}_4\phi_3$ in (3-11) can be summed by Andrews' q-analogue of Watson's ${}_3F_2$ sum [Gasper and Rahman 2004, (II.17)]. Thus, the ${}_4\phi_3$ is zero for k odd and, when k is replaced by $2k$, the ${}_4\phi_3$ is

$$\beta^{2k} q^{2nk+k} \frac{(q, -q^{1-n-2k}/\beta; q^2)_k}{(-\beta q^{n+2-2k}, \beta^2 q^{2n+2}; q^2)_k}.$$

Thus, the k-sum in (3-11) is ${}_2\phi_1(-\beta q^{n+2}, -\beta q^{n+1}; \beta^2q^{2n+2}; q^2, \alpha^2)$. Therefore,

$$\mathscr{E}_q(x;\alpha) = \frac{(\alpha^2;q^2)_\infty}{(q\alpha^2;q)_\infty} \sum_{n=0}^{\infty} \frac{\alpha^n q^{n^2/4}}{(\beta;q)_n} \times {}_2\phi_1\left(\begin{matrix} -\beta q^{n+2}, -\beta q^{n+1} \\ \beta^2 q^{2n+2} \end{matrix} \,\middle|\, q^2, \alpha^2\right) C_n(x;\beta \mid q). \tag{3-12}$$

By equating the left sides of (3-12) and (3-10), we establish the identity

$$\begin{aligned} J_\nu^{(2)}(2\alpha;q) &= \frac{\alpha^\nu(-\alpha^2;q^2)_\infty}{(q^{\nu+1};q)_\infty}\,{}_2\phi_1\left(\begin{matrix} -q^{\nu+2},\, -q^{\nu+1} \\ q^{2\nu+2} \end{matrix}\,\middle|\, q^2, -\alpha^2\right) \\ &= \frac{\alpha^\nu(q^{\nu+1}\alpha^2;q^2)_\infty}{(q^{\nu+1};q)_\infty}\,{}_2\phi_2\left(\begin{matrix} -q^{\nu+2},\, -q^{\nu+1} \\ q^{2\nu+2},\, q^{\nu+2}\alpha^2 \end{matrix}\,\middle|\, q^2, q^{\nu+1}\alpha^2\right), \end{aligned}$$

after applying the ${}_2\phi_1$ to ${}_2\phi_2$ transformation [Gasper and Rahman 2004, (III.4)]. The representation of $J_\nu^{(2)}$ as a ${}_2\phi_2$ is due to [Rahman 1987].

The double series in (3-11) also reduces to a single series in the case of continuous q-Jacobi polynomials, $t_2 = t_1 q^{1/2}$ and $t_4 = t_3 q^{1/2}$, yielding a result in [Ismail et al. 1996]. The details however are not lengthy and will be omitted.

4. Expansions of x^n and $(1 \pm x)^\rho$

Theorem 4.1. *The expansion*

$$\begin{aligned} (1-x)^\rho &= \frac{4}{\sqrt{\pi}}\, 2^\rho\, \Gamma(\rho+3/2) \\ &\times \sum_{k=0}^\infty \frac{1-\beta q^k}{1-\beta}\left(\sum_{j=0}^\infty \frac{(k+2j+1)\,(-\rho)_{k+2j}\,\beta^j\,(q/\beta;q)_j\,(q;q)_{k+j}}{(q,q)_j\,(q\beta;q)_{k+j}\,\Gamma(k+2j+\rho+3)}\right) C_k(x;\beta\mid q) \end{aligned} \tag{4-1}$$

holds for $-1 < x < 1$, $\rho > -1$ *and* $\beta \in (0, 1)$. *The expansion for* $(1+x)^\rho$ *is similar, since* $C_n(-x;\beta\mid q) = (-1)^n C_n(x;\beta\mid q)$.

Proof. Apply (2-2) with $\nu = 1$, then expand $C_k^1(x) = U_k(x) = C_k(x;q\mid q)$ in $C_j(x;\beta\mid q)$ by using (2-5), then rearrange the series. The expansion (2-2) holds for $\rho > -1$. The rearrangement is valid because the double series in the theorem converges absolutely for $\rho > -1$, in view of the asymptotic formula [Ismail 2009, (13.4.5)] and the well-known fact that $n^{b-a}\Gamma(n+a)/\Gamma(n+b) \to 1$ as $n \to +\infty$. □

It is interesting to note that, as $q \to 1$, the expansion (4-1) should reduce to (2-2). Indeed with $\beta = q^\nu$ the $q \to 1$ limit of the quantity in square brackets is a well-poised ${}_5F_4$ at $x = 1$, which can be summed, see Slater [Slater 1966, (III.12)]. So we could have discovered the abovementioned sum if it was not already known.

Theorem 4.2. *For nonnegative integers n we have the q-ultraspherical expansion*

$$\begin{aligned} x^n &= \frac{n!}{2^n}\sum_{m=0}^{\lfloor n/2\rfloor} \frac{1-\beta q^{n-2m}}{1-\beta}\, C_{n-2m}(x;\beta\mid q) \\ &\qquad \times \sum_{k=0}^{m} \frac{n+1-2k}{k!\,(n+1-k)!}\, \frac{\beta^{m-k}(q/\beta;q)_{m-k}(q;q)_{n-m-k}}{(q;q)_{m-k}(q\beta;q)_{n-m-k}}. \end{aligned} \tag{4-2}$$

Proof. The expansion (4-2) follows immediately from letting $\nu = 1$ in (2-1) then use (2-5) with $\gamma = 1$. $\square$

Note that

$$\frac{n!(n+1-2k)}{k!(n+1-k)!} = \binom{n}{k} - \binom{n}{k-1}.$$

With $\beta = q^\nu$, the limit of the k-sum in (4-2) as $q \to 1$ is a very well-poised ${}_4F_3$ at $x = -1$, which can be summed [Slater 1966, (III.11)].

5. Two bibasic integrals

In this section we give evaluations of the integral (5-2) and the more general integral (5-3). The proof uses the bibasic expansion

$$\text{(5-1)}\quad \frac{(q, qa^2; q)_\infty}{(qae^{i\theta}, qae^{-i\theta}; q)_\infty}(be^{i\theta}, be^{-i\theta}; p)_\infty \\ = \sum_{k=0}^{\infty} \frac{1-a^2q^{2k}}{1-a^2}\, \frac{(a^2, ae^{i\theta}, ae^{-i\theta}; q)_k}{(q, qae^{i\theta}, aqe^{-i\theta}; q)_k} (-1)^k q^{\binom{k+1}{2}} (abq^k, bq^{-k}/a; p)_\infty,$$

which is valid for $0 < p < q$, or $p = q$ and $|b| < |a|$ [Ismail and Stanton 2003].

Theorem 5.1. *We have the bibasic integral evaluation*

$$\text{(5-2)}\quad \int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty\, (be^{i\theta}, be^{-i\theta}; p)_\infty}{\prod_{j=1}^{5}(a_je^{i\theta}, a_je^{-i\theta}; q)_\infty}\, d\theta \\ = \frac{2\pi\, (a_2a_3a_4a_5/q; q)_\infty}{(q;q)_\infty \prod_{2\le r<s\le 5}(a_ra_s; q)_\infty}\, \frac{1}{(q, qa_1^2; q)_\infty} \\ \times \sum_{k=0}^{\infty} \frac{1-a_1^2q^{2k}}{1-a_1^2}\, \frac{(a_1^2; q)_k}{(q;q)_k}\, \frac{1-a_1^2q^{2k+1}}{\prod_{s=2}^{5}(1-a_1a_sq^k)} (-1)^k q^{\binom{k+1}{2}} \Big(a_1bq^k, \frac{bq^{-k}}{a_1}; p\Big)_\infty \\ \times {}_8W_7\Big(a_1^2q^{2k+1}; q, q^{k+1}\frac{a_1}{a_2}, q^{k+1}\frac{a_1}{a_3}, q^{k+1}\frac{a_1}{a_4}, q^{k+1}\frac{a_1}{a_5}; q, \frac{a_2a_3a_4a_5}{q}\Big) \\ = \frac{2\pi \prod_{j=2}^{5}(a_1a_2a_3a_4a_5/a_j; q)_\infty}{(q, a_1^2a_2a_3a_4a_5; q)_\infty \prod_{1\le r<s\le 5}(a_ra_s; q)_\infty} \\ \times \sum_{k=0}^{\infty} \frac{(a_1^2; q)_k\, (a_1^2a_2a_3a_4a_5; q)_{2k}}{(q;q)_k\, (a_1^2; q)_{2k}} \\ \times \prod_{j=2}^{5} \frac{(a_1a_j; q)_k}{(a_1a_2a_3a_4a_5/a_j; q)_k} (-1)^k q^{\binom{k+1}{2}} \Big(a_1bq^k, \frac{bq^{-k}}{a_1}; p\Big)_\infty \\ \times {}_8W_7\Big(a_1^2a_2a_3a_4a_5q^{2k-1}; a_1a_2q^k, a_1a_3q^k, a_1a_4q^k, a_1a_5q^k, \frac{a_2a_3a_4a_5}{q}; q, q\Big).$$

Proof. In view of (5-1), the left-hand side of (5-2) is

$$\frac{1}{(q, qa_1^2; q)_\infty}\sum_{k=0}^{\infty}\frac{1-a_1^2q^{2k}}{1-a_1^2}\frac{(a_1^2;q)_k}{(q;q)_k}(-1)^kq^{\binom{k+1}{2}}(a_1bq^k, bq^{-k}/a_1;p)_\infty$$
$$\times\int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta};q)_\infty\,(a_1q^{k+1}e^{i\theta}, a_1q^{k+1}e^{-i\theta};q)_\infty}{(a_1q^ke^{i\theta}, a_1q^ke^{-i\theta};q)_\infty\prod_{j=2}^5(a_je^{i\theta}, a_je^{-i\theta};q)_\infty}\,d\theta.$$

The first equality in (5-2) follows from (1-2). The second equality follows from the form of the Nassrallah–Rahman integral stated in [Gasper and Rahman 2004, (6.3.7)] with

$$f = a_1q^k. \qquad \square$$

When $p = q$, Theorem 5.1 should reduce to the Nassrallah–Rahman integral (1-2). This is not obvious, so we will indicate how it works. When $p = q$,

$$(-1)^kq^{\binom{k+1}{2}}(a_1bq^k, bq^{-k}/a_1;p)_\infty = (a_1b, b/a_1;q)_\infty\frac{b^k(qa_1/b;q)_k}{a^k(a_1b;q)_k}.$$

We use the second equation in (5-2) and write the ${}_8W_7$ as a sum over j. With $\ell = j+k$, the left-hand side of (5-2) becomes

$$\frac{2\pi(a_1b, b/a_1;q)_\infty\prod_{s=2}^5(a_1a_2a_3a_4a_5/a_s;q)_\infty}{(q, a_1^2a_2a_3a_4a_5;q)_\infty\prod_{1\le r<s\le 5}(a_ra_s;q)_\infty}$$
$$\times\sum_{\ell=0}^{\infty}\frac{1-a_1^2a_2a_3a_4a_5q^{2\ell-1}}{1-a_1^2a_2a_3a_4a_5/q}\frac{(a_2a_3a_4a_5/q, a_1^2a_2a_3a_4a_5/q;q)_\ell}{(q, qa_1^2;q)_\ell}q^\ell$$
$$\times\prod_{r=2}^5\frac{(a_1a_r;q)_\ell}{(a_1a_2a_3a_4a_5/a_r;q)_\ell}$$
$$\times\,{}_6W_5(a_1^2; qa_1/b, a_1^2a_2a_3a_4a_5q^{\ell-1}, q^{-\ell}; q, qb/a_1^2a_2a_3a_4a_5).$$

The ${}_6W_5$ can be summed by [Gasper and Rahman 2004, (II.20)], and the expression above reduces to the integral evaluation [Gasper and Rahman 2004, (6.3.7)].

The next theorem generalizes the evaluation of the moments of the Askey–Wilson weight function.

Theorem 5.2. *We have the integral evaluation*

$$\text{(5-3)}\quad \int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta};q)_\infty\,(be^{i\theta}, be^{-i\theta};p)_n}{\prod_{j=1}^4(a_je^{i\theta}, a_je^{-i\theta};q)_\infty}\,d\theta$$

$$\begin{aligned}
&= \frac{2\pi (a_1a_2a_3a_4;q)_\infty}{(q;q)_\infty \prod_{1\le j<k\le 4}(a_ja_k;q)_\infty} \frac{(a_1a_2, qa_1a_3, qa_1a_4;q)_n}{(q, qa_1^2, a_1a_2a_3a_4;q)_n} \\
&\quad \times \sum_{k=0}^{n} \frac{1-a_1^2q^{2k}}{1-a_1^2} \frac{(a_1^2, q^{-n};q)_k}{(q, a_1^2q^{n+1};q)_k} (a_1bq^k, bq^{-k}/a_1;p)_n \\
&\quad \times q^{k(n+1)} \frac{(1-a_1a_3)(1-a_1a_4)}{(1-a_1a_3q^k)(1-a_1a_4q^k)} \\
&\quad \times {}_4\phi_3\left(\begin{matrix} q^{k-n}, q, a_3a_4, a_1q^{k+1}/a_2 \\ a_1a_3q^{k+1}, a_1a_4q^{k+1}, q^{1-n}/a_1a_2 \end{matrix} \middle| \, q, q\right).
\end{aligned}$$

Proof. Observe that

$$\begin{aligned}
&(abq^k, bq^{-k}/a;p)_n \\
&\quad = (ab;p)_n \prod_{j=0}^{n-1}(1-ap^{-j}/b)q^{-kn}\Big(-\frac{b}{a}\Big)^n \prod_{j=0}^{n-1} \frac{(abp^j;q)_k\,(aqp^{-j}/b;q)_k}{(abp^j;q)_k\,(ap^{-j}/b;q)_k} \\
&\quad = (ab, b/a;p)_n q^{-kn} \prod_{j=0}^{n-1} \frac{(abp^j;q)_k\,(aqp^{-j}/b;q)_k}{(ap^{-j}/b, abp^j;q)_k}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{k=0}^{n} \frac{1-a_1^2q^{2k}}{1-a_1^2} \frac{(a_1^2, a_1e^{i\theta}, a_1e^{-i\theta}, q^{-n};q)_k}{(q, qa_1e^{-i\theta}, qa_1e^{i\theta}, a_1^2q^{n+1};q)_k} q^{k(n+1)} \Big(a_1bq^k, \frac{bq^{-k}}{a_1};p\Big)_n \\
&\qquad = (a_1b, b/a_1;p)_n \sum_{k=0}^{n} \frac{1-a_1^2q^{2k}}{1-a_1^2} \frac{(a_1^2, a_1e^{i\theta}, a_1e^{-i\theta}, q^{-n};q)_k}{(q, qa_1e^{i\theta}, qa_1e^{-i\theta}, a_1^2q^{n+1};q)_k} q^k \\
&\qquad\qquad \times \prod_{j=0}^{n-1} \frac{(qa_1p^{-j}/b, qa_1bp^j;q)_k}{(a_1bp^j, a_1p^{-j}/b;q)_k} \\
&\qquad = (a_1b, b/a_1;p)_n \frac{(qa_1^2, q;q)_n\,(be^{i\theta}, be^{-i\theta};p)_n}{(qa_1e^{i\theta}, qa_1e^{-i\theta};q)_n\,(a_1b, b/a_1;p)_n} \\
&\qquad = \frac{(q, qa_1^2;q)_n\,(be^{i\theta}, be^{-i\theta};p)_n}{(qa_1e^{i\theta}, qa_1e^{-i\theta};q)_n}.
\end{aligned}$$

So, we have

$$\begin{aligned}
&\frac{(q, qa_1^2;q)_n\,(be^{i\theta}, be^{-i\theta};p)_n}{(qa_1e^{i\theta}, qa_1e^{-i\theta};q)_n} \\
&\quad = \sum_{k=0}^{n} \frac{1-a_1^2q^{2k}}{1-a_1^2} \frac{(a_1^2, a_1e^{i\theta}, a_1e^{-i\theta}, q^{-n};q)_k}{(q, qa_1e^{-i\theta}, qa_1e^{i\theta}, a_1^2q^{n+1};q)_k} q^{k(n+1)} \Big(a_1bq^k, \frac{bq^{-k}}{a_1};p\Big)_n.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty \, (be^{i\theta}, be^{-i\theta}; p)_n}{\prod_{j=1}^4 (a_je^{i\theta}, a_je^{-i\theta}; q)_\infty} \, d\theta \\
&\quad = \frac{1}{(q, qa_1^2; q)_n} \sum_{k=0}^n \frac{1-a_1^2q^{2k}}{1-a_1^2} \frac{(a_1^2, q^{-n}; q)_k}{(q, a_1^2q^{n+1}; q)_k} q^{k(n+1)} \Big(a_1bq^k, \frac{bq^{-k}}{a_1}; p\Big)_n \\
&\qquad \times \int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(a_1q^{n+1}e^{i\theta}, a_1q^{n+1}e^{-i\theta}, a_1q^ke^{i\theta}, a_1q^ke^{-i\theta}; q)_\infty} \\
&\qquad\qquad \times \frac{(a_1q^{k+1}e^{i\theta}, a_1q^{k+1}e^{-i\theta}; q)_\infty}{\prod_{j=2}^4 (a_je^{i\theta}, a_je^{-i\theta}; q)_\infty} \, d\theta
\end{aligned}$$

Using [Gasper and Rahman 2004, (6.3.8)] and Watson's formula [Gasper and Rahman 2004, (III.18)], the integral in the equation above becomes

$$\begin{aligned}
&\frac{2\pi (a_1a_2a_3a_4; q)_\infty}{(q; q)_\infty \prod_{1 \le j < k \le 4} (a_ja_k; q)_\infty} \frac{(a_1a_2; q)_n (a_1a_3, a_1a_4; q)_{n+1}}{(a_1a_2a_3a_4; q)_n} \\
&\quad \times \frac{1}{(1-a_1a_3q^k)(1-a_1a_4q^k)} \, {}_4\phi_3\left(\begin{matrix} q^{k-n}, q, a_3a_4, a_1q^{k+1}/a_2 \\ a_1a_3q^{k+1}, a_1a_4q^{k+1}, q^{1-n}/a_1a_2 \end{matrix} \,\middle|\, q, q \right).
\end{aligned}$$

This completes the proof. ☐

We give a second proof of (5-3) because it has an idea which may be useful in other cases. The second proof uses the following recent result of [Ismail and Stanton 2010]:

$$\begin{aligned}
&\frac{(q, qa^2; q)_n}{(qae^{i\theta}, qae^{-i\theta}; q)_n} (be^{i\theta}, be^{-i\theta}; p)_n \\
&\quad = \sum_{k=0}^n \frac{1-a^2q^{2k}}{1-a^2} \frac{(q^{-n}, a^2, ae^{i\theta}, ae^{-i\theta}; q)_k}{(q, a^2q^{n+1}, aqe^{i\theta}, aqe^{-i\theta}; q)_k} q^{k(1+n)} (abq^k, bq^{-k}/a; p)_n.
\end{aligned} \tag{5-4}$$

Second proof of Theorem 5.2. In view of (5-4), the left-hand side of (5-3) is

$$\begin{aligned}
&\frac{1}{(q, qa_1^2; q)_n} \sum_{k=0}^n \frac{1-a_1^2q^{2k}}{1-a_1^2} \frac{(q^{-n}, a_1^2; q)_k}{(q, a_1^2q^{n+1}; q)_k} q^{k(1+n)} (a_1bq^k, bq^{-k}/a_1; p)_n \\
&\qquad \times \int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty \, (a_1q^{k+1}e^{i\theta}, a_1q^{k+1}e^{-i\theta})_{n-k}}{(a_1q^ke^{i\theta}, a_1q^ke^{-i\theta}; q)_\infty \prod_{j=2}^4 (a_je^{i\theta}, a_je^{-i\theta}; q)_\infty} \, d\theta.
\end{aligned}$$

This integral can be evaluated by (3-1) and equals

$$\frac{2\pi\,(a_1^2q^{2k+1}, q; q)_{n-k}\,(q^ka_1a_2a_3a_4; q)_\infty}{(q;q)_\infty\prod_{j=2}^4(q^ka_1a_j;q)_\infty\prod_{2\le r<s\le 4}(a_ra_s;q)_\infty} \times {}_4\phi_3\left(\begin{matrix} q^{k-n}, a_1a_2q^k, a_1a_3q^k, a_1a_4q^k \\ a_1^2q^{2k+1}, a_1a_2a_3a_4q^k, q^{-n}\end{matrix}\,\middle|\, q, q\right)$$

The application of the iterated Sears transformation [Gasper and Rahman 2004, (III.16)] reduces ${}_4\phi_3$ to

$$\frac{(a_1a_2q^k, a_1a_3q^{k+1}, a_1a_4q^{k+1}; q)_{n-k}}{(a_1^2q^{2k+1}, a_1a_2a_3a_4q^k, q; q)_{n-k}}$$

times the ${}_4\phi_3$ in (5-3). Simple manipulations now establish (5-3). □

Let $p=1$ and $\zeta = \frac{1}{2}(b+1/b)$. Then,

$$\begin{aligned}\text{(5-5)}\quad \int_0^\pi &\frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{\prod_{j=1}^4(a_je^{i\theta}, a_je^{-i\theta}; q)_\infty}(\cos\theta-\zeta)^n d\theta \\ &= \frac{2\pi(a_1a_2a_3a_4;q)_\infty}{(q;q)_\infty\prod_{1\le j<k\le 4}(a_ja_k;q)_\infty}\,\frac{(a_1a_2, qa_1a_3, qa_1a_4; q)_n}{(q, qa_1^2, a_1a_2a_3a_4; q)_n} \\ &\quad\times\sum_{k=0}^n \frac{1-a_1^2q^{2k}}{1-a_1^2}\,\frac{(a_1^2, q^{-n}; q)_k}{(q, a_1^2q^{n+1}; q)_k}\left(\tfrac{1}{2}(a_1q^k+q^{-k}/a_1)-\zeta\right)^n \\ &\quad\times q^k\frac{(1-a_1a_3)(1-a_1a_4)}{(1-a_1a_3q^k)(1-a_1a_4q^k)} \\ &\quad\times {}_4\phi_3\left(\begin{matrix} q^{k-n}, q, a_3a_4, a_1q^{k+1}/a_2 \\ a_1a_3q^{k+1}, a_1a_4q^{k+1}, q^{1-n}/a_1a_2\end{matrix}\,\middle|\, q, q\right).\end{aligned}$$

The special case $\zeta=0$ gives the Askey–Wilson moments

$$\begin{aligned}\text{(5-6)}\quad \int_{-1}^1 W(x;\mathbf{a})\,x^n\,dx &= \frac{(a_1a_2, qa_1a_3, qa_1a_4; q)_n}{(2a_1)^n(q, qa_1^2, a_1a_2a_3a_4; q)_n} \\ &\quad\times\sum_{k=0}^n \frac{1-a_1^2q^{2k}}{1-a_1^2}\,\frac{(a_1^2, q^{-n}; q)_k}{(q, a_1^2q^{n+1}; q)_k}\,(1+a_1^2q^{2k})^n \\ &\quad\times q^{k(n+1)}\frac{(1-a_1a_3)(1-a_1a_4)}{(1-a_1a_3q^k)(1-a_1a_4q^k)} \\ &\quad\times {}_4\phi_3\left(\begin{matrix} q^{k-n}, q, a_3a_4, a_1q^{k+1}/a_2 \\ a_1a_3q^{k+1}, a_1a_4q^{k+1}, q^{1-n}/a_1a_2\end{matrix}\,\middle|\, q, q\right),\end{aligned}$$

where W is the normalized weight function

$$W(x;\mathbf{a}) := \frac{(q;q)_\infty \prod_{1\le r<s\le 4}(a_r a_s;q)_\infty}{2\pi(a_1a_2a_3a_4;q)_\infty}\, w(x;\mathbf{a}). \tag{5-7}$$

The moments of the Askey–Wilson weight functions were first computed in the very interesting paper [Corteel and Williams 2007]. Corteel and Williams used purely combinatorial techniques and showed that the moments of the Askey–Wilson weight is a generating function for purely combinatorial objects. The Corteel-Williams formula is very different in nature from our (5-6), and a very interesting but difficult exercise is to show the equivalence of the two results.

6. The Andrews identities

We now prove both (1-4) and (1-5) using the ${}_5\phi_4$ to ${}_{12}\phi_{11}$ transformation [Gasper and Rahman 2004, (2.8.4)].

