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(p, p)-GALOIS REPRESENTATIONS ATTACHED TO AUTOMORPHIC FORMS ON GL_n

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We study the local reducibility at p of the p-adic Galois representation attached to a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$. In the case that the underlying Weil–Deligne representation is Frobenius semisimple and indecomposable, we analyze the reducibility completely. We use methods from p-adic Hodge theory, and work under a transversality assumption on the Hodge and Newton filtrations in the corresponding filtered module.

1. Introduction

Let $f = \sum_{n=1}^{\infty} a_n(f)q^n$ be a primitive elliptic modular cusp form of weight $k \geq 2$, level $N \geq 1$, and nebentypus $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$. Let K_f denote the number field generated by the Fourier coefficients of f. Fix an embedding of $\bar{\mathbb{Q}}$ into $\bar{\mathbb{Q}}_p$, and let $\wp$ be the prime of $\bar{\mathbb{Q}}$ determined by this embedding. Let $\wp$ also denote the induced prime of K_f, and let $K_{f,\wp}$ be the completion of K_f at $\wp$. For a global or local field F of characteristic 0, let G_F denote the absolute Galois group of F. There is a global Galois representation

$$\rho_{f,\wp} : G_{\mathbb{Q}} \to \mathrm{GL}_2(K_{f,\wp})$$

associated to f (and $\wp$) by Deligne which has the property that for all primes $\ell \nmid Np$,

$$\mathrm{trace}(\rho_{f,\wp}(\mathrm{Frob}_\ell)) = a_\ell(f) \quad \text{and} \quad \det(\rho_{f,\wp}(\mathrm{Frob}_\ell)) = \chi(\ell)\ell^{k-1}.$$

Thus $\det(\rho_{f,\wp}) = \chi\chi_{\mathrm{cyc},p}^{k-1}$, where $\chi_{\mathrm{cyc},p}$ is the p-adic cyclotomic character.

It is a well-known result of Ribet that the global representation $\rho_{f,\wp}$ is irreducible. However, if f is ordinary at $\wp$, that is, if $a_p(f)$ is a $\wp$-adic unit, then an important theorem of Wiles, valid more generally for Hilbert modular forms, says that the corresponding local representation is reducible.

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Theorem 1.1 [Wiles 1988]. *Let f be a $\wp$-ordinary primitive form as above. Then the restriction of $\rho_{f,\wp}$ to the decomposition subgroup $G_{\mathbb{Q}_p}$ is reducible. More precisely, there exists a basis in which*

$$\rho_{f,\wp}|_{G_{\mathbb{Q}_p}} \sim \begin{pmatrix} \chi_p \cdot \lambda(\beta/p^{k-1}) \cdot \chi_{\mathrm{cyc},p}^{k-1} & u \\ 0 & \lambda(\alpha) \end{pmatrix},$$

where $\chi = \chi_p \chi'$ is the decomposition of χ into its p and prime-to-p parts,

$$\lambda(x) : G_{\mathbb{Q}_p} \to K_{f,\wp}^{\times}$$

is the unramified character taking arithmetic Frobenius to x, and $u : G_{\mathbb{Q}_p} \to K_{f,\wp}$ is a continuous function; moreover, α is

(i) *the unit root of $X^2 - a_p(f)X + p^{k-1}\chi(p)$ if $p \nmid N$,*

(ii) *the unit $a_p(f)$ if $p \parallel N$, $p \nmid \mathrm{cond}\,\chi$, and $k = 2$,*

(iii) *the unit $a_p(f)$ if $p \mid N$ and $v_p(N) = v_p(\mathrm{cond}\,\chi)$,*

and β is given by $\alpha\beta = \chi'(p)p^{k-1}$.

Moreover, in case (ii), $a_p(f)$ is a unit if and only if $k = 2$, and one can show that $\rho_{f,\wp}|_{G_{\mathbb{Q}_p}}$ is irreducible when $k > 2$.

Urban has generalized Theorem 1.1 to the case of primitive Siegel modular cusp forms of genus 2. We briefly recall this result here. Let π be a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$ whose Archimedean component π_∞ belongs to the discrete series, with cohomological weights $(a, b; a+b)$ with $a \geq b \geq 0$. For each prime p, Laumon, Taylor and Weissauer have defined a four-dimensional Galois representation

$$\rho_{\pi,p} : G_{\mathbb{Q}} \to \mathrm{GL}_4(\bar{\mathbb{Q}}_p)$$

with standard properties. Let p be an unramified prime for π. Tilouine and Urban have generalized the notion of ordinariness for such primes p in three ways to what they call Borel ordinary, Siegel ordinary, and Klingen ordinary (these terms come from the underlying parabolic subgroups of $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$). In the Borel case, the p-ordinariness of π implies that the Hecke polynomial of π_p, namely

$$(X-\alpha)(X-\beta)(X-\gamma)(X-\delta),$$

has the property that the p-adic valuations of α, β, γ, and δ are 0, $b+1$, $a+2$, and $a+b+3$, respectively.

Theorem 1.2 [Urban 2005; Tilouine and Urban 1999]. *Say π is a Borel p-ordinary cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$ that is stable at ∞ with cohomological weights $(a, b; a+b)$. Then the restriction of $\rho_{\pi,p}$ to the decomposition subgroup $G_{\mathbb{Q}_p}$ is upper-triangular. More precisely, there is a basis in which*

$$\rho_{\pi,p}|_{G_{\mathbb{Q}_p}} \sim \begin{pmatrix} \lambda(\delta/p^{a+b+3}) \cdot \chi_{\mathrm{cyc},p}^{a+b+3} & * & * & * \\ 0 & \lambda(\gamma/p^{a+2}) \cdot \chi_{\mathrm{cyc},p}^{a+2} & * & * \\ 0 & 0 & \lambda(\beta/p^{b+1}) \cdot \chi_{\mathrm{cyc},p}^{b+1} & * \\ 0 & 0 & 0 & \lambda(\alpha) \end{pmatrix},$$

where $\lambda(x)$ is the unramified character that takes arithmetic Frobenius to x.

We remark that $\rho_{\pi,p}$ above is the contragredient of the one used in [Urban 2005] (we also use the arithmetic Frobenius in defining our unramified characters), so the theorem matches exactly with Corollary 1(iii) of that work. Similar results in the Siegel and Klingen cases can be found in [Urban 2005].

The local Galois representations appearing in Theorems 1.1 and 1.2 are sometimes referred to as (p, p)-Galois representations. The goal of this paper is to prove structure theorems for the local (p, p)-Galois representations attached to automorphic representations of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$ for any $n \geq 1$.

Let now π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$. We assume that the global p-adic Galois representation $\rho_{\pi,p}$ attached to π exists, and that it satisfies several natural properties; for example, it lives in a strictly compatible system of Galois representations, and satisfies local-global compatibility. Recently, much progress has been made on this front: such Galois representations have been attached to what are referred to as RAESDC (regular, algebraic, essentially self-dual, cuspidal) automorphic representations of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$ by Clozel, Harris, Kottwitz and Taylor, and for conjugate self-dual automorphic representations over CM fields these representations were shown by Taylor and Yoshida to satisfy local-global compatibility away from p.

The assumptions above allow us to specify the Weil–Deligne parameter at p. We study the (p, p)-Galois representation $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ attached to π, given this parameter. In fact, as the expert reader will note, since our methods are local, our results could equally well have been phrased purely in terms of this parameter.

A key tool in our analysis is the celebrated result of Colmez and Fontaine establishing an equivalence of categories between potentially semistable representations and filtered (φ, N)-modules with coefficients and descent data. Under some standard hypotheses, such as Assumption 3.6 that the Hodge and Newton filtrations are in general position in the corresponding crystal, we show that in several cases the corresponding local (p, p)-representation $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ has an upper-triangular form, and completely determines the diagonal characters. In other cases, and perhaps more interestingly, we give conditions under which this local representation is irreducible. For instance, we directly generalize the comment about irreducibility made just after Theorem 1.1. As a sample of our results, we state the following theorem, which is a collation of Theorems 5.7, 5.8, and 6.10.

Theorem 1.3 (indecomposable case). *Say π is a cuspidal automorphic representation of $\mathrm{GL}_{mn}(\mathbb{A}_{\mathbb{Q}})$ with infinitesimal character given by integers $-\beta_1 > \cdots > -\beta_{mn}$. Suppose the Weil–Deligne representation attached to π_p is Frobenius semisimple and indecomposable, that is,*

$$\mathrm{WD}(\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}) \sim \tau_m \otimes \mathrm{Sp}(n),$$

where τ_m is an irreducible representation of $W_{\mathbb{Q}_p}$ of dimension $m \geq 1$, and $\mathrm{Sp}(n)$ is the special representation for $n \geq 1$. Let Assumption 3.6 hold.

