Pacific Journal of Mathematics

K-GROUPS OF THE QUANTUM HOMOGENEOUS SPACE $\mathrm{SU}_q(n)/\mathrm{SU}_q(n-2)$

PARTHA SARATHI CHAKRABORTY AND S. SUNDAR

Volume 252 No. 2 August 2011

PACIFIC JOURNAL OF MATHEMATICS
Vol. 252, No. 2, 2011

K-GROUPS OF THE QUANTUM HOMOGENEOUS SPACE $\mathrm{SU}_q(n)/\mathrm{SU}_q(n-2)$

PARTHA SARATHI CHAKRABORTY AND S. SUNDAR

Dedicated to Prof. K. R. Parthasarathy on his 75th birthday

Quantum Stiefel manifolds were introduced by Vainerman and Podkolzin, who classified the irreducible representations of the C^*-algebras underlying such manifolds. We compute the K-groups of the quantum homogeneous spaces $\mathrm{SU}_q(n)/\mathrm{SU}_q(n-2)$ for $n \geq 3$. In the case $n = 3$, we show that K_1 is a free $\mathbb{Z}$-module, and the fundamental unitary for quantum SU(3) is part of a basis for K_1.

1. Introduction

Quantization of mathematical theories is a major theme of research today. The theories of quantum groups and noncommutative geometry are two prime examples in this program. Both these programs started in the early 1980s. In the setting of operator algebras, the theory of quantum groups was initiated independently in [Woronowicz 1987] and [Vaksman and Soibelman 1988], for the case of quantum SU(2). Later Woronowicz studied the family of compact quantum groups and obtained Tannaka-type duality theorems [Woronowicz 1988]. The notion of quantum subgroups and quantum homogeneous spaces soon followed [Podleś 1995].

The noncommutative differential geometry program of Alain Connes [1985] also started in the 1980s. In his interpretation, geometric data is encoded in elliptic operators or, more generally, in specific unbounded K-cycles, which he called spectral triples. It is natural to expect that, for compact quantum groups and their homogeneous spaces, there should be associated canonical spectral triples. Chakraborty and Pal [2003] showed that indeed that is the case for quantum SU(2). In fact for odd-dimensional quantum spheres, one can construct finitely summable spectral triples that display Poincaré duality [Chakraborty and Pal 2010].

In this connection, a natural question is, are these examples somewhat singular or can one in general construct finitely summable spectral triples with further

Chakraborty acknowledges financial support from Indian National Science Academy through its project "Noncommutative Geometry of Quantum Groups".
MSC2000: 46L80, 58B32.
Keywords: K-groups, quantum Stiefel manifolds, quantum groups, quantum homogeneous spaces.

properties like Poincaré duality, on quantum groups associated with Lie groups or their homogeneous spaces? Even though there are suggestions to construct such spectral triples [Neshveyev and Tuset 2010], their nontriviality as a K-cycle is not known. In fact, there are suggestions that, for quantum groups and their homogeneous spaces, one should look for a type-III formulation of noncommutative geometry. On this formulation also, there are currently two points of view, that of Alain Connes and Henri Moscovici [2008], and that of Carey–Phillips–Rennie [2010]. Therefore, to understand the true nature of the interplay between noncommutative geometry and quantum homogeneous spaces, it makes sense to take a closer look at these algebras.

The underlying C^*-algebras of these compact quantum groups were analyzed by Soibelman [1990] (also [Levendorskii and Soibelman 1991]) who described their irreducible representations. Exploiting their findings, Sheu went on to obtain composition sequences for these algebras. He initially obtained the results for $\mathrm{SU}_q(3)$ [Sheu 1991], and later extended them to the general $\mathrm{SU}_q(n)$ [Sheu 1997].

In this hierarchy of exploration, the next thing to look for would be K-groups; that is what we are looking for. But, instead of concentrating on quantum groups, we consider the quantum analogs of the Stiefel manifolds $\mathrm{SU}(n)/\mathrm{SU}(n-m)$, introduced by Podkolzin and Vainerman [1999]. Those authors have already described the structure of irreducible representations of the quantum Stiefel manifolds $\mathrm{SU}_q(n)/\mathrm{SU}_q(n-m)$. We take up the case of $\mathrm{SU}_q(n)/\mathrm{SU}_q(n-2)$ when $n \geq 3$. We obtain the composition sequences for these algebras and then, utilizing them, we compute the K-groups. More importantly, as we remarked earlier, applications towards noncommutative geometry require an explicit understanding of generators for these K-groups; during our calculation we also achieve that. Specializing to the case $n = 3$, we get the K-groups of quantum $\mathrm{SU}(3)$.

We should remark that these K-groups can be computed using the variant of KK-theory introduced by Nagy in [2000]. In fact, it is shown in [Nagy 1998] that $\mathrm{SU}_q(n)$ and $\mathrm{SU}(n)$ are KK-equivalent, but here we produce explicit generators, which is essential to test the nontriviality of K-cycles by computing the K-theory–K-homology pairing. To our knowledge, there are not many instances of K-theory calculations for compact quantum groups. Other than the paper by Nagy, there is another related work by McClanahan [1992], where he computes the K-groups of the universal C^*-algebra generated by the elements of a unitary matrix, and shows that the associated K_1 is generated by the defining unitary itself. This raises the question whether something similar holds for compact matrix quantum groups, namely, whether the defining unitary of a compact matrix quantum group is nontrivial in K_1. For quantum $\mathrm{SU}(2)$, this was remarked by Connes [2004]. Here, we not only prove that the defining unitary of quantum $\mathrm{SU}(3)$ is nontrivial, the K_1 is a free $\mathbb{Z}$-module, and the fundamental unitary for quantum $\mathrm{SU}(3)$ is part of a basis for K_1.

2. The quantum Stiefel manifolds and their irreducible representations

The quantum Stiefel manifold $S_q^{n,m}$ was introduced in [Podkolzin and Vainerman 1999]. Throughout, we assume that $q \in (0, 1)$. Recall that the C^*-algebra $C(\mathrm{SU}_q(n))$ is the universal unital C^*-algebra generated by n^2 elements u_{ij} satisfying the conditions

$$\sum_{k=1}^{n} u_{ik}u_{jk}^* = \delta_{ij}, \qquad \sum_{k=1}^{n} u_{ki}^* u_{kj} = \delta_{ij},$$

$$\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_n=1}^{n} E_{i_1 i_2 \dots i_n} u_{j_1 i_1} \dots u_{j_n i_n} = E_{j_1 j_2 \dots j_n},$$

where

$$E_{i_1 i_2 \dots i_n} := \begin{cases} 0 & \text{if } i_1, i_2, \dots i_n \text{ are not distinct,} \\ (-q)^{\ell(i_1, i_2, \dots, i_n)} & \text{otherwise,} \end{cases}$$

and where $\ell(\sigma)$ denotes the length of a permutation σ on $\{1, 2, \dots, n\}$. The C^*-algebra $C(\mathrm{SU}_q(n))$ has a compact quantum group structure with comultiplication given by

$$\Delta(u_{ij}) := \sum_k u_{ik} \otimes u_{kj}.$$

Let $1 \le m \le n-1$. Call v_{ij} the generators of $\mathrm{SU}_q(n-m)$. The map $\varphi : C(\mathrm{SU}_q(n)) \to C(\mathrm{SU}_q(n-m))$ defined by

$$\varphi(u_{ij}) := \begin{cases} v_{ij} & \text{if } 1 \le i,\ j \le n-m, \\ \delta_{ij} & \text{otherwise.} \end{cases} \tag{2-1}$$

is a surjective unital C^*-algebra homomorphism such that $\Delta \circ \varphi = (\varphi \otimes \varphi)\Delta$. In this way, the quantum group $\mathrm{SU}_q(n-m)$ is a subgroup of the quantum group $\mathrm{SU}_q(n)$. The C^*-algebra of the quotient $\mathrm{SU}_q(n)/\mathrm{SU}_q(n-m)$ is defined as

$$C\big(\mathrm{SU}_q(n)/\mathrm{SU}_q(n-m)\big) := \big\{a \in C(\mathrm{SU}_q(n)) :\ (\varphi \otimes 1)\Delta(a) = 1 \otimes a\big\}.$$

We refer to [Podkolzin and Vainerman 1999] for the proof of the following:

Proposition 2.1. *The C^*-algebra $C(\mathrm{SU}_q(n)/\mathrm{SU}_q(n-m))$ is generated by the last m rows of the matrix (u_{ij}), that is, by the set $\{u_{ij} : n-m+1 \le i \le n\}$.*

In [Podkolzin and Vainerman 1999], the quotient space $\mathrm{SU}_q(n)/\mathrm{SU}_q(n-m)$ is called a quantum Stiefel manifold and is denoted by $S_q^{n,m}$. We will use the same notation.

Before proceeding further, let us fix some notations. Let $\mathbb{N}$ be the set of nonnegative integers. Consider the number operator N and the left shift S on $\ell^2(\mathbb{N})$

defined on the standard orthonormal basis $\{e_n : n \geq 0\}$ by

$$Se_n := e_{n-1} \quad \text{and} \quad Ne_n := ne_n.$$

Note that N is an unbounded selfadjoint operator. We denote by τ the C^*-algebra generated by S. The C^*-algebra τ is nothing but the Toeplitz algebra.