Proof of (1-4). The limiting case $e \to 0$ of the ${}_5\phi_4$ to ${}_{12}\phi_{11}$ transformation (2.8.4) of [Gasper and Rahman 2004] is

$${}_4\phi_3\left(\begin{matrix} q^{-n}, & b, & c, & d \\ \frac{q^{1-n}}{b}, & \frac{q^{1-n}}{c}, & \frac{q^{1-n}}{d} & \end{matrix}\,\middle|\, q, q\right) = \frac{(\lambda^2 q^{n+1};q)_n}{(q\lambda;q)_n}(\lambda q^n)^{-n}$$

$$\times\ {}_{10}\phi_9\left(\begin{matrix} \lambda, q\sqrt{\lambda}, -q\sqrt{\lambda}, \lambda b q^n, \lambda c q^n, \lambda d q^n, q^{-\frac{n}{2}}, -q^{-\frac{n}{2}}, q^{\frac{1-n}{2}}, -q^{\frac{1-n}{2}}, \\ \sqrt{\lambda}, -\sqrt{\lambda}, \frac{q^{1-n}}{b}, \frac{q^{1-n}}{c}, \frac{q^{1-n}}{d}, \lambda q^{1+\frac{n}{2}}, -\lambda q^{1+\frac{n}{2}}, \lambda q^{\frac{1+n}{2}}, -\lambda q^{\frac{1+n}{2}} \end{matrix}\,\middle|\, q, \lambda q^{n+1}\right),$$

where $bcd\lambda = q^{1-2n}$. Thus, the ${}_4\phi_3$ above is

$$\begin{aligned} &\frac{(\lambda q^n)^{-n}}{(\lambda q;q)_n}\sum_{k=0}^{\lfloor n/2\rfloor}\frac{1-\lambda q^{2k}}{1-\lambda}\,\frac{(q^{-n};q)_{2k}\,(\lambda.\lambda b q^n, \lambda c q^n, \lambda d q^n;q)_k}{(q, q^{1-n}/b, q^{1-n}/c, q^{1-n}/d;q)_k} \\ &\qquad\qquad\times\frac{(\lambda^2 q^{n+1};q)_n}{(\lambda^2 q^{n+1};q)_{2k}}\,(\lambda q^{n+1})^k \\ &= \frac{(\lambda q^n)^{-n}}{(\lambda q;q)_n}\sum_{k=0}^{\lfloor n/2\rfloor}\frac{1-\lambda q^{2k}}{1-\lambda}\,\frac{(q^{-n};q)_{2k}\,(\lambda.\lambda b q^n, \lambda c q^n, \lambda d q^n;q)_k}{(q, q^{1-n}/b, q^{1-n}/c, q^{1-n}/d;q)_k} \\ &\qquad\qquad\times(\lambda^2 q^{n+1+2k};q)_{n-2k}\,(\lambda q^{n+1})^k, \end{aligned}$$

since $(a;q)_n/(a;q)_j = (aq^j;q)_{n-j}$. In the case of (1-4), we replace q by q^2, then replace b, c and d by a, b and q^{1-2n}/ab, respectively. These choices make $\lambda = q^{1-2n}$. Hence, the ${}_4\phi_3$ in (1-4) transforms to

$$\frac{q^{-n}}{(q^{3-2n};q^2)_n}\sum_{k=0}^{\lfloor n/2\rfloor}\frac{1-q^{1-2n+4k}}{1-q^{1-2n}}\,\frac{(q^{1-2n},qa,qb,q^{2-2n}/ab;q^2)_k}{(q^2,q^{2-2n}/a,q^{2-2n}/b,qab;q^2)_k}\times q^{3k}(q^{4-2n+4k};q^2)_{n-2k}\,(q^{-2n};q^2)_{2k}. \tag{6-1}$$

Since $(q^{4-2n+2k};q^2)_{n-2k}=q^{-2\binom{n-2k}{2}}(-q^2)^{n-2k}(q^{-2};q^2)_{n-2k}$ by [Gasper and Rahman 2004, (I.8)], we find that the summand of the series above vanishes, unless $0\le n-2k\le 1$, which implies that the only nonvanishing term is when $k=\lfloor n/2\rfloor$. Computing and simplifying this last term gives the right-hand side of (1-4). □

Proof of (1-5). The use of the easily verifiable identity

$$\frac{1-abq^{-1}}{1-ab^2q^{2n-3}}\,\frac{(b,q^{3-2n}/ab;q^2)_k}{(q^{4-2n}/b,ab/q;q^2)_k}-ab^2q^{2n-3}\frac{(b,q^{3-2n}/ab;q^2)_k}{(q^{2-2n}/b,qab;q^2)_k}=\frac{(b,q^{2-2n}/ab;q^2)_k}{(q^{4-2n}/b,qab;q^2)_k},$$

gives

$$\begin{aligned}{}_4\phi_3&\left(\begin{matrix}q^{-2n},a,b,q^{3-2n}/ab\\ q^{2-2n}/a,q^{4-2n}/b,qab\end{matrix}\;\middle|\;q^2,q^2\right)\\&=\frac{1-abq^{-1}}{1-ab^2q^{2n-3}}\,{}_4\phi_3\left(\begin{matrix}q^{-2n},a,b,q^{3-2n}/ab\\ q^{2-2n}/a,q^{4-2n}/b,ab/q\end{matrix}\;\middle|\;q^2,q^2\right)\\&\qquad-ab^2q^{2n-3}{}_4\phi_3\left(\begin{matrix}q^{-2n},a,b,q^{3-2n}/ab\\ q^{2-2n}/a,q^{2-2n}/b,qab\end{matrix}\;\middle|\;q^2,q^2\right),\end{aligned}\tag{6-2}$$

yielding two balanced and nearly-poised series of the second kind on the right-hand side. Now we use the Watson transformation formula [Gasper and Rahman 2004, (III.18)] to transform the right-hand side of into ${}_8\phi_7$ series. Thus

$$\begin{aligned}{}_4\phi_3&\left(\begin{matrix}q^{-2n},a,b,\dfrac{q^{3-2n}}{ab}\\ \dfrac{q^{2-2n}}{a},\dfrac{q^{4-2n}}{b},qab\end{matrix}\;\middle|\;q^2,q^2\right)=\frac{1-abq^{-1}}{1-ab^2q^{2n-3}}\,\frac{(a/q,b/q;q^2)_n}{(1/q,ab/q;q^2)_n}\\&\times{}_8\phi_7\left(\begin{matrix}q^{1-2n},q^{-n+5/2},-q^{-n+5/2},q^{-2n},a,qa,\dfrac{b}{q},b\\ q^{-n+1/2},-q^{-n+1/2},q^3,\dfrac{q^{3-2n}}{a},\dfrac{q^{2-2n}}{a},\dfrac{q^{4-2n}}{b},\dfrac{q^{3-2n}}{b}\end{matrix}\;\middle|\;q^2,\frac{q^{6-2n}}{a^2b^2}\right)\\-ab^2q^{2n-3}\frac{(b/q,ab;q^2)_n}{(1/q,a;q^2)_n}&{}_8\phi_7\left(\begin{matrix}q^{1-2n},q^{-n+5/2},-q^{-n+5/2},q^{-2n},b,qb,\dfrac{q^{2-2n}}{ab},\dfrac{q^{3-2n}}{ab}\\ q^{-n+1/2},-q^{-n+1/2},q^3,\dfrac{q^{3-2n}}{b},\dfrac{q^{2-2n}}{b},qab,ab\end{matrix}\;\middle|\;q^2,a^2\right).\end{aligned}$$

The crucial formula to use now is the quadratic transformation formula [Gasper and Rahman 2004, (3.5.10)], that after some simplification, gives

$$
\begin{aligned}
&{}_4\phi_3\left(\begin{matrix} q^{-2n}, a, b, q^{3-2n}/ab \\ q^{2-2n}/a, q^{4-2n}/b, qab \end{matrix}\,\middle|\, q^2, q^2\right) \\
&= \frac{(q^{2-2n};q)_{2n}}{(q^{2-2n}/a;q)_{2n}} \frac{(abq^{n-2}, -abq^{n-2};q)_n}{(bq^{n-2}, -bq^{n-2};q)_n} a^{-2n} \frac{1-abq^{-1}}{1-ab^2q^{2n-3}} \\
&\quad\times \sum_{k=0}^{n} \frac{1+q^{2-2n+2k}/b}{1+q^{2-2n}/b} \frac{(-q^{2-2n}/b, q^{2-n}/b, -q^{2-n}/b, a;q)_k}{(q, q^{3-n}/b, -q^{3-n}/b, -q^{3-2n}/a;q)_k} \frac{q^{2k}(q^{-2n};q^2)_k}{a^k(q^{2-2n};q^2)_k} \\
&- abq^{3n-3} \frac{(b/q, ab, a^2;q^2)_n}{(1/q, a, a^2b^2q^2;q^2)_n} \frac{(q^{2-2n};q)_{2n}}{(q^{2-2n}/b;q)_{2n}} \\
&\quad\times \sum_{k=0}^{n} \frac{1+abq^{2k}}{1+ab} \frac{(-ab, b, abq^{n-1}, -abq^{n-1};q)_k}{(q, -qa, b, abq^{n+1}, -abq^{n+1};q)_k} \frac{q^{2k}(q^{-2n};q^2)_k}{b^k(q^{2-2n};q^2)_k}
\end{aligned}
$$

However, in each of the two series above there is the common factor

$$
\frac{(q^{2-2n};q)_{2n}(q^{-2n};q^2)_k}{(q^{2-2n};q^2)_k} = (q^{2-2n+2k};q^2)_{n-k}(q^{-2n};q^2)_k(q^{3-2n};q^2)_n,
$$

which vanishes unless $k = n$. So, the only term that survives in each is the one term with $k = n$. Combining the two terms after a lot of messy but straightforward simplifications, we obtain (1-5). □

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CHARACTERIZING ALMOST PRÜFER v-MULTIPLICATION DOMAINS IN PULLBACKS

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Let I be an ideal of an integral domain T, let $\varphi : T \to T/I$ be the projection, let D be an integral domain contained in T/I, and let $R = \varphi^{-1}(D)$. We characterize when R is an almost Prüfer v-multiplication domain, an almost valuation domain, and an almost Prüfer domain, in the context of pullbacks.

1. Introduction

Let I be an ideal of an integral domain T, let $\varphi : T \to T/I$ be the natural projection, let D be an integral domain contained in T/I, and let $k = qf(D)$ be the quotient field of D. Let $R = \varphi^{-1}(D)$ be the integral domain arising from the following pullback of canonical homomorphisms:

$$\begin{tikzcd} R \arrow[r] \arrow[d] & D \arrow[d] \\ T \arrow[r] & T/I \end{tikzcd}$$

It is well-known that $D = R/I$ and that I is a prime ideal of R. Notice that I is a common ideal of R and T, and hence T is an overring of R. We assume that R is properly contained in T, and we refer to this as a pullback diagram of type ($\triangle$). For the diagram ($\triangle$), if $qf(D) \subseteq T/I$, then we refer to this as a diagram of type ($\triangle'$). For the diagram ($\triangle$), if I is a prime ideal of T and $qf(D) = qf(T/I)$, then we refer to this as a diagram of type ($\triangle^*$). Here $qf(T/I)$ denotes the quotient field of T/I. For the diagram ($\triangle$), if $I = M$ is a maximal ideal of T, we refer to this as a diagram of type ($\triangle_M$). For the diagram ($\triangle_M$), if $qf(D) = T/M$, then we refer to this as a diagram of type ($\triangle_M^*$).

Pullbacks are an important tool in constructing interesting examples and counterexamples. They have become so important that in recent years there have been many papers devoted to ring- and ideal-theoretic properties in pullback domains.

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For more details on pullbacks, see [Mimouni 2004; Houston and Taylor 2007; Fontana and Gabelli 1996; Gilmer 1972; Gabelli and Houston 1997].

Zafrullah [1985] began a general theory of almost factoriality and introduced the notion of an almost GCD-domain. Zafrullah defined R to be an almost GCD-domain (AGCD-domain for short) if for each $a, b \in D \setminus \{0\}$, there is a positive integer $n = n(a, b)$ such that $a^n D \cap b^n D$ is principal (or equivalently, $(a^n, b^n)_v$ is principal). Anderson and Zafrullah [AZ 1991] introduced several classes of integral domains related to almost GCD-domains, including almost Bézout domains (AB-domains), almost Prüfer domains (AP-domains), and almost valuation domains (AV-domains). As in [AZ 1991], an integral domain R is an AB-domain if for each $a, b \in D \setminus \{0\}$, there is a positive integer $n = n(a, b)$ such that (a^n, b^n) is principal; while R is an AP-domain if for each $a, b \in D \setminus \{0\}$, there is a positive integer $n = n(a, b)$ such that (a^n, b^n) is invertible. Following [AZ 1991], an integral domain R is said to be an AV-domain if for each $a, b \in D \setminus \{0\}$, there is a positive integer $n = n(a, b)$ such that $a^n | b^n$ or $b^n | a^n$. Similarly, in [Li 2012] we defined an integral domain R to be an almost Prüfer v-multiplication domain (APVMD) if for each $a, b \in R \setminus \{0\}$, there is a positive integer $n = n(a, b)$ such that $a^n D \cap b^n D$ is t-invertible, or equivalently, (a^n, b^n) is t-invertible. Recall that an integral domain R is said to be a Prüfer v-multiplication domain (PVMD) if each $a, b \in R \setminus \{0\}$, (a, b) is t-invertible. The class of APVMDs includes a lot of important rings, such as AV-domains, AB-domains, AGCD-domains, AP-domains, PVMDs, and so on.

Anderson and Zafrullah [1991, Theorem 4.9] proved that D is an AB-domain (respectively, AP-domain) if and only if $R = D + Xk[X]$ is an AB-domain (respectively, AP-domain). However, we notice that the $(D + Xk[X])$-construction is a special case of the pullback of type $(\triangle_M)$. Mimouni [2004, Theorem 2.2] generalized these results and proved that for the diagram $(\triangle_M)$, R is an AP-domain if and only if T and D are AP-domains and the extension $k \subseteq T/M$ is a root extension. He also gave a similar characterization for AV-domains. Mimouni [2004, Corollary 2.6] continued to show that for the diagram $(\triangle_M)$, assuming that $D = k$ is a field, then R is an AB-domain if and only if T is an AB-domain and the extension $k \subseteq T/M$ is a root extension. In [Li 2012, Theorem 3.10], we proved that D is an APVMD if and only if $R = D + Xk[X]$ is an APVMD.

From this we notice that the characterization of AV-domains and AP-domains is known only in the context of the special pullback of type $(\triangle_M)$, and that the study of APVMDs is only in the $(D + Xk[X])$-construction, a special case of type $(\triangle_M)$. So the main purpose of this paper is to characterize APVMDs in pullbacks in greater generality and to generalize the characterization of AV-domains and AP-domains for the pullback of type $(\triangle_M)$ to that for the pullback of type $(\triangle')$.

In Section 2, we mainly prove that in the pullback of type $(\triangle_M)$, R is an APVMD if and only if D and T are APVMDs, T_M is an AV-domain, and the extension

$qf(D) \subseteq T/M$ is a root extension. Using this fact, we give Example 2.2 to show that an APVMD is not necessarily a PVMD. We also show that for the diagram $(\triangle_M^*)$, R is an APVMD if and only if D and T are APVMDs and T_M is an AV-domain. Using this result, we prove that D is an APVMD if and only if $R = D + Xk[\![X]\!]$ is an APVMD.

In Section 3, we mainly indicate that for the diagram $(\triangle')$, if T is an AV-domain, then R is an APVMD if and only if D is an APVMD and the extension $qf(D) \subseteq T/I$ is a root extension. We prove that for the diagram $(\triangle')$, R is an AV-domain if and only if T and D are AV-domains and the extension $k = qf(D) \subseteq T/I$ is a root extension. We also show that for the diagram $(\triangle')$, assuming that T is an AV-domain, then R is an AP-domain if and only if D is an AP-domain and the extension $k = qf(D) \subseteq T/I$ is a root extension.

Following [Zafrullah 1988, p. 95], assume that D is the ring of entire functions and S is the multiplicative set generated by the principal primes of D; then D is integrally closed, and hence $R = D + XD_S[X]$ is integrally closed, but $R = D + XD_S[X]$ is not a PVMD. Because an integrally closed APVMD is a PVMD by [Li 2012, Theorem 2.4], R is not an APVMD. Consider the following pullback:

$$\begin{tikzcd}
R = D + XD_S[X] \arrow[r] \arrow[d] & D \arrow[d] \\
T = D_S[X] \arrow[r] & D_S \cong T/I
\end{tikzcd}$$

Here I denotes $XD_S[X]$. The example indicates that $qf(D) = qf(T/I)$, D and T are APVMDs, I is principal in T, and $T = D_S[X]$ is a PVMD. It follows that T_I is an AV-domain by [Li 2012, Theorem 2.3]. However, R is not an APVMD. The pullback above belongs to the pullback of type $(\triangle^*)$. Therefore, for the diagram $(\triangle^*)$, without some other assumption on T, D or T/I, there is no hope of proving that R is an APVMD even when T and D are APVMDs and T_I is an AV-domain. So in Section 4, we prove that in a pullback of type $(\triangle^*)$, if $T = (I_v : I_v)$, then R is an APVMD if and only if T is an APVMD, T_I is an AV-domain, and for each nonzero prime ideal $\bar{P}$ of D, either (1) $D_{\bar{P}}$ and $T_{\varphi^{-1}(D \setminus \bar{P})}$ are AV-domains, or (2) there exists a finitely generated ideal A of D such that $A \subseteq \bar{P}$, $A^{-1} \cap E = D$, and $(\varphi^{-1}(\bar{P})T)_t = T$.

For details on star operations, see [Gilmer 1972, Sections 32 and 34].

2. Pullbacks of type $(\triangle_M)$

We begin with the characterization of APVMDs in a pullback of type $(\triangle_M)$.

Theorem 2.1. *For the diagram* $(\triangle_M)$, *R is an APVMD if and only if D and T are APVMDs*, *T_M is an AV-domain*, *and the extension $qf(D) \subseteq T/M$ is a root extension.*

Proof. ($\Rightarrow$) Assume that R is an APVMD. Let $x, y \in D \setminus \{0\}$; then $\varphi(a) = x$ and $\varphi(b) = y$ for some $a, b \in R \setminus M$. Because R is an APVMD, there is a positive integer $n = n(a, b)$ such that (a^n, b^n) is t-invertible in R. By [Wang 2006, Theorem 10.3.11], $(\varphi(a^n), \varphi(b^n))$ is t-invertible in D. Because $(x^n, y^n) = (\varphi(a)^n, \varphi(b)^n) = (\varphi(a^n), \varphi(b^n))$, it follows that (x^n, y^n) is t-invertible in D. Thus D is an APVMD. Let $c, d \in T \setminus \{0\}$. Because T and R have the same quotient field, there is an element $r \in R \setminus \{0\}$ with $rc, rd \in R$. Then $((rc)^n, (rd)^n)R$ is a t-invertible ideal of R for some positive integer n. According to [Wang 2006, Theorem 10.3.11], $((rc)^n, (rd)^n)T$ is t-invertible in T. It is well-known that $((rc)^n, (rd)^n)T = r^n(c^n, d^n)T$, so $(c^n, d^n)T$ is t-invertible in T. Therefore T is an APVMD. As we know, M is a v-ideal of R. Then R_M is an AV-domain by [Li 2012, Theorem 2.3]. By [Wang 2006, Theorem 10.2.2], we have the pullback

$$\begin{tikzcd}
R_M \arrow[r] \arrow[d] & D_{R\setminus M} \arrow[d] \\
T_M \arrow[r] & T/M
\end{tikzcd}$$

By [Mimouni 2004, Theorem 2.2], T_M and $D_{R\setminus M}$ are AV-domains and the extension $qf(D) = qf(D_{R\setminus M}) \subseteq T/M$ is a root extension.

($\Leftarrow$) Let P be a maximal t-ideal of R.

Case 1. Suppose that $M \nsubseteq P$. By [Wang 2006, Theorem 10.2.4(3)], there is a prime ideal Q of T with $P = Q \cap R$. Clearly, $M \nsubseteq Q$. In fact $P \nsubseteq M$. Because M is a v-ideal of R, M is a t-ideal of R. As the maximality, $P \nsubseteq M$. So $Q \nsubseteq M$. Hence Q is incomparable to M. According to [Fontana et al. 1998, Lemma 3.3], Q is a maximal t-ideal of T. Since T is an APVMD, T_Q is an AV-domain. By [Wang 2006, Theorem 10.2.1(6)], $R_P = T_Q$. Hence R_P is an AV-domain.

Case 2. Suppose that $M \subseteq P$. There exists a prime ideal p of D such that $P = \varphi^{-1}(p)$. Because P is a t-ideal of R, $P = P_t$. Then $\varphi^{-1}(p) = (\varphi^{-1}(p))_t = \varphi^{-1}(p_t)$ by [Wang 2006, Theorem 10.3.5(3)]. So $p = p_t$. Thus p is a t-ideal of D. Since D is an APVMD, D_p is an AV-domain. In this case, consider the following pullback:

$$\begin{tikzcd}
R_P \arrow[r] \arrow[d] & D_p \arrow[d] \\
T_M \arrow[r] & T/M
\end{tikzcd}$$

Since T_M and D_p are AV-domains and the extension $qf(D) \subseteq T/M$ is a root extension, R_P is an AV-domain by [Mimouni 2004, Theorem 2.2]. Therefore R is an APVMD. $\square$

Gabelli and Houston [1997, Theorem 4.13] showed that for the diagram $(\triangle_M)$, R is a PVMD if and only if T and D are PVMDs, $k = T/M$, and T_M is a valuation domain. Using this result and Theorem 2.1, we can easily get the following result.

Example 2.2. Let $R = K + XL[X]$, where K and L are fields, $K \subseteq L$, and for some prime p, $L^p \subseteq K$. Consider the pullback

$$\begin{tikzcd}
K + XL[X] \arrow[r] \arrow[d] & K \arrow[d] \\
L[X] \arrow[r] & L
\end{tikzcd}$$

Then R is an APVMD but not a PVMD. Thus an APVMD need not be a PVMD.

Corollary 2.3. *For the diagram* $(\triangle^*_M)$, *R is an APVMD if and only if D and T are APVMDs and T_M is an AV-domain.*

Proof. It easily follows from Theorem 2.1 and [Mimouni 2004, Lemma 2.3]. □

Corollary 2.4. *For the diagram* $(\triangle^*_M)$, *R is an AP-domain if and only if D and T are AP-domains.*

Proof. (⇒) It follows from [Mimouni 2004, Theorem 2.2].

(⇐) Let P be a maximal ideal of R.

Case 1. Suppose that $M \not\subseteq P$. By [Wang 2006, Theorems 10.2.4(3) and 10.2.1(6)], there is a prime ideal Q of T with $P = Q \cap R$ and $R_P = T_Q$. Since T is an AP-domain, T_Q is an AV-domain by [AZ 1991, Theorem 5.8]. Hence R_P is an AV-domain.

Case 2. Suppose that $M \subseteq P$. There exists a prime ideal p of D such that $P = \varphi^{-1}(p)$. Since D is an AP-domain, D_p is an AV-domain. In this case, consider the pullback

$$\begin{tikzcd}
R_P \arrow[r] \arrow[d] & D_p \arrow[d] \\
T_M \arrow[r] & T/M
\end{tikzcd}$$

Since T_M and D_p are AV-domains and $qf(D) = qf(D_p) = T/M$, R_P is an AV-domain by [Mimouni 2004, Lemma 2.3]. Therefore R is an AP-domain. □

Proposition 2.5. *For the diagram* $(\triangle_M)$, *suppose that (T, M) is a quasilocal domain and $D = k$ is a proper field of T/M. Then R is an APVMD if and only if R is an AV-domain.*

Proof. (⇐) It easily follows from their definitions.

(⇒) Assume that D is a field. Since $D = R/M$, M is a maximal ideal of R. Because T is quasilocal, R is quasilocal by [Wang 2006, Corollary 10.2.1]. Also

$M = (R : T)$ is a v-ideal of R. Hence M is the unique maximal t-ideal of R. Therefore $R = R_M$ is an AV-domain. □

In [Li 2012, Theorem 3.10], we considered the polynomial ring case and proved that D is an APVMD if and only if $R = D + Xk[X]$ is an APVMD. Similarly, we consider the power series ring case and get the following result.

Corollary 2.6. *Let D be an integral domain with quotient field k. Then D is an APVMD if and only if $R = D + Xk[\![X]\!]$ is an APVMD.*

Proof. Consider the pullback

$$\begin{tikzcd}
R = D + Xk[\![X]\!] \arrow[r] \arrow[d] & D \arrow[d] \\
T = k[\![X]\!] \arrow[r] & k = k[\![X]\!]/Xk[\![X]\!]
\end{tikzcd}$$

$T = k[\![X]\!]$ is a UFD, so T is an APVMD. The rest follows from Corollary 2.3. □

3. Pullbacks of type (Δ')

Mimouni [2004] considered the pullbacks of type ($\triangle_M$) in AP-domains and AV-domains. He proved that for the diagram ($\triangle_M$), R is an AV-domain (respectively AP-domain) if and only if T and D are AV-domains (respectively AP-domains) and the extension $k \subseteq T/M$ is a root extension. We generalize these results for the special pullback of type ($\triangle_M$) to those for the pullback of type ($\triangle'$).