(i) *Suppose $m = 1$ and $\tau_1 = \chi_0 \cdot \chi'$ is a character, where χ_0 is the ramified part, and χ' is an unramified character mapping arithmetic Frobenius to α.*

  (a) *If π is ordinary at p (i.e., $v_p(\alpha) = -\beta_1$), then the β_i are necessarily consecutive integers, and*

$$\rho_{\pi,p}|_{G_{\mathbb{Q}_p}} \sim \begin{pmatrix} \chi_0 \cdot \lambda\left(\frac{\alpha}{p^{v_p(\alpha)}}\right) \cdot \chi_{\mathrm{cyc},p}^{-\beta_1} & * & * \\ 0 & \chi_0 \cdot \lambda\left(\frac{\alpha}{p^{v_p(\alpha)}}\right) \cdot \chi_{\mathrm{cyc},p}^{-\beta_1-1} & * \\ & & \ddots & \\ 0 & 0 & \chi_0 \cdot \lambda\left(\frac{\alpha}{p^{v_p(\alpha)}}\right) \cdot \chi_{\mathrm{cyc},p}^{-\beta_1-(n-1)} \end{pmatrix},$$

  where $\lambda(x)$ is the unramified character taking arithmetic Frobenius to x.

  (b) *If π is not p-ordinary, $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ is irreducible.*

(ii) *Suppose $m \geq 2$. Then $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ is irreducible.*

The theorem gives complete information about the reducibility of the (p, p)-Galois representation in the indecomposable case (under Assumption 3.6). In particular, the image of the (p, p)-representation tends to be either in a minimal parabolic subgroup or a maximal parabolic subgroup of GL_n. While this is forced in the GL_2 setting, it is somewhat surprising that the image does not lie in any "intermediate" parabolic subgroups even in the GL_n setting. We also point out that parts (i)(b) and (ii) of the theorem imply that the global representation $\rho_{\pi,p}$ is irreducible (see also [Taylor and Yoshida 2007, Corollary B] for the case of conjugate self-dual representations over CM fields).

The theorem is proved in Sections 5 and 6 using methods from p-adic Hodge theory. Recall that the category of Weil–Deligne representations is equivalent to the category of (φ, N)-modules. In Section 6, we classify the (φ, N)-submodules of the (φ, N)-module associated with the indecomposable Weil–Deligne representation in the theorem. This classification plays a key role in analyzing the (p, p)-representation once the Hodge filtration is introduced. Along the way, we take a slight detour to write down explicitly the filtered (φ, N)-module attached to an m-dimensional "unramified supercuspidal" representation, since this might be a useful addition to the literature (see [Ghate and Mézard 2009] for the two-dimensional case).

The term "indecomposable case" in the discussion above refers to the standard fact that every Frobenius semisimple indecomposable Weil–Deligne representation has the form stated in the theorem. Some partial results in the decomposable case, where the Weil–Deligne representation is a direct sum of indecomposables, can be found in Section 8 of [Ghate and Kumar 2010]. The principal series case is treated completely in Section 4 of the present paper. In the spherical case our results overlap with those in D. Geraghty's thesis [2010], and we thank T. Gee for pointing this out to us.

We also refer to [Ghate and Kumar 2010, §3] for another proof of Theorem 1.1 along the lines of this paper. The original proof used Dieudonné theory only in weight 2 and then Hida theory [1986] (see also [Banerjee et al. 2010]) to extend to higher weights. Of the remaining sections, Section 2 recalls some useful facts from p-adic Hodge theory, whereas Section 3 recalls some general facts and conjectures about Galois representations associated with automorphic representations of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$.

2. p-adic Hodge theory

We start by recalling some results we need from p-adic Hodge theory. For the basic definitions in the subject, e.g., of Fontaine's ring $\mathbf{B}_{\mathrm{st}}$, filtered (φ, N)-modules with coefficients and descent data, and Newton and Hodge numbers, see [Fontaine 1994; Savitt 2005; Fontaine and Ouyang; Ghate and Mézard 2009, §2]. Also, see [Ghate and Kumar 2010, §2] for proofs.

Newton and Hodge numbers. We start by stating some facts about Newton and Hodge numbers, which do not seem to be in the literature when the coefficients are not necessarily $\mathbb{Q}_p$.

Let F and E be two finite field extensions of $\mathbb{Q}_p$, and assume that all the conjugates of F are contained in E.

Lemma 2.1 (Newton number). *Suppose D is a filtered (φ, N, F, E)-module of rank n such that the action of φ is E-semisimple, that is, there exists a basis $\{e_1, \dots, e_n\}$ of D such that $\varphi(e_i) = \alpha_i e_i$, for some $\alpha_i \in E^\times$. Then*

$$t_N(D) = [E : \mathbb{Q}_p] \cdot \sum_{i=1}^{n} v_p(\alpha_i).$$

Lemma 2.2 (Hodge number). *Suppose D is a filtered (φ, N, F, E)-module of rank n. Then*

$$t_H(D) = [E : \mathbb{Q}_p] \cdot \sum_{i \in \mathbb{Z}} i \cdot \mathrm{rank}_{F \otimes_{\mathbb{Q}_p} E}\ \mathrm{gr}^i D_F.$$

Remark. By the last two lemmas, one can drop the common factor of $[E : \mathbb{Q}_p]$ when checking the admissibility of a filtered (φ, N, F, E)-module.

Lemma 2.3. *Let D_1 and D_2 be filtered (φ, N, F, E)-modules of rank r_1 and r_2, respectively. Assume that the action of φ on D_1 and D_2 is semisimple. Then*

$$t_N(D_1 \otimes D_2) = \operatorname{rank}(D_1)\, t_N(D_2) + \operatorname{rank}(D_2)\, t_N(D_1),$$
$$t_H(D_1 \otimes D_2) = \operatorname{rank}(D_1)\, t_H(D_2) + \operatorname{rank}(D_2)\, t_H(D_1).$$

Remark. The formulas above are well-known if $E = \mathbb{Q}_p$.

Potentially semistable representations. Let E and F be two finite extensions of $\mathbb{Q}_p$, and let V be a finite dimensional vector space over E.

Definition. A representation $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}(V)$ is said to be *semistable* over F or *F-semistable*, if

$$\dim_{F_0} D_{\mathrm{st},F}(V) = \dim_{F_0}(\mathbf{B}_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_F} = \dim_{\mathbb{Q}_p} V,$$

where $F_0 = \mathbf{B}_{\mathrm{st}}^{G_F}$. If such an F exists, ρ is said to be a potentially semistable representation. If $F = \mathbb{Q}_p$, we say that ρ is semistable.

Remark. If ρ is F-semistable, ρ is F'-semistable for any finite extension of F'/F. Hence we may and do assume that F is Galois over $\mathbb{Q}_p$.

The following fundamental theorem plays a key role in subsequent arguments.

Theorem 2.4 [Colmez and Fontaine 2000]. *There is an equivalence of categories between F-semistable representations $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(E)$ with Hodge–Tate weights $-\beta_n \leq \cdots \leq -\beta_1$ and admissible filtered (φ, N, F, E)-modules D of rank n over $F_0 \otimes_{\mathbb{Q}_p} E$ such that the jumps in the Hodge filtration $\mathrm{Fil}^i D_F$ on $D_F := F \otimes_{F_0} D$ are at $\beta_1 \leq \cdots \leq \beta_n$.*

The equivalence of categories in the theorem is induced by Fontaine's functor $D_{\mathrm{st},F}$. The Frobenius φ, monodromy N, and filtration on $\mathbf{B}_{\mathrm{st}}$ induce the corresponding structures on $D_{\mathrm{st},F}(V)$. There is also an induced action of $\mathrm{Gal}(F/\mathbb{Q}_p)$ on $D_{\mathrm{st},F}(V)$.

As an illustration of the power of the theorem we recall a useful and well-known fact:

Corollary 2.5. *Every potentially semistable character $\chi : G_{\mathbb{Q}_p} \to E^\times$ is of the form $\chi = \chi_0 \cdot \lambda(a_0) \cdot \chi^i_{\mathrm{cyc},p}$, where χ_0 is a finite order character of $\mathrm{Gal}(F/\mathbb{Q}_p)$ for a cyclotomic extension F of $\mathbb{Q}_p$, $-i \in \mathbb{Z}$ is the Newton number of $D_{st,F}(\chi)$, and $\lambda(a_0)$ is the unramified character that takes arithmetic Frobenius to the unit $a_0 = p^{-i}/a \in \mathcal{O}_E^\times$, where $a = \varphi(v)/v$ for any nonzero vector v in $D_{st,F}(\chi)$.*

Weil–Deligne representations. We now recall the definition of the Weil–Deligne representation associated with an F-semistable representation $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(E)$, due to Fontaine. We assume that $F/\mathbb{Q}_p$ is Galois and $F \subseteq E$. Let W_F denote the Weil group of F. For any (φ, N, F, E)-module D, we have the decomposition

$$D \simeq \prod_{i=1}^{[F_0:\mathbb{Q}_p]} D_i, \tag{2-1}$$

where $D_i = D \otimes_{(F_0 \otimes_{\mathbb{Q}_p} E, \sigma^i)} E$, and σ is the arithmetic Frobenius of $F_0/\mathbb{Q}_p$.