The irreducible representations of the C^*-algebra $C(S_q^{n,m})$ was described in [Podkolzin and Vainerman 1999]. First, we recall the irreducible representations of $C(\mathrm{SU}_q(n))$ as in [Soibelman 1990]. The one-dimensional representations of $C(\mathrm{SU}_q(n))$ are parametrized by the torus $\mathbb{T}^{n-1}$. We consider $\mathbb{T}^{n-1}$ as a subset of $\mathbb{T}^n$ under the inclusion $(t_1, t_2, \ldots t_{n-1}) \to (t_1, t_2, \ldots, t_{n-1}, t_n)$, where $t_n := \prod_{i=1}^{n-1} \bar{t}_i$. For $t := (t_1, t_2, \ldots, t_n) \in \mathbb{T}^{n-1}$, let $\tau_t : C(\mathrm{SU}_q(n)) \to \mathbb{C}$ be defined as

$$\tau_t(u_{ij}) := t_{n-i+1}\delta_{ij}.$$

Then, τ_t is a $*$-algebra homomorphism. The set $\{\tau_t : t \in \mathbb{T}^{n-1}\}$ is a complete set of mutually inequivalent one-dimensional representations of $C(\mathrm{SU}_q(n))$.

Denote the transposition $(i, i+1)$ by s_i. The map $\pi_{s_i} : C(\mathrm{SU}_q(n)) \to B(\ell^2(\mathbb{N}))$, defined on the generators u_{rs} by

$$\pi_{s_i}(u_{rs}) := \begin{cases} \sqrt{1-q^{2N+2}}\,S & \text{if } r=i,\ s=i, \\ -q^{N+1} & \text{if } r=i,\ s=i+1, \\ q^N & \text{if } r=i+1,\ s=i, \\ S^*\sqrt{1-q^{2N+2}} & \text{if } r=i+1,\ s=i+1, \\ \delta_{ij} & \text{otherwise,} \end{cases}$$

is a $*$-algebra homomorphism. For any two representations φ and ξ of $C(\mathrm{SU}_q(n))$, let $\varphi * \xi := (\varphi \otimes \xi)\Delta$. For $\omega \in S_n$, let $\omega = s_{i_1}s_{i_2}\ldots s_{i_k}$ be a reduced expression. Then, the representation $\pi_\omega := \pi_{s_{i_1}} * \pi_{s_{i_2}} * \cdots * \pi_{s_{i_k}}$ is an irreducible representation. Up to unitary equivalence, the representation π_ω is independent of the reduced expression. For $t \in \mathbb{T}^{n-1}$ and $\omega \in S_n$ let $\pi_{t,\omega} := \tau_t * \pi_\omega$. We refer to [Soibelman 1990] for the proof of the following:

Theorem 2.2. $\{\pi_{t,\omega} : t \in \mathbb{T}^{n-1}, \omega \in S_n\}$ *is a complete set of mutually inequivalent irreducible representations of* $C(\mathrm{SU}_q(n))$.

The irreducible representations of $C(S_q^{n,m})$ were studied in [Podkolzin and Vainerman 1999]. We recall them here. Embed $\mathbb{T}^m$ into $\mathbb{T}^{n-1}$ via the map

$$t = (t_1, t_2, \ldots, t_m) \to (t_1, t_2, \ldots, t_m, 1, 1, \ldots, 1, t_n),$$

where $t_n := \prod_{i=1}^m \bar{t}_i$. For a permutation $\omega \in S_n$, let ω^s be the permutation in the coset $S_{n-m}\omega$ with the least possible length. We denote the restriction of the representation $\pi_{t,\omega}$ to the subalgebra $C(S_q^{n,m})$ by $\pi_{t,\omega}$ itself.

Theorem 2.3 [Podkolzin and Vainerman 1999]. *The set* $\{\pi_{t,\omega^s} : t \in \mathbb{T}^m, \omega \in S_n\}$ *is a complete set of mutually inequivalent irreducible representations of* $C(S_q^{n,m})$.

3. Composition sequences

In this section, we derive certain exact sequences analogous to that of [Sheu 1997, Theorem 4]. We then apply the six-term sequence in K-theory to compute the K-groups of $C(S_q^{n,2})$.

Lemma 3.1. *Let* $t \in \mathbb{T}^m$ *and* $\omega := s_{n-1}s_{n-2}\dots s_{n-k}$. *The image of* $C(S_q^{n,m})$ *under the homomorphism* $\pi_{t,\omega}$ *contains the algebra of compact operators* $\mathcal{K}(\ell^2(\mathbb{N}^k))$.

Proof. Since $\pi_{t,\omega}(C(S_q^{n,m})) = \pi_\omega(C(S_q^{n,m}))$, it is enough to show that $\mathcal{K}(\ell^2(\mathbb{N}^k)) \subset \pi_\omega(C(S_q^{n,m}))$. We prove this result by induction on n. Since

$$\pi_\omega(u_{nn}) := S^*\sqrt{1-q^{2N+2}} \otimes 1,$$

it follows that $S \otimes 1 \in \pi_\omega(C(S_q^{n,m})$. Hence, $\mathcal{K}(\ell^2(\mathbb{N})) \otimes 1 \subset \pi_w(C(S_q^{n,m})$, and the result is true when $n = 2$.

Next, observe that $(p \otimes 1)\,\pi_\omega(u_{n,i}) := p \otimes \pi_{\omega'}(v_{n-1,i})$ for $1 \le i \le n-1$, where $\omega' := s_{n-2}s_{n-3}\dots s_{n-k}$ and (v_{ij}) denotes the generators of $C(\mathrm{SU}_q(n-1))$. Hence, $\pi_\omega(C(S_q^{n,m}))$ contains the algebra $p \otimes \pi_{\omega'}(C(S_q^{n-1,m}))$. Now, by the induction hypothesis, it follows that $\pi_\omega(C(S_q^{n,m}))$ contains $p \otimes \mathcal{K}(\ell^2(\mathbb{N}^{k-1}))$. Since $\pi_\omega(C(S_q^{n,m}))$ contains both $\mathcal{K}(\ell^2(\mathbb{N})) \otimes 1$ and $p \otimes \mathcal{K}(\ell^2(\mathbb{N}^{k-1}))$, it follows that $\pi_\omega(C(S_q^{n,m}))$ contains the algebra of compact operators, which completes the proof. □

Let w be a word on $s_1, s_2, \dots, s_n$, say, $w := s_{i_1}s_{i_2}\dots s_{i_n}$ (not necessarily a reduced expression). Define $\psi_w := \pi_{s_{i_1}} * \pi_{s_{i_2}} * \dots \pi_{s_{i_r}}$ and, for $t \in \mathbb{T}^n$, let $\psi_{t,w} := \tau_t * \psi_w$. Observe that the image of $\psi_{t,w}$ is contained in $\tau^{\otimes r}$. We prove that, if w' is a subword of w, then $\psi_{t,w'}$ factors through $\psi_{t,w}$.

Proposition 3.2. *Let* $w = w_1 s_k w_2$ *be a word on* $s_1, s_2, \dots, s_n$. *Denote the word* w_1w_2 *by* w' *and let* $t \in \mathbb{T}^m$ *be given. There exists a* $*$*-homomorphism*

$$\varepsilon : \psi_{t,w}(C(S_q^{n,m})) \to \psi_{t,w'}(C(S_q^{n,m}))$$

such that $\psi_{t,w'} = \varepsilon \circ \psi_{t,w}$.

Proof. If $\ell(u)$ denotes the length of a word u on $s_1, s_2, \dots, s_n$, then $\psi_{t,w}(C(S_q^{n,m}))$ is contained in $\tau^{\otimes \ell(w_1)} \otimes \tau \otimes \tau^{\otimes \ell(w_2)}$. Let ε denote the restriction of $1 \otimes \sigma \otimes 1$ to $\psi_{t,w}(C(S_q^{n,m}))$, where $\sigma : \tau \to \mathbb{C}$ is the homomorphism for which $\sigma(S) = 1$.

$$\psi_{t,w}(u_{rs}) = \sum_{j_1,j_2} \psi_{t,w_1}(u_{rj_1}) \otimes \pi_{s_k}(u_{j_1j_2}) \otimes \psi_{w_2}(u_{j_2s}).$$

Since $\sigma(\pi_{s_k}(u_{j_1 j_2})) = \delta_{j_1 j_2}$, it follows that

$$\varepsilon \circ \psi_{t,w}(u_{rs}) = \sum_j \psi_{t,w_1}(u_{rj}) \otimes \psi_{w_2}(u_{js}) = \psi_{t,w'}(u_{rs}).$$

This completes the proof. □

Let w be a word on $s_1, s_2, \ldots s_n$. Then, for $n-m+1 \leq i \leq n$ and $1 \leq j \leq n$, the map $\mathbb{T}^m : t \to \psi_{t,w}(u_{ij}) \in \tau^{\otimes \ell(w)}$ is continuous. Thus, we get a homomorphism $\chi_w : C(S_q^{n,m}) \to C(\mathbb{T}^m) \otimes \tau^{\otimes \ell(w)}$ such that $\chi_w(a)(t) = \psi_{t,w}(a)$ for all $a \in C(S_q^{n,m})$.