Lemma 3.1. *For the diagram ($\triangle'$), if R is an AP-domain (resp. AGCD-domain), then the extension $k = qf(D) \subseteq T/I$ is a root extension.*

Proof. Assume that R is an AP-domain (resp. AGCD-domain). By way of contradiction, suppose that the extension $k \subseteq T/I$ is not a root extension. So there is $\lambda \in T/I$ such that λ^n is not in k for each positive integer n. Set $\lambda = \varphi(a)$ for some $a \in T \setminus I$. Let b be a nonzero fixed element of I. Since R is an AP-domain (resp. AGCD-domain), $((ab)^n, b^n)$ is invertible (resp. $((ab)^n, b^n)_v$ is principal) for some positive integer n. Let J denote $((ab)^n, b^n)$. Then $JJ^{-1} = R$ (resp. $J_v = cR$ for some $c \in R$). By [Wang 2006, Example 8.1.10(1)], $J^{-1} = (ab)^{-n}R \cap b^{-n}R$. Let $f \in J^{-1}$; then $f = (ab)^{-n} f_1 = b^{-n} f_2$ for some $f_1, f_2 \in R$. Thus $a^{-n} f_1 = f_2$ and so $f_1 = a^n f_2$. If f_2 is not in I, then $\varphi(f_2) \in D \setminus \{0\}$. Hence $\varphi(f_1) = \varphi(a^n f_2) = \varphi(a)^n \varphi(f_2) = \lambda^n \varphi(f_2)$. So $\lambda^n \in qf(D) = k$, a contradiction. Therefore $f_2 \in I$. So $J^{-1} \subseteq b^{-n} I$. We claim $b^{-n} I \subseteq J^{-1}$. Let $z \in I$ and $x \in J$ and write $x = \alpha(ab)^n + \beta b^n$ for some $\alpha, \beta \in R$. Then $(b^{-n}z)x = (b^{-n}z)(\alpha(ab)^n + \beta b^n) = z\alpha a^n + z\beta \subseteq I \subseteq R$, so $b^{-n}z \in J^{-1}$. Then $b^{-n} I \subseteq J^{-1}$. Therefore $b^{-n} I = J^{-1}$. So $J_v = b^n I^{-1}$. Since $JJ^{-1} = R$ (resp. $J_v = cR$), we have $1 = g_1h_1 + \cdots + g_mh_m$ for $g_1, \ldots, g_m \in J$, $h_1, \ldots, h_m \in J^{-1}$ (resp. $b^n I^{-1} = cR$). For each $i \in \{1, 2, \ldots, m\}$, write $g_i =$

$\alpha_i(ab)^n + \beta_i b^n$ and $h_i = b^{-n} f_i$, where $\alpha_i, \beta_i \in R$, $f_i \in I$. Then we have $1 = g_1h_1+\cdots+g_mh_m = (\alpha_1(ab)^n+\beta_1 b^n)(b^{-n} f_1)+\cdots+(\alpha_m(ab)^n+\beta_m b^n)(b^{-n} f_m) = (\alpha_1 a^n + \beta_1) f_1 + \cdots + (\alpha_m a^n + \beta_m) f_m \in I$, which is absurd. (Respectively, for each $y \in I^{-1}$, $TyI \subseteq yI \subseteq R$, so $Ty \in I^{-1}$, hence $T \subseteq (I^{-1} : I^{-1})$. Then $R \subset T \subseteq (I^{-1} : I^{-1}) = (b^n I^{-1} : b^n I^{-1}) = (J^{-1} : J^{-1}) = (cR : cR) = R$, which is absurd.) Therefore the extension $k \subseteq T/I$ is a root extension. □

Lemma 3.2. *For the diagram* ($\triangle'$), *assume that $D = k$ is a field. Then R is an AV-domain if and only if T is an AV-domain and the extension $k \subseteq T/I$ is a root extension.*

Proof. ($\Rightarrow$) It follows from Lemma 3.1 and the fact that T is an overring of R.

($\Leftarrow$) Let $x \in qf(R)$; then $x \in qf(T)$. Since T is an AV-domain, there is a positive integer $n = n(x)$ such that $x^n \in T$ or $x^{-n} \in T$. Assume that, for example, $x^n \in T$. If $x^n \in I$, then $x^n \in R$. If $x^n \in T \setminus I$, then $\varphi(x)^n = \varphi(x^n) \in T/I \setminus \{0\}$. Since the extension $k \subseteq T/I$ is a root extension, there is a positive integer m such that $\varphi(x^{nm}) = \varphi(x)^{nm} \in k$. Hence $x^{nm} \in \varphi^{-1}(k) = R$. It follows that R is an AV-domain. □

Theorem 3.3. *For the diagram* ($\triangle'$), *R is an AV-domain if and only if T and D are AV-domains and the extension $k = qf(D) \subseteq T/I$ is a root extension.*

Proof. ($\Rightarrow$) By [AZ 1991, Lemma 4.5], T is an AV-domain as an overring of R; and by [AZ 1991, Theorem 4.10], $D = R/I$ is an AV-domain. Also by Lemma 3.1, the extension $k = qf(D) \subseteq T/I$ is a root extension.

($\Leftarrow$) We use the fact that the diagram ($\triangle'$) splits into two parts as follows:

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & D \arrow[d] \\
R_0 = \varphi^{-1}(k) \arrow[r] \arrow[d] & k = R_0/I \arrow[d] \\
T \arrow[r] & T/I
\end{tikzcd}$$

Consider the second part of this diagram:

$$\begin{tikzcd}
R_0 \arrow[r] \arrow[d] & k \arrow[d] \\
T \arrow[r] & T/I
\end{tikzcd}$$

Since T is an AV-domain and the extension $k \subseteq T/I$ is a root extension, by Lemma 3.2 R_0 is an AV-domain. The first part of the diagram —

$$\begin{tikzcd} R \arrow[r] \arrow[d] & D \arrow[d] \\ R_0 \arrow[r] & k \end{tikzcd}$$

— is a pullback diagram of type $(\triangle^*_M)$. Since D and R_0 are AV-domains, R is an AV-domain by [Mimouni 2004, Lemma 2.3]. □

Lemma 3.4. *For the diagram* $(\triangle)$, *let* $Q(A) = \{x \in T \mid xI \subseteq A\}$ *for an ideal* A *of* R. *Then if* P *is a prime ideal of* R *and* $I \nsubseteq P$, *then* $Q(P)$ *is a prime ideal of* T, $P = Q(P) \cap R$ *and* $R_P = T_{Q(P)}$.

Proof. Let $I \nsubseteq P$, let $x, y \in T$, and let $xy \in Q(P)$. Then $xyI^2 \subseteq xyI \subseteq P$. Since $xI, yI \subseteq I \subseteq R$ and P is a prime ideal of R, we have $xI \subseteq P$ or $yI \subseteq P$. So $x \in Q(P)$ or $y \in Q(P)$. Thus $Q(P)$ is a prime ideal of T. We claim $P = Q(P) \cap R$. Because $PI \subseteq P$, we have $P \subseteq Q(P) \cap R$. Let $x \in Q(P) \cap R$; then $xI \subseteq P$. Since $I \nsubseteq P$, we have $x \in P$. Hence $Q(P) \cap R \subseteq P$. Thus $P = Q(P) \cap R$. Next we show that $R_P = T_{Q(P)}$. It easily follows that $R_P \subseteq T_{Q(P)}$. For the reverse inclusion, let $x \in T_{Q(P)}$. Then $x = z_1/z_2$ for some $z_1 \in T$, $z_2 \in T \setminus Q(P)$. Since $I \nsubseteq P$, there exists $u \in I \setminus P$. Of course $u \in I \setminus Q(P)$. Then $uz_1 \in I \subseteq R$, $uz_2 \in I \setminus Q(P) \subseteq R \setminus Q(P)$. Thus $uz_2 \in R \setminus P$. So $x = uz_1/uz_2 \in R_P$. Thus $T_{Q(P)} \subseteq R_P$, so $R_P = T_{Q(P)}$. □

Theorem 3.5. *For the diagram* $(\triangle')$, *assume that* T *is an AV-domain. Then* R *is an APVMD if and only if* D *is an APVMD and the extension* $k = qf(D) \subseteq T/I$ *is a root extension.*

Proof. As in Theorem 3.3, we consider the diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & D \arrow[d] \\ R_0 = \varphi^{-1}(k) \arrow[r] \arrow[d] & k = R_0/I \arrow[d] \\ T \arrow[r] & T/I \end{tikzcd}$$

($\Leftarrow$) Since T is an AV-domain, R_0 is an AV-domain by Lemma 3.2. Because D is an APVMD, by Corollary 2.3 R is an APVMD.

($\Rightarrow$) Assume that R is an APVMD; by Corollary 2.3 D and R_0 are APVMDs and $(R_0)_I$ is an AV-domain. Set $S = R \setminus I$. Then $R_S = R_I$ and $(R_0)_I = (R_0)_S$. By

[Houston and Taylor 2007, Lemma 1.2], consider the pullback

$$\begin{tikzcd}
(R_0)_S \arrow[r] \arrow[d] & k = k_{\varphi(S)} \arrow[d] \\
T_S \arrow[r] & (T/I)_{\varphi(S)}
\end{tikzcd}$$

As $(R_0)_S = (R_0)_I$ is an AV-domain, the extension $k \subseteq (T/I)_{\varphi(S)}$ is a root extension by Lemma 3.2. So the extension $k \subseteq T/I$ is a root extension. □

Theorem 3.6. *For the diagram* $(\triangle')$, *assume that T is an AV-domain. Then R is an AP-domain if and only if D is an AP-domain and the extension $k = qf(D) \subseteq T/I$ is a root extension.*

Proof. ($\Leftarrow$) As in Theorem 3.3, we consider the diagram

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & D \arrow[d] \\
R_0 = \varphi^{-1}(k) \arrow[r] \arrow[d] & k = R_0/I \arrow[d] \\
T \arrow[r] & T/I
\end{tikzcd}$$

Since T is an AV-domain, R_0 is an AV-domain by Lemma 3.2. Then R is an AP-domain by Corollary 2.4.

($\Rightarrow$) Assume that R is an AP-domain; then $D = R/I$ is an AP-domain by [AZ 1991, Theorem 4.10]. Also by Lemma 3.1, the extension $k \subseteq T/I$ is a root extension. □

4. Pullbacks of type (Δ*)

Lemma 4.1. *For a diagram* $(\triangle^*)$, *R is an AV-domain if and only if T and D are AV-domains.*

Proof. The proof is similar to that of Lemma 3.2.

($\Rightarrow$) If R is an AV-domain, so are its homomorphic image of D and its overring T.

($\Leftarrow$) Let $x \in qf(R)$; then $x \in qf(T)$. Since T is an AV-domain, there is a positive integer $n = n(x)$ such that $x^n \in T$ or $x^{-n} \in T$. Assume that, for example, $x^n \in T$. If $x^n \in I$, then $x^n \in R$. If $x^n \in T \setminus I$, then $\varphi(x)^n = \varphi(x^n) \in T/I \setminus \{0\} \subseteq qf(T/I) = qf(D)$. Since D is an AV-domain, there is a positive integer m such that $\varphi(x)^{nm} \in D$. Hence $x^{nm} \in \varphi^{-1}(D) = R$. It follows that R is an AV-domain. □

Proposition 4.2. *Let R be an integral domain and I a nonzero ideal of R. If R is an APVMD, then $(I_v : I_v)$ is an APVMD.*

Proof. Set $T=(I_v:I_v)$. Assume that $x, y \in T=(I_v:I_v)$. Choose a fixed element $a \in I_v$. Then $ax, ay \in I_v \subseteq R$. Since R is an APVMD, there is a positive integer $n=n(ax, ay)$ such that $((ax)^n, (ay)^n)$ is t-invertible in R. Let J denote $((ax)^n, (ay)^n)$. So $(JJ^{-1})_t=R$. There is a finitely generated ideal $H \subseteq JJ^{-1} \subseteq R$ such that $H_v=R$. By [Houston and Taylor 2007, Lemma 2.3], $(I_v:I_v)$ is t-linked over R. Then $(HT)_v=T$. So $(JJ^{-1}T)_t=T$. Thus $(a^n(x^n, y^n)J^{-1}T)_t=\big(((ax)^n, (ay)^n)J^{-1}T\big)_t=T$. So (x^n, y^n) is t-invertible in T. Therefore $T=(I_v:I_v)$ is an APVMD. $\square$

Proposition 4.3. *For a diagram* $(\triangle^*)$, *if R is an APVMD, then I is a prime t-ideal of both R and T.*

Proof. We claim R_I is an AV-domain, and thus I is a t-ideal of R. Let $x, y \in R\setminus\{0\}$. If $(x^n, y^n)(x^n, y^n)^{-1} \subseteq I$ for each positive integer n, then $((x^n, y^n)(x^n, y^n)^{-1})^{-1} \supseteq I^{-1} \supseteq T \supsetneq R$, which contradicts that R is an APVMD. Hence there exists a positive integer n such that $(x^n, y^n)(x^n, y^n)^{-1} \nsubseteq I$. Thus $((x^n, y^n)(x^n, y^n)^{-1})R_I=R_I$. So $(x^n, y^n)R_I$ is invertible in R_I. Since R_I is quasilocal, $(x^n, y^n)R_I$ is principal. Then R_I is an AV-domain. So IR_I is a maximal t-ideal of R_I. By [Kang 1989, Lemma 3.17], $I=IR_I \cap R$ is a t-ideal of R. Since $qf(D)=qf(T/I)$, we have $R_I=T_I$ by [Houston and Taylor 2007, Lemma 1.2]. So T_I is an AV-domain. Then IT_I is a maximal t-ideal of T. Therefore I is a prime t-ideal of T. $\square$

Houston and Taylor [2007, Theorem 2.8] characterized the PVMD-property in a pullback of type $(\triangle^*)$. Similarly, we are ready to study the APVMD-property in a pullback of type $(\triangle^*)$. For convenience, let E denote T/I.

Theorem 4.4. *For a diagram* $(\triangle^*)$, *assume that $T=(I_v:I_v)$. Then R is an APVMD if and only if T is an APVMD and T_I is an AV-domain, and for each nonzero prime ideal $\bar{P}$ of D, either*

(1) *$D_{\bar{P}}$ and $T_{\varphi^{-1}(D\setminus\bar{P})}$ are AV-domains, or*

(2) *there is a finitely generated ideal A of D such that $A \subseteq \bar{P}$, $A^{-1} \cap E=D$, and $(\varphi^{-1}(\bar{P})T)_t=T$.*

Proof. $(\Rightarrow)$ Assume that R is an APVMD. By Proposition 4.2, $T=(I_v:I_v)$ is an APVMD. Also, T_I is an AV-domain by Proposition 4.3. Let $\bar{P}$ be a prime ideal of D, and let $P=\varphi^{-1}(\bar{P})$.

Case 1. If P is a t-ideal of R, then R_P is an AV-domain. By [Houston and Taylor 2007, Lemma 1.2], we have the pullback

$$\begin{tikzcd}
R_P \arrow[r] \arrow[d] & D_{\varphi(R\setminus P)}=D_{\bar{P}} \arrow[d] \\
T_{R\setminus P}=T_{\varphi^{-1}(D\setminus\bar{P})} \arrow[r] & E_{\varphi(S)}=E_{D\setminus\bar{P}}
\end{tikzcd}$$

By Lemma 4.1, $D_{\bar{P}}$ and $T_{R\setminus P} = T_{\varphi^{-1}(D\setminus\bar{P})}$ are AV-domains.

Case 2. Suppose that P is not a t-ideal of R. Since R is an APVMD, it is a UMT-domain by [Li 2012, Theorem 3.8]. By [Fontana et al. 1998, Corollary 1.6], $P_t = R$. Hence there is a finitely generated ideal $J \subseteq P$ such that $J^{-1} = R$. Since T is t-linked over R by [Houston and Taylor 2007, Lemma 2.3], we have $(JT)^{-1} = T$. So $(\varphi^{-1}(\bar{P})T)_t = (PT)_t = T$. Now let $A = \varphi(J)$ and $e \in A^{-1} \cap E$. Then $\varphi(t) = e$ for some $t \in T$ and $eA \subseteq D$. Hence $\varphi^{-1}(eA) \subseteq \varphi^{-1}(D) = R$. Also, $\varphi^{-1}(eA) = \varphi^{-1}(e)\varphi^{-1}(A) = \varphi^{-1}(\varphi(t))\varphi^{-1}(\varphi(J)) \supseteq tJ$. So $tJ \subseteq R$. Then $t \in J^{-1} = R$. Thus $e = \varphi(t) \in D$. Therefore $A^{-1} \cap E = D$.

($\Leftarrow$) Let P be a maximal t-ideal of R. It suffices to show that R_P is an AV-domain.

Case 1. Assume that $I \nsubseteq P$. By Lemma 3.4, there is a prime ideal Q of T such that $P = Q \cap R$ and $R_P = T_Q$. By Proposition 4.3, we know that I is a prime t-ideal of R. Then $(PT)_t \neq T$ by [Houston and Taylor 2007, Lemma 2.6]. Hence $PT \subseteq Q_1$ for some prime t-ideal Q_1 of T. Since $T = (I_v : I_v)$ is t-linked over R by [Houston and Taylor 2007, Lemma 2.3], it follows that $(Q_1 \cap R)_t \neq R$. However, $P \subseteq Q_1 \cap R$ and P is a maximal t-ideal of R. It follows that $Q = Q_1$. Then Q is t-ideal of T. Therefore $R_P = T_Q$ is an AV-domain.

Case 2. Assume that $I \subseteq P$. Let $\bar{P}$ denote $\varphi(P)$. By way of contradiction, suppose that condition (2) of the hypothesis holds: there is a finitely generated ideal A of D such that $A \subseteq \bar{P}$, $A^{-1} \cap E = D$, and $(\varphi^{-1}(\bar{P})T)_t = (PT)_t = T$. Then $A = \varphi(J_1)$ and $(J_2T)^{-1} = T$ for some finitely generated ideals J_1, J_2 of R. Also $J_1 + J_2 \subseteq P$. Set $J = J_1 + J_2$. Then $J^{-1} \subseteq J_2^{-1}$. Let $x \in J_2^{-1}$; then $xJ_2 \subseteq R$, and hence $xJ_2T \subseteq T$. So $x \in (J_2T)^{-1} = T$. So $J^{-1} \subseteq J_2^{-1} \subseteq T$. Since $J \subseteq P$ and P is a prime t-ideal of R, then $J^{-1} \neq R$. Otherwise, if $J^{-1} = R$, then $R = J_v \subseteq P_t = P$, a contradiction. So $R \subsetneqq J^{-1}$. Therefore, there is an element $t \in J^{-1} \setminus R$ with $tJ \subseteq R$. So $\varphi(t)A \subseteq \varphi(t)\varphi(J_1) \subseteq \varphi(t)\varphi(J) = \varphi(tJ) \subseteq D$. Then $\varphi(t) \in A^{-1} \cap E = D$. So $t \in R$, a contradiction. Hence condition (1) must hold. Localize the diagram at P and consider the pullback

$$\begin{tikzcd}
R_P \arrow[r] \arrow[d] & D_{\varphi(R\setminus P)} = D_{\bar{P}} \arrow[d] \\
T_{R\setminus P} = T_{\varphi^{-1}(D\setminus\bar{P})} \arrow[r] & E_{\varphi(S)} = E_{D\setminus\bar{P}}
\end{tikzcd}$$

By Lemma 4.1, R_P is an AV-domain. Therefore, R is an APVMD. □

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WHITNEY UMBRELLAS AND SWALLOWTAILS

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Dedicated to Professor Shyuichi Izumiya on the occasion of his sixtieth birthday

We introduce map germs of pedal unfolding type and the notion of normalized Legendrian map germs. We show that the fundamental theorem of calculus provides a natural one-to-one correspondence between Whitney umbrellas of pedal unfolding type and normalized swallowtails.

1. Introduction

The map germ

$$f(x,y)=(xy,x^2,y) \tag{1}$$

is known as the normal form of Whitney umbrella, after Whitney's pioneering works [1943; 1944]. Compose the germ (1) with the coordinate transformations

$$h_s(x,y)=(x,x^2+y) \quad \text{and} \quad h_t(X,Y,Z)=(X,-Z,-Y+Z),$$

where (X,Y,Z) are the standard coordinates of the target space $\mathbb{R}^3$. This leads to the map germ

$$g(x,y)=h_t\circ f\circ h_s(x,y)=(x^3+xy,-x^2-y,y). \tag{2}$$

Set

$$\begin{aligned} G(x,y)&=\left(\int_0^x (x^3+xy)dx,\int_0^x (-x^2-y)dx,y\right) \\ &=\left(\tfrac{1}{4}x^4+\tfrac{1}{2}x^2y,-\tfrac{1}{3}x^3-xy,y\right). \end{aligned} \tag{3}$$

Compose the map germ (3) with the scaling transformations

$$H_s(x,y)=(x,\tfrac{1}{6}y) \quad \text{and} \quad H_t(X,Y,Z)=(12X,12Y,6Z)$$

to obtain the map germ

$$H_t\circ G\circ H_s(x,y)=(3x^4+x^2y,-4x^3-2xy,y), \tag{4}$$

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known as the normal form of the swallowtail [Bruce and Giblin 1992, page 129].

Two C^∞ map germs $\varphi, \psi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ are said to be $\mathscr{A}$-equivalent if there exist germs of C^∞ diffeomorphisms

$$h_s : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0) \quad \text{and} \quad h_t : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0),$$

such that $\psi = h_t \circ \varphi \circ h_s$. A C^∞ map germ $\varphi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is called a *Whitney umbrella* if it is $\mathscr{A}$-equivalent to (1); it is called a *swallowtail* if it is $\mathscr{A}$-equivalent to (4). As seen above, the Whitney umbrella (1) produces the swallowtail (4) via (2) and (3). By the converse procedure, the swallowtail (4) produces the Whitney umbrella (1).

It is impossible to produce a swallowtail by integrating (1) directly. This is because the discriminant set of (4) is not diffeomorphic to the discriminant set of

$$(x, y) \mapsto \left(\int_0^x xy\,dx, \int_0^x x^2 dx, y \right). \tag{5}$$

Note that the form (2) may be written as follows:

$$g(x, y) = \big(x(x^2 + y), -(x^2 + y), y\big) = \big(b(-x, -(x^2 + y)), y\big),$$

where $b(X, Y) = (XY, Y)$ (b stands for "blowdown").

Definition 1.1. (i) A C^∞ map germ $\varphi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ having the following form is said to be of *pedal unfolding type*.

$$\varphi(x, y) = \big(n(x, y)p(x, y), p(x, y), y\big) = \big(b(n(x, y), p(x, y)), y\big), \tag{6}$$

where $n : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ is a C^∞ function germ, such that

$$\frac{\partial n}{\partial x}(0, 0) \neq 0 \quad \text{and} \quad p : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0) \text{ is a } C^\infty \text{ function germ.}$$

(ii) For a C^∞ map germ of pedal unfolding type

$$\varphi(x, y) = \big(n(x, y)p(x, y), p(x, y), y\big),$$

set

$$\mathscr{I}(\varphi)(x, y) = \left(\int_0^x n(x, y)p(x, y)dx, \int_0^x p(x, y)dx, y \right).$$

The map germ $\mathscr{I}(\varphi) : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is called the *integration of* φ.

(iii) A C^∞ map germ $\Phi : (\mathbb{R}^m, 0) \to (\mathbb{R}^{m+1}, 0)$ is called a *Legendrian map germ* if there exists a germ of C^∞ vector field $\nu_\Phi : (\mathbb{R}^m, 0) \to T_1\mathbb{R}^{m+1}$ along Φ such that

$$\frac{\partial \Phi}{\partial x_1}(x_1, \dots, x_m) \cdot \nu_\Phi(x_1, \dots, x_m) = \cdots = \frac{\partial \Phi}{\partial x_m}(x_1, \dots, x_m) \cdot \nu_\Phi(x_1, \dots, x_m) = 0$$

and the map germ $L_\Phi : (\mathbb{R}^m, 0) \to T_1\mathbb{R}^{m+1}$ defined by

$$L_\Phi(x_1, \dots, x_m) = \bigl(\Phi(x_1, \dots, x_m), \nu_\Phi(x_1, \dots, x_m)\bigr)$$

is nonsingular. L_Φ is called a *Legendrian lift of* Φ. (Here the dot stands for the scalar product of two vectors of $T_{\Phi(x,y)}\mathbb{R}^{m+1}$, and $T_1\mathbb{R}^{m+1}$ is the unit tangent bundle of $\mathbb{R}^{m+1}$.) The C^∞ vector field ν_Φ is called a *unit normal vector field* of Φ.

(iv) A Legendrian map germ $\Phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is said to be *normalized* if it has the form

$$\Phi(x, y) = (\Phi_1(x, y), \Phi_2(x, y), y) \tag{7}$$

with

$$\frac{\partial \Phi_2}{\partial x}(0, 0) = 0 \tag{8}$$

and if, furthermore,

$$\nu_\Phi(0, 0) = \frac{\partial}{\partial X} \quad \text{or} \quad \nu_\Phi(0, 0) = -\frac{\partial}{\partial X}. \tag{9}$$

(v) For a normalized Legendrian map germ $\Phi(x, y) = (\Phi_1(x, y), \Phi_2(x, y), y)$, set

$$\mathcal{D}(\Phi)(x, y) = \left(\frac{\partial \Phi_1}{\partial x}(x, y), \frac{\partial \Phi_2}{\partial x}(x, y), y\right).$$

The map germ $\mathcal{D}(\Phi) : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is called the *differential* of Φ.