Definition 2.6 (Weil–Deligne representation). Let $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(E)$ be an F-semistable representation. Let D be the corresponding filtered module. Noting $W_{\mathbb{Q}_p}/W_F = \mathrm{Gal}(F/\mathbb{Q}_p)$, we let

$$g \in W_{\mathbb{Q}_p} \text{ act on } D \text{ by } (g \bmod W_F) \circ \varphi^{-\alpha(g)},$$

where the image of g in $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ is the $\alpha(g)$-th power of the arithmetic Frobenius at p. We also have an action of N via the monodromy operator on D. These actions induce a Weil–Deligne action on each D_i in (2-1), and the resulting Weil–Deligne representations are all isomorphic. This isomorphism class is defined to be the Weil–Deligne representation $\mathrm{WD}(\rho)$ associated with ρ.

Remark. If $F/\mathbb{Q}_p$ is totally ramified and $\mathrm{Frob}_p \in W_{\mathbb{Q}_p}$ is the arithmetic Frobenius, then observe that $\mathrm{WD}(\rho)(\mathrm{Frob}_p)$ acts by φ^{-1}.

Lemma 2.7. *Let $\rho : \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}_n(E)$ be a potentially semistable representation. If $\mathrm{WD}(\rho)$ is irreducible, so is ρ.*

Compatible systems. We recall the notion of a strictly compatible system of Galois representations following [Khare and Wintenberger 2009, §5], where it was used to great effect in the two-dimensional case. Let F be a number field, ℓ a prime, and $\rho : G_F \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$ a continuous global Galois representation.

Definition. Say that ρ is *geometric* if it is unramified outside a finite set of primes of F and its restrictions to the decomposition groups at primes above ℓ are potentially semistable.

A geometric representation defines, for every prime q of F, a representation of the Weil–Deligne group at q, denoted by WD_q, with values in $\mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$, well-defined up to conjugacy. For q of characteristic not ℓ, the definition is classical, and comes from the theory of Deligne–Grothendieck, and for q of characteristic ℓ, the definition comes from Fontaine theory (Definition 2.6).

Definition. Let L be a number field. An L-rational, n-dimensional *strictly compatible system* of geometric representations (ρ_ℓ) of G_F is the collection of data consisting of:

(1) For each prime ℓ and each embedding $i : L \hookrightarrow \bar{\mathbb{Q}}_\ell$, a continuous, semisimple representation $\rho_\ell : G_F \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$ that is geometric.

(2) For each prime q of F, an F-semisimple (Frobenius semisimple) representation r_q of the Weil–Deligne group WD_q with values in $\mathrm{GL}_n(L)$ such that

- r_q is unramified for all q outside a finite set;
- for each ℓ and each $i : L \hookrightarrow \bar{\mathbb{Q}}_\ell$, the Frobenius semisimple Weil–Deligne representation $\mathrm{WD}_q \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$ associated with $\rho_\ell|_{D_q}$ is conjugate to r_q (via the embedding $i : L \hookrightarrow \bar{\mathbb{Q}}_\ell$); and
- there are n distinct integers $\beta_1 < \cdots < \beta_n$ such that ρ_ℓ has Hodge–Tate weights $\{-\beta_1, \ldots, -\beta_n\}$. (The minus signs arise since the weights are the negatives of the jumps in the Hodge filtration on the associated filtered module.)

The existence of strictly compatible systems attached to classical cusp forms is well-known [Katz and Messing 1974; Saito 1997]. For general cuspidal automorphic representations, we will not use the full strength of this definition. In fact we only use it to obtain information about the Weil–Deligne parameter at p. Our results could equally well be stated using this parameter as the *starting point* of our analysis.

3. The case of GL_n

The goal of this paper is to prove various generalizations of Theorem 1.1 for the local (p, p)-Galois representations attached to automorphic representations of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$. In this section we collect together some facts about such automorphic representations and their Galois representations needed for the proof. The main results we need are the local Langlands correspondence [Henniart 2000; Harris and Taylor 2001] and the existence of strictly compatible systems of Galois representations attached to cuspidal automorphic representations of GL_n (much progress has been made on this by Clozel, Harris, Kottwitz, and Taylor [Clozel et al. 2008]).

Local Langlands correspondence. We will need a few results concerning the local Langlands correspondence. We follow [Kudla 1994] in our exposition, noting that that article follows [Rodier 1982], which in turn is based on the original work of Bernstein and Zelevinsky.

Let F be a complete non-Archimedean local field of residue characteristic p, let $n \geq 1$, and let $G = \mathrm{GL}_n(F)$. For a partition $n = n_1 + n_2 + \cdots + n_r$ of n, let P be the corresponding parabolic subgroup of G, let M be the Levi subgroup of P, and N the unipotent radical of P. Let δ_P denote the modulus character of the adjoint action of M on N. If $\sigma = \sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_r$ is a smooth representation of M on V, we let

$$I_P^G(\sigma) = \{f : G \to V \mid f \text{ smooth on } G \text{ and } f(nmg) = \delta_P^{1/2}(m)(\sigma(m)(f(g))\}$$

for $n \in N$, $m \in M$, and $g \in G$. The group G acts on functions in $I_P^G(\sigma)$ by right translation and $I_P^G(\sigma)$ is the usual induced representation of σ. It is an admissible representation of finite length.

A result of Bernstein and Zelevinsky says that if all the σ_i are supercuspidal and σ is irreducible, smooth and admissible, then $I_P^G(\sigma)$ is reducible if and only if $n_i = n_j$ and $\sigma_i = \sigma_j(1)$ for some $i \neq j$. For the partition $n = m + m + \cdots + m$ (r times), and for a supercuspidal representation σ of $\mathrm{GL}_m(F)$, call the data

$$(\sigma, \sigma(1), \cdots, \sigma(r-1)) = [\sigma, \sigma(r-1)] = \Delta$$

a segment. Clearly $I_P^G(\Delta)$ is reducible. By [Kudla 1994, Theorem 1.2.2], the induced representation $I_P^G(\Delta)$ has a unique irreducible quotient $Q(\Delta)$ that is essentially square-integrable.

Two segments

$$\Delta_1 = [\sigma_1, \sigma_1(r_1 - 1)] \quad \text{and} \quad \Delta_2 = [\sigma_2, \sigma_2(r_2 - 1)]$$

are said to be linked if $\Delta_1 \not\subseteq \Delta_2$, $\Delta_2 \not\subseteq \Delta_1$, and $\Delta_1 \cup \Delta_2$ is a segment. We say that Δ_1 precedes Δ_2 if Δ_1 and Δ_2 are linked and if $\sigma_2 = \sigma_1(k)$ for some $k \in \mathbb{N}$.

Theorem 3.1 (Langlands classification). *Let $\Delta_1, \ldots, \Delta_r$ be segments such that if $i < j$ then Δ_i does not precede Δ_j.*

(1) *The induced representation $I_P^G(Q(\Delta_1) \otimes \cdots \otimes Q(\Delta_r))$ admits a unique irreducible quotient $Q(\Delta_1, \cdots, \Delta_r)$, called the Langlands quotient. Moreover, r and the segments Δ_i up to permutation are uniquely determined by the Langlands quotient.*

(2) *Every irreducible admissible representation of $\mathrm{GL}_n(F)$ is isomorphic to some $Q(\Delta_1, \cdots, \Delta_r)$.*

(3) *The induced representation $I_P^G(Q(\Delta_1) \otimes \cdots \otimes Q(\Delta_r))$ is irreducible if and only if no two of the segments Δ_i and Δ_j are linked.*

So much for the automorphic side. We now turn to the Galois side. Recall that a representation of W_F is said to be Frobenius semisimple if arithmetic Frobenius acts semisimply. An admissible representation of the Weil–Deligne group of F is one for which the action of W_F is Frobenius semisimple. Let $\mathrm{Sp}(r)$ denote the Weil–Deligne representation of order r with the usual definition. When $F = \mathbb{Q}_p$, there is a basis $\{f_i\}$ of $\mathrm{Sp}(r)$ for which $\varphi f_i = p^{i-1} f_i$, and $N f_i = f_{i-1}$ for $i > 1$ and $N f_1 = 0$. It is well-known that every indecomposable admissible representation of W_F is of the form $\tau \otimes \mathrm{Sp}(r)$, where τ is an irreducible admissible representation of W_F and $r \geq 1$. Moreover (cf. [Rohrlich 1994, §5, Corollary 2]), every admissible representation of W_F is of the form

$$\bigoplus_i \tau_i \otimes \mathrm{Sp}(r_i),$$

where the τ_i are irreducible admissible representations of W_F and the $r_i \in \mathbb{N}$.