Remark 3.3. Clearly, for a word w on $s_1, s_2, \ldots s_n$, the representations $\psi_{t,w}$ factors through χ_w. One can also prove, as in Proposition 3.2, that if w' is a subword of w, then $\chi_{w'}$ factors through χ_w.

Let us introduce some notation. Denote by $\omega_{j,i}$ the permutation $s_j s_{j-1} \ldots s_i$ for $j \geq i$. If $j < i$, let $\omega_{j,i} := 1$. For $1 \leq k \leq n$, let $\omega_k := \omega_{n-m,1}\omega_{n-m+1,1} \ldots \omega_{n-1,n-k+1}$.

Theorem 3.4. *The homomorphism $\chi_{\omega_n} : C(S_q^{n,m}) \to C(\mathbb{T}^m) \otimes \tau^{\otimes \ell(\omega_n)}$ is faithful.*

Proof. If $\omega_0 \in S_n$ then ω_0^s(the representative in $S_{n-m}\omega_0$ with the shortest length) is a subword of ω_n. By Remark 3.3, it follows that every irreducible representation of $C(S_q^{n,m})$ factors through χ_{ω_n}. Hence, χ_{ω_n} is faithful. This completes the proof. □

For $1 \leq k \leq n$, let $C(S_q^{n,m,k}) := \chi_{\omega_k}(C(S_q^{n,m}))$. Then,

$$C(S_q^{n,m,k}) \subset C(S_q^{n,m,1}) \otimes \tau^{\otimes (k-1)}.$$

For $2 \leq k \leq n$, let σ_k denote the restriction of $(1 \otimes 1^{\otimes(k-2)} \otimes \sigma)$ to $C(S_q^{n,m,k})$. The image of σ_k is $C(S_q^{n,m,k-1})$. We determine the kernel of σ_k in the next proposition. We need the following two lemmas.

Lemma 3.5. *The algebra $\chi_{\omega_{n-1,n-k}}(C(S_q^{n,1}))$ contains $C^*(t_1) \otimes \mathcal{K}(\ell^2(\mathbb{N}^k))$, which is isomorphic to $C(\mathbb{T}) \otimes \mathcal{K}(\ell^2(\mathbb{N}^k))$.*

Proof. Note that $\chi_{\omega_{n-1,n-k}}(u_{nn}) = t_1 \otimes S^* \sqrt{1-q^{2N+2}} \otimes 1$. Hence it follows that the operator

$$1 \otimes \sqrt{1-q^{2N+2}} \otimes 1 = \chi_{\omega_{n-1,n-k}}(u_{nn}^* u_{nn})$$

lies in the algebra $\chi_{\omega_{n-1,n-k}}(C(S_q^{n,1}))$. As $\sqrt{1-q^{2N+2}}$ is invertible, $t_1 \otimes S^* \otimes 1 \in \chi_{\omega_{n-1,n-k}}(C(S_q^{n,1}))$. Thus, the projection $1 \otimes p \otimes 1$ is in the algebra $C(S_q^{n,1,k+1})$. Observe that, for $1 \leq s \leq n-1$, one has

$$(1 \otimes p \otimes 1)\, \chi_{\omega_{n-1,n-k}}(u_{ns}) = t_1 \otimes p \otimes \pi_{\omega_{n-2,n-k}}(v_{n-1,s}), \tag{3-1}$$

where (v_{ij}) are the generators of $C(\mathrm{SU}_q(n-1))$. If $n = 2$, then $k = 1$, and what we have shown is that $C(S_q^{2,1,2})$ contains $t_1 \otimes S^*$ and $t_1 \otimes p$. Hence, $C^*(t_1) \otimes \mathcal{K}$ is contained in the algebra $C(S_q^{2,1,2})$.

We can now complete the proof by induction on n. Equation (3-1) shows that $C^*(t_1)\otimes p\otimes \mathcal{K}^{\otimes(k-1)}$ is contained in the algebra $C(S_q^{n,1,k+1})$. Also, $t_1\otimes S^*\otimes 1\in C(S_q^{n,1,k+1})$. It follows that $C^*(t_1)\otimes \mathcal{K}^{\otimes k}$ is contained in the algebra $C(S_q^{n,1,k+1})$. This completes the proof. □

Lemma 3.6. *Given $1\le s\le n$, there exist compact operators x_s, y_s such that $x_s\pi_{\omega_{n-1,n-k}}(u_{js})y_s=\delta_{js}(p\otimes p\otimes\cdots\otimes p)$, where $p:=1-S^*S$.*

Proof. Let $1\le s\le n$ be given. Note that the operator

$$\omega_{n-1,n-k}(u_{ss})=z_1\otimes z_2\otimes\cdots\otimes z_k,$$

where $z_i\in\{1,\sqrt{1-q^{2N+2}}S,\,S^*\sqrt{1-q^{2N+2}}\}$. Define x_i, y_i by

$$x_i:=\begin{cases} p & \text{if } z_i=1,\\ p & \text{if } z_i=\sqrt{1-q^{2N+2}}S,\\ (1-q^2)^{-\frac{1}{2}}pS & \text{if } z_i=S^*\sqrt{1-q^{2N+2}};\end{cases}$$

$$y_i:=\begin{cases} p & \text{if } z_i=1,\\ (1-q^2)^{-\frac{1}{2}}S^*p & \text{if } z_i=\sqrt{1-q^{2N+2}}S,\\ p & \text{if } z_i=S^*\sqrt{1-q^{2N+2}}.\end{cases}$$

Then, $x_iz_iy_i=p$ for $1\le i\le k$. Now, let $x_s:=x_1\otimes x_2\otimes\ldots x_k$ and $y_s:=y_1\otimes y_2\otimes\ldots y_k$. Then,

$$x_s\,\chi_{\omega_{n-1,n-k}}(u_{ss})=\underbrace{p\otimes p\otimes\cdots\otimes p}_{k\text{ times}}.$$

Let $j\neq s$ be given. Then, $\chi_{\omega_{n-1,n-k}}(u_{js})=a_1\otimes a_2\otimes\cdots\otimes a_k$ where

$$a_i\in\{1,\sqrt{1-q^{2N+2}}\,S,\,S^*\sqrt{1-q^{2N+2}},\,-q^{N+1},\,q^N\}.$$

Since $j\neq s$, there exists an i such that $a_i\in\{q^N,-q^{N+1}\}$. Let r be the largest integer for which $a_r\in\{q^N,-q^{N+1}\}$. Then, $z_r\neq 1$ and hence $x_ra_ry_r=0$. Thus, $x_s\,\chi_{\omega_{n-1,n-k}}(u_{js})\,y_s=0$, which completes the proof. □

Proposition 3.7. *Let $2\le k\le n$. Then, $C(S_q^{n,m,1})\otimes\mathcal{K}(\ell^2(\mathbb{N}))^{\otimes(k-1)}$ is contained in the algebra $C(S_q^{n,m,k})$. Moreover, the kernel of the homomorphism σ_k is exactly $C(S_q^{n,m,1})\otimes\mathcal{K}(\ell^2(\mathbb{N}))^{\otimes(k-1)}$. We have the exact sequence*

$$0\longrightarrow C(S_q^{n,m,1})\otimes\mathcal{K}^{\otimes(k-1)}\longrightarrow C(S_q^{n,m,k})\xrightarrow{\sigma_k}C(S_q^{n,m,k-1})\longrightarrow 0.$$

Proof. First, we prove that $C(S_q^{n,m,1})\otimes\mathcal{K}^{\otimes(k-1)}$ is contained in $C(S_q^{n,m,k})$. For $a\in C(S_q^{n,1})$, one has $\chi_{\omega_k}(a)=1\otimes\chi_{\omega_{n-1,n-k+1}}(a)$, and it follows from Lemma 3.5 that $C(S_q^{n,m,k})$ contains $1\otimes\mathcal{K}(\ell^2(\mathbb{N}^{k-1}))$. Let $n-m+1\le r\le m$ and $1\le s\le n$ be given. Note that

$$\chi_{\omega_k}(u_{rs}) = \sum_{j=1}^{n} \chi_{\omega_1}(u_{rj}) \otimes \pi_{\omega_{n-1,n-k+1}}(u_{js}).$$

By Lemma 3.6, there exist $x_s, y_s \in C(S_q^{n,m,k})$ such that

$$x_s\,\chi_{\omega_k}(u_{rs})\,y_s := \chi_{\omega_1}(u_{rs}) \otimes p^{\otimes(k-1)},$$

where $p^{\otimes(k-1)} := p \otimes p \otimes \cdots \otimes p$. Thus, we have shown that $C(S_q^{n,m,k})$ contains $1 \otimes \mathcal{K}^{\otimes(k-1)}$ and $C(S_q^{n,m,1}) \otimes p^{\otimes(k-1)}$. Hence, $C(S_q^{n,m,k})$ contains $C(S_q^{n,m,1}) \otimes \mathcal{K}^{\otimes(k-1)}$.