We showed in [Nishimura 2010] that any germ $\varphi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ of a one-parameter pedal unfolding of a spherical pedal curve has the form (6). Hence, a map germ φ having the form (6) is said to be of pedal unfolding type. As shown in [Nishimura 2010], not only nonsingular map germs, but also Whitney umbrellas may be realized as germs of one-parameter pedal unfoldings of spherical pedal curves. For more information on Legendrian map germs, see [Arnold et al. 1985; Izumiya 1987; Zakalyukin 1976; 1983]. Note that both (3) and (10) are normalized Legendrian map germs.

Proposition 1.2. (i) *If* $\varphi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ *is a* C^∞ *map germ of pedal unfolding type*, $\mathcal{I}(\varphi)$ *is a normalized Legendrian map germ.*

(ii) *If* $\Phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ *is a normalized Legendrian map germ*, $\mathcal{D}(\Phi)$ *is a map germ of pedal unfolding type.*

Set

$$\mathcal{W} = \{\varphi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \text{ Whitney umbrella of pedal unfolding type}\},$$

$$\mathcal{S} = \{\Phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \text{ normalized swallowtail}\}.$$

The main purpose of this paper is to show the following:

Theorem 1.3. (i) *The map* $\mathcal{I} : \mathcal{W} \to \mathcal{S}$ *defined by* $\mathcal{W} \ni \varphi \mapsto \mathcal{I}(\varphi) \in \mathcal{S}$ *is well-defined and bijective.*

(ii) *The map* $\mathcal{D} : \mathcal{S} \to \mathcal{W}$ *defined by* $\mathcal{S} \ni \Phi \mapsto \mathcal{D}(\Phi) \in \mathcal{W}$ *is well-defined and bijective.*

Incidentally, we show Theorem 1.4. A C^∞ map germ $\Phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is called a *cuspidal edge* if Φ is $\mathcal{A}$-equivalent to the following:

$$(x, y) \mapsto \left(\tfrac{1}{3}x^3, \tfrac{1}{2}x^2, y\right). \tag{10}$$

Set

$$\mathcal{N} = \{\varphi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \ \text{nonsingular map germ of pedal unfolding type}\},$$
$$\mathcal{C} = \{\Phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \ \text{normalized cuspidal edge}\}.$$

Theorem 1.4. (i) *The map* $\mathcal{I} : \mathcal{N} \to \mathcal{C}$ *defined by* $\mathcal{N} \ni \varphi \mapsto \mathcal{I}(\varphi) \in \mathcal{C}$ *is well-defined and bijective.*

(ii) *The map* $\mathcal{D} : \mathcal{C} \to \mathcal{N}$ *defined by* $\mathcal{C} \ni \Phi \mapsto \mathcal{D}(\Phi) \in \mathcal{N}$ *is well-defined and bijective.*

Any stable map germ $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is either a Whitney umbrella or nonsingular, and any Legendrian stable singularity $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is either a cuspidal edge or a swallowtail (see [Arnold et al. 1985], for example). Theorems 1.3 and 1.4 can thus be regarded as a "fundamental theorem of calculus" for stable map germs $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ and Legendrian stable singularities $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$.

Based on Theorems 1.3 and 1.4, it is natural to ask:

Question 1.5. (i) Let $\varphi_1, \varphi_2 : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be two C^∞ map germs of pedal unfolding type. Suppose that φ_1 is $\mathcal{A}$-equivalent to φ_2. Is $\mathcal{I}(\varphi_1)$ necessarily $\mathcal{A}$-equivalent to $\mathcal{I}(\varphi_2)$?

(ii) Let $\Phi_1, \Phi_2 : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be two normalized Legendrian map germs. Suppose that Φ_1 is $\mathcal{A}$-equivalent to Φ_2. Is $\mathcal{D}(\Phi_1)$ necessarily $\mathcal{A}$-equivalent to $\mathcal{D}(\Phi_2)$?

In Section 2, several preparations for the proofs of Theorems 1.3 and 1.4 and the proof of Proposition 1.2 are given. Theorems 1.3 and 1.4 are proved in Section 3 and Section 4 respectively.

2. Preliminaries

Function germs with two variables and map germs with two variables. Let $\mathcal{E}_2$ be the set of C^∞ function germs $(\mathbb{R}^2, 0) \to \mathbb{R}$, and let m_2 be the subset of $\mathcal{E}_2$

consisting of C^∞ function germs $(\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$. The set $\mathscr{E}_2$ has a natural $\mathbb{R}$-algebra structure. For a C^∞ map germ $\varphi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, let $\varphi^* : \mathscr{E}_2 \to \mathscr{E}_2$ be the $\mathbb{R}$-algebra homomorphism defined by $\varphi^*(u) = u \circ \varphi$. Set $Q(\varphi) = \mathscr{E}_2/\varphi^* m_2 \mathscr{E}_2$. Then, $Q(\varphi)$ is an $\mathbb{R}$-algebra. A special case of [Mather 1969, Theorem 2.1] follows.

Proposition 2.1. *Let $p : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be a C^∞ function germ.*

(i) *The $\mathbb{R}$-algebra $Q(p(x, y), y)$ is isomorphic to $Q(x^2, y)$ if and only if*

$$\frac{\partial p}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial^2 p}{\partial x^2}(0,0) \neq 0.$$

(ii) *The $\mathbb{R}$-algebra $Q(p(x, y), y)$ is isomorphic to $Q(x, y)$ if and only if*

$$(x, y) \mapsto (p(x, y), y)$$

is a germ of C^∞ diffeomorphism.

Definition 2.2 [Mond 1985]. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation of the form $T(s, \lambda) = (-s, \lambda)$. Two C^∞ function germs $p_1, p_2 : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ are said to be *$\mathscr{K}^T$-equivalent* if there exists a germ of C^∞ diffeomorphism

$$h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$$

of the form $h \circ T = T \circ h$, and a C^∞ function germ $M : (\mathbb{R}^2, (0, 0)) \to \mathbb{R} - \{0\}$ of the form $M \circ T = M$, such that $p_1 \circ h(x, y) = M(x, y) p_2(x, y)$.

Theorem 2.3 [Mond 1985]. *Two C^∞ map germs $\varphi_i : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ $(i = 1, 2)$ of the form*

$$\varphi_i(x, y) = (x p_i(x^2, y), x^2, y), \quad \textit{where } p_i(x^2, y) \notin m_2^\infty \quad (i = 1,\ 2)$$

are $\mathscr{A}$-equivalent if and only if the function germs $p_i(x^2, y)$ are $\mathscr{K}^T$-equivalent. Here,

$$m_2^\infty = \left\{ q : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0) \;\middle|\; \frac{\partial^{i+j} q}{\partial x^i \partial y^j}(0, 0) = 0 \textit{ for all } i, j \in \{0\} \cup \mathbb{N} \right\}.$$

From this and the Malgrange preparation theorem [Arnold et al. 1985], we have:

Corollary 2.4. *Two C^∞ map germs $\varphi_i : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ $(i = 1, 2)$ of the form*

$$\varphi_i(x, y) = (n_i(x, y) p_i(x^2, y), x^2, y),$$

where $p_i(x^2, y) \notin m_2^\infty$ and $(\partial n_i/\partial x)(0, 0) \neq 0$ for $i = 1, 2$, are $\mathscr{A}$-equivalent if and only if the function germs $p_i(x^2, y)$ are $\mathscr{K}^T$-equivalent.

Map germs of pedal unfolding type. Let $\varphi : I \times J \to \mathbb{R}^3$ be a representative of a given C^∞ map germ of pedal unfolding type, where I, J are sufficiently small intervals containing the origin of $\mathbb{R}$. We may put $\varphi(x, y) = \big(n(x, y)p(x, y), p(x, y)\big)$. Set

$$\Phi(x, y) = (\Phi_1(x, y), \Phi_2(x, y), y) = \left(\int_0^x n(x, y)p(x, y)dx, \int_0^x p(x, y)dx, y\right)$$

and

$$\tilde{\mu}_\Phi(x, y) = \frac{\partial}{\partial X} - n(x, y)\frac{\partial}{\partial Y}.$$

Since $\tilde{\mu}_\Phi(x, y) \neq 0$ for any $x \in I$ and $y \in J$, for any fixed $y \in J$ we may define the map germ $L_{\Phi,y} : (\mathbb{R}, 0) \to T_1\mathbb{R}^2$ as

$$L_{\Phi,y}(x) = \left((\Phi_1(x, y), \Phi_2(x, y)), \frac{\tilde{\mu}_\Phi(x, y)}{\|\tilde{\mu}_\Phi(x, y)\|}\right),$$

where $T_1\mathbb{R}^2$ is the unit tangent bundle of $\mathbb{R}^2$. Then, since φ is a representative of a map germ of pedal unfolding type, we have:

Lemma 2.5. *For any $y \in J$, $L_{\Phi,y} : (\mathbb{R}, 0) \to T_1\mathbb{R}^2$ is a Legendrian lift of the map germ $x \mapsto (\Phi_1(x, y), \Phi_2(x, y))$.*

This implies:

Lemma 2.6. *For any $y \in J$, the map germ $\widetilde{\Phi}_y : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ defined by $\widetilde{\Phi}_y(x) = (\Phi_1(x, y), \Phi_2(x, y))$ is a Legendrian map germ.*

Next, set

$$\tilde{\nu}_\Phi(x, y) = \tilde{\mu}_\Phi(x, y) - \left(\frac{\partial \Phi_1}{\partial y}(x, y) - n(x, y)\frac{\partial \Phi_2}{\partial y}(x, y)\right)\frac{\partial}{\partial Z}.$$

Lemma 2.7. *For any $x \in I$ and $y \in J$,*

$$\tilde{\nu}_\Phi(x, y) \cdot \frac{\partial \Phi}{\partial x}(x, y) = 0, \quad \tilde{\nu}_\Phi(x, y) \cdot \frac{\partial \Phi}{\partial y}(x, y) = 0.$$

Since $\tilde{\nu}_\Phi(x, y) \neq 0$ for any $x \in I$ and $y \in J$, we may define the map germ

$$L_\Phi : (\mathbb{R}^2, 0) \to T_1\mathbb{R}^3$$

as

$$L_\Phi(x, y) = \left(\Phi(x, y), \frac{\tilde{\nu}_\Phi(x, y)}{\|\tilde{\nu}_\Phi(x, y)\|}\right).$$

Then Lemma 2.7 implies successively:

Lemma 2.8. *$L_\Phi : (\mathbb{R}^2, 0) \to T_1\mathbb{R}^3$ is a Legendrian lift of Φ.*

Lemma 2.9. *$\Phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is a Legendrian map germ.*

Normalized Legendrian map germs. Let $\Phi : U \to \mathbb{R}^3$ be a representative of a given normalized Legendrian map germ $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$, where U is a sufficiently small neighborhood of the origin of $\mathbb{R}^2$. We assume that the origin of $\mathbb{R}^2$ is a singular point of Φ. By condition (7) of the definition of normalized Legendrian map germs, we may assume that Φ has the form

$$\Phi(x, y) = \big(\Phi_1(x, y), \Phi_2(x, y), y\big).$$

Since Φ is a representative of a Legendrian map germ, we have the following:

Lemma 2.10. *There exists a C^∞ vector field ν_Φ along Φ,*

$$\nu_\Phi(x, y) = n_1(x, y)\frac{\partial}{\partial X} + n_2(x, y)\frac{\partial}{\partial Y} + n_3(x, y)\frac{\partial}{\partial Z},$$

such that

(i) $$n_1(x, y)\frac{\partial \Phi_1}{\partial x}(x, y) + n_2(x, y)\frac{\partial \Phi_2}{\partial x}(x, y) = 0;$$

(ii) $$n_1(x, y)\frac{\partial \Phi_1}{\partial y}(x, y) + n_2(x, y)\frac{\partial \Phi_2}{\partial y}(x, y) + n_3(x, y) = 0;$$

(iii) *the map $L_\Phi : U \to T_1\mathbb{R}^3$ defined by $L_\Phi(x, y) = (\Phi(x, y), \nu_\Phi(x, y))$ is an immersion.*

Condition (9) in the definition of normalized Legendrian map germs gives:

Lemma 2.11. *For the vector field ν_Φ, $n_1(0, 0) \neq 0$ and $n_2(0, 0) = n_3(0, 0) = 0$.*

By Lemma 2.10(i) and Lemma 2.11, we have the following equality of function germs:

$$\frac{\partial \Phi_1}{\partial x}(x, y) = -\frac{n_2(x, y)}{n_1(x, y)}\frac{\partial \Phi_2}{\partial x}(x, y). \tag{11}$$

This, together with condition (8) in the definition of normalized Legendrian maps, implies that

$$\mathcal{D}(\Phi)(0, 0) = (0, 0, 0). \tag{12}$$

The next lemma is clear:

Lemma 2.12. *The function germs n and p given by*

$$n(x, y) = -\frac{n_2(x, y)}{n_1(x, y)} \quad \textit{and} \quad p(x, y) = \frac{\partial \Phi_2}{\partial x}(x, y)$$

are of class C^∞, and satisfy $\mathcal{D}(\Phi)(x, y) = \big(n(x, y)p(x, y), p(x, y), y\big)$.

Lemma 2.13. *The function germ n satisfies $n(0, 0) = 0$ and $\dfrac{\partial n}{\partial x}(0, 0) \neq 0$.*

Proof. By Lemma 2.11, we have $n(0,0)=0$. Assume, for a contradiction, that $\partial n/\partial x$ vanishes at the origin; then so does $\partial n_2/\partial x$. At the same time, by differentiating both sides of the equality in Lemma 2.10(ii) with respect to x, we have

$$n_1(0,0)\frac{\partial^2\Phi_1}{\partial x\partial y}(0,0)+\frac{\partial n_3}{\partial x}(0,0)=0. \tag{13}$$

Because Φ is a normalized Legendrian map germ such that the origin of $\mathbb{R}^2$ is a singular point of Φ, we obtain $(\partial n_3/\partial x)(0,0)\neq 0$, which together with (13) gives

$$\frac{\partial^2\Phi_1}{\partial x\partial y}(0,0)\neq 0.$$

From (8), (11), and Lemma 2.11 we have a contradiction. □

Definition 2.14. Let $\Phi:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ be a Legendrian map germ, and let ν_Φ be a unit normal vector field of Φ given in the definition of Legendrian map germs. The C^∞ function germ $LJ_\Phi:(\mathbb{R}^2,0)\to\mathbb{R}$ defined by

$$LJ_\Phi(x,y)=\det\left(\frac{\partial\Phi}{\partial x}(x,y),\ \frac{\partial\Phi}{\partial y}(x,y),\ \nu_\Phi(x,y)\right)$$

is called the *Legendrian Jacobian* of Φ.

Note that if ν_Φ satisfies the conditions of unit normal vector field of Φ, then $-\nu_\Phi$ also satisfies them. Thus, the sign of $LJ_\Phi(x,y)$ depends on the particular choice of unit normal vector field ν_Φ. The Legendrian Jacobian of Φ is also called the *signed area density function* [Saji et al. 2009b]. Although it is reasonable to call LJ_Φ the area density function from the viewpoint of investigating the singular surface $\Phi(U)$ (U is a sufficiently small neighborhood of the origin of $\mathbb{R}^2$), it is also reasonable to call it the Legendrian Jacobian from the viewpoint of investigating the singular map germ Φ.

Let $\Phi:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ be a normalized Legendrian map germ and ν_Φ a unit normal vector field of Φ. Write

$$\begin{aligned}\Phi(x,y)&=\big(\Phi_1(x,y),\Phi_2(x,y),y\big),\\ \nu_\Phi(x,y)&=n_1(x,y)\frac{\partial}{\partial X}+n_2(x,y)\frac{\partial}{\partial Y}+n_3(x,y)\frac{\partial}{\partial Z}.\end{aligned}$$

By Lemma 2.11, we may set

$$\tilde{\nu}_\Phi(x,y)=\frac{\partial}{\partial X}+\frac{n_2(x,y)}{n_1(x,y)}\frac{\partial}{\partial Y}+\frac{n_3(x,y)}{n_1(x,y)}\frac{\partial}{\partial Z}.$$

We now give a formula for the Legendrian Jacobian. We start with the cross product (vector product)

$$\frac{\partial\Phi}{\partial x}(x,y)\times\frac{\partial\Phi}{\partial y}(x,y)=\frac{\partial\Phi_2}{\partial x}(x,y)\tilde{\nu}_\Phi(x,y).$$

This gives

$$LJ_\Phi(x,y)=\frac{(\partial\Phi_2/\partial x)(x,y)}{n_1(x,y)}. \tag{14}$$

Proof of Proposition 1.2. (i) Set

$$\mathcal{I}(\varphi)=\Phi(x,y)=(\Phi_1(x,y),\Phi_2(x,y),y)\,.$$

By Lemma 2.9, Φ is a Legendrian map germ. Thus, it is sufficient to show that (8) and (9) in the definition of normalized Legendrian map germs are satisfied.

Set $\varphi(x,y)=\bigl(n(x,y)p(x,y),p(x,y),y\bigr)$. By the definition of map germs of pedal unfolding type, we have $n(0,0)=0$ and $p(0,0)=0$. It follows that

$$\frac{\partial\Phi_2}{\partial x}(0,0)=p(0,0)=0.$$

Thus, condition (8) is satisfied. By Lemma 2.8, the germ L_Φ given by

$$L_\Phi(x,y)=\left(\Phi(x,y),\frac{\tilde{\nu}_\Phi(x,y)}{\|\tilde{\nu}_\Phi(x,y)\|}\right)$$

is a germ of Legendrian lift of Φ, where

$$\tilde{\nu}_\Phi(x,y)=\frac{\partial}{\partial X}-n(x,y)\frac{\partial}{\partial Y}-\left(\frac{\partial\Phi_1}{\partial y}(x,y)-n(x,y)\frac{\partial\Phi_2}{\partial y}(x,y)\right)\frac{\partial}{\partial Z}.$$

Since $n(0,0)=0$ and

$$\frac{\partial\Phi_1}{\partial y}(0,0)=\int_0^0\frac{\partial np}{\partial y}(x,0)\,dx=0,$$

we have

$$\frac{\tilde{\nu}_\Phi(0,0)}{\|\tilde{\nu}_\Phi(0,0)\|}=\frac{\partial}{\partial X}.$$

Thus, condition (9) is satisfied, proving part (i) of the proposition.

Proposition 1.2(ii) follows from (12), Lemma 2.12, and Lemma 2.13. □

3. Proof of Theorem 1.3

Suppose that both $\mathcal{I}:\mathcal{W}\to\mathcal{S}$ and $\mathcal{D}:\mathcal{S}\to\mathcal{W}$ are well-defined. By the fundamental theorem of calculus, we have $\mathcal{D}\circ\mathcal{I}(\varphi)=\varphi$ for all $\varphi\in\mathcal{W}$, and $\mathcal{I}\circ\mathcal{D}(\Phi)=\Phi$ for all $\Phi\in\mathcal{S}$. That is, both $\mathcal{I}$ and $\mathcal{D}$ are bijective. Therefore, in order to complete the proof, it is sufficient to show that both $\mathcal{I}$ and $\mathcal{D}$ are well-defined.

Proof that $\mathcal{I}:\mathcal{W}\to\mathcal{S}$ is well-defined. Let $\varphi(x,y)=(n(x,y)p_\varphi(x,y),p_\varphi(x,y),y)$ be an element of $\mathcal{W}$. Set $\Phi=\mathcal{I}(\varphi)$. Then Φ is a normalized Legendrian map germ by Proposition 1.2. Let g be the Whitney umbrella of pedal unfolding type — see (2) in Section 1:

$$g(x, y) = (xp_g(x, y), p_g(x, y), y) = (x(x^2 + y), -x^2 - y, y).$$

Lemma 3.1. *There exists a germ of C^∞ diffeomorphism $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that h has the form $h(x, y) = (h_1(x, y), h_2(y))$ and $p_\varphi \circ h(x, y)$ is $x^2 + y$ or $-(x^2 + y)$.*

Proof. Since φ is a Whitney umbrella of pedal unfolding type, we have

$$Q(p_\varphi(x, y), y) \cong Q(\varphi) \cong Q(g) \cong Q(x^2, y).$$

Thus, we may set $p_\varphi(x, 0) = a_2x^2 + o(x^2)$ $(a_2 \neq 0)$ by Proposition 2.1. By the Morse lemma with parameters [Bruce and Giblin 1992], there exists a germ of C^∞ diffeomorphism $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that h has the form

$$h(x, y) = (h_1(x, y), h_2(y)) \quad \text{and} \quad p_\varphi \circ h(x, y) = \pm(x^2 + q(y))$$

by a certain C^∞ function germ $q : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$. Since φ is $\mathcal{A}$-equivalent to g, by Corollary 2.4, $\pm(x^2 + q(y))$ is $\mathcal{K}^T$-equivalent to p_g and thus $q : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ is a germ of C^∞ diffeomorphism. Lemma 3.1 follows. □

Set $G = \mathcal{I}(g)$. Then, G has the form of (3) from Section 1, which is a normalized swallowtail. Since G is normalized, $\partial/\partial x$ is the null vector field for G defined in [Kokubu et al. 2005; Saji et al. 2009a], that is,

$$\frac{\partial G}{\partial x}(x, y) = 0$$

holds for any (x, y) which is a singular point of G. Since G is a swallowtail, we have, by [Saji et al. 2009a, Corollary 2.5],

$$LJ_G(0, 0) = \frac{\partial LJ_G}{\partial x}(0, 0) = 0, \quad \frac{\partial^2 LJ_G}{\partial x^2}(0, 0) \neq 0, \quad Q\Big(LJ_G, \frac{\partial LJ_G}{\partial x}\Big) \cong Q(x, y).$$

On the other hand, by (14) and Lemma 3.1, there exists a germ of C^∞ diffeomorphism $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ and a C^∞ function germ $\xi : (\mathbb{R}^2, 0) \to \mathbb{R}$, such that h has the form $h(x, y) = \big(h_1(x, y), h_2(y)\big)$, $\xi(0, 0) \neq 0$ and we have

$$LJ_\Phi \circ h(x, y) = \xi(x, y) LJ_G(x, y).$$

Because $\partial/\partial x$ is the null vector field for Φ (this is because Φ is normalized), LJ_Φ satisfies

$$LJ_\Phi(0, 0) = \frac{\partial LJ_\Phi}{\partial x}(0, 0) = 0, \quad \frac{\partial^2 LJ_\Phi}{\partial x^2}(0, 0) \neq 0, \quad Q\Big(LJ_\Phi, \frac{\partial LJ_\Phi}{\partial x}\Big) \cong Q(x, y).$$

Hence, Φ is a swallowtail by [Saji et al. 2009a, Corollary 2.5], and we have proved that $\mathcal{I} : \mathcal{W} \to \mathcal{S}$ is well-defined. □

Proof that $\mathscr{D} : \mathscr{S} \to \mathscr{W}$ is well-defined.. Let Φ be an element of $\mathscr{S}$. Then, by Proposition 2.1, $\mathscr{D}(\Phi)$ is of pedal unfolding type; we must show that it is a Whitney umbrella.

Lemma 3.2. *For the Legendrian Jacobian LJ_Φ, we have*

$$LJ_\Phi(0,0) = \frac{\partial LJ_\Phi}{\partial x}(0,0) = 0, \qquad \frac{\partial^2 LJ_\Phi}{\partial x^2}(0,0) \neq 0, \qquad Q\Big(LJ_\Phi, \frac{\partial LJ_\Phi}{\partial x}\Big) \cong Q(x,y).$$

Proof. Since Φ is normalized, $\partial/\partial x$ is the null vector field. Since Φ is a swallowtail, Lemma 3.2 follows from [Saji et al. 2009a, Corollary 2.5]. □

Since $\mathscr{D}(\Phi)$ is of pedal unfolding type, there exists a C^∞ function germ $n : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ such that

$$\frac{\partial n}{\partial x}(0,0) \neq 0 \quad \text{and} \quad \frac{\partial \Phi_1}{\partial x}(x,y) = n(x,y)\frac{\partial \Phi_2}{\partial x}(x,y),$$

where $\Phi(x,y) = (\Phi_1(x,y), \Phi_2(x,y), y)$. Set $p_\varphi = \partial\Phi_2/\partial x$. By (14), Lemma 3.2, and the Morse lemma with parameters, there is a germ of C^∞ diffeomorphism $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that h has the form $h(x,y) = (h_1(x,y), h_2(y))$ and $p_\varphi \circ h(x,y) = \pm(x^2 + y)$. Then, by Corollary 2.4, $\mathscr{D}(\Phi)$ is $\mathscr{A}$-equivalent to g. □

4. Proof of Theorem 1.4

As with Theorem 1.3, it is sufficient to show that both $\mathscr{I} : \mathscr{N} \to \mathscr{C}$ and $\mathscr{D} : \mathscr{C} \to \mathscr{N}$ are well-defined.

Proof that $\mathscr{I} : \mathscr{N} \to \mathscr{C}$ is well-defined. Let $\varphi(x,y) = (n(x,y)p_\varphi(x,y), p_\varphi(x,y), y)$ be an element of $\mathscr{N}$. Set $\Phi = \mathscr{I}(\varphi)$. Then, since φ is of pedal unfolding type, Φ is a normalized Legendrian map germ by Proposition 1.2. Let g be the nonsingular map germ of pedal unfolding type defined by $g(x,y) = (x^2, x, y)$.