Theorem 3.2 (local Langlands correspondence: [Harris and Taylor 2001, VII.2.20; Henniart 2000; Kutzko 1980]). *There is a bijection between isomorphism classes of irreducible admissible representations of* $\mathrm{GL}_n(F)$ *and isomorphism classes of admissible n-dimensional representations of* W_F.

The correspondence is given as follows. The key point is to construct a bijection $\Phi_F : \sigma \mapsto \tau = \Phi_F(\sigma)$ between the set of isomorphism classes of supercuspidal representations of $\mathrm{GL}_n(F)$ and the set of isomorphism classes of irreducible admissible representations of W_F. This was done in [Henniart 2000] and [Harris and Taylor 2001]. Then, to $Q(\Delta)$, for the segment $\Delta = [\sigma, \sigma(r-1)]$, one associates the indecomposable admissible representation $\Phi_F(\sigma)\otimes\mathrm{Sp}(r)$ of the Weil–Deligne group of F. More generally, to the Langlands quotient $Q(\Delta_1, \cdots, \Delta_r)$, where $\Delta_i = [\sigma_i, \sigma_i(r_i-1)]$, for $i = 1$ to r, one associates the admissible representation $\bigoplus_i \Phi_F(\sigma_i)\otimes\mathrm{Sp}(r_i)$ of the Weil–Deligne group of F.

***Automorphic forms on* $\mathbf{GL}_n$.** The Harish-Chandra isomorphism identifies the center $\mathfrak{z}_n$ of the universal enveloping algebra of the complexified Lie algebra $\mathfrak{gl}_n$ of GL_n, with the algebra $\mathbb{C}[X_1, X_2, \cdots, X_n]^{S_n}$, where the symmetric group S_n acts by permuting the X_i. Given a multiset $H = \{x_1, x_2, \ldots, x_n\}$ of n complex numbers one obtains an infinitesimal character of $\mathfrak{z}_n$ given by $\chi_H : X_i \mapsto x_i$.

Cuspidal automorphic forms with infinitesimal character χ_H (or more simply just H) are smooth functions $f : \mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A}_\mathbb{Q}) \to \mathbb{C}$ satisfying the usual finiteness condition under a maximal compact subgroup, a cuspidality condition, and a growth condition, for which we refer the reader to [Taylor 2004]. In addition, if $z \in \mathfrak{z}_n$, then $z \cdot f = \chi_H(z) f$. The space of such functions is denoted by

$$\mathcal{A}^\circ_H(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})).$$

This space is a direct sum of irreducible admissible $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q}^{(\infty)}) \times (\mathfrak{gl}_n, O(n))$-modules each occurring with multiplicity one, and these irreducible constituents are referred to as cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$ with infinitesimal character χ_H. Let π be such an automorphic representation. By a result of Flath, π is a restricted tensor product $\pi = \bigotimes'_p \pi_p$ [Bump 1997, Theorem 3.3.3] of local automorphic representations.

Galois representations. Let π be an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$ with infinitesimal character χ_H, where H is a multiset of integers. The following very strong, but natural, conjecture seems to be part of the folklore.

Conjecture 3.3. *Let H consist of n distinct integers. There is a strictly compatible system of Galois representations $(\rho_{\pi,\ell})$ associated with π, with Hodge–Tate weights H, such that local-global compatibility holds.*

Here local-global compatibility means that the underlying semisimplified Weil–Deligne representation at p in the compatible system (which is independent of the residue characteristic ℓ of the coefficients by hypothesis) corresponds to π_p via the local Langlands correspondence. Considerable evidence towards this conjecture is available for self-dual representations, thanks to the work of Clozel, Kottwitz, Harris, and Taylor. We quote the following theorem from [Taylor 2004], referring to that paper for the original references (e.g., [Clozel 1991]).

Theorem 3.4 [Taylor 2004, Theorem 3.6]. *Let H consist of n distinct integers. Suppose that the contragredient representation $\pi^\vee = \pi \otimes \psi$ for some character $\psi : \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$, and suppose that for some prime q, the representation π_q is square-integrable. Then there is a continuous representation*

$$\rho_{\pi,\ell} : G_\mathbb{Q} \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$$

such that $\rho_{\pi,\ell}|_{G_{\mathbb{Q}_\ell}}$ is potentially semistable with Hodge–Tate weights given by H, and such that for any prime $p \neq \ell$, the semisimplification of the Weil–Deligne representation attached to $\rho_{\pi,\ell}|_{G_{\mathbb{Q}_p}}$ is the same as the Weil–Deligne representation associated by the local Langlands correspondence with π_p, except possibly for the monodromy operator.

Subsequent work of Taylor and Yoshida [2007] shows that the two Weil–Deligne representations in the theorem above are in fact the same (i.e., the monodromy operators also match).

In any case, for the rest of this paper we shall *assume* that Conjecture 3.3 holds. In particular, we assume that the Weil–Deligne representation at p associated with a p-adic member of the compatible system of Galois representations attached to π using Fontaine theory is the same as the Weil–Deligne representation at p attached to an ℓ-adic member of the family, for $\ell \neq p$.

A variant. A variant of the result above can be found in [Clozel et al. 2008]. We state it now, using the notation and terminology from §4.3 of that reference.

Say π is an RAESDC (regular, algebraic, essentially self-dual, cuspidal) automorphic representation if π is a cuspidal automorphic representation such that

- $\pi^\vee = \pi \otimes \chi$ for some character $\chi : \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$, and
- π_∞ has the same infinitesimal character as some irreducible algebraic representation of GL_n.

Let $a \in \mathbb{Z}^n$ satisfy

$$a_1 \geq \cdots \geq a_n. \tag{3-1}$$

Let Ξ_a denote the irreducible algebraic representation of GL_n with highest weight a. We say that an RAESDC automorphic representation π has weight a if π_∞ has the same infinitesimal character as $\Xi_a^\vee$; in this case there is an integer w_a such that $a_i + a_{n+1-i} = w_a$ for all i.

Let S be a finite set of primes of $\mathbb{Q}$. For $v \in S$ let ρ_v be an irreducible square-integrable representation of $\mathrm{GL}_n(\mathbb{Q}_v)$. Say that an RAESDC representation π has type $\{\rho_v\}_{v \in S}$ if for each $v \in S$, π_v is an unramified twist of $\rho_v^\vee$.

With this setup, Clozel, Harris, and Taylor attached a Galois representation to an RAESDC π.

Theorem 3.5 [Clozel et al. 2008, Proposition 4.3.1]. *Let $\iota : \bar{\mathbb{Q}}_\ell \simeq \mathbb{C}$. Let π be an* RAESDC *automorphic representation as above of weight a and type $\{\rho_v\}_{v \in S}$. There is a continuous semisimple Galois representation $r_{\ell,\iota}(\pi) : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$ such that*:

(1) $r_{\ell,\iota}(\pi)|_{G_{\mathbb{Q}_p}}^{\mathrm{ss}} = (r_\ell(\iota^{-1}\pi_p)^\vee)(1-n)^{\mathrm{ss}}$ *for every prime $p \nmid \ell$.* (Here r_ℓ is the reciprocity map defined in [Harris and Taylor 2001].)

(2) *If $\ell = p$, then the restriction $r_{\ell,\iota}(\pi)|_{G_{\mathbb{Q}_p}}$ is potentially semistable and if π_p is unramified, it is crystalline, with Hodge–Tate weights $-(a_j + n - j)$ for $j = 1, \ldots, n$.*

The Newton and Hodge filtrations. Let $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ be the (p, p)-representation attached to an automorphic representation π, and let D be the corresponding filtered (φ, N, F, E)-module (for suitable choices of F and E).

There are two natural filtrations on D_F, the Hodge filtration $\mathrm{Fil}^i D_F$ and the Newton filtration defined by ordering the slopes of the crystalline Frobenius (the valuations of the roots of φ). To keep the analysis of the structure of the (p, p)-representation $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ within reasonable limits, we make this assumption:

Assumption 3.6. The Newton filtration on D_F is in general position with respect to the Hodge filtration $\mathrm{Fil}^i D_F$.

Here, if V is a space and $\mathrm{Fil}_1^i V$ and $\mathrm{Fil}_2^j V$ are two filtrations on V, we say they are in general position if each $\mathrm{Fil}_1^i V$ is as transverse as possible to each $\mathrm{Fil}_2^j V$. We remark that the condition above is in some sense generic since two random filtrations on a space tend to be in general position.