Clearly, σ_k vanishes on $C(S_q^{n,m,1}) \otimes \mathcal{K}^{\otimes(k-1)}$. Let π be an irreducible representation of $C(S_q^{n,m,k})$ which vanishes on the ideal $C(S_q^{n,m,1}) \otimes \mathcal{K}^{\otimes(k-1)}$. Then, $\pi \circ \chi_{\omega_k}$ is an irreducible representation of $C(S_q^{n,m})$ and hence $\pi \circ \chi_{\omega_k} = \pi_{t,\omega}$ for $t \in \mathbb{T}^m$ and some ω of the form $\omega_{n-m,i_1}\omega_{n-m+1,i_2}\cdots\omega_{n-1,i_{n-m}}$. Since $\pi \circ \chi_{\omega_k}(u_{n,n-k+1}) = 0$, it follows that $\pi_{t,w}(u_{n,n-k+1}) = 0$. However, $\pi_{t,\omega}(u_{n,n-k+1}) = t_n(1 \otimes \pi_{\omega_{n-1,i_{n-m}}}(u_{n,n-k+1}))$ and hence $i_{n-m} > n-k+1$. In other words, ω is a subword of ω_{k-1}. Therefore, $\pi \circ \chi_{\omega_k}$ factors through $\chi_{\omega_{k-1}}$ and so there exists a representation ρ of $C(S_q^{n,m,k-1})$ such that $\pi \circ \chi_{\omega_k} = \rho \circ \chi_{\omega_{k-1}}$. Since $\chi_{\omega_{k-1}} = \sigma_k \circ \chi_{\omega_k}$, it follows that $\pi = \rho \circ \sigma_k$.

We have shown that every irreducible representation of $C(S_q^{n,m,k})$ which vanishes on the ideal $C(S_q^{n,m,1}) \otimes \mathcal{K}^{\otimes(k-1)}$ factors through σ_k. Hence, the kernel of σ_k is exactly the ideal $C(S_q^{n,m,1}) \otimes \mathcal{K}^{\otimes(k-1)}$. This completes the proof. □

We apply the six-term exact sequence in K-theory to the exact sequence in Proposition 3.7 to compute the K-groups of $C(S_q^{n,2,k})$ for $1 \le k \le n$. In the next section, we briefly recall the product operation in K-theory.

4. The operation P

The algebras that we consider will be nuclear. So, no problem arises with regard to tensor products. Let A and B be C^*-algebras. We have the product maps

$$K_0(A) \otimes K_0(B) \to K_0(A \otimes B), \qquad K_1(A) \otimes K_0(B) \to K_1(A \otimes B),$$
$$K_0(A) \otimes K_1(B) \to K_1(A \otimes B), \qquad K_1(A) \otimes K_1(B) \to K_0(A \otimes B).$$

The first map is defined as $[p] \otimes [q] \to [p \otimes q]$; the second one, as $[u] \otimes [p] \to [u \otimes p + 1 - 1 \otimes p]$; and the third one likewise. The fourth map is defined using Bott periodicity and the first product; we describe it briefly, referring the reader to [Connes 1981, Appendix] for details.

Let $h : \mathbb{T}^2 \to P_1(\mathbb{C}) := \{p \in \mathrm{Proj}(M_2(\mathbb{C})) : \mathrm{trace}(p) = 1\}$ be a degree-one map. Given unitaries $u \in M_p(A)$ and $v \in M_q(B)$, the product $[u] \otimes [v]$ is given by

$[h(u, v)] - [e_0]$, where

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(M_{pq}(A \otimes B))$$

and $h(u, v)$ is the matrix with entries $h_{ij}(u \otimes 1, 1 \otimes v)$. We denote the image of $[x] \otimes [y]$ by $[x] \otimes [y]$ itself. Let A be a unital commutative C^*-algebra. Then, the multiplication $m : A \otimes A \to A$ is a C^*-algebra homomorphism. Hence, we get a map at the K-theory level from $K_1(A) \otimes K_1(A)$ to $K_0(A)$.

Suppose U and V are two commuting unitaries in a C^*-algebra B. If $A := C^*(U, V)$, then A is commutative. Define

$$P(U, V) := K_0(m)([U] \otimes [V]),$$

which is an element in $K_0(A)$. By composing with the inclusion map, we can think of it as an element in $K_0(B)$. From the formula of [Connes 1981] that we just recalled, the following properties are clear:

(1) If U and V are commuting unitaries in A, and p is a rank-one projection in $\mathcal{K}$, then we have $P(U \otimes p + 1 - 1 \otimes p, V \otimes p + 1 - 1 \otimes p) := P(U, V) \otimes p$.

(2) If U and V are commuting unitaries, and p is a projection that commutes with U and V, then $P(U, Vp + 1 - p) = P(Up + 1 - p, Vp + 1 - p)$.

(3) If $\varphi : A \to B$ is a unital homomorphism, and U and V are commuting unitaries in A, then $K_0(\varphi)(P(U, V)) = P(\varphi(U), \varphi(V))$.

(4) If U is a unitary in A, then $P(U, U) = 0$. Since $P_1(\mathbb{C})$ is simply connected, the matrix $h(U, U)$ is path-connected to a rank-one projection in $M_2(\mathbb{C})$. Hence, $P(U, U) = 0$.

We need the following lemma in the six-term computation. Let $z_1 \otimes 1$ and $1 \otimes z_2$ be the generating unitaries of $C(\mathbb{T}) \otimes C(\mathbb{T})$. Then, $K_0(C(\mathbb{T}^2))$ is isomorphic to $\mathbb{Z}^2$ and is generated by 1 and $P(z_1 \otimes 1, 1 \otimes z_2)$.

Lemma 4.1. *Consider the exact sequence*

$$0 \longrightarrow C(\mathbb{T}) \otimes \mathcal{K} \longrightarrow C(\mathbb{T}) \otimes \tau \longrightarrow C(\mathbb{T}) \otimes C(\mathbb{T}) \longrightarrow 0$$

and the six-term sequence in K-theory

$$\begin{array}{ccccc} K_0(C(\mathbb{T}) \otimes \mathcal{K}) & \longrightarrow & K_0(C(\mathbb{T}) \otimes \tau) & \longrightarrow & K_0(C(\mathbb{T}) \otimes C(\mathbb{T})) \\ {\scriptstyle\partial}\uparrow & & & & \downarrow{\scriptstyle\delta} \\ K_1(C(\mathbb{T}) \otimes C(\mathbb{T})) & \longleftarrow & K_1(C(\mathbb{T}) \otimes \tau) & \longleftarrow & K_1(C(\mathbb{T}) \otimes \mathcal{K}). \end{array}$$

The subgroup generated by $\delta(P(z_1 \otimes 1, 1 \otimes z_2))$ coincides with the group generated by $z_1 \otimes p + 1 - 1 \otimes p$, which is $K_1(C(\mathbb{T}) \otimes \mathcal{K}) \cong \mathbb{Z}$.

Proof. The Toeplitz map $\varepsilon : \tau \to C(\mathbb{T})$ induces an isomorphism at the K_0-level. Thus, by the Künneth theorem, it follows that the image of $K_0(1 \otimes \varepsilon)$ is $K_0(C(\mathbb{T})) \otimes K_0(C(\mathbb{T}))$, which is the subgroup generated by [1]. The inclusion $0 \to \mathcal{K} \to \tau$ induces the zero map at the K_0 level and hence, again by the Künneth theorem, the inclusion $0 \to C(\mathbb{T}) \otimes \mathcal{K} \to C(\mathbb{T}) \otimes \tau$ induces the zero map at the K_1-level. Thus, the image of δ is $K_1(C(\mathbb{T}) \otimes \mathcal{K})$, which completes the proof. □

Corollary 4.2. *Let* $0 \longrightarrow I \longrightarrow A \xrightarrow{\varphi} B \longrightarrow 0$ *be a short exact sequence of* C^**-algebras. Consider the six-term sequence in* K*-theory*

$$\begin{array}{ccccc} K_0(I) & \longrightarrow & K_0(A) & \xrightarrow{K_0(\varphi)} & K_0(B) \\ {\scriptstyle\partial}\uparrow & & & & \downarrow{\scriptstyle\delta} \\ K_1(B) & \xleftarrow[K_1(\varphi)]{} & K_1(A) & \longleftarrow & K_1(I). \end{array}$$

Suppose that U *and* V *are two commuting unitaries in* B *such that there exists a unitary* X *and an isometry* Y *with* $\varphi(X) = U$ *and* $\varphi(Y) = V$. *If* X *and* Y *commute, then the subgroup generated by* $\delta(P(U, V))$ *coincides with the subgroup generated by the unitary* $X(1 - YY^*) + YY^*$ *in* $K_1(I)$.