Lemma 4.1. *There exists a germ of C^∞ diffeomorphism $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ having the form $h(x,y) = (h_1(x,y), h_2(y))$ and such that $p_\varphi \circ h(x,y) = x$.*

Proof. Since φ is nonsingular and of pedal unfolding type, we have

$$Q(p_\varphi(x,y), y) \cong Q(\varphi) \cong Q(g) \cong Q(x,y).$$

Thus, $(p_\varphi(x,y), y)$ is a germ of C^∞ diffeomorphism by Proposition 2.1. From the form of $(p_\varphi(x,y), y)$, its inverse map germ $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ has the form $h(x,y) = (h_1(x,y), h_2(y))$. Since h is the inverse map germ of $(p_\varphi(x,y), y)$, it follows that $p_\varphi \circ h(x,y) = x$. □

Since Φ is normalized, $\partial/\partial x$ is the null vector field for Φ. By Lemma 4.1 and (14), we have

$$\frac{\partial LJ_\Phi}{\partial x}(0,0) \neq 0.$$

Thus, the null vector field $\partial/\partial x$ is transverse to $\{(x, y) \mid LJ_\Phi(x, y) = 0\}$ at $(0, 0) \in \mathbb{R}^2$. Hence, Φ is a cuspidal edge by [Kokubu et al. 2005, Proposition 1.3], showing that $\mathcal{I} : \mathcal{N} \to \mathcal{C}$ is well-defined. □

Proof that $\mathcal{D} : \mathcal{C} \to \mathcal{N}$ *is well-defined.* Let Φ be an element of $\mathcal{C}$. By Proposition 1.2, $\mathcal{D}(\Phi)$ is of pedal unfolding type.

Lemma 4.2. *The Legendrian Jacobian* LJ_Φ *satisfies*

$$LJ_\Phi(0,0) = 0 \quad \text{and} \quad \frac{\partial LJ_\Phi}{\partial x}(0,0) \neq 0.$$

Proof. Since $\partial/\partial x$ is the null vector field for Φ and Φ is a cuspidal edge, Lemma 4.2 follows from [Saji et al. 2009a, Corollary 2.5]. □

Since $\mathcal{D}(\Phi)$ is of pedal unfolding type, there exists a C^∞ function germ $n : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ such that

$$\frac{\partial n}{\partial x}(0,0) \neq 0 \quad \text{and} \quad \frac{\partial \Phi_1}{\partial x}(x, y) = n(x, y)\frac{\partial \Phi_2}{\partial x}(x, y),$$

where $\Phi(x, y) = (\Phi_1(x, y), \Phi_2(x, y), y)$. Set $p_\varphi = \partial\Phi_2/\partial x$. By Lemma 4.2 and (14), the map germ $(x, y) \mapsto (p_\varphi(x, y), y)$ is a germ of a C^∞ diffeomorphism. Thus, $\mathcal{D}(\Phi)$ is nonsingular. Since $\mathcal{D}(\Phi)(0,0) = (0,0,0)$, we have proved that $\mathcal{D} : \mathcal{C} \to \mathcal{N}$ is well-defined. □

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THE KOSZUL PROPERTY AS A TOPOLOGICAL INVARIANT AND MEASURE OF SINGULARITIES

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Cassidy, Phan and Shelton have associated to any regular cell complex X a quadratic K-algebra $R(X)$. They gave a combinatorial solution to the question of when this algebra is Koszul. The algebra $R(X)$ is a combinatorial invariant but not a topological invariant. We show that, nevertheless, the property that $R(X)$ be Koszul is a topological invariant.

In the process, we establish some conditions on the types of local singularities that can occur in cell complexes X such that $R(X)$ is Koszul, and more generally in cell complexes that are pure and connected by codimension-one faces.

1. Introduction

Let X be a finite regular cell complex of dimension d, and let $\boldsymbol{K}$ be a field. Following Cassidy, Phan and Shelton (see [Cassidy et al. 2010], henceforth abbreviated [CPS]) we will associate to X, under certain global assumptions, a quadratic K-algebra $R(X)$ (defined below). The focus of [CPS] is to determine the combinatorial properties required for this algebra to be *Koszul*. The primary focus of this paper is to show that the Koszul property is actually a topological invariant, even though the algebra is not. In the process we see that our global assumptions also imply some restrictions on singularities of appropriate spaces X.

After a definition of our two technical assumptions we can state our main theorem. Our complexes will be finite throughout. We will generally not restate this hypothesis.

Definition 1. Let X be a regular cell complex of dimension d.

(1) X is pure if X is the closure of its open d-cells.

(2) A *pure*, finite regular cell complex X, is *connected through codimension-one faces* if the space $X - X^{(d-2)}$ is path connected (where $X^{(d-2)}$ is the $(d-2)$-skeleton of X).

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Theorem 2. *Let X be a pure regular cell complex of dimension d, connected through codimension-one faces. Then $R(X)$ is Koszul (for the field $\boldsymbol{K}$) if and only if both of the following conditions hold.*

(1) $\tilde{H}_i(X; \boldsymbol{K}) = 0$ *for* $i < d$.

(2) $H_i(X, X - \{p\}; \boldsymbol{K}) = 0$ *for each* $p \in X$ *and each* $i < d$.

Because our hypotheses on the cell complex structure and on homology are obviously homeomorphism invariant, Theorem 2 shows that the Koszul property for $R(X)$ is a homeomorphism invariant. Also, $H_i(X, X - \{p\}; \boldsymbol{K})$ depends only on the open cell containing p, so (2) involves checking only finitely many conditions. We point out, however, that one does not have any nice homotopy invariance:

Example 3. There are homotopy equivalent, pure regular cell complexes X and Y of dimension three such that $R(X)$ is not Koszul but $R(Y)$ is Koszul. For instance, let Y the union of two 3-cells attached by some 2-dimensional face (so $R(Y)$ is Koszul by Theorem 2) and let X be the complex in Example 22, taken from [CPS, Example 5.9]. X is homotopy equivalent to Y since one gets a space homeomorphic to Y by collapsing the contractible subcomplex a of X to a point. But [CPS] shows that $R(X)$ is not Koszul (as one can also see by Theorem 2).

Although our argument does not make direct use of the definition of $R(X)$, we review that definition here in the interest of self-containedness.

Let P be any finite ranked poset with minimal element $\bar{0}$. For each $x \in P$ let $s_1(x) = \{y \in P \mid y < x, \operatorname{rank}(x) - \operatorname{rank}(y) = 1\}$ (the elements immediately below x in the ranked poset). We define $R(P)$ to be the quadratic $\boldsymbol{K}$-algebra on generators r_x, $x \in P - \{\bar{0}\}$ with relations

$$r_x r_y = 0 \text{ for all } y \notin s_1(x)$$

and

$$r_x \sum_{z \in s_1(x)} r_z = 0 \text{ for all } x.$$

The set of all closed cells of a regular cell complex, together with the empty set, form a finite ranked poset under set inclusion. Following [CPS], this is denoted $\bar{P}(X)$. (In this poset the rank of a cell is one more than its dimension, so that the rank of the empty set is 0).

If we assume that X is pure, we may adjoin one additional (maximal) element to the poset $\bar{P}(X)$. The resulting poset, which is denoted $\hat{P}(X)$ is still a ranked poset. If we also assume that X is connected through codimension-one faces, then $\hat{P}(X)$ has the combinatorial property known as *uniform* (cf. [Gelfand et al. 2005]). Then we define $R(X)$ to be $R(\hat{P}(X))$. While $R(\bar{P}(X))$ is always Koszul under the hypotheses that X is pure and connected through codimension-one faces (see

[CPS] and [Retakh et al. 2010]), the Koszul property for $R(X)$ is substantially more subtle. In [CPS, Theorem 5.3] a precise statement was given in terms of the combinatorial cell structure of X describing when $R(X)$ is Koszul (refer to Theorem 5 below).

We note that there are results of a similar nature about different sorts of combinatorially derived algebras in the literature. One example of results in a similar spirit but of a different nature can be found in [Reiner and Stamate 2010].

2. CPS cohomology and local homology

We fix X, a finite regular CW complex of dimension d. We begin by recalling the definitions of the groups $H_X(n,k)$ from [CPS, Section 4], which we will write as $H_k^n(X)$.

Assign orientations to each cell of X. If β is an n cell and α is an $n+1$ cell, let $[\alpha:\beta]$ be the incidence number of β in α. Because X is regular, this is either 0, 1 or -1. These incidence numbers are usually defined in the context of cellular homology so that, if $C_*(X)$ is the cellular chain complex of X and α is an $n+1$ cell,

$$d(\alpha) = \sum_{n-1 \text{ cells } \beta} [\alpha:\beta]\beta.$$

Because X is finite and we have a chosen basis for the cellular chains, (given by the cells of X) we have an isomorphism between the cellular chains and the cellular cochains of X. We consider the cochains in dimension n to be generated by the basis dual to the cellular basis for $C_n(X)$, but we will use the same notation. That is, an n-cell α when considered as a generator of $C^n(X)$ has value 1 on α considered as a generator of $C_n(X)$, and value 0 on other cells generating $C_n(X)$. With this identification, the coboundary map $\delta : C^n(X) \to C^{n+1}(X)$ is given by

$$\delta(\alpha) = \sum_{n+1 \text{ cells } \beta} [\beta:\alpha]\beta.$$

We define $C_k^n(X)$ to be the submodule of $C^n(X) \otimes C_k(X)$ generated by $\alpha \otimes \beta$, such that $\beta \subseteq \partial\alpha$ (that is, the cell associated to β is a subset of the boundary of the cell associated to α). Then, d induces a differential $C_k^n(X) \to C_{k-1}^n(X)$ and δ induces a differential $C_k^n(X) \to C_k^{n+1}(X)$.

Definition 4 [CPS, Definition 4.1]. For each k and n, let

$$L_k^n(X) = \operatorname{coker}(C_{k+1}^n(X) \xrightarrow{d} C_k^n(X)).$$

Then, $L_k^*(X)$ is a cochain complex with differential induced by δ. The CPS cohomology groups of X are defined by

$$H_k^n(X) = H^n(L_k^*(X)).$$

These cohomology groups are defined with coefficients in $\mathbb{Z}$. We write $H^n_k(X; R)$ to denote the same groups when calculated with coefficients in a commutative coefficient ring R.

Theorem 5 [CPS, Theorem 5.3]. *Let X be a pure regular cell complex of dimension d, connected by codimension-one faces. Then, the $\boldsymbol{K}$-algebra $R(X)$ is Koszul if and only if $H^n_k(X, \boldsymbol{K}) = 0$ for $0 \leq k < n < d$.*

For our purposes, it is convenient to present a reformulation of Theorem 5 in terms of relative cohomology groups involving the stars of the cells of X.

Definition 6. The *star* of a cell σ in a regular cell complex X is

$$\mathrm{st}(\sigma) = \{y \in X : y \text{ is in some open cell whose closure contains } \sigma\}.$$

We note that $\mathrm{st}(\sigma)$ is an open subset of X. We also use $\mathrm{st}^l(\sigma)$ to denote the union of the open cell σ with all open cells in $\mathrm{st}(\sigma)$ of dimension $\leq l$.

Theorem 7. *Let X be a pure regular cell complex of dimension d, connected by codimension-one faces. Then, the $\boldsymbol{K}$-algebra $R(X)$ is Koszul if and only if*

(1) $\tilde{H}^n(X, \boldsymbol{K}) = 0$ *for* $n < d$,

(2) *For every k-cell σ and $k + 1 < n < d$, $H^n(X, X - \mathrm{st}(\sigma); \boldsymbol{K}) = 0$.*

Remark 1. Theorem 7 is a reformulation of [CPS, Corollary 5.8], where the condition $k + 1 < n$ was inadvertently omitted.

As we will see in the next section, the cohomology groups $H^n(X, X - st(\sigma))$ can be replaced by the *local homology* groups $H_n(X, X - \{x\})$ for any $x \in \sigma$ (see Lemma 10). This suggests the following definition.

Definition 8. We define the set S_n (relative to the ring of coefficients R) by $x \in S_n$ if $H_i(X, X - \{x\}; R) = 0$ for $i \leq n$ and $H_{n+1}(X, X - \{x\}; R) \neq 0$.

Now we can state and prove a proposition equivalent to Theorem 2. We will leave certain technical aspects of the proof to the subsequent two sections, as well as a more extensive discussion of the structure and significance of the sets S_n.

Proposition 9. *Let X be a pure regular cell complex of dimension d which is connected through codimension-one faces. Then, $R(X)$ is Koszul if and only if*

(1) $\tilde{H}^n(X, \boldsymbol{K}) = 0$ *for* $n < d$

(2) *The sets S_k (relative to $\boldsymbol{K}$) are empty for* $0 \leq k \leq d - 2$.

Proof. In this proof all homology and cohomology groups should be computed relative to the field $\boldsymbol{K}$. We suppress this from the notation.

We must see that condition (2) of Theorem 7 and condition (2) of Proposition 9 are equivalent under the hypotheses on X and condition (1). Suppose the sets S_k are empty for $0 \leq i \leq d - 2$. Then by Lemma 10, $H_i(X, X - \mathrm{st}(\sigma)) = 0$ for every cell σ and every $i < d$. The same follows by the universal coefficient theorem for $H^i(X, X - \mathrm{st}(\sigma))$.

Conversely, assume that for some $i \leq d - 2$, the set S_i is not empty. Let m be minimal such that $S_m \neq \varnothing$. By Lemma 14 and Proposition 17, S_m is a union of cells and must contain a cell α of dimension $k < m$. By Lemma 10, $H^{m+1}(X, X - st(\alpha))$ does not vanish, contradicting (2) of Theorem 7. $\square$

3. Preliminary homotopy results

Let X be a regular cell complex of dimension d. If $x \in X$, we write $\sigma(x)$ for the unique open cell of X containing x. The following is a standard lemma of piecewise linear topology.

Lemma 10. *Given a cell* σ, $\mathrm{st}(\sigma)$ *is contractible (and in fact has a strong deformation retract to* σ*). Also, given any point* $x \in \sigma$, *there is a strong deformation retract*

$$X - \{x\} \to X - \mathrm{st}(\sigma(x)).$$

Proof. To see that $\mathrm{st}(\sigma)$ has a strong deformation retract to σ, we want a homotopy

$$H : \mathrm{st}(\sigma) \times I \to \mathrm{st}(\sigma).$$

Of course $H|_{\sigma \times I}$ will just be the projection to σ.

Now suppose H has been defined on the subset of $\mathrm{st}(\sigma)$ consisting of σ together with other open cells up through cells of dimension l. Since X was a regular cell complex, for each open $l + 1$ cell E of $\mathrm{st}(\sigma)$, H is defined on a contractible subset of the boundary, $E' \subseteq \partial E$. So H is defined on $W = E \times \{0\} \cup E' \times I$. The pair $(E \times I, W)$ has the homotopy extension property (see [Hatcher 2002, page 23]), so we use that to define H on $E \times I$.

To define our retract

$$X - \{x\} \to X - \mathrm{st}(\sigma(x)),$$

we begin by noting there is a strong deformation retract $\overline{\sigma(x)} - \{x\}$ to $\partial\overline{\sigma(x)}$. Now assume the homotopy is defined on $\mathrm{st}^l(\sigma(x)) - \{x\}$ (and of course is the identity on $X - \mathrm{st}(\sigma(x))$). Note that $\mathrm{st}^0(\sigma(x)) = \sigma(x)$.

Let E be the closure of an $l + 1$ cell of $\mathrm{st}(\sigma(x))$. Since the pair

$$((E - \{x\}) \times I, (E - \{x\}) \times \{0\} \cup (\partial E - \{x\}) \times I)$$

has the homotopy extension property, extend H across $E - \{x\}$. Continue until the homotopy is defined on all of $X - \{x\}$. $\square$

Corollary 11. *Given a cell* σ *there is a strong deformation retract*

$$X - \sigma \to X - \mathrm{st}(\sigma),$$

such that if $x \in E$ *for some cell* E, *then the image of* $x \times I$ *is in* $\overline{E}$, *and meets no cells of* $\mathrm{st}(\sigma)$ *other than* E.

Proof. Apply the strong deformation retract of Lemma 10 to the space $X - \sigma$. $\square$

As an application of Corollary 11 we have:

Corollary 12. *Let* D *be an open* n*-cell of* X *and* σ *a* 0*-cell in* ∂D. *Then,*

$$X - (\sigma \cup D) \simeq X - \sigma.$$

Proof. Apply the homotopy from Corollary 11 to the space $X - (\sigma \cup D)$. This gives a retract of $X - (\sigma \cup D)$ to $X - (\mathrm{st}(\sigma))$, and since that space is also a retract of $X - \sigma$, we get $X - (\sigma \cup D) \simeq X - \sigma$. $\square$

Proposition 13. *Let* X *be the realization of a simplicial complex* Δ, *and let* A *be a closed* i*-simplex in* Δ' (*the first barycentric subdivision of* Δ). *Let* v *be the vertex that* A *shares with an* i*-simplex of* Δ. *Then,*

$$X - \{v\} \simeq X - A.$$

Proof. In the complex given by Δ we can construct the deformation retract of $X - \{v\}$ to $X - \mathrm{st}(v)$ (where $\mathrm{st}(v)$ is defined using the simplicial complex Δ)

$$H : (X - \{v\}) \times I \to X - \{v\}$$

explicitly by using barycentric coordinates in each simplex of Δ.

Specifically, if σ is a simplex of Δ not containing v, then $H(p, t) = p$ for $p \in \sigma$.

If σ does contain v, let the vertices of σ be $v = v_0, v_1, \ldots, v_k$. Then a typical point of $\sigma - \{v\}$ is given by $sv_0 + (1-s)\sum_{i=1}^k a_i v_i$, where $\sum_{i=1}^k a_i = 1$. Then,

$$H\Big(sv_0 + (1-s)\sum_{i=1}^{k} a_i v_i, t\Big) = (1-t)sv_0 + (1 - s + ts)\sum_{i=1}^{k} a_i v_i.$$

Applying this homotopy to $X - A$ gives a deformation retract to $X - \mathrm{st}(v)$. So $X - \{v\} \simeq X - A$. $\square$

4. Singularities detected by local homology

The singular sets S_n are composed of cells of dimension less than or equal to n. We continue to assume that X is a finite regular cell complex of dimension d. Throughout this section we assume further that X is pure. Recall that we refer to $H_*(X, X - x)$ as the local homology at x. Since X is locally contractible, we can choose a contractible neighborhood of x, say U. Then, by excision, we have

$$H_*(X, X - \{x\}) \cong H_*(U, U - \{x\}) \cong \tilde{H}_{*-1}(U - \{x\}).$$

From this we see that any x in the interior of a d-cell of X has local homology $H_*(X, X-x) \cong \tilde{H}(S^d)$ and $x \in S_{d-1}$. Similarly, if x is on the boundary of exactly one d-cell then $H_*(X, X-x)=0$ and x is none of the sets S_k. So if we think of X as a singular manifold with boundary, the point with neighborhoods homeomorphic to $\mathbb{R}^d$ or the corresponding half-space is not in S_k when $k < d-1$. The sets S_i, $0 \le i \le d-2$, form a stratification of those singularities of X that are detected by local homology.

Of course it is also possible for X to be topologically singular and still have local homology zero in dimensions below d at every point. A simple but illustrative example (for $d = 1$) is the space consisting of three copies of the unit interval identified at one end point:

$$\big([0,1] \times \{a,b,c\}\big) \big/ \{(0,a) \sim (0,b) \sim (0,c)\}.$$

The identification point is a singular point and has no local homology below dimension one. This singularity is still detected by local homology of course, but not until dimension one.

As is well known, there are also spaces with singularities so that all the local homology groups are those of a manifold. A standard source of examples is the suspension of any homology sphere which isn't actually a sphere.

We begin by showing that the sets S_n put restrictions on the cell structure of X. Recall first that S_n does not depend on the cellular structure of X (Definition 8). Nevertheless, we have the following.

Lemma 14. *The set S_n is a union of open cells* (*in any cell structure on X*).

Proof. If x is in some open cell D, then $X-D \simeq X-\{x\}$ by Lemma 10 together with Corollary 11. So applying the same argument to $x' \in D$ and using the appropriate long exact sequences, we get

$$H_*(X, X-\{x\}) \cong H_*(X, X-D) \cong H_*(X, X-\{x'\}). \qquad \square$$

Lemma 15. *If $x \in S_n$ for $n < d-1$, then x must be in the interior of a cell of dimension n or lower.*

Note that this fact depends on X being pure. For example, If we take X to be the union of a two cell and a one cell at a vertex, then points in the interior of the one cell will be in S_0. Geometrically, $x \in S_n$ says that if we take a contractible neighborhood of x and remove x from that neighborhood then the resulting set is no longer n-connected.

Proof. Recall $st^k(\sigma)$ is the union of σ and the open cells of dimension k and lower that are contained in $\mathrm{st}(\sigma)$. This is the same as $\mathrm{st}(\sigma)$ within the space $X^{(k)}$ if σ is a cell of dimension less than or equal to k.

Suppose x is in the interior of a cell of dimension $k < d$. We have a commutative square of spaces

$$\begin{tikzcd}
X^{(k+1)} - \{x\} \arrow[r] \arrow[d] & X^{(k+1)} - \mathrm{st}^{(k+1)}(\sigma(x)) \arrow[d] \\
X - \{x\} \arrow[r] & X - \mathrm{st}(\sigma(x)).
\end{tikzcd}$$

The horizontal maps are homotopy equivalences by Lemma 10. The spaces on the right are subcomplexes of X, and the right hand vertical map is inclusion of the $k+1$-skeleton. So, by cellular approximation, all maps induce isomorphisms in H_i for $i < k+1$.

From the long exact sequence of a pair, it follows that

$$H_i(X^{(k+1)}, X^{(k+1)} - \{x\}) \to H_i(X, X - \{x\})$$

is an isomorphism for $i \leq k$. Now let $U = \mathrm{st}^{(k+1)}(\sigma(x))$, which is an open neighborhood of x in $X^{(k+1)}$. U consists of the open k-cell containing x and any open $k+1$ cells that have that k-cell as a face. So, U looks like a finite collection of $k+1$-cells identified along part of their boundary, and x is in that part of the common boundary. It follows that $U - \{x\}$ is homotopy equivalent to a wedge of k-spheres (one fewer than the number of $k+1$-cells attached to $\sigma(x)$).

So,

$$H_i(X^{(k+1)}, X^{(k+1)} - \{x\}) = H_i(U, U - \{x\})$$

is 0 for $i < k+1$ (and is free abelian on one fewer generator than the number of $k+1$-cells attached to $\sigma(x)$ for $i = k+1$).

It follows that x is not in S_n for $n < k$. □

See the appendix for examples where S_n contains the interiors of cells of dimension strictly smaller than n.

The implications of connectivity by codimension-one faces. We have already assumed the global topological condition: X is pure. Our final goal is to understand the effect of the extra global topological condition: connected by codimension-one faces. Under that condition we can prove a remarkable strengthening of Lemma 15 (see Proposition 17 and its Corollary). We require one technical lemma.

Lemma 16. *Let X be a pure regular cell complex of dimension d. Let $n < d$. Suppose $S_0 = \cdots = S_{n-1} = \varnothing$, $\tilde{H}_k(X) = 0$ for $k < d$, and D is an open n-cell of S_n with $S_n \cap \partial D = \varnothing$. Let $Y = X - D$.*

Let $A \subseteq \partial D$ be a subspace homeomorphic to D^i (the closed i disk) and also a subcomplex of ∂D under some cell structure on ∂D which subdivides the given cell structure.

Then, $\tilde{H}_j(Y - A) = 0$ for $j < n+1-i$.

Proof. The proof is by a double induction with the outer induction on i and the inner induction on the number of i-cells in A, which we'll denote by r.

Let i be 0. Note that $r = 1$ by our hypotheses that $A \cong D^0$. Then by Corollary 12, $Y - A = X - (A \cup D) \simeq X - A$. Since A is a single point in ∂D, and by hypothesis that point isn't in $S_0 \cup \cdots \cup S_n$, we have $\tilde{H}_j(X - A) = 0$ for $j < n + 1$.

Now suppose the lemma is established for $i - 1 \geq 0$. Consider first the following special case. Subdivide the cell complex structure on ∂D so that it is a simplicial complex. Then take the first barycentric subdivision of that simplicial complex. Let A be the closure of an i-cell in that complex, so $A \cong D^i$.

Let v be the vertex that A shares with the i-simplex (before subdivision) that A is part of. By Proposition 13, $Y - A \simeq Y - \{v\}$, which is in turn homotopy equivalent to $X - \{v\}$ by the previous case. So $\tilde{H}_j(Y - A) = 0$ for $j < n + 1$.

Now let A be as in the hypotheses, with the additional assumption that it is a subcomplex of a barycentric subdivision of a simplicial subdivision of ∂D, as in the special case above. Suppose A has $r + 1$ cells, and that the lemma is true in the case of r cells. Write $A = A' \cup A''$ where A' is a single cell, A'' has r-cells, and $A' \cap A''$ is homeomorphic to D^{i-1}.