(Quasi)ordinary representations. As mentioned earlier, our goal is to prove that the (p, p)-Galois representation attached to π is upper triangular in several cases. To this end it is convenient to recall some terminology (see, e.g., [Greenberg 1994, p. 152] or [Ochiai 2001, Definition 3.1]).

Definition. Let F be a number field. A p-adic representation V of G_F is called *ordinary* (respectively *quasiordinary*) if the following conditions are satisfied:

(1) For each place v of F over p, there is a decreasing filtration of G_{F_v}-modules $\cdots \mathrm{Fil}_v^i V \supseteq \mathrm{Fil}_v^{i+1} V \supseteq \cdots$ such that $\mathrm{Fil}_v^i V = V$ for $i \ll 0$ and $\mathrm{Fil}_v^i V = 0$ for $i \gg 0$.

(2) For each v and i, I_v acts on $\mathrm{Fil}_v^i V / \mathrm{Fil}_v^{i+1} V$ via the character $\chi_{\mathrm{cyc},p}^i$, where $\chi_{\mathrm{cyc},p}$ is the p-adic cyclotomic character (respectively, there exists an open subgroup of I_v acting on $\mathrm{Fil}_v^i V / \mathrm{Fil}_v^{i+1} V$ via $\chi_{\mathrm{cyc},p}^i$).

4. Principal series

Let π be an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$ with infinitesimal character H, for a set of distinct integers H. Let π_p denote the local automorphic representation of $\mathrm{GL}_n(\mathbb{Q}_p)$. In this section we study the behavior of the (p, p)-Galois representation assuming that π_p is in the principal series.

Spherical case. Assume that π_p is an unramified principal series representation. Since π_p is a spherical representation of $\mathrm{GL}_n(\mathbb{Q}_p)$, there exist unramified characters $\chi_1, \ldots, \chi_n$ of $\mathbb{Q}_p^\times$ such that π_p is the Langlands quotient $Q(\chi_1, \ldots, \chi_n)$. We can parametrize the isomorphism class of this representation by the Satake parameters $\alpha_1, \ldots, \alpha_n$ for $\alpha_i = \chi_i(\omega)$, where ω is a uniformizer for $\mathbb{Q}_p$.

Note that $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ is crystalline with Hodge–Tate weights H. Let D be the corresponding filtered φ-module, having a filtration with jumps $\beta_1 < \beta_2 < \cdots < \beta_n$ (so that the Hodge–Tate weights H are $-\beta_1 > \cdots > -\beta_n$).

Definition 4.1. An automorphic representation π is *p-ordinary* if $\beta_i + v_p(\alpha_i) = 0$ for all $i = 1, \ldots, n$. (In particular, the $v_p(\alpha_i)$ are integers.)

Theorem 4.2 (spherical case). *Suppose that π is an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$ with infinitesimal character given by the integers $-\beta_1 > \cdots > -\beta_n$ and such that π_p is in the unramified principal series with Satake parameters $\alpha_1, \ldots, \alpha_n$. If π is p-ordinary, then*

$$\rho_{\pi,p}|_{G_{\mathbb{Q}_p}} \sim$$

$$\begin{pmatrix} \lambda\Big(\frac{\alpha_1}{p^{v_p(\alpha_1)}}\Big)\cdot\chi_{\mathrm{cyc},p}^{-\beta_1} & * & * & * \\ 0 & \lambda\Big(\frac{\alpha_2}{p^{v_p(\alpha_2)}}\Big)\cdot\chi_{\mathrm{cyc},p}^{-\beta_2} & * & * \\ & & \ddots & \\ 0 & 0 & \lambda\Big(\frac{\alpha_{n-1}}{p^{v_p(\alpha_{n-1})}}\Big)\cdot\chi_{\mathrm{cyc},p}^{-\beta_{n-1}} & * \\ 0 & 0 & 0 & \lambda\Big(\frac{\alpha_n}{p^{v_p(\alpha_n)}}\Big)\cdot\chi_{\mathrm{cyc},p}^{-\beta_n} \end{pmatrix}.$$

In particular, $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ *is ordinary.*

Proof. Since π_p is p-ordinary, we have $v_p(\alpha_n) < v_p(\alpha_{n-1}) < \cdots < v_p(\alpha_1)$. By strict compatibility, the characteristic polynomial of the inverse of crystalline Frobenius of D_n is equal to $\prod_i (X - \alpha_i)$.

Since the $v_p(\alpha_i)$ are distinct, there exists a basis of eigenvectors of D_n for the operator φ, say $\{e_i\}$, with corresponding eigenvalues $\{\alpha_i^{-1}\}$. For $1 \le i \le n$, let D_i be the φ-submodule generated by $\{e_1, \ldots, e_i\}$. Since D_n is admissible we know that $t_H(D_i) \le t_N(D_i)$ for all $i = 1, \ldots, n$.

The filtration on D_n is

$$\cdots \subseteq 0 \subsetneq \mathrm{Fil}^{\beta_n}(D_n) \subseteq \cdots \subsetneq \mathrm{Fil}^{\beta_1}(D_n) = D_n \subseteq \cdots .$$

Since D_n is admissible, we have

$$\sum_{i=1}^{n} \beta_i = -\sum_{i=1}^{n} v_p(\alpha_i). \tag{4-1}$$

By Assumption 3.6, the jumps in the induced filtration on D_{n-1} are $\beta_1, \ldots, \beta_{n-1}$. By (4-1), we have

$$t_H(D_{n-1}) = \sum_{i=1}^{n-1} \beta_i = -\sum_{i=1}^{n-1} v_p(\alpha_i) = t_N(D_{n-1}),$$

since $\beta_n = -v_p(\alpha_n)$. This implies that D_{n-1} is admissible. Moreover, D_n/D_{n-1} is also admissible since $t_H(D_n/D_{n-1}) = \beta_n$ and $t_N(D_n/D_{n-1}) = -v_p(\alpha_n)$ since φ acts on D_n/D_{n-1} by α_n^{-1}. Therefore, the Galois representation $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ looks like

$$\rho \sim \begin{pmatrix} \rho_{n-1} & * \\ 0 & \lambda\left(\dfrac{\alpha_n}{p^{v_p(\alpha_n)}}\right) \cdot \chi_{\mathrm{cyc},p}^{-\beta_n} \end{pmatrix},$$

where ρ_{n-1} is the $(n-1)$-dimensional representation of $G_{\mathbb{Q}_p}$ corresponding to D_{n-1}.

Successive application of this argument to $D_{n-1}, D_{n-2}, \ldots, D_1$ yields the result. □

Variant, following [Clozel et al. 2008]. Let π now be an RAESDC representation of weight a as in Theorem 3.5, and let π_p denote the local p-adic automorphic representation associated with π. For any $i = 1, \ldots, n$, set $\beta'_{n+1-i} := a_i + n - i$, where the a_i's are as in (3-1). We have $\beta'_n > \beta'_{n-1} > \cdots > \beta'_1$, and the Hodge–Tate weights are $-\beta'_n < -\beta'_{n-1} < \cdots < -\beta'_1$.

Assume that π_p is in the unramified principal series, so $\pi_p = Q(\chi_1, \chi_2, \ldots, \chi_n)$, where the χ_i are unramified characters of $\mathbb{Q}_p^\times$. Set $\alpha'_i = \chi_i(\omega) p^{(n-1)/2}$. Let $t_p^{(j)}$

be the eigenvalue of $T_p^{(j)}$ on $\pi_p^{\mathrm{GL}_n(\mathbb{Z}_p)}$, where $T_p^{(j)}$ is the j-th Hecke operator as in [Clozel et al. 2008], and $\pi_p^{\mathrm{GL}_n(\mathbb{Z}_p)}$ is spanned by a $\mathrm{GL}_n(\mathbb{Z}_p)$-fixed vector, unique up to a constant. We would like to compute the right-hand side of the equality in Theorem 3.5(1). By [Clozel et al. 2008, Corollary 3.1.2], in the spherical case, one has

$$\begin{aligned}(r_\ell(\iota^{-1}\pi_p)^\vee)(1-n)(\mathrm{Frob}_p^{-1}) &= \prod_i (X-\alpha_i') \\ &= X^n - t_p^{(1)}X^{n-1} + \cdots + (-1)^j p^{j(j-1)/2} t_p^{(j)} X^{n-j} + \cdots + (-1)^n p^{n(n-1)/2} t_p^{(n)},\end{aligned}$$

where Frob_p^{-1} is geometric Frobenius. Let s_j denote the j-th elementary symmetric polynomial. Then, by the equation above, for any $j = 1, \dots, n$, we have

$$p^{j(j-1)/2} t_p^{(j)} = s_j(\alpha_i') = p^{j(n-1)/2} s_j(\chi_i(p)),$$

and hence $t_p^{(j)} = s_j(\chi_i(p)) p^{j(n-j)/2}$. In this setting, we have:

Definition 4.3. An automorphic representation π is *p-ordinary* if $\beta_i' + v_p(\alpha_i') = 0$ for all $i = 1, \dots, n$.