Proof. Since $C(\mathbb{T})$ is the universal C^*-algebra generated by a unitary, and τ is the universal C^*-algebra generated by an isometry, there exists homomorphisms $\Phi : C(\mathbb{T}) \otimes \tau \to A$ and $\Psi : C(\mathbb{T}) \otimes C(\mathbb{T}) \to B$ such that

$$\Phi(z_1 \otimes 1) := X, \quad \Phi(1 \otimes S^*) := Y, \quad \Psi(z_1 \otimes 1) := U, \quad \Psi(1 \otimes z_2) := V.$$

We have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C(\mathbb{T}) \otimes \mathcal{K} & \longrightarrow & C(\mathbb{T}) \otimes \tau & \longrightarrow & C(\mathbb{T}) \otimes C(\mathbb{T}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\Phi} & & \downarrow{\scriptstyle\Phi} & & \downarrow{\scriptstyle\Psi} & & \\ 0 & \longrightarrow & I & \longrightarrow & A & \xrightarrow{\varphi} & B & \longrightarrow & 0. \end{array}$$

By the functoriality of δ and P, it follows that

$$\delta(P(U, V)) = K_1(\Phi)(\delta(P(z_1 \otimes 1, 1 \otimes z_2))).$$

By Lemma 4.1, it follows that the subgroup generated by $\delta(P(U, V))$ is the subgroup generated by $\Phi(z_1 \otimes p + 1 - 1 \otimes p)$ in $K_1(I)$. Note that $\Phi(z_1 \otimes p + 1 - 1 \otimes p) = X(1 - YY^*) + YY^*$. This completes the proof. □

5. K-groups of $C(S_q^{n,2,k})$ for $k < n$

In this section, we compute the K-groups of $C(S_q^{n,2,k})$ for $1 \leq k < n$, by applying the six-term sequence in K-theory to the exact sequence in Proposition 3.7. We fix some notation. If q is a projection in $\ell^2(\mathbb{N})$ then q_r denotes the projection $q \otimes q \otimes \cdots \otimes q$ (r factors) in $\ell^2(\mathbb{N}^r)$. We define the unitaries U_k, V_k, u_k, v_k by

$$\begin{aligned} U_k &:= t_1 \otimes 1_{n-2} \otimes p_{k-1} + 1 - 1 \otimes 1_{n-2} \otimes p_{k-1}, \\ V_k &:= t_2 \otimes p_{n-2} \otimes 1_{k-1} + 1 - 1 \otimes p_{n-2} \otimes 1_{k-1}, \\ u_k &:= t_1 \otimes p_{n-2} \otimes p_{k-1} + 1 - 1 \otimes p_{n-2} \otimes p_{k-1}, \\ v_k &:= t_2 \otimes p_{n-2} \otimes p_{k-1} + 1 - 1 \otimes p_{n-2} \otimes p_{k-1}. \end{aligned}$$

Note that the operators U_k, V_k, u_k, v_k lies in the algebra $C(S_q^{n,2,k})$. Indeed,

$$\begin{aligned} U_k &= 1_{\{1\}}(u_{n,n-k+1}u^*_{n,n-k+1})\, u_{n,n-k+1} + 1 - 1_{\{1\}}(u_{n,n-k+1}u^*_{n,n-k+1}), \\ V_k &= 1_{\{1\}}(u_{n-1,1}u^*_{n-1,1})\, u_{n-1,1} + 1 - 1_{\{1\}}(u_{n-1,1}u^*_{n-1,1}), \\ u_k &= 1_{\{1\}}(u_{n,n-k+1}u^*_{n,n-k+1}u_{n-1,1}u^*_{n-1,1})\, u_{n,n-k+1} + 1 \\ &\qquad - 1_{\{1\}}(u_{n,n-k+1}u^*_{n,n-k+1}u_{n-1,1}u^*_{n-1,1}), \\ v_k &= 1_{\{1\}}(u_{n,n-k+1}u^*_{n,n-k+1}u_{n-1,1}u^*_{n-1,1})u_{n-1,1} + 1 \\ &\qquad - 1_{\{1\}}(u_{n,n-k+1}u^*_{n,n-k+1}u_{n-1,1}u^*_{n-1,1}). \end{aligned}$$

Note that the unitaries U_n, u_n and v_n lie in the algebra $C(S_q^{n,2,n})$. We start with the computation of the K-groups of $C(S_q^{n,2,1})$.

Lemma 5.1. *The K-groups $K_0(C(S_q^{n,2,1}))$ and $K_1(C(S_q^{n,2,1}))$ are both isomorphic to $\mathbb{Z}^2$. In fact, $[U_1]$ and $[V_1]$ form a $\mathbb{Z}$-basis for $K_1(C(S_q^{n,2,1}))$, while $[1]$ and $P(u_1, v_1)$ form a $\mathbb{Z}$-basis for $K_0(C(S_q^{n,2,1}))$.*

Proof. First, note that $C(S_q^{n,2,1})$ is generated by $t_1 \otimes 1_{n-2}$ and $t_2 \otimes \pi_{\omega_{n-2,1}}(u_{n-1,j})$ for $1 \le j \le n-1$. The C^*-algebra generated by $\{t_2 \otimes \pi_{\omega_{n-2,1}}(u_{n-1,j}) : 1 \le j \le n-1\}$ is isomorphic to $C(S_q^{2n-3})$. Hence, $C(S_q^{n,2,1})$ is isomorphic to $C(\mathbb{T}) \otimes C(S_q^{2n-3})$. Also, $K_0(C(S_q^{2n-3})$ and $K_1(C(S_q^{2n-3}))$ are both isomorphic to $\mathbb{Z}$, with $[1]$ generating $K_0(C(S_q^{2n-3}))$, and $[t_2 \otimes p_{n-2} + 1 - 1 \otimes p_{n-2}]$ generating $K_1(C(S_q^{2n-3}))$.

Now, by the Künneth theorem for the tensor product of C^*-algebras (see [Blackadar 1986]), it follows that $C(S_q^{n,2,1})$ has both K_1 and K_0 isomorphic to $\mathbb{Z}^2$, with $[U_1]$ and $[V_1]$ generating $K_1(C(S_q^{n,2,1}))$, and $[1]$ and

$$P(t_1 \otimes 1_{n-2}, t_2 \otimes p_{n-2} + 1 - 1 \otimes p_{n-2})$$

generating $K_0(C(S_q^{n,2,1}))$. The projection $1 \otimes p_{n-2} = 1_{\{1\}}(\chi_{\omega_{n-2,1}}(u_{n-1,1}u^*_{n-1,1}))$ is in $C(S_q^{n,2,1})$ and commutes with the unitaries $t_1 \otimes 1_{n-2}$ and $t_2 \otimes p_{n-2} + 1 - 1 \otimes p_{n-2}$. Hence,

$$P(t_1 \otimes 1_{n-2}, t_2 \otimes p_{n-2} + 1 - 1 \otimes p_{n-2}) = P(u_1, v_1).$$

This completes the proof. □

Proposition 5.2. *Let $1 \leq k < n$. The K-groups $K_0(C(S_q^{n,2,k}))$ and $K_1(C(S_q^{n,2,k}))$ are both isomorphic to $\mathbb{Z}^2$ and, in particular, $[U_k]$ and $[V_k]$ form a $\mathbb{Z}$-basis for $K_1(C(S_q^{n,2,k}))$, while $[1]$ and $P(u_k, v_k)$ form a $\mathbb{Z}$-basis for $K_0(C(S_q^{n,2,k}))$.*

Proof. We prove this result by induction on k. The case $k = 1$ is just Lemma 5.1. Assume the result to be true for k. From Proposition 3.7, we have the short exact sequence

$$0 \longrightarrow C(S_q^{n,2,1}) \otimes \mathcal{K}^{\otimes(k)} \longrightarrow C(S_q^{n,2,k+1}) \xrightarrow{\sigma_{k+1}} C(S_q^{n,2,k}) \longrightarrow 0,$$

which gives rise to the following six-term sequence in K-theory:

$$\begin{array}{ccccc} K_0(C(S_q^{n,2,1}) \otimes \mathcal{K}^{\otimes k}) & \longrightarrow & K_0(C(S_q^{n,2,k+1})) & \xrightarrow{K_0(\sigma_{k+1})} & K_0(C(S_q^{n,2,k})) \\ \partial \uparrow & & & & \downarrow \delta \\ K_1(C(S_q^{n,2,k}) & \xleftarrow[K_1(\sigma_{k+1})]{} & K_1(C(S_q^{n,2,k+1})) & \longleftarrow & K_1(C(S_q^{n,2,1}) \otimes \mathcal{K}^{\otimes k}). \end{array}$$

To compute the six-term sequence, we determine δ and ∂. Since $\sigma_{k+1}(V_{k+1}) = V_k$, it follows that $\partial([V_k]) = 0$. Since $C(S_q^{n,2,k+1})$ contains the algebra $C(S_q^{n,2,1}) \otimes \mathcal{K}^{\otimes k}$, it follows that the operator

$$\tilde{X} := t_1 \otimes 1_{n-2} \otimes \underbrace{q^N \otimes q^N \otimes \dots q^N}_{(k-1)\text{ times}} \otimes S^*$$

is in the algebra $C(S_q^{n,2,1})$; indeed, the difference $X - \chi_{\omega_{k+1}}(u_{n,n-k+1})$ lies in the ideal $C(S_q^{n,2,1}) \otimes \mathcal{K}^{\otimes k}$. Let

$$X := 1_{\{1\}}(\tilde{X}^* \tilde{X}) \tilde{X} + 1 - 1_{\{1\}}(\tilde{X}^* \tilde{X}).$$

Then, X is an isometry such that $\sigma_{k+1}(X) = U_k$ and hence

$$\partial([U_k]) = [1 - X^* X] - [1 - X X^*].$$

Thus, $\partial([U_k]) = -[1 \otimes 1_{n-2} \otimes p_k]$. The image of ∂ is the subgroup of $K_0(C(S_q^{n,2,1}) \otimes \mathcal{K}^{\otimes k})$ generated by $[1 \otimes 1_{n-2} \otimes p_k]$, while its kernel is $[V_k]$.