We look at the Mayer–Vietoris sequence for

$$Y - (A' \cap A'') = (Y - A') \cup (Y - A''),$$

which gives

$$\begin{aligned} H_{j+1}(Y - A') \oplus H_{j+1}(Y - A'') \to H_{j+1}(Y - (A' \cap A'')) \to H_j(Y - A) \\ \to H_j(Y - A') \oplus H_j(Y - A''). \end{aligned}$$

By our two inductive hypotheses (on $i - 1$ and r), we have $H_{j+1}(Y - (A' \cap A'')) = 0$ for $j + 1 < n + 1 - (i - 1)$ (or $j < n + 1 - i$), and $H_j(Y - A'') = H_j(Y - A') = 0$ for $j < n + 1 - i$.

It follows that $H_j(Y - A) = 0$ for $j < n + 1 - i$. By induction on r, this holds for any A which is an appropriate subcomplex of our subdivision (of the cell structure on ∂D).

Finally, if $A \subseteq \partial D$ is any appropriate subcomplex of a subdivision of ∂D so that $A \cong D^i$, then A is also an appropriate subcomplex of a finer subdivision of ∂D which is itself a barycentric subdivision of a simplicial complex. So our special case covers this subcomplex A of ∂D. □

Proposition 17. *Let X be a complex as above. In addition, assume that $\tilde{H}_i(X) = 0$ for $i < d$, and that X is connected through codimension-one faces. If there is an $n < d - 1$ so that $S_n \neq \varnothing$, then there is some point in some such S_n which is in an open cell of dimension smaller than n.*

Proof. Let n be minimal so that $S_n \neq \varnothing$. If there is no such n, or if $n \geq d-1$, we're done. So assume $n < d-1$. By Lemmas 14 and 15 S_n must contain an open cell D of dimension n at most. If D has dimension less than n, then we are done. So assume D has dimension n. Let $Y = X - D$. From the hypothesis $\tilde{H}_k(X) = 0$ for $k < d$, we get $H_n(Y) = H_{n+1}(X, Y) \neq 0$.

We wish to prove that $S_n \cap \partial D \neq \varnothing$. Choose a sequence of subsets A^i, B^i, $i = 0, \dots, n-1$ subcomplexes of ∂D (or of some subdivision) so that

- $A^{n-1} \cup B^{n-1} = \partial D \cong S^{n-1}$,
- $A^i \cup B^i \cong S^i$,
- $A^i \cap B^i = A^{i-1} \cup B^{i-1}$.

Notice that A^0 and B^0 are distinct singleton sets.

Assume $S_n \cap \partial D = \varnothing$. Consider

$$Y = (Y - A^0) \cup (Y - B^0). \tag{18}$$

The space $Y - A^0 \simeq X - A^0$ by Corollary 12, so since the point of A^0 is not in $S_0 \cup \cdots \cup S_n$, we get $\tilde{H}_j(Y - A^0) = 0$ for $j \leq n$, and of course the same result for $\tilde{H}_J(Y - B^0)$.

Then, in the Mayer–Vietoris sequence for (18), we get

$$H_n(Y) \cong H_{n-1}(Y - (A^0 \cup B^0)).$$

We do a similar analysis for

$$Y - (A^0 \cup B^0) = (Y - A^1) \cup (Y - B^1). \tag{19}$$

We have $\tilde{H}_j(Y - A^1) = 0$ for $j < n + 1 - 1 = n$ by Lemma 16.

Then, in the Mayer–Vietoris sequence for (19), we get

$$H_{n-1}(Y - (A^0 \cup B^0)) \cong H_{n-2}(Y - (A^1 \cup B^1)).$$

Similarly, we have

$$Y - (A^{k-1} \cup B^{k-1}) = (Y - A^k) \cup (Y - B^k). \tag{20}$$

Lemma 16 tells us that $\tilde{H}_j(Y - A^k) = 0$ for $j < n + 1 - k$ (and a similar result for B^k).

So, by the Mayer–Vietoris sequence for (20), we get

$$\tilde{H}_{n-k}(Y - (A^{k-1} \cup B^{k-1})) \cong \tilde{H}_{n-k-1}(Y - (A_k \cup B_k)).$$

Assembling this information, we get

$$0 \neq H_n(Y) = \tilde{H}_0(Y - (A^{n-1} \cup B^{n-1})) = \tilde{H}_0(X - \overline{D}).$$

The hypothesis that X is connected through codimension-one faces tells us that $\tilde{H}_0(X - \overline{D}) = 0$ unless (possibly) D has codimension 1. But D was assumed to have dimension $n < d - 1$, so we have a contradiction to our assumption that $S_n \cap \partial D = \varnothing$. □

Corollary 21. *Suppose X is a pure regular cell complex of dimension d, connected through codimension-one faces, and with $\tilde{H}_i(X) = 0$ for $i < d$.*

If for each $0 \le i < d - 1$, S_i contains no cells of dimension less than i, then for $0 \le i < d - 1$, each S_i is empty.

Appendix: Examples of singularities

As is clear from the above, $x \in S_n$ does not determine the dimension of the open cell containing x. For example, S_{d-1} contains all open d cells and all interior open $d - 1$ cells. Similarly, the 3-dimensional complex X of Example 22 below has points $x \in S_1$ so that x is in an open 1-cell, and also at least one $x \in S_1$ which is in an open 0-cell.

Below we give examples of contractible cell complexes connected through codimension-one faces, where (regardless of the chosen cell structure) the singular set S_r, for some r, is composed of cells of varying dimensions.

Example 22. Let T_1 be a 3-simplex with vertices $\{v_0, \dots, v_3\}$ and T_2 be a 3-simplex with vertices $\{w_0, \dots, w_3\}$. Define X by

$$(T_1 \sqcup T_2)/\sim,$$

where the relation $\sim$ is given by identifying the 2-simplex spanned by $\{v_0, v_1, v_2\}$ linearly (preserving the order of the vertices) with that spanned by $\{w_0, w_1, w_2\}$, and by identifying the 1-simplex spanned by $\{v_0, v_3\}$ with that spanned by $\{w_0, w_3\}$ (again preserving the order of simplices).

If we just made the first identification we would have something homeomorphic to a 3-disk. With both identifications, we have something homotopy equivalent to a 3-disk since it is a homotopy equivalence to identify the contractible subcomplex consisting of the closed 1-cell that is the image of the 1-simplex spanned by $\{v_0, v_3\}$ to a point.

In this space, S_1 is the open 1-cell that is the image of the open 1-simplex spanned by $\{v_0, v_3\}$ (or equivalently by $\{w_0, w_3\}$) together with the image of v_0. The point that is the image of v_0 will be a 0-cell in any cell structure on X, so that S_1 (in this example) will always have points belonging to 0-cells.

Example 23. We can mimic Example 22 in higher dimensions. For example to create a space of dimension 4 such that S_1 must necessarily contain both points of 1-cells and 0-cells, we can define X as follows. Let T_1 be a 4-simplex with vertices

$\{v_0, \dots, v_4\}$, and let T_2 be a 4-simplex with vertices $\{w_0, \dots, w_4\}$.

$$X = (T_1 \sqcup T_2)/\sim,$$

where the relation $\sim$ is given by linearly identifying the 3-simplex spanned by $\{v_0, v_1, v_2, v_3\}$ (while preserving the order of the vertices) with that spanned by $\{w_0, w_1, w_2, w_3\}$, and by identifying the 1-simplex spanned by $\{v_0, v_4\}$ with that spanned by $\{w_0, w_4\}$ (again preserving the order of simplices).

Then S_1 is analogous to the previous example; it contains the image of the 1-simplex spanned by $\{v_0, v_4\}$ together with the image of v_0. $S_2 = \varnothing$. As in the previous example, the point that is the image of v_0 will be a 0-cell in any cell structure on X, and the rest of the points of S_1 will be in 1-cells or 0-cells.

Example 24. We can also create a complicated S_2.

Let T_1 be a 4-simplex with vertices $\{v_0, \dots, v_4\}$, and let T_2 be a 4-simplex with vertices $\{w_0, \dots, w_4\}$.

$$X = (T_1 \sqcup T_2)/\sim,$$

where the relation $\sim$ is given by linearly identifying the 3-simplex spanned by $\{v_0, v_1, v_2, v_3\}$ (while preserving the order of the vertices) with that spanned by $\{w_0, w_1, w_2, w_3\}$, and by identifying the 2-simplex spanned by $\{v_0, v_1, v_4\}$ with that spanned by $\{w_0, w_1, w_4\}$ (preserving the order of simplices).

Then $S_1 = \varnothing$, but S_2 consisted of the open 2-cell that is the image of the simplex $\{v_0, v_1, v_4\}$ together with the open 1-cell that is the image of the simplex $\{v_0, v_1\}$. This subset of X will be a union of a nonzero number of open 2-cells and a nonzero number of open 1-cells in any cell complex on X.

Example 25. It is also possible to create a complex X of dimension 4, where S_2 will be a union of a nonzero number of 2-cells, a nonzero number of 1-cells and a nonzero number of 0-cells for any cell structure on X.

We begin by creating the subcomplex most of which will become S_2. We will glue three 2-simplices together along a common edge. So let A be the simplicial complexes with vertices a, b, c, d, e, 2-simplices $\{a, d, e\}$, $\{b, d, e\}$, $\{c, d, e\}$. This determines the 1-simplices, and, thus, the 1-simplex common to the three 2-simplices is $\{d, e\}$. This is illustrated by the three shaded 2-simplices in Figure 1.

Next we attach three 4-simplices to A by attaching adjacent 2-faces (sharing a 1-face) of each 4-simplex to pairs of 2-simplices in A. Let T_1 be the 4-simplex with vertices $\{u_0, \dots, u_3\}$, let T_2 have vertices $\{v_0, \dots, v_3\}$ and let T_3 have vertices $\{w_0, \dots, w_3\}$.

We identify the 2-face spanned by u_0, u_1, u_2 with the simplex e, d, a, and the 2-face spanned by u_0, u_1, u_3 with the simplex e, d, c (preserving the given order in both cases).

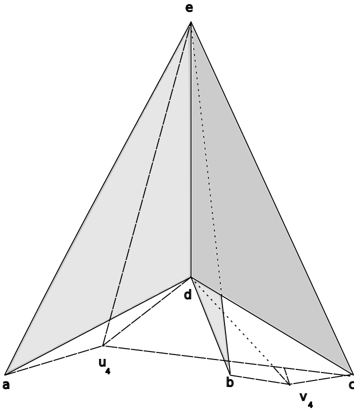

Figure 1. A with $\{u_0, \dots, u_3\}$ and $\{v_0, \dots, v_3\}$ attached.

Similarly for v_0, v_1, v_2 we use e, d, c and for v_0, v_1, v_3 we use e, d, b. Finally, for w_0, w_1, w_2 we use e, d, b and for w_0, w_1, w_3 we use e, d, a. We sketch part of this complex in Figure 1, but note that we have no realistic way to sketch the 4-simplices involved, so we are only showing a two dimensional sketch of a three dimensional picture of the space.

Of course this is contractible, and S_2 consists of the three open 2-simplices from A together with the open 1-simplex e, d. But this space is not connected through codimension-one faces. To fix this, we add a last 4-simplex with vertices $\{x_0, \dots, x_4\}$. We identify the face with vertices $x_0, \dots, x_3$ with u_1, u_2, u_3, u_4, the face x_0, x_2, x_3, x_4 with v_1, v_2, v_3, v_4 and the face x_0, x_1, x_3, x_4 with w_1, w_3, w_2, w_4.

We'll call the resulting space X. X is now dimension 4, contractible and connected through codimension-one faces. $S_0 = S_1 = \varnothing$ and S_2 is the union of the open 2-cells of A together with the open 1-cells $\{a, d\}$, $\{b, d\}$, $\{c, d\}$ and the 0-cell $\{d\}$. In any cell structure on X, $\{d\}$ will be a zero cell, and all but finitely many points of the open 1-cells we just listed will be in open 1-cells. Of course almost all points in the interiors of the 2-cells listed will be in open 2-cells.

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A COMPLETELY POSITIVE MAP ASSOCIATED WITH A POSITIVE MAP

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We show that each positive map from $B(K)$ to $B(H)$ is a scalar multiple of a map of the form $\mathrm{Tr} - \psi$ with ψ completely positive. This is used to give necessary and sufficient conditions for maps to be $\mathscr{C}$-positive for a large class of mapping cones; in particular, we apply the results to k-positive maps.

Introduction

In [Skowronek and Størmer 2010], we studied several norms on positive maps from $B(K)$ into $B(H)$, where K and H are finite-dimensional Hilbert spaces. These norms were very useful in the study of maps of the form $\mathrm{Tr} - \lambda\psi$, where Tr is the usual trace on $B(K)$, $\lambda > 0$, and ψ is a completely positive map of $B(K)$ into $B(H)$. Herein we shall see that every positive map is a positive scalar multiple of a map of the above form with $\lambda = 1$; hence the results in that reference are applicable to all positive maps. In particular, they yield a simple criterion for some maps to be k-positive but not $(k+1)$-positive. As an illustration, we give a new proof that the Choi map of $B(\mathbb{C}^3)$ into itself is atomic, that is, not the sum of a 2-positive and a 2-copositive map.

$\mathscr{C}$-positive maps

Let K and H be finite-dimensional Hilbert spaces. We denote by $B(B(K), B(H))$ (resp. $B(B(K), B(H))^+$) the linear (resp. positive linear) maps of $B(K)$ into $B(H)$. In the case $K = H$, we write $P(H) = B(B(H), B(H))^+$. Following [Størmer 1986], we say that a closed cone $\mathscr{C} \subset P(H)$ is a *mapping cone* if $\alpha \circ \phi \circ \beta \in \mathscr{C}$ for all $\phi \in \mathscr{C}$ and $\alpha, \beta \in CP$ — the completely positive maps in $P(H)$. A map ϕ in $B(B(K), B(H))$ defines a linear functional $\tilde{\phi}$ on $B(K) \otimes B(H)$, identified with $B(K \otimes H)$ in the sequel, by $\tilde{\phi}(a \otimes b) = \mathrm{Tr}(\phi(a) b^t)$, where Tr is the usual trace on $B(H)$ and t denotes the transpose. Let $P(B(K), \mathscr{C})$ denote the closed cone

$$P(B(K), \mathscr{C}) = \{a \in B(K \otimes H) : \iota \otimes \alpha(a) \geq 0 \;\text{ for all }\; \alpha \in \mathscr{C}\},$$

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where ι denotes the identity map on $B(K)$. Then a map $\phi \in B((B(K), B(H))$ is said to be *$\mathscr{C}$-positive* if $\tilde{\phi}$ is positive on $P(B(K), \mathscr{C})$. We denote by $\mathscr{P}_{\mathscr{C}}$ the cone of $\mathscr{C}$-positive maps.

If (e_{ij}) is a complete set of matrix units for $B(K)$, then the *Choi matrix* for a map ϕ is

$$C_\phi = \sum e_{ij} \otimes \phi(e_{ij}) \in B(K \otimes H).$$

By [Størmer 2008; 2009], the transpose C_ϕ^t of C_ϕ is the density operator for $\tilde{\phi}$, and by [Choi 1975], ϕ is completely positive if and only if $C_\phi \geq 0$ if and only if $\tilde{\phi} \geq 0$ as a linear functional on $B(K \otimes H)$. When $\mathscr{C} = CP$, we have $P(B(K), CP) = B(K \otimes H)^+$, so ϕ is CP-positive if and only if ϕ is completely positive.

If $\mathscr{C}_1 \subset \mathscr{C}_2$ are two mapping cones on $B(H)$, then $P(B(K), \mathscr{C}_1) \supset P(B(K), \mathscr{C}_2)$, because if $\iota \otimes \alpha(a) \geq 0$ for all $\alpha \in \mathscr{C}_2$, then the same inequality holds for all $\alpha \in \mathscr{C}_1$. Thus $\tilde{\phi} \geq 0$ on $P(B(K), \mathscr{C}_1)$ implies $\tilde{\phi} \geq 0$ on $P(B(K), \mathscr{C}_2)$, so $\mathscr{P}_{\mathscr{C}_1} \subset \mathscr{P}_{\mathscr{C}_2}$.

Let $\mathscr{C}$ be a mapping cone on $B(H)$. Let $\mathscr{P}_{\mathscr{C}}^o$ be the *dual cone* of $\mathscr{P}_{\mathscr{C}}$ defined as

$$\mathscr{P}_{\mathscr{C}}^o = \{\phi \in B(B(K), B(H)) : \operatorname{Tr}(C_\phi C_\psi) \geq 0 \text{ for all } \psi \in \mathscr{P}_{\mathscr{C}}\}.$$

Thus, if $\mathscr{C}_1 \subset \mathscr{C}_2$ then $\mathscr{P}_{\mathscr{C}_1}^o \supset \mathscr{P}_{\mathscr{C}_2}^o$. In the particular case when $\mathscr{C} \supset CP$, we thus get $\mathscr{P}_{\mathscr{C}}^o \subset \mathscr{P}_{CP}^o = CP(K, H)$ — the completely positive maps of $B(K)$ into $B(H)$.

As in [Skowronek and Størmer 2010], $\mathscr{C}$ defines a norm on $B(B(K), B(H))$ by

$$\|\phi\|_{\mathscr{C}} = \sup\{|\operatorname{Tr}(C_\phi C_\psi)| : \psi \in \mathscr{P}_{\mathscr{C}}^o, \operatorname{Tr}(C_\psi) = 1\}.$$

In the special case when $\mathscr{C} \supset CP$, it follows that

$$\|\phi\|_{\mathscr{C}} = \sup |\rho(C_\phi)|,$$

where the sup is taken over all states ρ on $B(K \otimes H)$ with density operator C_ψ with $\psi \in \mathscr{P}_C^o$. Let $\phi \in B(B(K), B(H))$ be a self-adjoint map, that is, $\phi(a)$ is self-adjoint for a self-adjoint. Then C_ϕ is a self-adjoint operator, and so is a difference $C_\phi^+ - C_\phi^-$ of two positive operators with orthogonal supports. Let $c \geq 0$ be the smallest positive number such that $c1 \geq C_\phi$. Then $c = \|C_\phi^+\|$. Hence, if $c \neq 0$, there exists a map $\phi_{cp} \in B(B(K), B(H))$ such that the Choi matrix for ϕ_{cp} equals $1 - c^{-1} C_\phi$, which is a positive operator. Thus, if we let Tr denote the map $x \mapsto \operatorname{Tr}(x)1$, then ϕ_{cp} is completely positive, and $c^{-1}\phi = \operatorname{Tr} - \phi_{cp}$, since $C_{\operatorname{Tr}} = 1$, as is easily shown. Combining this discussion with [Skowronek and Størmer 2010, Proposition 2], we get the following theorem.

Theorem 1. *Let ϕ be a self-adjoint map of $B(K)$ into $B(H)$. Then if $-\phi$ is not completely positive, we have:*

(i) *There exists a completely positive map $\phi_{cp} \in B(B(K), B(H))$ such that*

$$\|C_\phi^+\|^{-1} \phi = \operatorname{Tr} - \phi_{cp}.$$

(ii) *If $\mathscr{C}$ is a mapping cone on $B(H)$ containing CP, then ϕ is $\mathscr{C}$-positive if and only if*

$$1 \geq \|\phi_{cp}\|_{\mathscr{C}} = \sup \rho(C_{\phi_{cp}}),$$

where the sup *is taken over all states ρ on $B(K \otimes H)$ with density operator C_ψ with $\psi \in \mathscr{P}^o_{\mathscr{C}}$.*

We did not need to take the absolute value of $\rho(C_{\phi_{cp}})$ because $C_{\phi_{cp}} \geq 0$ and $\psi \in \mathscr{P}^o_{\mathscr{C}} \subset CP$.

We next spell out the theorem for some well-known mapping cones. Recall that a map ϕ is *decomposable* if $\phi = \phi_1 + \phi_2$ with ϕ_1 completely positive and ϕ_2 copositive, that is, $\phi_2 = t \circ \psi$ with ψ completely positive. Also recall that a state ρ on $B(K \otimes H)$ is a *PPT -state* if $\rho \circ (\iota \otimes t)$ is also a state.

Corollary 2. *Let $\phi \in B(B(K), B(H))$ be a self-adjoint map. Then we have*:

(i) *ϕ is positive if and only if $\rho(C_{\phi_{cp}}) \leq 1$ for all separable states ρ on $B(K \otimes H)$.*

(ii) *ϕ is decomposable if and only if $\rho(C_{\phi_{cp}}) \leq 1$ for all PPT-states ρ on $B(K \otimes H)$.*

(iii) *ϕ is completely positive if and only if $\rho(C_{\phi_{cp}}) \leq 1$ for all states ρ on $B(K \otimes H)$.*

Proof. (i) That ϕ is positive is the same as saying that ϕ is $P(H)$-positive. Since the dual cone of $P(H)$ is the cone of separable states, (i) follows.

(ii) A state ρ is PPT if and only if its density operator is of the form C_ψ with ψ a map that is both positive and copositive [Størmer 2008, Proposition 4]. But the dual of those maps is the cone of decomposable maps [Skowronek et al. 2009]. Thus (ii) follows from the theorem.

(iii) This follows because the dual cone of the completely positive maps is the cone of completely positive maps, and the density operator for a state is positive; hence the corresponding map ψ is completely positive. $\square$

k-positive maps

A map $\phi \in B(B(K), B(H))$ is said to be *k-positive* if

$$\phi \otimes \iota \in B(B(K \otimes L), B(H \otimes L))^+$$

whenever L is a k-dimensional Hilbert space. The k-positive maps in $P(H)$ form a mapping cone P_k containing CP. Denote by $P_k(K, H)$ the cone of k-positive maps in $B(B(K), B(H))$. Then we have (see also [Itoh 1987]):

Lemma 3. $\mathscr{P}_{P_k} = P_k(K, H)$.

Proof. We have $P_k^o = SP_k$, the k-superpositive maps in $P(H)$, which is the mapping cone generated by maps of the form AdV defined by $AdV(a) = VaV^*$, where

$V \in B(H)$, rank $V \leq k$ [Skowronek et al. 2009]. By [Størmer 2009], the dual cone of $\mathcal{P}_{P_k^o}$ is given by

$$\mathcal{P}^o_{P_k^o} = \{\phi \in B(B(K), B(H)) : AdV \circ \phi \in CP(K, H) \text{ for all } V \in B(H), \text{rank } V \leq k\}.$$

By [Skowronek 2010, Theorem 3] or [Skowronek and Størmer 2010, Theorem 2], it follows that $\mathcal{P}^o_{P_k^o} = P_k(K, H)$. By [Størmer 1986, Theorem 3.6], $\mathcal{P}_{P_k}$ is generated by maps of the form $\alpha \circ \beta$ with $\alpha \in P_k, \beta \in CP(K, H)$. Let $AdV \circ \gamma$, $AdV \in SP_k, \gamma \in CP(K, H)$ be a generator for $\mathcal{P}_{P_k^o}$. Then

$$\mathrm{Tr}(C_{\alpha\circ\beta} C_{AdV\circ\gamma}) = \mathrm{Tr}(C_{AdV^*\circ\alpha\circ\beta} C_\gamma) \geq 0,$$

since $AdV^* \circ \alpha$ is completely positive because $\alpha \in P_k$ and rank $V \leq k$. Since the above inequality holds for the generators of the two cones, it follows that $\mathcal{P}_{P_k} = \mathcal{P}^o_{P_k^o} = P_k(K, H)$, completing the proof of the lemma. ☐

It follows from the above description of $\mathcal{P}^o_{P_k}$ that the states with density operators C_ψ, $\psi \in \mathcal{P}^o_{P_k}$, are the same as the vector states generated by vectors in the Schmidt class $S(k)$, that is, the vectors $y = \sum_{i=1}^k x_i \otimes y_i$, $x_i \in K$, $y_i \in H$, where the x_i and y_i are not necessarily all $\neq 0$.

Theorem 4. *Let* $\phi \in B(B(K), B(H))^+$. *Then we have*:

(i) ϕ *is k-positive if and only if* $\sup_{x \in S(k), \|x\|=1} (C_{\phi_{cp}} x, x) \leq 1$.

(ii) *Suppose* $k < \min(\dim K, \dim H)$, *and that there exists a unit vector* $y = \sum_{i=1}^k x_i \otimes y_i \in S(k)$ *such that* $y \perp C_\phi y \notin X \otimes Y$, *where* $X = \mathrm{span}(x_i)$, $Y = \mathrm{span}(y_i)$. *Then* ϕ *is not* $(k+1)$*-positive.*

In order to prove the theorem we first prove a lemma.