Again, if π is p-ordinary, the $v_p(\alpha_i')$ are integers.

By strict compatibility, crystalline Frobenius has its characteristic polynomial exactly as above. The next theorem is proved like Theorem 4.2.

Theorem 4.4 (spherical case, variant). *Let π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$ of weight a, as in Theorem 3.5. Let $r_{p,\iota}(\pi)$ be the corresponding p-adic Galois representation, with Hodge–Tate weights $-\beta_{n+1-i}' := a_i + n - i$, for $i = 1, \dots, n$. Suppose π_p is in the principal series with Satake parameters $\alpha_1, \dots, \alpha_n$, and set $\alpha_i' = \alpha_i p^{(n-1)/2}$. If π is p-ordinary, then*

$$r_{p,\iota}(\pi)|_{G_{\mathbb{Q}_p}} \sim \begin{pmatrix} \lambda\Big(\frac{\alpha_1'}{p^{v_p(\alpha_1')}}\Big)\cdot\chi_{\mathrm{cyc},p}^{-\beta_1'} & * & * & * \\ 0 & \lambda\Big(\frac{\alpha_2'}{p^{v_p(\alpha_2')}}\Big)\cdot\chi_{\mathrm{cyc},p}^{-\beta_2'} & * & * \\ & & \ddots & \\ 0 & 0 & \lambda\Big(\frac{\alpha_{n-1}'}{p^{v_p(\alpha_{n-1}')}}\Big)\cdot\chi_{\mathrm{cyc},p}^{-\beta_{n-1}'} & * \\ 0 & 0 & 0 & \lambda\Big(\frac{\alpha_n'}{p^{v_p(\alpha_n')}}\Big)\cdot\chi_{\mathrm{cyc},p}^{-\beta_n'} \end{pmatrix}.$$

In particular, $r_{p,\iota}(\pi)|_{G_{\mathbb{Q}_p}}$ is ordinary.

Theorem 4.4 was also obtained by D. Geraghty in the course of proving modularity lifting theorems for GL_n (see [Geraghty 2010, Lemma 2.7.7 and Corollary 2.7.8]). We thank T. Gee for pointing this out to us.

Ramified principal series case. Returning to the case where π is an automorphic representation with infinitesimal character H, we assume now that the automorphic representation $\pi_p = Q(\chi_1, \dots, \chi_n)$, where the χ_i are possibly ramified characters of $\mathbb{Q}_p^\times$.

By the local Langlands correspondence, we think of the χ_i as characters of the Weil group $W_{\mathbb{Q}_p}$. In particular, the restriction of the χ_i to the inertia group have finite image. By strict compatibility,

$$\mathrm{WD}(\rho)|_{I_p} \simeq \bigoplus_i \chi_i|_{I_p}.$$

The characters $\chi_i|_{I_p}$ factor through $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{nr}}(\zeta_{p^m})/\mathbb{Q}_p^{\mathrm{nr}}) \simeq \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^m})/\mathbb{Q}_p)$ for some $m \geq 1$. Denote $\mathbb{Q}_p(\zeta_{p^m})$ by F. Observe that F is a finite abelian totally ramified extension of $\mathbb{Q}_p$. Let $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}} : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(E)$ be the corresponding (p, p)-representation. Note that $\rho_{\pi,p}|_{G_F}$ is crystalline.

Let D_n be the corresponding filtered module. Then $D_n = Ee_1 + \cdots + Ee_n$, where $g \in \mathrm{Gal}(F/\mathbb{Q}_p)$ acts by χ_i on e_i. A short computation shows that $\varphi(e_i) = \alpha_i^{-1} e_i$, where $\alpha_i = \chi_i(\omega_F)$ for ω_F a uniformizer of F.

Using Corollary 2.5, and following the proof of Theorem 4.2, we obtain:

Theorem 4.5 (ramified principal series). *Say $\pi_p = Q(\chi_1, \dots, \chi_n)$ is in the ramified principal series. If π is p-ordinary,*

$$\rho_{\pi,p}|_{G_{\mathbb{Q}_p}} \sim \begin{pmatrix} \chi_1 \cdot \lambda\left(\frac{\alpha_1}{p^{v_p(\alpha_1)}}\right) \cdot \chi_{\mathrm{cyc},p}^{-\beta_1} & * & & * \\ 0 & \chi_2 \cdot \lambda\left(\frac{\alpha_2}{p^{v_p(\alpha_2)}}\right) \cdot \chi_{\mathrm{cyc},p}^{-\beta_2} & & * \\ 0 & 0 & \ddots & * \\ & & & \chi_n \cdot \lambda\left(\frac{\alpha_n}{p^{v_p(\alpha_n)}}\right) \cdot \chi_{\mathrm{cyc},p}^{-\beta_n} \end{pmatrix}.$$

In particular, $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ is quasiordinary.

5. The Steinberg case

In this section we treat the case where the Weil–Deligne representation attached to π_p is a twist of the special representation $\mathrm{Sp}(n)$. The case of unramified twists occupies most of the section; ramified twists are the subject of Theorem 5.8 at the end.

We start with the case where the Weil–Deligne representation attached to π_p is of the form $\chi \otimes \mathrm{Sp}(n)$, where χ is an unramified character.

Let D be the filtered $(\varphi, N, \mathbb{Q}_p, E)$-module attached to $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$. Thus D is a vector space over E. Note $N^n = 0$ and $N^{n-1} \neq 0$ so that there is a basis $\{f_n, f_{n-1}, \dots, f_1\}$ of D with $f_{i-1} := N f_i$ for $1 < i \leq n$ and $N f_1 = 0$, i.e.,

$$f_n \overset{N}{\mapsto} f_{n-1} \overset{N}{\mapsto} \cdots \overset{N}{\mapsto} f_1 \overset{N}{\mapsto} 0.$$

Say χ takes arithmetic Frobenius to α. Since $N\varphi = p\varphi N$, we may assume that $\varphi(f_i) = \alpha_i^{-1} f_i$ for all $i = 1, \dots, n$, where $\alpha_i^{-1} = p^{i-1}/\alpha$. When $\alpha = 1$, D reduces to the Weil–Deligne representation $\mathrm{Sp}(n)$ mentioned after Theorem 3.1.

For each $1 \leq i \leq n$, let D_i denote the subspace $\langle f_i, \dots, f_1 \rangle$. Clearly, $\dim(D_i) = i$ and $D_1 \subsetneq D_2 \subsetneq \cdots \subsetneq D_n$. One can easily prove:

Lemma 5.1. *For every integer $1 \leq r \leq n$, there is a unique N-submodule of D, of rank r, namely D_r.*

Let $\beta_n > \cdots > \beta_1$ be the jumps in the Hodge filtration on D. We assume that the Hodge filtration is in general position with respect to the Newton filtration given by the D_i (Assumption 3.6). An example of such a filtration is

$$\langle f_n \rangle \subsetneq \langle f_n, f_{n-1} \rangle \subsetneq \cdots \subsetneq \langle f_n, f_{n-1}, \dots, f_2 \rangle \subsetneq \langle f_n, f_{n-1}, \dots, f_1 \rangle.$$

The following elementary lemma plays an important role in later proofs.

Lemma 5.2. *Let m be a natural number. Let $\{a_i\}_{i=1}^n$ be an increasing sequence of integers such that $|a_{i+1} - a_i| = m$. Let $\{b_i\}_{i=1}^n$ be another increasing sequence of integers, such that $|b_{i+1} - b_i| \geq m$. Assume that $\sum_i a_i = \sum_i b_i$. If $a_n = b_n$ or $a_1 = b_1$, then $a_i = b_i$ for all i.*

The same holds with "decreasing" instead of "increasing".

Proof. Let us prove the lemma when $a_n = b_n$ and the a_i are increasing. The proof in the other cases is similar. We have

$$m(n-1+n-2+\cdots+1) \leq \sum_{i=1}^{n} (b_n - b_i) = \sum_{i=1}^{n} (a_n - a_i) = m(n-1+n-2+\cdots+1).$$

The first equality follows from $a_n = b_n$. From the equation above, we see that $b_n - b_i = a_n - a_i$ for every $1 \leq i \leq n$. Since $a_n = b_n$, we have $a_i = b_i$ for every $1 \leq i \leq n$. □

By Lemma 5.1, the D_i are the only (φ, N)-submodules of D. The following proposition shows that if two consecutive submodules D_i and D_{i+1} are admissible, all the D_i are admissible.