Next, we compute δ. Since $\sigma_{k+1}(1) = 1$, it follows that $\delta([1]) = 0$. Let

$$Y := (1 \otimes p_{n-2} \otimes 1_k)(1 \otimes 1_{n-2} \otimes p_{k-1} \otimes 1) \tilde{X} + 1 - 1 \otimes p_{n-2} \otimes p_{k-1} \otimes 1.$$

Since $1 \otimes p_{n-2} \otimes 1 = 1_{\{1\}}(\chi_{\omega_k}(u_{n-1,1}^* u_{n-1,1}))$ and $1 \otimes 1_{n-2} \otimes p_{k-1} = 1_{\{1\}}(\tilde{X}^* \tilde{X})$, it follows that $Y \in C(S_q^{n,2,k+1})$. Also,

$$Y = t_1 \otimes p_{n-2} \otimes p_{k-1} \otimes S^* + 1 - 1 \otimes p_{n-2} \otimes p_{k-1} \otimes 1.$$

Note that Y is an isometry such that $\sigma_{k+1}(Y) = u_k$. One has $\sigma_{k+1}(v_{k+1}) = v_k$. Observe that Y and v_{k+1} commute. By Lemma 4.1, it follows that the image of δ is the subgroup generated by $[v_{k+1}(1 - YY^*) + YY^*] = [V_1 \otimes p_k + 1 - 1 \otimes p_k]$.

This computation with the six-term sequence implies that $K_0(C(S_q^{n,2,k+1}))$ is isomorphic to $\mathbb{Z}^2$ and is generated by $P(u_1, v_1) \otimes p_k = P(u_k, v_k)$ and $[1]$. Also, the group $K_1(C(S_q^{n,2,k+1}))$ is isomorphic to $\mathbb{Z}^2$ and is generated by $[V_{k+1}]$ and $[U_1 \otimes p_k + 1 - 1 \otimes p_k] = [U_{k+1}]$. This completes the proof. □

6. K-groups of $C(S_q^{n,2})$

In this section, we compute the K-groups of $C(S_q^{n,2})$. We start with a few observations.

Lemma 6.1. *In the permutation group S_n, one has $\omega_{n-2,1}\,\omega_{n-1,1} = \omega_{n-1,1}\,\omega_{n-1,2}$.*

Proof. First, note that $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, and $s_i s_j = s_j s_i$ if $|i-j| \geq 2$. Hence, $\omega_{n-1,k}\,\omega_{n-1,1} = \omega_{n-1,k+1}\,\omega_{n-1,1} s_{k+1}$. The result follows by induction on k. □

We denote the representation $\chi_{\omega_{n-1,1}} * \pi_{\omega_{n-1,2}}$ by $\tilde{\chi}_{\omega_n}$. Since $\omega_{n-1,1}\,\omega_{n-1,2}$ is a reduced expression for ω_n, the representations $\tilde{\chi}_{\omega_n}$ and χ_{ω_n} are equivalent. Let U be a unitary such that $U \chi_{\omega_n}(\,\cdot\,)\, U^* = \tilde{\chi}_{\omega_n}(\,\cdot\,)$. It is clear that

$$\tilde{\chi}_{\omega_n}(C(S_q^{n,2})) \subset C(\mathbb{T}^m) \otimes \tau \otimes \tau^{\otimes \ell(\omega_{n-1})}.$$

Let $\tilde{\sigma}_n$ denote the restriction of $1 \otimes \sigma \otimes 1^{\otimes(2(n-2)}$ to $\tilde{\chi}_{\omega_n}(C(S_q^{n,2}))$. Since

$$\tilde{\sigma}_n(\tilde{\chi}_{\omega_n}(u_{ij})) = \chi_{\omega_{n-1}}(u_{ij}),$$

we have the commutative diagram

$$\begin{array}{ccc} \chi_{\omega_n}(C(S_q^{n,2})) & \xrightarrow{\;U(\,\cdot\,)\,U^*\;} & \tilde{\chi}_{\omega_n}(C(S_q^{n,2})). \\ & {\scriptstyle \sigma_n}\searrow \quad \swarrow{\scriptstyle \tilde{\sigma}_n} & \\ & C(S_q^{n,2,n-1}) & \end{array}$$

Lemma 6.2. *There exists a coisometry $X \in \chi_{\omega_n}(C(S_q^{n,2}))$ such that $\sigma_n(X) = V_{n-1}$ and $X^*X = 1 - 1_{\{1\}}(\chi_{\omega_n}(u_{n1}^* u_{n1}))$.*

Proof. From the commutative diagram above, it is enough to show that there exists a coisometry $\widetilde{X} \in \tilde{\chi}_{\omega_n}(C(S_q^{n,2}))$ such that

$$\widetilde{\sigma_n}(X) = V_{n-1} \quad \text{and} \quad X^*X = 1 - 1_{\{1\}}(\widetilde{\chi}_{\omega_n}(u_{n1}^* u_{n1}).$$

Note that

$$\widetilde{\chi}_{\omega_n}(u_{n-1,1}^* u_{n-1,1} - q^2 u_{n1} u_{n1}) = 1 \otimes 1 \otimes \underbrace{q^{2N} \otimes q^{2N} \otimes \dots q^{2N}}_{(n-2)\text{ times}} \otimes 1_{n-2}.$$

Hence the projection $1 \otimes 1 \otimes p_{n-2} \otimes 1_{n-2} = 1_{\{1\}}(\widetilde{\chi}_{\omega_n}(u^*_{n-1,1}u_{n-1,1} - q^2 u^*_{n1}u_{n1}))$ is in the algebra $\widetilde{\chi}_{\omega_n}(C(S_q^{n,2}))$. Now let $Y := (1 \otimes 1 \otimes p_{n-2} \otimes 1_{n-2})\widetilde{\chi}_{\omega_n}(u_{n-1,1})$. Then,

$$Y := t_2 \otimes \sqrt{1-q^{2N+2}}\, S \otimes p_{n-2} \otimes 1_{n-2}.$$

Hence, the operator $Z := t_2 \otimes S \otimes p_{n-2} \otimes 1_{n-2}$ is in the algebra $\widetilde{\chi}_{\omega_n}(C(S_q^{n,2}))$. Now, let $\widetilde{X} := Z + 1 - ZZ^*$. Then, $\widetilde{X}$ is a coisometry such that $\widetilde{\sigma}_n(\widetilde{X}) = V_{n-1}$ and $\widetilde{X}^*\widetilde{X} = 1 - 1 \otimes p_{n-1} \otimes 1_{n-2}$, which is $1 - 1_{\{1\}}(\widetilde{\chi}_{\omega_n}(u^*_{n1}u_{n1}))$. This completes the proof. □

Observe that the operator

$$\tilde{Z} := t_1 \otimes 1_{n-1} \otimes \underbrace{q^N \otimes q^N \otimes \dots q^N}_{(n-2)\text{ times}} \otimes S^*$$

lies in the algebra $C(S_q^{n,2,n})$, since the difference $\tilde{Z} - \chi_{\omega_n}(u_{n,2})$ lies in the ideal $C(S_q^{n,2,1}) \otimes \mathcal{K}^{\otimes(n-1)}$. Let $Z := 1_{\{1\}}(\tilde{Z}^*\tilde{Z})\,\tilde{Z}$ and $Y_n := Z + 1 - Z^*Z$. Then,

$$Z_n = t_1 \otimes 1_{n-2} \otimes p_{n-2} \otimes S^*, \tag{6-1}$$

$$Y_n = t_1 \otimes 1_{n-2} \otimes p_{n-2} \otimes S^* + 1 - 1 \otimes 1_{n-2} \otimes p_{n-2} \otimes 1. \tag{6-2}$$

Hence, Y is an isometry and $YY^* = 1 - 1_{\{1\}}(\chi_{\omega_n}(u^*_{n1}u_{n1}))$. If X is a coisometry in $C(S_q^{n,2,n})$ such that $\sigma_n(X) = v_{n-1}$ and $X^*X := 1 - 1_{\{1\}}(\chi_{\omega_n}(u^*_{n1}u_{n1}))$, then XY is a unitary. (The existence of such an X was shown in Lemma 6.2.)