Lemma 5. *Let A be a self-adjoint operator in* $B(K \otimes H)$. *Suppose* $y = \sum_{i=1}^k x_i \otimes y_i$ *satisfies* $(Ay, y) = 1$, *and* $Ay \notin X \otimes Y$ *with* X, Y *as in Theorem 4. Then there exist a unit product vector* $x \perp X \otimes Y$ *and* $s \in (0, 1)$ *such that*

$$\big(A(sx + (1-s^2)^{1/2} y), sx + (1-s^2)^{1/2} y\big) > 1.$$

Proof. Because $Ay \notin X \otimes Y$, there exists a product vector $x \perp X \otimes Y$ such that $\mathrm{Re}(x, Ay) > 0$. Let $s \in (-1, 1)$ and $t = t(s) = (1-s^2)^{1/2}$, and let f denote the function

$$f(s) = (A(sx + ty),\ s + ty) = s^2(Ax, x) + t^2(Ay, y) + 2st\ \mathrm{Re}(Ax, y).$$

Because $(Ay, y) = 1$, we get

$$f'(0) = 2\,\mathrm{Re}(Ax, y) > 0.$$

Therefore, for $s > 0$ and near 0 we have $(A(sx + ty),\ s + ty) > f(0) = 1$, proving the lemma. ☐

Proof of Theorem 4. (i) is a direct consequence of Theorem 1, since, as noted in the proof of Lemma 3, the vector states ω_x with $x \in S(k)$ generate the set of states with density operators C_ψ with $\psi \in \mathcal{P}^o_{P_k}$.

(ii) By Theorem 1, we have $C_{\phi_{cp}} = 1 - \|C^+_\phi\|^{-1} C_\phi$, so that $(C_{\phi_{cp}} y, y) = 1$, using the assumption that $C_\phi y \perp y$. Furthermore, $C_{\phi_{cp}} y = y - \|C^+_\phi\|^{-1} C_\phi y$. Since $C_\phi y \notin X \otimes Y$, we have $C_{\phi_{cp}} y \notin X \otimes Y$. Thus by Lemma 5, there exist a unit product vector $x \perp X \otimes Y$ and $s, t = (1-s^2)^{1/2} > 0$ such that $(C_{\phi_{cp}}(sx+ty), sx+ty) > 1$. Since $sx + ty$ is a unit vector in $S(k+1)$, ϕ is not $(k+1)$-positive by part (i), completing the proof of the theorem. □

Example. We illustrate the above results by an application to the Choi map $\phi \in B(B(C^3), B(C^3))$ defined by

$$\phi((x_{ij})) = \begin{bmatrix} x_{11}+x_{33} & -x_{12} & -x_{13} \\ -x_{21} & x_{11}+x_{22} & -x_{23} \\ -x_{31} & -x_{32} & x_{22}+x_{33} \end{bmatrix}.$$

We have $C_{t\circ\phi} = (\iota \otimes t) C_\phi$. So if $y = x \otimes x$ with $x = 3^{-1/2}(1, 1, 1) \in C^3$, then $(C_\phi y, y) = (C_{t\circ\phi} y, y) = 0$, and $C_\phi y \neq 0 \neq C_{t\circ\phi} y$. Hence, by Theorem 4, neither ϕ nor $t \circ \phi$ is 2-positive, that is, ϕ is neither 2-positive nor 2-copositive. Since ϕ is an extremal positive map of $B(C^3)$ into itself [Choi and Lam 1977], ϕ cannot be the sum of a 2-positive and a 2-copositive map, and hence ϕ is atomic, a result first proved in [Tanahashi and Tomiyama 1988], and then extended to more general maps by others (see [Ha 1998] for references).

The Choi map ϕ also yields an example of a PPT-state on $B(C^3) \otimes B(C^3)$, which is not separable. Indeed, in [Størmer 1982] we gave an example of a positive matrix in A in $B(C^3) \otimes B(C^3)$ such that its partial transpose $t \otimes \iota(A)$ is also positive, and that $\phi \otimes \iota(A)$ is not positive. Then A cannot be of the form $\sum A_i \otimes B_i$ with A_i and B_i positive, and hence the state $\rho(x) = \operatorname{Tr}(A)^{-1} \operatorname{Tr}(Ax)$ is PPT but not separable. An example of a PPT state on $B(C^3) \otimes B(C^3)$ that is not separable was later exhibited in [Horodecki 1997].

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CLASSIFICATION OF EMBEDDED PROJECTIVE MANIFOLDS SWEPT OUT BY RATIONAL HOMOGENEOUS VARIETIES OF CODIMENSION ONE

KIWAMU WATANABE

We give a classification of embedded smooth projective varieties swept out by rational homogeneous varieties whose Picard number and codimension are one.

1. Introduction

A central problem in the theory of polarized varieties is to classify smooth projective varieties admitting special varieties A as ample divisors. In [Watanabe 2008], we investigated this problem in the case where A is a homogeneous variety. On the other hand, related to the classification problem of polarized varieties, several authors have studied the structure of embedded projective varieties swept out by special varieties [Beltrametti and Ionescu 2008; Muñoz and Solá Conde 2009; Sato 1997; Watanabe 2010]. Inspired by these results, we give a classification of embedded smooth projective varieties swept out by rational homogeneous varieties whose Picard number and codimension are one. Our main result is:

Theorem 1.1. *Let $X \subset \mathbb{P}^N$ be a complex smooth projective variety of dimension $n \geq 3$ and A an $(n-1)$-dimensional rational homogeneous variety with* $\mathrm{Pic}(A) \cong \mathbb{Z}[\mathcal{O}_A(1)]$. *Assume that X satisfies either of the following properties.*

(a) *Through a general point $x \in X$, there is a subvariety $Z_x \subset X$ such that $(Z_x, \mathcal{O}_{Z_x}(1))$ is isomorphic to $(A, \mathcal{O}_A(1))$.*

(b) *There is a subvariety $Z \subset X$ such that $(Z, \mathcal{O}_Z(1))$ is isomorphic to $(A, \mathcal{O}_A(1))$ and the normal bundle $N_{Z/X}$ is nef.*

Then we have one of the following:

(i) *X is a projective space $\mathbb{P}^n$;*

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(ii) *X is a quadric hypersurface Q^n;*

(iii) *X is the Grassmannian of lines $G(1, \mathbb{P}^m)$;*

(iv) *X is an E_6 variety $E_6(\omega_1)$, where $E_6(\omega_1) \subset \mathbb{P}^{26}$ is the projectivization of the highest weight vector orbit in the 27-dimensional irreducible representation of a simple algebraic group of Dynkin type E_6; or*

(v) *X admits an extremal contraction of a ray $\varphi : X \to C$ to a smooth curve whose general fibers are projectively equivalent to $(A, \mathcal{O}_A(1))$.*

In cases (i)–(iv), *the corresponding rational homogeneous variety A is one of those in Theorem 2.2.*

We outline the proof of Theorem 1.1. A significant step is to show the existence of a covering family $\mathcal{H}$ of lines on X induced from lines on rational homogeneous varieties of codimension one (Claim 3.2). Then we see that the rationally connected fibration associated to $\mathcal{H}$ is an extremal contraction of the ray $\mathbb{R}_{\geq 0}[\mathcal{H}]$. By applying results from [Watanabe 2008], we obtain our theorem. In this paper, we work over the field of complex numbers.

2. Preliminaries

We denote a simple linear algebraic group of Dynkin type G simply by G, and for a dominant integral weight ω of G, the minimal closed orbit of G in $\mathbb{P}(V_\omega)$ by $G(\omega)$, where V_ω is the irreducible representation space of G with highest weight ω. For example, $E_6(\omega_1)$ is the minimal closed orbit of an algebraic group of type E_6 in $\mathbb{P}(V_{\omega_1})$, where ω_1 is the first fundamental dominant weight in the standard notation of Bourbaki [1968]. For any rational homogeneous variety X of Picard number one, there exists a simple linear algebraic group G and a dominant integral weight ω of G such that X is isomorphic to $G(\omega)$. A rational homogeneous variety A is a Fano variety, that is, the anticanonical divisor of A is ample. If the Picard number of A is one, we have $\mathrm{Pic}(A) \cong \mathbb{Z}[\mathcal{O}_A(1)]$, where $\mathcal{O}_A(1)$ is a very ample line bundle on A. We recall two results on rational homogeneous varieties.

Theorem 2.1 [Hwang and Mok 2005, Main Theorem; 1998, 5.2]. *Let A be a rational homogeneous variety of Picard number one. Let $\rho : X \to Z$ be a smooth proper morphism between two varieties. Suppose for some point y on Z, the fiber X_y is isomorphic to A. Then, for any point z on Z, the fiber X_z is isomorphic to A.*

Theorem 2.2 [Watanabe 2008]. *Let X be a smooth projective variety and A a rational homogeneous variety of Picard number one. If A is an ample divisor on X, (X, A) is isomorphic to $(\mathbb{P}^n, \mathbb{P}^{n-1})$, $(\mathbb{P}^n, Q^{n-1})$, (Q^n, Q^{n-1}), $(G(2, \mathbb{C}^{2l}), C_l(\omega_2))$ or $(E_6(\omega_1), F_4(\omega_4))$.*

For a numerical polynomial $P(t) \in \mathbb{Q}[t]$, write $\mathrm{Hilb}_{P(t)}(X)$ for the Hilbert scheme of X relative to $P(t)$. More generally, for an m-tuple of numerical polynomials $\mathbb{P}(t) := (P_1(t), \dots, P_m(t))$, denote by $\mathrm{FH}_{\mathbb{P}(t)}(X)$ the flag Hilbert scheme of X relative to $\mathbb{P}(t)$ [Sernesi 2006, Section 4.5]. For the Hilbert polynomial of a line $P_1(t)$, an irreducible component of $\mathrm{Hilb}_{P_1(t)}(X)$ is called a *family of lines* on X. Let $\mathrm{Univ}(X)$ be the universal family of $\mathrm{Hilb}(X)$ with the associated morphisms $\pi : \mathrm{Univ}(X) \to \mathrm{Hilb}(X)$ and $\iota : \mathrm{Univ}(X) \to X$. For a subset V of $\mathrm{Hilb}(X)$, $\iota(\pi^{-1}(V))$ is denoted by $\mathrm{Locus}(V) \subset X$. A *covering family of lines* $\mathcal{K}$ means an irreducible component of $F_1(X)$ satisfying $\mathrm{Locus}(\mathcal{K}) = X$. For a covering family of lines, we have the following fibration.

Theorem 2.3 [Campana 1992; Kollár et al. 1992]. *Let $X \subset \mathbb{P}^N$ be a smooth projective variety and $\mathcal{K}$ a covering family of lines. Then there exists an open subset $X^0 \subset X$ and a proper morphism $\varphi : X^0 \to Y^0$ with connected fibers onto a normal variety, such that any two points on the fiber of φ can be joined by a connected chain of finite $\mathcal{K}$-lines.*

We shall call the morphism φ a *rationally connected fibration with respect to $\mathcal{K}$.*

Theorem 2.4 [Bonavero et al. 2007, Theorem 2]. *Under the conditions and notation of Theorem 2.3, assume that $3 \geq \dim Y^0$. Then $\mathbb{R}_{\geq 0}[\mathcal{K}]$ is extremal in the sense of Mori theory and the associated contraction yields a rationally connected fibration with respect to $\mathcal{K}$.*

3. Proof of Theorem 1.1

For a subset $V \subset X$, denote the closure by $\overline{V}$. Let $P_1(t)$, $P_2(t)$ be the Hilbert polynomials of a line $(A, \mathcal{O}_A(1))$ and set $\mathbb{P}(t) := (P_1(t), P_2(t))$. We denote the natural projections by

$$p_i : \mathrm{FH}_{\mathbb{P}(t)}(X) \to \mathrm{Hilb}_{P_i(t)}(X), \text{ where } i = 1, 2. \tag{1}$$

Let $\mathcal{H}$ be the open subscheme of $\mathrm{Hilb}_{P_2(t)}(X)$ parametrizing smooth subvarieties of X with Hilbert polynomial $P_2(t)$. Now we work under the assumption that X satisfies (a) or (b) in Theorem 1.1.

Claim 3.1. *In both cases* (a) *and* (b), *there exists a curve $C \subset \mathcal{H}$ that contains a point o corresponding to a subvariety isomorphic to $(A, \mathcal{O}_A(1))$.*

Proof. If the assumption (a) holds, there exists an irreducible component $\mathcal{H}_0$ of $\mathcal{H}$ that contains $o := [Z_x]$ for some $x \in X$ and satisfies $\overline{\mathrm{Locus}(\mathcal{H}_0)} = X$. Then we can take a curve $C \subset \mathcal{H}_0$ that contains o. If the assumption (b) holds, we see that $h^1(N_{Z/X}) = 0$ and $h^0(N_{Z/X}) \geq 1$. Since there is no obstruction in the deformation of Z in X, it turns out that $\mathcal{H}$ is smooth at $o := [Z]$ and $\dim_{[Z]} \mathcal{H} \geq 1$. Then we can also take a curve $C \subset \mathcal{H}_0$ that contains o. □

From now on, we shall not use the assumptions (a) and (b) except through the property proved in Claim 3.1. $\overline{\mathrm{Locus}(C)} = X$. Denote by $\mathcal{H}_0$ an irreducible component of $\mathcal{H}$ that contains C. For the universal family $\pi : \mathcal{U}_0 \to \mathcal{H}_0$ and the normalization $\nu : \tilde{C} \to C \subset \mathcal{H}_0$, we denote $\tilde{C} \times_{\mathcal{H}_0} \mathcal{U}_0$ by $\mathcal{U}_{\tilde{C}}$ and a natural projection by $\tilde{\pi} : \mathcal{U}_{\tilde{C}} \to \tilde{C}$. Let $(\mathcal{U}_{\tilde{C}})_{\mathrm{red}}$ be the reduced scheme associated to $\mathcal{U}_{\tilde{C}}$ and $\Pi : (\mathcal{U}_{\tilde{C}})_{\mathrm{red}} \to \tilde{C}$ the composition of $\tilde{\pi}$ and $(\mathcal{U}_{\tilde{C}})_{\mathrm{red}} \to \mathcal{U}_{\tilde{C}}$. Then we have the following diagram:

$$\begin{tikzcd}
(\mathcal{U}_{\tilde{C}})_{\mathrm{red}} \arrow[r] \arrow[dr, "\Pi"'] & \mathcal{U}_{\tilde{C}} \arrow[r] \arrow[d, "\tilde{\pi}"'] & \mathcal{U}_0 \arrow[d, "\pi"'] \\
 & \tilde{C} \arrow[r, "\nu"'] & \mathcal{H}_0
\end{tikzcd}$$

Now we have an isomorphism between scheme theoretic fibers

$$\tilde{\pi}^{-1}(p) \cong \pi^{-1}(\nu(p))$$

for any closed point $p \in \tilde{C}$. In particular, $\tilde{\pi}^{-1}(p)$ is a smooth projective variety and $\tilde{\pi}^{-1}(\tilde{o}) \cong A$ for a point $\tilde{o} \in \tilde{C}$ corresponding to $o \in C$. Moreover, a natural morphism $\Pi^{-1}(p) \to \tilde{\pi}^{-1}(p)$ is a homeomorphic closed immersion for any closed point $p \in \tilde{C}$. Since $\tilde{\pi}^{-1}(p)$ is reduced, we see that $\Pi^{-1}(p) \cong \tilde{\pi}^{-1}(p)$. Thus we conclude that Π is a proper flat morphism whose fibers on closed points are smooth projective varieties, that is, a proper smooth morphism. Because Π admits a central fiber $\tilde{\Pi}^{-1}(\tilde{o}) \cong A$, it follows that every fiber $\tilde{\Pi}^{-1}(\tilde{p})$ is isomorphic to A from Theorem 2.1. Hence it turns out that every fiber of π over a closed point in C is isomorphic to A. Let consider a constructible subset $p_1(p_2^{-1}(C)) \subset \mathrm{Hilb}_{P_1(t)}(X)$. Since C parametrizes subvarieties isomorphic to $(A, \mathcal{O}_A(1))$ which is covered by lines, we see that

$$\overline{\mathrm{Locus}(p_1(p_2^{-1}(C)))} = X.$$

Claim 3.2. *There exists a covering family of lines $\mathcal{K}$ on X satisfying the following property: Through a general point $x \in X$, there is a subvariety $S_x \subset X$ such that $(S_x, \mathcal{O}_{S_x}(1)) \cong (A, \mathcal{O}_A(1))$ and any line lying in S_x is a member of $\mathcal{K}$.*

Proof. Take an irreducible component $\mathcal{K}^0$ of $p_1(p_2^{-1}(C))$ such that $\overline{\mathrm{Locus}(\mathcal{K}^0)} = X$. Through a general point x on X, there is a line $[l_x]$ in $\mathcal{K}^0$ that is not contained in any irreducible component of $p_1(p_2^{-1}(C))$ except $\mathcal{K}^0$. There is also a subvariety $[S_x]$ in C containing l_x. Because $p_1(p_2^{-1}([S_x]))$ is the Hilbert scheme of lines on S_x, it is irreducible [Landsberg and Manivel 2003, Theorem 4.3; Strickland 2002, Theorem 1]). Therefore $p_1(p_2^{-1}([S_x]))$ is contained in an irreducible component of $p_1(p_2^{-1}(C))$. Since $p_1(p_2^{-1}([S_x]))$ contains $[l_x]$, this implies that $p_1(p_2^{-1}([S_x]))$ is contained in $\mathcal{K}^0$. Thus we put $\mathcal{K}$ as an irreducible component of $\mathrm{Hilb}_{P_1(t)}(X)$ containing $\mathcal{K}^0$. $\square$

Two points on $S_x \cong A$ can be joined by a connected chain of lines in $\mathcal{H}$. This implies that the relative dimension of the rationally connected fibration $\varphi : X \cdots \to Y$ with respect to $\mathcal{H}$ is at least $n-1$. According to Theorem 2.4, $\mathbb{R}_{\geq 0}[\mathcal{H}]$ spans an extremal ray of $NE(X)$ and φ is its extremal contraction. In particular, φ is a morphism that contracts S_x to a point. If $\dim Y = 0$, we see that the Picard number of X is one. This implies that S_x is a very ample divisor on X. From Theorem 2.2, X is $\mathbb{P}^n$, Q^n, $G(1, \mathbb{P}^m)$ or $E_6(\omega_1)$. If $\dim Y = 1$, then Y is a smooth curve C and a general fiber of φ coincides with S_x. Therefore φ is an A-fibration on a smooth curve C. Hence Theorem 1.1 holds.

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NOTE ON THE RELATIONS IN THE TAUTOLOGICAL RING OF $\mathcal{M}_g$

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We give some nontrivial relations in the tautological ring of $\mathcal{M}_g$. These are derived from some new geometric relations obtained by localization on the moduli of stable quotients, which was recently introduced by A. Marian, D. Oprea and R. Pandharipande.

1. Introduction

We denote by $\mathcal{M}_g$ the moduli space of smooth curves of genus $g \geq 2$ over an algebraically closed field. Let $\pi : \mathcal{C}_g \to \mathcal{M}_g$ be its tautological family and ω_π be the dualizing sheaf. We denote by $\mathbb{E} = \pi_*\omega_\pi$ the Hodge bundle. Define $\kappa_i = \pi_*(c_1(\omega_\pi)^{i+1}) \in A^i(\mathcal{M}_g)$, $\lambda_i = c_i(\mathbb{E})$, and in particular, $k_0 = 2g-2$, $k_{-1} = 0$. The tautological ring $R^*(\mathcal{M}_g)$ is defined to be the subring generated by λ-classes and κ-classes. By Mumford's formula [1983], the tautological ring is in fact generated by the κ-classes $\kappa_1, \dots, \kappa_{g-2}$.

Faber [1999] proposed a series of remarkable conjectures about the structure of $R^*(\mathcal{M}_g)$:

(a) The tautological ring $R^*(\mathcal{M}_g)$ is Gorenstein with socle in degree $g-2$, and when an isomorphism $R^{g-2}(\mathcal{M}_g) = \mathbb{Q}$ is fixed, the natural pairing

$$R^i(\mathcal{M}_g) \times R^{g-2-i}(\mathcal{M}_g) \to R^{g-2}(\mathcal{M}_g) = \mathbb{Q}$$

is perfect.

(b) The $[g/3]$ classes $\kappa_1, \dots, \kappa_{[g/3]}$ generate the ring $R^*(\mathcal{M}_g)$, with no relations in degrees $\leq [g/3]$.

(c) Let $\sum_{j=1}^n d_j = g-2$ and $d_j \geq 0$. Then

$$\sum_{\sigma \in S_n} \kappa_\sigma = \frac{(2g-3+n)!}{(2g-2)!!\prod_{j=1}^n (2d_j+1)!!}\kappa_{g-2}, \tag{1}$$

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where κ_σ is defined as follows: write the permutation $\sigma = \beta_1 \dots \beta_{\nu(\sigma)}$, where we think of the symmetric group S_n as acting on the n-tuple $(d_1, \dots, d_n)$. Denote by $|\beta|$ the sum of the elements of a cycle β. Then $\kappa_\sigma = \kappa_{|\beta_1|}\kappa_{|\beta_2|} \dots \kappa_{|\beta_{\nu(\sigma)}|}$.

By now there are many works related to Faber's conjecture. Looijenga [1995] illustrated that

$$\dim R^k(\mathcal{M}_g) = 0, \quad k > g-2, \quad \text{and} \quad \dim R^{g-2}(\mathcal{M}_g) \le 1.$$

Faber [1999] proved that actually $\dim R^{g-2}(\mathcal{M}_g) = 1$ and thus $R^*(\mathcal{M}_g)$ has the Gorenstein property. But the perfect pairing conjecture is still open.

Part (b) of the conjecture was proved independently by Morita [2003] and Ionel [2005] with different methods.

Part (c) of the conjecture (that is, Faber's intersection number conjecture) is equivalent to a closed formula of the $\lambda_g\lambda_{g-1}$ Hodge integral, itself a consequence of the degree-zero Virasoro conjecture for surfaces [Getzler and Pandharipande 1998]. A short and direct proof of that integral formula can be found in [Liu and Xu 2009]. Recently, Buryak and Shadrin [2009] gave another combinatoric approach to this problem.

Thus, only the perfect pairing conjecture is open in Faber's original conjecture. Liu and Xu [2010] proved some effective recursive relations in the top-degree tautological ring $R^{g-2}(\mathcal{M}_g)$ based on Faber's intersection number conjecture. We know that it is important to find explicit relations in the tautological ring independent of genus. Faber [1999] also proposed a conjecture that all the tautological relations can be generated by the Brill–Noether method. Recently, Faber and Pandharipande have found some counterexamples when $g \ge 24$, and thus Faber's approach may not produce all tautological relations starting from $g = 24$. The Brill–Noether method is an effective way to calculate the tautological relations. Ionel [2005] found some explicit relations in dimension $a = g+b+1-2d$ for each $d \ge 2$, $g \ge 2$ and $b \ge 0$. As an application, Ionel gave a proof for Part (b) of Faber's conjecture.

Marian, Oprea and Pandharipande [2009] obtained a vanishing theorem via a localization technique on the moduli space of stable quotients. Their result leads to some new geometric relations in the tautological ring. They computed these three special cases of their new geometric relation:

Case 1. If $a = 0$, $b = 1$, and $c = 2k$ (for $k \ge 1$), then in $R^{g-2d-1+2k}(\mathcal{M}_g)$,

$$\rho_*(c_{g-d-1+2k}(\tilde{\mathbb{F}}_d)) = 0. \tag{2}$$

Case 2. If $a = 1$, $b = 1$, and $c = 2k$ (for $k \ge 1$), then in $R^{g-2d+2k}(\mathcal{M}_g)$,

$$\rho_*\big(2(K_1 + \cdots + K_d) \cdot c_{g-d-1+2k}(\tilde{\mathbb{F}}_d) + (2g-2) \cdot c_{g-d+2k}(\tilde{\mathbb{F}}_d)\big) = 0. \tag{3}$$

Case 3. If $a = 2$, $b = 0$, and $c = 2k$ (for $k \geq 1$), then in $R^{g-2d+2k}(\mathcal{M}_g)$,

$$\rho_*\big(-2(K_1 + \cdots + K_d - 2\Delta \cdot c_{g-d-1+2k}(\tilde{\mathbb{F}}_d) + 2d \cdot c_{g-d+2k}(\tilde{\mathbb{F}}_d))\big) = 0. \tag{4}$$

Combining (3) and (4), we have

$$\rho_*\big(2\Delta \cdot c_{g-d-1+2k}(\tilde{\mathbb{F}}_d) + (g+d-1)c_{g-d+2k}(\tilde{\mathbb{F}}_d)\big) = 0 \;\text{ in }\; R^{g-2d+2k}(\mathcal{M}_g). \tag{5}$$

The notation in these formulas is explained in Section 2.

In this note, applying the method of [Ionel 2005], we derive new relations for the tautological ring in $\mathcal{M}_g$ from (2) and (3), which can be considered a partial generalization of the main results in [Ionel 2005]. We also show that (5) is equivalent to (3).

Our main results are given by the following proposition.

Proposition 1.1. *For each $g, d \geq 2$ and $k \geq 1$, formula* (2) *is equivalent to*

$$\Big[\exp\Big(\frac{1}{t}\pi_* G(tK, w)\Big)\Big]_{t^{g-2d-1+2k}w^d} = 0, \tag{6}$$

and (3) *is equivalent to*

$$\Big[\exp\Big(\frac{1}{t}\pi_* G(tK, w)\Big)\pi_*\big((2wG_w(tK, w) + 1)K\big)\Big]_{t^{g-2d+2k}w^d} = 0. \tag{7}$$

Here $G(x, w)$ (as in [Ionel 2005, Definition 2.1]) *is the unique formal power series in x and w that satisfies the recursive formula*

$$xwG_{ww} = w(G_w)^2 + (1-x)G_{ww} - 1 \tag{8}$$

with

$$G(x, 0) = -\sum_{a=2}^{\infty} \frac{B_a}{a(a-1)} x^a, \tag{9}$$

where B_a denotes the Bernoulli numbers.