Proposition 5.3. *Suppose there exists an integer $1 \leq i \leq n$ such that both D_i and D_{i+1} are admissible. Then each D_r, for $1 \leq r \leq n$, is admissible. Moreover, the β_i are consecutive integers.*

Proof. Since D_i and D_{i+1} are admissible, we have the equalities

$$\beta_1+\beta_2+\cdots+\beta_i=-\sum_{r=1}^{i} v_p(\alpha_r) \quad \text{and} \quad \beta_1+\beta_2+\cdots+\beta_{i+1}=-\sum_{r=1}^{i+1} v_p(\alpha_r), \tag{5-1}$$

whose difference gives

$$-v_p(\alpha_{i+1})=\beta_{i+1}. \tag{5-2}$$

Define $a_r=-v_p(\alpha_r)$ and $b_r=\beta_r$ for $1 \leq r \leq n$. Hence,

$$\begin{aligned} a_n > \cdots > a_{i+2} > a_{i+1} > a_i > \cdots > a_1, \\ b_n > \cdots > b_{i+2} > b_{i+1} > b_i > \cdots > b_1. \end{aligned} \tag{5-3}$$

By (5-2), $a_{i+1}=b_{i+1}$. By Lemma 5.2 and (5-1), we have $a_r=b_r$ for all $1 \leq r \leq i+1$.

Since D_n is admissible,

$$t_H(D_n)=\sum_{r=1}^{n}\beta_r=-\sum_{r=1}^{n} v_p(\alpha_r)=t_N(D_n). \tag{5-4}$$

From (5-1) and (5-4), we have

$$\sum_{r=i+1}^{n}\beta_r=-\sum_{r=i+1}^{n} v_p(\alpha_r).$$

Again, by (5-3) and Lemma 5.2, we have $a_r=b_r$ for all $i+1 \leq r \leq n$. Hence $\beta_r=-v_p(\alpha_r)$ for all $1 \leq r \leq n$. This shows that all the other D_i's are admissible. Also, the β_i are consecutive integers since the $v_p(\alpha_i)$ are consecutive integers. □

Corollary 5.4. *Keeping the notation as above, the admissibility of D_1 or D_{n-1} implies the admissibility of all other D_i.*

Theorem 5.5. *Assume that the Hodge filtration on D is in general position with respect to the D_i (Assumption 3.6). Then the crystal D is either irreducible or reducible, in which case each D_i, for $1 \leq i \leq n$, is admissible.*

Proof. If D is irreducible, we are done. If not, there exists an i, such that D_i is admissible. If D_{i-1} or D_{i+1} is admissible, then by Proposition 5.3, all the D_r are admissible. So, it is enough to consider the case where neither D_{i-1} nor D_{i+1} is admissible (and D_i is admissible). We have

$$\beta_1 + \beta_2 + \cdots + \beta_{i-1} < -\sum_{r=1}^{r=i-1} v_p(\alpha_r), \tag{5-5a}$$

$$\beta_1 + \beta_2 + \cdots + \beta_{i} = -\sum_{r=1}^{r=i} v_p(\alpha_r), \tag{5-5b}$$

$$\beta_1 + \beta_2 + \cdots + \beta_{i+1} < -\sum_{r=1}^{r=i+1} v_p(\alpha_r). \tag{5-5c}$$

Subtracting (5-5b) from (5-5a), we get $-\beta_i < v_p(\alpha_i)$. Subtracting (5-5b) from (5-5c), we get $\beta_{i+1} < -v_p(\alpha_{i+1}) = -v_p(\alpha_i) + 1$. Adding these inequalities, we obtain $\beta_{i+1} - \beta_i < 1$. But this is a contradiction, since $\beta_{i+1} > \beta_i$. This proves the theorem. □

Definition 5.6. Say π is *p-ordinary* if $\beta_1 + v_p(\alpha) = 0$.

If π is p-ordinary, D_1 is admissible, so the flag $D_1 \subset D_2 \subset \cdots \subset D_n$ is an admissible flag by Theorem 5.5 (an easy check shows that if π is p-ordinary, Assumption 3.6 holds automatically).

Applying the discussion above to the local Galois representation $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$, we obtain:

Theorem 5.7 (unramified twist of Steinberg representation). *Say π is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$ with infinitesimal character given by the integers $-\beta_1 > \cdots > -\beta_n$. Suppose that π_p is an unramified twist of the Steinberg representation, that is, $\mathrm{WD}(\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}) \sim \chi \otimes \mathrm{Sp}(n)$, where χ is the unramified character mapping arithmetic Frobenius to α. If π is ordinary at p (that is, $v_p(\alpha) = -\beta_1$), then the β_i are necessarily consecutive integers and*

$$\rho_{\pi,p}|_{G_{\mathbb{Q}_p}} \sim \begin{pmatrix} \lambda\left(\frac{\alpha}{p^{v_p(\alpha)}}\right)\cdot\chi_{\mathrm{cyc},p}^{-\beta_1} & * & & * \\ 0 & \lambda\left(\frac{\alpha}{p^{v_p(\alpha)}}\right)\cdot\chi_{\mathrm{cyc},p}^{-\beta_1-1} & & * \\ & & \ddots & \\ 0 & 0 & & \lambda\left(\frac{\alpha}{p^{v_p(\alpha)}}\right)\cdot\chi_{\mathrm{cyc},p}^{-\beta_1-(n-1)} \end{pmatrix},$$

where $\lambda(\alpha/p^{v_p(\alpha)})$ is an unramified character that takes arithmetic Frobenius to $\alpha/p^{v_p(\alpha)}$, and in particular, $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ is ordinary. If π is not p-ordinary and Assumption 3.6 holds, then $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ is irreducible.

Proof. By strict compatibility, D is the filtered $(\varphi, N, \mathbb{Q}_p, E)$-module attached to $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$. If π is p-ordinary, we are done and the characters on the diagonal are determined by Corollary 2.5.

If π is not p-ordinary, D is irreducible. Indeed, if D is reducible, then by Theorem 5.5, all D_i, and in particular D_1, are admissible, so π is p-ordinary. $\square$

Theorem 5.8 (ramified twist of Steinberg representation). *Let the notation and hypotheses be as in Theorem 5.7, except that this time assume that*

$$\mathrm{WD}(\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}) \sim \chi \otimes \mathrm{Sp}(n),$$

where χ is an arbitrary, possibly ramified, character. Write $\chi = \chi_0 \cdot \chi'$ where χ_0 is the ramified part of χ, and χ' is an unramified character taking arithmetic Frobenius to α. If π is p-ordinary ($\beta_1 = -v_p(\alpha)$), then the β_i are consecutive integers and

$$\rho_{\pi,p}|_{G_{\mathbb{Q}_p}} \sim \begin{pmatrix} \chi_0 \cdot \lambda\left(\frac{\alpha}{p^{v_p(\alpha)}}\right) \cdot \chi_{\mathrm{cyc},p}^{-\beta_1} & * & * \\ 0 & \chi_0 \cdot \lambda\left(\frac{\alpha}{p^{v_p(\alpha)}}\right) \cdot \chi_{\mathrm{cyc},p}^{-\beta_1-1} & * \\ & & \ddots & \\ 0 & 0 & \chi_0 \cdot \lambda\left(\frac{\alpha}{p^{v_p(\alpha)}}\right) \cdot \chi_{\mathrm{cyc},p}^{-\beta_1-(n-1)} \end{pmatrix}.$$

If π is not p-ordinary and Assumption 3.6 holds, then $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ is irreducible.

Proof. Let F be a totally ramified abelian (cyclotomic) extension of $\mathbb{Q}_p$ such that $\chi_0|_{I_F} = 1$. Then the reducibility of $\rho_{\pi,p}|_{G_F}$ over F can be shown exactly as in Theorem 5.7, and the theorem over $\mathbb{Q}_p$ follows using the descent data of the underlying filtered module. If π is not p-ordinary, then by arguments similar to those used in proving Theorem 5.7, $\rho_{\pi,p}|_{G_F}$ is irreducible, so that $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ is also irreducible. $\square$

6. Supercuspidal ⊗ Steinberg

We now turn to the case where the Weil–Deligne representation attached to π_p is indecomposable. Thus we assume that $\mathrm{WD}(\rho_{\pi,p}|_{G_{\mathbb{Q}_p}})$ is Frobenius semisimple and is of the form $\tau \otimes \mathrm{Sp}(n)$, where τ is an irreducible m-dimensional representation corresponding to a supercuspidal representation of GL_m for $m \geq 1$, and $\mathrm{Sp}(n)$ for $n \geq 1$ denotes the usual special representation.

(φ, N)-submodules. We start by classifying the (φ, N, F, E)-submodules of D, the crystal attached to the local representation $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$, when $\mathrm{WD}(\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}) = \tau \otimes \mathrm{Sp}(n)$ for $m \geq 1$ and $n \geq 1$. This will play a key role in the study of the structure of $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$, when taking the Hodge filtration on D into account.