Proposition 6.3. *The K-groups $K_0(C(S_q^{n,2})$ and $K_1(C(S_q^{n,2})$ are both isomorphic to $\mathbb{Z}^2$. In particular,*

1. *the projections* $[1]$ *and* $P(u_n, v_n)$ *generate* $K_0(C(S_q^{n,2}))$;
2. *the unitaries U_n and XY_n generate $K_1(C(S_q^{n,2}))$, where X is a coisometry in $C(S_q^{n,2})$ such that $\sigma_n(X) = V_{n-1}$ and $X^*X = 1 - 1_{\{1\}}(u^*_{n1}u_{n1})$, while Y_n is as in (6-2)*

Proof. By Proposition 3.7, we have the exact sequence

$$0 \longrightarrow C(S_q^{n,2,1}) \otimes \mathcal{K}^{\otimes(n-1)} \longrightarrow C(S_q^{n,2,n}) \xrightarrow{\sigma_n} C(S_q^{n,2,n-1}) \longrightarrow 0,$$

which gives rise to the six-term sequence in K-theory

$$\begin{array}{ccccc} K_0(C(S_q^{n,2,1}) \otimes \mathcal{K}^{\otimes n-1}) & \longrightarrow & K_0(C(S_q^{n,2,n})) & \xrightarrow{K_0(\sigma_n)} & K_0(C(S_q^{n,2,k})) \\ \partial \uparrow & & & & \downarrow \delta \\ K_1(C(S_q^{n,2,n-1})) & \xleftarrow[K_1(\sigma_n)]{} & K_1(C(S_q^{n,2,n})) & \longleftarrow & K_1(C(S_q^{n,2,1}) \otimes \mathcal{K}^{\otimes n-1}). \end{array}$$

Now, we evaluate ∂ and δ to compute the six-term sequence. Since $[U_{n-1}]$ and $[V_{n-1}]$ generate $K_1(C(S_q^{n,2,n-1}))$, it follows that $[U_{n-1}]$ and $[V_{n-1}U_{n-1}]$ generate $K_1(C(S_q^{n,2,n-1}))$. As XY_n is a unitary with $\sigma_n(XY_n) = V_{n-1}U_{n-1}$, it follows that $\partial([V_{n-1}U_{n-1}]) = 0$. Next, Y_n is an isometry with $\sigma_n(Y_n) = U_{n-1}$. Hence $\partial([U_{n-1}]) = [1 - Y^*Y] - [1 - YY^*]$. Thus, $\partial([U_{n-1}]) = -[1 \otimes 1_{n-2} \otimes p_{n-1}]$.

Now, we compute δ. Since $\sigma_n(1) = 1$, it follows that $\delta([1]) = 0$. One observes that $p_{n-2} \otimes S^*\pi_{\omega_{n-1,1}}(u_{j1}) = 0$ if $j > 1$. Hence,

$$Z_n\,\chi_{\omega_n}(u_{n-1,1}) = t_1t_2 \otimes p_{n-2} \otimes p_{n-2} \otimes \sqrt{1-q^{2N+2}},$$

where Z_n is as defined in (6-1). The operator $R_n := t_1t_2 \otimes p_{n-2} \otimes p_{n-2} \otimes 1$ lies in the algebra $C(S_q^{n,2,n})$, since the difference $R_n - Z_n\,\chi_{\omega_n}(u_{n-1,1})$ lies in the ideal $C(\mathbb{T}^2) \otimes \mathcal{K}^{\otimes(2n-3)}$. Hence, the projection $1 \otimes p_{n-2} \otimes p_{n-2} \otimes 1$ lies in the algebra $C(S_q^{n,2,n})$. Now, define

$$\begin{aligned} S_n &:= R_n + 1 - R_nR_n^*, \\ T_n &:= (1 \otimes p_{n-2} \otimes p_{n-2} \otimes 1)Z_n + 1 - 1 \otimes p_{n-2} \otimes p_{n-2} \otimes 1. \end{aligned}$$

Then, S_n is a unitary and T_n is an isometry such that $\sigma_n(S_n) = u_{n-1}v_{n-1}$ and $\sigma_n(T_n) = u_{n-1}$. Moreover, S_n and T_n commute. Note that $P(u_{n-1}, v_{n-1}) = P(u_{n-1}, u_{n-1}v_{n-1})$. By Lemma 4.1, the image of δ is the subgroup generated by $S_n(1 - T_nT_n^*) + T_nT_n^*$ in $K_1(C(S_q^{n,2,1}) \otimes \mathcal{K}^{\otimes(n-1)})$. Now,

$$S_n(1 - T_nT_n^*) + T_nT_n^* = t_1t_2 \otimes p_{n-2} \otimes p_{n-1} + 1 - 1 \otimes p_{n-2} \otimes p_{n-1}.$$

Since $1 \otimes p_{n-2}$ is trivial in $K_0(C(S_q^{2n-3}))$, the unitary $t_1 \otimes p_{n-2} + 1 - 1 \otimes p_{n-2}$ is trivial in $K_1(C(S_q^{n,2,1})) = K_1(C(\mathbb{T}) \otimes C(S_q^{2n-3}))$. One has $[S_n(1-T_nT_n^*)+T_nT_n^*] = [V_1 \otimes p_{n-1} + 1 - 1 \otimes p_{n-1}]$ in $K_1(C(S_q^{n,2,1}) \otimes \mathcal{K}^{\otimes(n-1)})$.

This computation, with the exactness of the six-term sequence, completes the proof. □

7. *K*-groups of quantum SU(3)

In this section, we show that when $n = 3$ the unitary XY_n in Proposition 6.3 can be replaced by the fundamental 3×3 matrix (u_{ij}) of $C(\mathrm{SU}_q(3))$. First, note that for $n = 3$ we have $C(S_q^{n,2}) = C(\mathrm{SU}_q(3))$, since $C(\mathrm{SU}_q(1)) = \mathbb{C}$. The embedding $\mathrm{SU}_q(1) \subseteq \mathrm{SU}_q(3)$ is given by the counit. The quotient $C(\mathrm{SU}_q(3)/\,\mathrm{SU}_q(1))$ becomes isomorphic with $C(\mathrm{SU}_q(3))$. In [Sheu 1997], the algebra $C(S_q^{3,2,1})$ is denoted $C(U_q(2))$. Then, $C(U_q(2)) = C(\mathbb{T}) \otimes C(\mathrm{SU}_q(2))$. Let $\mathrm{ev}_1 : C(\mathbb{T}) \to \mathbb{C}$ be the evaluation at the point 1. Then, $\varphi = (\mathrm{ev}_1 \otimes 1)\,\sigma_2\sigma_3$, where $\varphi : C(\mathrm{SU}_q(3)) \to C(\mathrm{SU}_q(2))$ is the subgroup homomorphism defined in (2-1).

Proposition 7.1. *The K-group $K_1(C(\mathrm{SU}_q(3))$ is isomorphic to $\mathbb{Z}^2$, generated by the unitary $U_3 := t_1 \otimes p \otimes p + 1 - 1 \otimes p \otimes p$ and the fundamental unitary $U = (u_{ij})$*

Proof. By Proposition 6.3, we know that $K_1(C(\mathrm{SU}_q(3))$ is isomorphic to $\mathbb{Z}^2$ and generated by $[U_3]$ and $[XY_3]$, where X is a coisometry such that $\sigma_3(X) = V_2$ and $X^*X = 1 - 1_{\{1\}}(\chi_{\omega_3}(u_{31}^* u_{31}))$. Observe that $\varphi(X) = t_2 \otimes p + 1 - 1 \otimes p$ and $\varphi(Y_3) = 1$. Hence, $\varphi(XY_3) = t_2 \otimes p + 1 - 1 \otimes p$. Also note that

$$\varphi(U_3) = 0 \quad \text{and} \quad \varphi(U) = \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix},$$

where u denote the fundamental unitary of $C(\mathrm{SU}_q(2))$. Since $K_1(C(\mathrm{SU}_q(2))$ is isomorphic to $\mathbb{Z}$, the proof is complete if we show that $t_2 \otimes p + 1 - 1 \otimes p$ and $[u]$ represent the same element in $K_1(C(\mathrm{SU}_q(2))$; we do this in the next lemma. □

Denote by u_q the 2×2 fundamental unitary $u = (u_{ij})$ of $C(\mathrm{SU}_q(2))$. Consider the representation $\chi_{s_1} : C(\mathrm{SU}_q(2)) \to B(\ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{N}))$. We let the unitary t act on $\ell^2(\mathbb{Z})$ as the right shift, that is, $te_n = e_{n+1}$. Let $\{e_{n,m} : n \in \mathbb{Z}, m \in \mathbb{N}\}$ be the standard orthonormal basis for the Hilbert space $\ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{N})$. For an integer k, denote by P_k the orthogonal projection onto the closed subspace spanned by $\{e_{n,m} : n + m \le k\}$, and set $F_k := 2P_k - 1$. Note that F_k is a selfadjoint unitary.

Proposition 7.2. *For any integer k, the triple $(\chi_{s_1}, \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{N}), F_k)$ is an odd Fredholm module for $C(\mathrm{SU}_q(2))$, and we have the pairing*

(1) $\langle [u_q], F_k \rangle = -1$,

(2) $\langle t \otimes p + 1 - 1 \otimes p, F_k \rangle = -1$, *where* $p = 1 - S^* S$.