Theorem 1.2. *For each $g, d \geq 2$ and $k \geq 1$, formulas* (6) *and* (7) *give the following relations in $R^{g-2d-1+2k}(\mathcal{M}_g)$ and $R^{g-2d+2k}(\mathcal{M}_g)$, respectively*:

$$\Big[(1+4u)^{k-1} \exp\Big(-\sum_{a=1}^{\infty} x^a \kappa_a \sum_{j=0}^{a} c_{a,j} u^j\Big)\Big]_{x^{g-2d-1+2k}u^d} = 0, \tag{10}$$

$$\Big[(1+4u)^{k-1} \exp\Big(-\sum_{a=1}^{\infty} x^a \kappa_a \sum_{j=0}^{a} c_{a,j} u^j\Big) \times \Big(g - 1 - \sum_{a=0}^{\infty} x^{a+1} \kappa_{a+1} \sum_{j=0}^{a} q_{a,j} u^{j+1}\Big)\Big]_{x^{g-2d+2k}u^d} = 0. \tag{11}$$

Here the positive integers $q_{k,j}$ (as in [Ionel 2005, Definition 1.3]) *are defined recursively for* $k \geq j \geq 0$ *by the relation*

$$q_{k,j} = (2k+4j-2)q_{k-1,j-1} + (j+1)q_{k-1,j} + \sum_{m=0}^{k-1}\sum_{l=0}^{j-1} q_{m,l}q_{k-1-m,j-1-l}, \tag{12}$$

with initial condition $q_{0,0} = 1$; *and the coefficients* $c_{k,j}$, *for* $k \geq 1$ *and* $k \geq j \geq 0$, *by the relation*

$$q_{k,j} = (2k+4j)c_{k,j} + (j+1)c_{k,j+1}, \tag{13}$$

for all $k \geq 1$ *and* $k \geq j \geq 0$.

When $k = 1$, formulas (10) and (11) are just [Ionel 2005, (1.10) and (1.9)] with $b = 0$ and $b = 1$ respectively.

Theorem 1.3. *Formula* (5) *is equivalent to formula* (3).

2. Preliminaries

In this section, we introduce the notations and results that we use in this paper. We denote by $\mathscr{C}_g^d$ the d-fold of not necessarily distinct fiber products of $\mathscr{C}_g$ over $\mathcal{M}_g$, parametrizing smooth curves of genus g with n-tuples of necessary distinct points, that is, $\mathscr{C}_g^d = \{(C, x_1, \dots, x_d) | x_i \in C\}$. Let $\rho : \mathscr{C}_g^d \to \mathcal{M}_g$ be the map forgetting all the points. Then ρ is the composition of morphisms π_i: $\mathscr{C}_g^i \to \mathscr{C}_g^{i-1}$ forgetting the i-th point, $\rho = \pi_1\pi_2 \dots \pi_d$; here $\pi_1 = \pi$.

There are some natural classes in $A^1(\mathscr{C}_g^d)$: $K_i = p_i^*(c_1(\omega_\pi))$ where p_i is the i-th projection from $\mathscr{C}_g^d$ to $\mathscr{C}_g$. K_1 is written as K in the following. D_{ij} is the diagonal class of $\mathscr{C}_g^d$ where the points $x_i = x_j$.

Faber [1999] collected the following ρ-rules, due to Harris and Mumford [1982]:

Formularium. (a) *Every monomial in the classes*

$$K_i (1 \leq i \leq d) \text{ and } D_{ij} (1 \leq i < j \leq d)$$

on $\mathscr{C}_g^d$ *can be rewritten as monomial* M *pulled back from* $\mathscr{C}_g^{d-1}$ *times either a single diagonal* D_{id} *or a power* K_d^l *of* K_d *by a repeated application of the following substitution rules*:

$$\begin{aligned} D_{id}D_{jd} &\to D_{ij}D_{id} \quad (i < j < d), \\ D_{id}^2 &\to -K_i D_{id} \quad (i < d), \\ K_d D_{id} &\to K_i D_{id} \quad (i < d). \end{aligned}$$

(b) *For* M *a monomial pulled back from* $\mathscr{C}_g^{d-1}$,

$$\begin{aligned} (\pi_d)_*(M \cdot D_{id}) &= M, \\ (\pi_d)_*(M \cdot K_d^l) &= M \cdot \rho^*(k_{l-1}). \end{aligned}$$

For convenience, Ionel [2005] introduced the more general classes in $A^*(\mathscr{C}_g^d)$. If $1 \le i_1 < \cdots < i_k \le d$ is a sequence of integers, let $D_{i_1,\dots,i_k}$ be the class of the stratum of $\mathscr{C}_g^d$, where all the points x_{i_l} are equal for $l = 1, \dots, k$. Given an unordered partition $\{J_1, \dots, J_k\}$ of $\{x_1, \dots, x_d\}$, we denote by $\Delta_{J_1,\dots,J_k} = \prod_{i=1}^d D_{J_i}$ the codimension $d-k$ multidiagonal in $\mathscr{C}_d^d$, where all points in each J_i are equal. Given such a stratum $\Delta_{J_1,\dots,J_k}$, we denote by x_{J_i} any one of the points of J_i, and by K_{J_i} its corresponding K-class; also $|J_i| > 0$ denotes the number of points in J_i. If I, J are two subsets of $\{1, \dots, d\}$ with $I \cap J \neq \varnothing$, then

$$D_I \cdot D_J = (-K_I)^{|I\cap J|-1} D_{I\cup J}. \tag{14}$$

Let $f(x_1, \dots, x_d) \in \mathbb{Q}[x_1, \dots, x_d]$ be an arbitrary polynomial. Then by (14) and the ρ-rules, we have

$$\rho_*(\Delta_{J_1,\dots,J_k} \cdot f(K_1, \dots, K_d)) = \pi_* f(K, \dots, K). \tag{15}$$

Denote by $\mathbb{F}_d = (\pi_{d+1})_*\big(\mathcal{O}_{\mathscr{C}_g^{d+1}}(D_{1,d+1} + \cdots + D_{d,d+1})/\mathcal{O}_{\mathscr{C}_g^{d+1}}\big)$ the jet bundle at d points, and let $\mathbb{E}^\vee$ be the dual of the Hodge bundle. By direct calculation,

$$c(\mathbb{F}_d) = c(\mathbb{F}_{d-1})(1 - K_d + D_{1,d} + \cdots + D_{d-1,d}).$$

Let $\tilde{\mathbb{F}}_d = \rho^*\mathbb{E}^\vee - \mathbb{F}_d$. The geometric relation formulated in [Ionel 2005] is

$$(2d + 2g - 2) \cdot \rho_*(c_{g+1-d}(\tilde{\mathbb{F}}_d) K_1^b) = (d-1)\kappa_{b-1} \cdot \rho_*(D_{12} \cdot c_{g+1-d}(\tilde{\mathbb{F}}_d)), \tag{16}$$

for $d \ge 2$, $g \ge 2$ and $b \ge 0$.

Via the main formula [Ionel 2005, Proposition 2.3], we have

$$\begin{aligned} c_t(\tilde{\mathbb{F}}_d) &= \frac{\rho^*(c_t(\mathbb{E}^\vee))}{c_t(\mathbb{F}_d)} \\ &= \rho^* \exp\Big(-\sum_{a=1}^{\infty} \frac{B_{a+1}}{a(a+1)} \kappa_a t^a\Big) \cdot \sum_{r=0}^{\infty} \sum_{\{J_1,\dots,J_r\}} t^{d-r} \Delta_{J_1,\dots,J_r} \prod_{i=1}^{r} H_{|J_i|}(tK_{J_i}), \end{aligned} \tag{17}$$

where the last sum is over all (unordered) partitions $\{J_1, \dots, J_r\}$ of $\{x_1, \dots, x_d\}$, and the formal power series

$$G(x, w) = \sum_{d=0}^{\infty} H_d(x) \frac{w^d}{d!} = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} a_{kj} x^k w^j \tag{18}$$

satisfies (8) and (9). The main result of [Ionel 2005] is that relation (16) gives the following relation in $R^{g+1+b-2d}(\mathcal{M}_g)$:

(19)

$$\Big[\exp\Big(-\sum_{a=1}^{\infty} x^a \kappa_a \sum_{j=0}^{a} c_{a,j} u^j\Big) \cdot \Big(\kappa_{b-1} - 2\sum_{a=0}^{\infty} \kappa_{a+b} x^{a+1} \sum_{j=0}^{a} q_{a,j} u^{j+1}\Big)\Big]_{x^{g+2-2d}u^d} = 0,$$

where $g, d \geq 2$ and $b \geq 0$. For $b = 0$, this relation simplifies to

$$\left[\exp\Bigl(-\sum_{a=1}^{\infty} x^a \kappa_a \sum_{j=0}^{a} c_{a,j} u^j\Bigr)\right]_{x^{g+1-2d}u^d} = 0. \tag{20}$$

As an application of the relation (19), Part (b) of Faber's conjecture is proved.

Marian, Oprea and Pandharipande [Marian et al. 2009] obtained a vanishing theorem via a localization technique on the moduli space of stable quotients. We describe their main statement for the reader's convenience.

Given an element $[C, \hat{p}_1, \dots, \hat{p}_d] \in \mathcal{C}_g^d$, there is a canonically associated stable quotient

$$0 \to \mathbb{O}_C\Bigl(-\sum_{j=1}^{d} \hat{p}_j\Bigr) \to \mathbb{O}_C \to Q \to 0. \tag{21}$$

Consider the universal curve $\pi : U \to \mathcal{C}_g^d$ with universal quotient sequence

$$0 \to S_U \to \mathbb{O}_U \to Q_U \to 0$$

obtained from (21). Let $\overline{\mathbb{F}}_d = -R\pi_*(S_U^*) \in K(\mathcal{C}_g^d)$ be the class in K-theory. With some computations, we have

$$c(\overline{\mathbb{F}}_d) = c(\tilde{\mathbb{F}}_d) = \frac{\rho^*(c(\mathbb{E}^\vee))}{c(\mathbb{F}_d)}. \tag{22}$$

Consider the proper morphism

$$\nu : Q_g(\mathbb{P}^1, d) \to \mathcal{M}_g.$$

The universal curve

$$\Pi : U \to Q_g(\mathbb{P}^1, d)$$

carries the basic divisor classes $s = c_1(S_U^*)$ and $\omega = c_1(\omega_\pi)$.

Let $c > 0$ and $a, b \geq 0$. Then by (22), the geometric relation in the tautological ring shows that [Marian et al. 2009, Proposition 5]

$$\rho_*\Bigl(\Pi_*(s^a\omega^b) \cdot c_{g-d-1+c}(\tilde{\mathbb{F}}_d) + (-1)^{g-d-1}\bigl[\Pi_*((s-1)^a\omega^b) \cdot c_-(\tilde{\mathbb{F}}_d)\bigr]^{g-d-2+a+b+c}\Bigr) = 0 \tag{23}$$

in $R^*(\mathcal{M}_g)$, where $c_-(\tilde{\mathbb{F}})$ denotes the total Chern class of $\tilde{\mathbb{F}}_d$ evaluated at -1. Then they obtained the following three special cases of (23).

Case 1. If $a = 0$, $b = 1$, and $c = 2k$ (for $k \geq 1$), then in $R^{g-2d-1+2k}(\mathcal{M}_g)$,

$$\rho_*(c_{g-d-1+2k}(\tilde{\mathbb{F}}_d)) = 0. \tag{24}$$

Case 2. If $a=1$, $b=1$, and $c=2k$ (for $k\geq 1$), then in $R^{g-2d+2k}(\mathcal{M}_g)$,

$$\rho_*\big(2(K_1+\cdots+K_d)\cdot c_{g-d-1+2k}(\tilde{\mathbb{F}}_d)+(2g-2)\cdot c_{g-d+2k}(\tilde{\mathbb{F}}_d)\big)=0. \tag{25}$$

Case 3. If $a=2$, $b=0$, and $c=2k$ (for $k\geq 1$), then in $R^{g-2d+2k}(\mathcal{M}_g)$,

$$\rho_*\big(-2(K_1+\cdots+K_d-2\Delta\cdot c_{g-d-1+2k}(\tilde{\mathbb{F}}_d)+2d\cdot c_{g-d+2k}(\tilde{\mathbb{F}}_d)\big)=0, \tag{26}$$

where $\Delta=\sum_{1\leq i<j\leq d}D_{ij}$.

Combining (25) and (26), we have

$$\rho_*\big(2\Delta\cdot c_{g-d-1+2k}(\tilde{\mathbb{F}}_d)+(g+d-1)c_{g-d+2k}(\tilde{\mathbb{F}}_d)\big)=0 \text{ in } R^{g-2d+2k}(\mathcal{M}_g). \tag{27}$$

In the next section, we show how to get similar results with (19) and (20) by the same combinatorial method as in [Ionel 2005].

3. Proof of the main results

With a minor modification of [Ionel 2005, Lemma 2.5], we have:

Lemma 3.1. *In terms of the generating function $G(x,w)$ defined by* (8) *and* (9), *we have*

$$\sum_{d=0}^{\infty}\frac{w^d t^{-d}}{d!}\rho_* c_t(\tilde{\mathbb{F}}_d)=\exp\Big(\frac{1}{t}\rho_* G(tK,w)\Big) \tag{28}$$

and

$$\sum_{d=1}^{\infty}\frac{w^{d-1}t^{-d}}{d!}\rho_*(c_t(\tilde{\mathbb{F}}_d)K_j)=\exp\Big(\frac{1}{t}\rho_* G(tK,w)\Big)\cdot\frac{1}{t}\rho_*\big(G_w(tK,w)K_j\big), \tag{29}$$

for $j=1,\ldots,d$.

Proof. The $c_t(\tilde{\mathbb{F}}_d)$, after being pushed forward by ρ, depends only on the lengths l_i of sets J_i. By (17), and some combinatoric enumeration, it is easy to get (28) and (29); see [Ionel 2005] for details. ☐

Therefore, by Lemma 3.1, identities (2) and (3) give rise to (6) and (7) in Proposition 1.1, respectively.

In order to get Theorem 1.2, we need to better understand the structure of the function $G(x,w)$ defined by (8) and (9). Ionel [2005, Lemmas 3.1 and 3.2] obtained the following expansions for $G_w(x,w)$ and $G(x,w)$:

$$\begin{aligned}(30)\quad & G_w(x,w)\\ &=\frac{-1+\sqrt{1+4w}}{2w}+\frac{x}{1+4w}+\sum_{k=1}^{\infty}\sum_{j=0}^{k}x^{k+1}q_{k,j}(-w)^j(1+4w)^{-j-k/2-1},\end{aligned}$$

where the coefficients $q_{k,j}$ are defined by (12) and

$$G(x,w)=G(0,w)+\frac{x}{4}\ln(1+4w)-\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}x^{k+1}c_{k,j}(-w)^j(1+4w)^{-j-k/2}, \tag{31}$$

where the coefficients $c_{k,j}$ are related to the coefficients $q_{k,j}$ by (13). Also, we need the variable transformation formula used in [Ionel 2005].

Lemma 3.2 [Ionel 2005, Lemma 3.3]. *Let $P(x,w)$ be a formal power series in x and w. Denote by $\hat{P}(y,u)$ the formal power series in y and u obtained from $P(x,w)$ after the change of variables $u=-w/(1+4w)$ and $y=x/\sqrt{1+4w}$.*

$$[P(x,w)]_{x^a w^d}=(-1)^d[(1+4u)^{(a+2d-2)/2}\hat{P}(y,u)]_{y^a u^d}. \tag{32}$$

By the expansion (31),

$$\frac{1}{t}\pi_*G(tK,w)=\frac{\kappa_0}{4}\ln(1+4w)-\sum_{a=1}^{\infty}t^a\kappa_a\sum_{j=0}^{\infty}c_{a,j}(-w)^j(1+4w)^{-j-a/2}. \tag{33}$$

Using the change of variables,

$$t\to(1+4w)^{\frac{1}{2}}y,\quad w\to\frac{-u}{1+4u},\quad (1+4w)\to\frac{1}{1+4u}, \tag{34}$$

we have

$$\exp\Big(\frac{1}{t}\pi_*(G(tK,w))\Big)=(1+4u)^{-\kappa_0/4}\exp\Big(-\sum_{a=1}^{\infty}y^a\kappa_a\sum_{j=0}^{\infty}c_{a,j}u^j\Big). \tag{35}$$

Similarly, by the expansion (30),

$$\begin{aligned}
&\pi_*\big((2wG_w(tK,w)+1)K\big)\\
&\quad=\pi_*\Big((1+4w)^{1/2}K+\frac{2w}{1+4w}tK^2\\
&\qquad+\sum_{a=1}^{\infty}\sum_{j=0}^{a}t^{a+1}K^{a+2}q_{a,j}(-w)^j2w(1+4w)^{-j-a/2-1}\Big)\\
&\quad=(1+4w)^{\frac{1}{2}}\Big(\kappa_0-2\sum_{a=0}^{\infty}t^{a+1}\kappa_{a+1}\sum_{j=0}^{a}q_{a,j}(-w)^{j+1}(1+4w)^{-j-(a+1)/2-1}\Big).
\end{aligned}$$

By the change of variables

$$t\to(1+4w)^{\frac{1}{2}}y,\quad w\to\frac{-u}{1+4u},\quad (1+4w)\to\frac{1}{1+4u}, \tag{36}$$

we get

$$\begin{aligned}(37)\quad \pi_*\big((2wG_w(tK,w)+1)K\big)\\ = (1+4u)^{-1/2}\Big(\kappa_0 - 2\sum_{a=0}^{\infty} y^{a+1}\kappa_{a+1}\sum_{j=0}^{a} q_{a,j}u^{j+1}\Big).\end{aligned}$$

By Lemma 3.2, (35) and (37), we have

$$\begin{aligned}&\Big[\exp\Big(\frac{1}{t}\pi_*G(tK,w)\Big)\Big]_{t^{g-2d-1+2k}w^d}\\ &\qquad = (-1)^d\Big[(1+4u)^{k-1}\exp\Big(-\sum_{a=1}^{\infty} y^a\kappa_a\sum_{j=0}^{\infty} c_{a,j}u^j\Big)\Big]_{y^{g-2d-1+2k}u^d},\\ &\Big[\exp\Big(\frac{1}{t}\pi_*G(tK,w)\Big)\pi_*\big((2wG_w(tK,w)+1)K\big)\Big]_{t^{g-2d+2k}w^d}\\ &\qquad = \Big[(1+4u)^{k-1}\exp\Big(-\sum_{a=1}^{\infty} x^a\kappa_a\sum_{j=0}^{a} c_{a,j}u^j\Big)\\ &\qquad\qquad \Big(g-1-\sum_{a=0}^{\infty} x^{a+1}\kappa_{a+1}\sum_{j=0}^{a} q_{a,j}u^{j+1}\Big)\Big]_{x^{g-2d+2k}u^d}.\end{aligned}$$

By formulas (6) and (7), Theorem 1.2 is proved.

Theorem 3.3. *Formula* (3) *is equivalent to formula* (5).

Proof. By the definition of $\tilde{\mathbb{F}}_d$,

$$\pi_d^*(c_t(\tilde{\mathbb{F}}_{d-1})) = c_t(\tilde{\mathbb{F}}_d)(1 - tK_d + tD_{1,d} + \cdots + tD_{d-1,d}).$$

After being pushed forward by ρ,

$$0 = \rho_*\pi_d^*(c_t(\tilde{\mathbb{F}}_{d-1})) = \rho_*(c_t(\tilde{\mathbb{F}}_d)) - t\rho_*(K_d\cdot c_t(\tilde{\mathbb{F}}_d)) + (d-1)t\rho_*(D_{1,d}\cdot c_t(\tilde{\mathbb{F}}_d)).$$

In particular,

$$(d-1)\big[\rho_*(D_{1,d}\cdot c_t(\tilde{\mathbb{F}}_d))\big]_{t^{g-d-1+2k}} = \big[\rho_*(K_d\cdot c_t(\tilde{\mathbb{F}}_d))\big]_{t^{g+d-1+2k}} - \big[\rho_*(c_t(\tilde{\mathbb{F}}_d))\big]_{t^{g-d+2k}}.$$

In fact

$$\rho_*(D_{i,j}\cdot c_t(\tilde{\mathbb{F}}_d)) = \rho_*(D_{1,d}\cdot c_t(\tilde{\mathbb{F}}_d)),$$

and the equivalence of formulas (3) and (5) is deduced from the identity

$$\begin{aligned}2\big[\rho_*(\Delta\cdot c_t(\tilde{\mathbb{F}}_d))\big]_{t^{g-d-1+2k}} &= d(d-1)\big[\rho_*(D_{1,d}\cdot c_t(\tilde{\mathbb{F}}_d))\big]_{t^{g-d-1+2k}}\\ &= d\big[\rho_*(K_d\cdot c_t(\tilde{\mathbb{F}}_d))\big]_{t^{g-d-1+2k}} - d\big[\rho_*(c_t(\tilde{\mathbb{F}}_d))\big]_{t^{g-d+2k}}\\ &= -(g-1)\big[\rho_*(c_t(\tilde{\mathbb{F}}_d))\big]_{t^{g-d+2k}} - d\big[\rho_*(c_t(\tilde{\mathbb{F}}_d))\big]_{t^{g-d+2k}}\\ &= -(g+d-1)\big[\rho_*(c_t(\tilde{\mathbb{F}}_d))\big]_{t^{g-d+2k}}.\end{aligned}$$

□

Example 3.4. We give some low genus examples for Theorem 1.2. By the recursion relations (12) and (13) of constants $q_{k,j}$, $c_{k,j}$, we get

$$\begin{aligned} &q_{0,0}=1,\\ &q_{1,0}=1, \quad q_{1,1}=5,\\ &q_{2,0}=1, \quad q_{2,1}=18, \quad q_{2,2}=60,\\ &q_{3,0}=1, \quad q_{3,1}=47, \quad q_{3,2}=442, \quad q_{3,3}=1105,\\ &\dots \end{aligned}$$

and

$$\begin{aligned} &c_{1,0}=\tfrac{1}{12}, \quad c_{1,1}=\tfrac{5}{6},\\ &c_{2,0}=0, \quad c_{2,1}=1, \quad c_{2,2}=5,\\ &c_{3,0}=-\tfrac{1}{360}, \quad c_{3,1}=\tfrac{61}{60}, \quad c_{3,2}=\tfrac{221}{12}, \quad c_{3,3}=\tfrac{1105}{18},\\ &\dots \end{aligned}$$

Taking $g=5$, $d=3$, $k=2$, formula (10) gives a relation in $R^2(\mathcal{M}_5)$:

$$\tfrac{25}{18}\kappa_1^2-20\kappa_2=0. \tag{38}$$

Taking $g=6$, $d=3$, $k=2$, formula (10) gives a relation in $R^3(\mathcal{M}_6)$:

$$-\tfrac{275}{1296}\kappa_1^3+\tfrac{55}{6}\kappa_1\kappa_2-\tfrac{2431}{18}\kappa_3=0. \tag{39}$$

It is easy to check that the relations (38) and (39) match the results in [Faber 1999].

We have written a Maple program to calculate more relations through Theorem 1.2. Unfortunately, it is difficult to determine if they contain the new tautological relations in high genus beyond those obtained by Faber.

4. Conclusion

The new relations in the tautological ring obtained in this note, that is, formulas (10) and (11), can be regarded as a partial generalization of formula (19) (which is [Ionel 2005, (1.9)]) in the special cases $b=0$, 1.

When $b=0$, formula (19) is just (20):

$$\Bigl[\exp\Bigl(-\sum_{a=1}^{\infty}x^a\kappa_a\sum_{j=0}^{a}c_{a,j}u^j\Bigr)\Bigr]_{x^{g+1-2d}u^d}=0,$$

which is the special case of formula (10) with $k=1$.

For $b=1$, (19) becomes

$$\Bigl[\exp\Bigl(-\sum_{a=1}^{\infty}x^a\kappa_a\sum_{j=0}^{a}c_{a,j}u^j\Bigr)\cdot\Bigl(g-1-\sum_{a=0}^{\infty}\kappa_{a+1}x^{a+1}\sum_{j=0}^{a}q_{a,j}u^{j+1}\Bigr)\Bigr]_{x^{g+2-2d}u^d}=0,$$

which is the special case of (11) with $k = 1$.

However, our results can not cover formula (19) when $b \geq 2$. In this note, we only consider three special cases of (23) to deduce our main results. We hope that one can obtain a more general formula, like (11), from formula (23), with the same method.[1]

As mentioned in the introduction, it is important to find explicit relations in the tautological ring in studying Faber's conjecture. From this note, we see that the stable quotient method introduced by Marian, Oprea and Pandharipande [2009] provides a new and effective way to obtain the relations in the tautological ring. With this method, recently, Pandharipande [2009a; 2009b] introduced the κ ring of the moduli of curves of compact type and studied its algebraic structure. In our further study, we hope to find more applications of the stable quotient method.

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