Proof. It is not difficult to show that $C(\mathrm{SU}_q(2))$ is generated by $t \otimes S$ and $t \otimes p$. Now, one can see that $[t\otimes S, P_k] = 0$ and $[t\otimes p, P_k]$ is a finite-rank operator. Hence, the triple $(\chi_{s_1}, \ell^2(\mathbb{Z})\otimes\ell^2(\mathbb{N}), F_k)$ is an odd Fredholm module for $C(\mathrm{SU}_q(2))$. Since $C(\mathrm{SU}_q(2))$ is generated by $t \otimes S$ and $t \otimes p$, it follows that $u_p \in C(\mathrm{SU}_q(2))$ for every $p > 0$. Also, as $p \to 0$, u_p approaches u in norm, where u is given by

$$u := \begin{pmatrix} t \otimes S & 0 \\ \bar{t} \otimes p & \bar{t} \otimes S^* \end{pmatrix}.$$

Hence, $[u_q] = [u]$ in $K_1(C(\mathrm{SU}_q(2)))$. It is now easy to check that $\langle [u], F_k \rangle = -1$ and $\langle [t \otimes p + 1 - 1 \otimes p], F_k \rangle = -1$. This completes the proof. □

References

[Blackadar 1986] B. Blackadar, *K-theory for operator algebras*, Mathematical Sciences Research Institute Publications **5**, Springer, New York, 1986. MR 88g:46082 Zbl 0597.46072

[Carey et al. 2010] A. L. Carey, J. Phillips, and A. Rennie, "Twisted cyclic theory and an index theory for the gauge invariant KMS state on the Cuntz algebra O_n", *J. K-Theory* **6**:2 (2010), 339–380. MR 2011j:19011 Zbl 1220.19004 arXiv 0801.4605

[Chakraborty and Pal 2003] P. S. Chakraborty and A. Pal, "Equivariant spectral triples on the quantum SU(2) group", *K-Theory* **28**:2 (2003), 107–126. MR 2004k:58042 Zbl 1028.58005

[Chakraborty and Pal 2010] P. S. Chakraborty and A. Pal, "Equivariant spectral triples and Poincaré duality for $SU_q(2)$", *Trans. Amer. Math. Soc.* **362**:8 (2010), 4099–4115. MR 2011j:58046 Zbl 1198.58003

[Connes 1981] A. Connes, "An analogue of the Thom isomorphism for crossed products of a C^*-algebra by an action of $\mathbf{R}$", *Adv. in Math.* **39**:1 (1981), 31–55. MR 82j:46084 Zbl 0461.46043

[Connes 1985] A. Connes, "Noncommutative differential geometry", *Inst. Hautes Études Sci. Publ. Math.* 62 (1985), 257–360. MR 87i:58162

[Connes 2004] A. Connes, "Cyclic cohomology, quantum group symmetries and the local index formula for $SU_q(2)$", *J. Inst. Math. Jussieu* **3**:1 (2004), 17–68. MR 2005f:58044 Zbl 1074.58012

[Connes and Moscovici 2008] A. Connes and H. Moscovici, "Type III and spectral triples", pp. 57–71 in *Traces in number theory, geometry and quantum fields*, edited by S. Albeverio et al., Aspects of Mathematics **E38**, Friedr. Vieweg, Wiesbaden, 2008. MR 2010b:58036 Zbl 1159.46041

[Levendorskii and Soibelman 1991] S. Levendorskiĭ and Y. Soibelman, "Algebras of functions on compact quantum groups, Schubert cells and quantum tori", *Comm. Math. Phys.* **139**:1 (1991), 141–170. MR 92h:58020

[McClanahan 1992] K. McClanahan, "C^*-algebras generated by elements of a unitary matrix", *J. Funct. Anal.* **107**:2 (1992), 439–457. MR 93j:46062 Zbl 0777.46033

[Nagy 1998] G. Nagy, "Deformation quantization and K-theory", pp. 111–134 in *Perspectives on quantization* (South Hadley, MA, 1996), edited by L. A. Coburn and M. A. Rieffel, Contemp. Math. **214**, Amer. Math. Soc., Providence, RI, 1998. MR 99b:46107 Zbl 0903.46068

[Nagy 2000] G. Nagy, "Bivariant K-theories for C^*-algebras", *K-Theory* **19**:1 (2000), 47–108. MR 2001b:46112 Zbl 0949.46036

[Neshveyev and Tuset 2010] S. Neshveyev and L. Tuset, "The Dirac operator on compact quantum groups", *J. Reine Angew. Math.* **641** (2010), 1–20. MR 2011k:58037 Zbl 1218.58020 arXiv math.OA/0703161

[Podkolzin and Vainerman 1999] G. B. Podkolzin and L. I. Vainerman, "Quantum Stiefel manifold and double cosets of quantum unitary group", *Pacific J. Math.* **188**:1 (1999), 179–199. MR 2000c:17029 Zbl 0952.17014

[Podleś 1995] P. Podleś, "Symmetries of quantum spaces. Subgroups and quotient spaces of quantum SU(2) and SO(3) groups", *Comm. Math. Phys.* **170**:1 (1995), 1–20. MR 96j:58013 Zbl 0853.46074

[Sheu 1991] A. J.-L. Sheu, "The structure of twisted SU(3) groups", *Pacific J. Math.* **151**:2 (1991), 307–315. MR 93c:46130 Zbl 0713.22008

[Sheu 1997] A. J. L. Sheu, "Compact quantum groups and groupoid C^*-algebras", *J. Funct. Anal.* **144**:2 (1997), 371–393. MR 98e:46090 Zbl 0932.17016

[Soibelman 1990] Y. S. Soĭbel'man, "Algebra of functions on a compact quantum group and its representations", *Algebra i Analiz* **2**:1 (1990), 190–212. MR 91i:58053a

[Vaksman and Soibelman 1988] L. L. Vaksman and Y. S. Soĭbel'man, "An algebra of functions on the quantum group SU(2)", *Funktsional. Anal. i Prilozhen.* **22**:3 (1988), 1–14, 96. MR 90f:17019 Zbl 0661.43001

[Woronowicz 1987] S. L. Woronowicz, "Twisted SU(2) group. An example of a noncommutative differential calculus", *Publ. Res. Inst. Math. Sci.* **23**:1 (1987), 117–181. MR 88h:46130

[Woronowicz 1988] S. L. Woronowicz, "Tannaka–Kreĭn duality for compact matrix pseudogroups. Twisted SU(N) groups", *Invent. Math.* **93**:1 (1988), 35–76. MR 90e:22033 Zbl 0664.58044

Received July 23, 2010.

Partha Sarathi Chakraborty
The Institute of Mathematical Sciences
Chennai 600113
India
parthac@imsc.res.in

S. Sundar
The Institute of Mathematical Sciences
Chennai 600113
India
sundarsobers@gmail.com

PACIFIC JOURNAL OF MATHEMATICS

http://www.pjmath.org

Founded in 1951 by
E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

EDITORS

V. S. Varadarajan (Managing Editor)
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
pacific@math.ucla.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Darren Long
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
long@math.ucsb.edu

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Robert Finn
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
finn@math.stanford.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Alexander Merkurjev
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
merkurev@math.ucla.edu

Jonathan Rogawski
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
jonr@math.ucla.edu

PRODUCTION

pacific@math.berkeley.edu

Silvio Levy, Scientific Editor
Matthew Cargo, Senior Production Editor

SUPPORTING INSTITUTIONS

ACADEMIA SINICA, TAIPEI
CALIFORNIA INST. OF TECHNOLOGY
INST. DE MATEMÁTICA PURA E APLICADA
KEIO UNIVERSITY
MATH. SCIENCES RESEARCH INSTITUTE
NEW MEXICO STATE UNIV.
OREGON STATE UNIV.
STANFORD UNIVERSITY
UNIV. OF BRITISH COLUMBIA
UNIV. OF CALIFORNIA, BERKELEY
UNIV. OF CALIFORNIA, DAVIS
UNIV. OF CALIFORNIA, LOS ANGELES
UNIV. OF CALIFORNIA, RIVERSIDE
UNIV. OF CALIFORNIA, SAN DIEGO
UNIV. OF CALIF., SANTA BARBARA
UNIV. OF CALIF., SANTA CRUZ
UNIV. OF MONTANA
UNIV. OF OREGON
UNIV. OF SOUTHERN CALIFORNIA
UNIV. OF UTAH
UNIV. OF WASHINGTON
WASHINGTON STATE UNIVERSITY

These supporting institutions contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.

See inside back cover or www.pjmath.org for submission instructions.

The subscription price for 2011 is US $420/year for the electronic version, and $485/year for print and electronic.
Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. Prior back issues are obtainable from Periodicals Service Company, 11 Main Street, Germantown, NY 12526-5635. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and the Science Citation Index.

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

Copyright ©2011 by Pacific Journal of Mathematics

PACIFIC JOURNAL OF MATHEMATICS

Volume 252 No. 2 August 2011