Recall from [Breuil and Schneider 2007, Proposition 4.1] that there is an equivalence of categories between (φ, N)-modules with coefficients and descent data, and

Weil–Deligne representations. We write D_τ for the (φ, N)-modules corresponding to τ, and likewise $D_{\mathrm{Sp}(n)}$ for those corresponding to $\mathrm{Sp}(n)$, and so on.

Theorem 6.1. *All the (φ, N, F, E)-submodules of $D = D_\tau \otimes D_{\mathrm{Sp}(n)}$ are of the form $D_\tau \otimes D_{\mathrm{Sp}(r)}$ for some $1 \leq r \leq n$.*

The case that $m = 1$ was treated in the previous section (twist of Steinberg), and the case $n = 1$ is vacuously true.

Assume first that τ is an induced representation of dimension m of the form $\mathrm{Ind}_{W_K}^{W_p} \chi$, where K is a p-adic field such that $[K : \mathbb{Q}_p] = m$, and χ is a character of W_K. This is known to always hold if $(p, m) = 1$ or $p > m$. For simplicity, we shall assume that K is the unique unramified extension of $\mathbb{Q}_p$, namely $\mathbb{Q}_{p^m}$, and refer to τ in this case as an unramified supercuspidal representation. Let σ be the generator of $\mathrm{Gal}(\mathbb{Q}_{p^m}/\mathbb{Q}_p)$, and let I_{p^m} denote the inertia subgroup of $\mathbb{Q}_{p^m}$. Then

$$\tau|_{I_p = I_{p^m}} \simeq (\mathrm{Ind}_{W_{p^m}}^{W_p} \chi)|_{I_{p^m}} \simeq \bigoplus_{i=1}^{m} \chi^{\sigma^i}|_{I_{p^m}}.$$

Since τ is irreducible, by Mackey's criterion, we have $\chi \neq \chi^{\sigma^i}$ for all i, on W_{p^m} and also on I_{p^m}. Moreover, $\chi^{\sigma^j} \neq \chi^{\sigma^i}$ for any $i \neq j$.

Following the methods of [Ghate and Mézard 2009], it is possible to explicitly write down the crystal D_τ whose underlying Weil–Deligne representation is the unramified supercuspidal representation τ above. For the details we refer the reader to [Ghate and Kumar 2010, §7.2]. In particular, one may write down appropriate finite extensions F_0 and F of $\mathbb{Q}_p$, so that D_τ is a free rank m module over $F_0 \otimes E$ with basis e_i, $i = 1, \ldots, m$, such that D_τ is given by

$$\begin{aligned} &\varphi(e_i) = t_m^{-1/m} e_i, \quad N(e_i) = 0, \quad \sigma(e_i) = e_{i+1}, \\ &g(e_i) = (1 \otimes \chi^{\sigma^{i-1}}(g))(e_i), \quad g \in I(F/K) \end{aligned} \tag{6-1}$$

for all $1 \leq i \leq m$ and some constant $t_m \in \mathbb{O}_E$. When $m = 2$, this (φ, N)-module is exactly the one given in [Ghate and Mézard 2009, §3.3], though the e_i used here differ by a scalar from the e_i used there.

Recall that the module $D_{\mathrm{Sp}(n)}$ has a basis $\{f_n, f_{n-1}, \ldots, f_1\}$, with properties as in Section 5. Using (6-1) and a basis of $D_\tau \otimes D_{\mathrm{Sp}(r)}$ of the form $e_i \otimes f_j$, it is possible to give an explicit proof of Theorem 6.1, when τ is an unramified supercuspidal representation of dimension m [Ghate and Kumar 2010, §7.2.5]. However, it is also possible to prove the theorem for general τ, independently of any explicit formulas. We give this proof now. The following general lemma is useful:

Lemma 6.2. *The theory of Jordan canonical forms can be extended to nilpotent operators on free finite-rank $(F_0 \otimes E)$-modules. We call the number of blocks in the Jordan decomposition of the monodromy operator N as the index of N.*

Proof. One simply extends the usual theory of Jordan canonical forms on each projection under (2-1) to modules over $F_0 \otimes E$-modules. □

Returning to our situation:

Lemma 6.3. *There are no rank r (φ, N, F, E)-submodules of $D = D_\tau \otimes D_{\mathrm{Sp}(n)}$ on which N acts trivially for $1 \leq r \leq m-1$.*

Proof. Suppose there exists such a module, say $\tilde{D}$, of rank $r < m$. Since N acts trivially on $\tilde{D}$, we have $\tilde{D} \subseteq D_\tau \otimes \langle f_1 \rangle = D_\tau \otimes D_{\mathrm{Sp}(1)} \simeq D_\tau$. But τ is irreducible, so D_τ is irreducible by Lemma 2.7, a contradiction. □

Corollary 6.4. *The index of N on a (φ, N, F, E)-submodule of D is m.*

Proof of Theorem 6.1. Let D' be a (φ, N, F, E)-submodule of $D = D_\tau \otimes D_{\mathrm{Sp}(n)}$. By the corollary above, there are m blocks in the Jordan canonical form of N on D'. Without loss of generality, assume that the blocks have sizes $r_1 \leq r_2 \leq \cdots \leq r_m$ with $\sum_{i=1}^{m} r_i = \text{rank } D'$. Suppose $w_1, \ldots, w_m$ are the corresponding basis vectors in D' such that the order of nilpotency of N on w_i is r_i, so that the $N^j(w_i)$ form a basis of D'. If all the r_i are equal to say r, an easy argument shows $D' = D_\tau \otimes D_{\mathrm{Sp}(r)}$. We show that this is indeed the case.

Suppose towards a contradiction that $r_i \neq r_{i+1}$ for some $1 \leq i < m$. For $1 \leq i \leq n$, let D_i denote the submodule $D_\tau \otimes \mathrm{Ker}(N^i) = D_\tau \otimes D_{\mathrm{Sp}(i)}$. Now, arrange the basis vectors $N^j w_k$ of D' as follows:

$$
\begin{array}{r}
w_1,\ Nw_1,\ \ldots,\ N^{r_1-1}w_1,\\
\vdots\\
w_k,\ Nw_k,\ \ldots,\ N^{r_k-1}w_k,\\
w_{k+1},\ Nw_{k+1},\ \ldots,\ N^{r_{k+1}-r_1-1}w_{k+1},\ N^{r_{k+1}-r_1}w_{k+1},\ \ldots,\ N^{r_{k+1}-1}w_{k+1},\\
\vdots\\
w_m,\ Nw_m,\ N^2w_m,\ \ldots,\ N^{r_m-r_1-1}w_m,\ N^{r_m-r_1}w_m,\ \ldots,\ N^{r_m-1}w_m.
\end{array}
$$

With respect to this arrangement, denote the span of the vectors in the last i columns by A_i. Since $r_i \neq r_{i+1}$, the rank of the space A_{r_i+1}/A_{r_i} is less than m. Moreover, A_{r_i+1}/A_{r_i} is a subspace of D_{r_i+1}/D_{r_i}; that is, there is an inclusion of (φ, N, F, E)-modules $A_{r_i+1}/A_{r_i} \hookrightarrow D_{r_i+1}/D_{r_i}$. Now

$$
\begin{aligned}
D_{r_i+1}/D_{r_i} &= (D_\tau \otimes D_{\mathrm{Sp}(r_i+1)})/(D_\tau \otimes D_{\mathrm{Sp}(r_i)})\\
&\simeq D_\tau \otimes (D_{\mathrm{Sp}(r_i+1)}/D_{\mathrm{Sp}(r_i)}) \simeq D_\tau \otimes D_{\mathrm{Sp}(1)} \simeq D_\tau.
\end{aligned}
$$

All the isomorphisms above are isomorphisms of (φ, N, F, E)-modules over $F_0 \otimes E$. By Lemma 6.3, the inclusion above is not possible! Hence all the r_i are indeed equal. This finishes the proof of Theorem 6.1. □

Filtration on $D = D_\tau \otimes D_{\mathrm{Sp}(n)}$. We can now apply the discussion above to write down the structure of the (p, p)-Galois representation attached to a cuspidal automorphic representation of $\mathrm{GL}_{mn}(\mathbb{A}_{\mathbb{Q}})$.

We start with some remarks. Suppose D_1 and D_2 are two admissible filtered modules. It is well-known (see [Totaro 1996]) that the tensor product $D_1 \otimes D_2$ is also admissible. The difficulty in proving this lies in the fact that one does not have much information about the structure of the (φ, N)-submodules of the tensor product. If they are of the form $D' \otimes D''$, where D' and D'' are admissible (φ, N)-submodules of D_1 and D_2 respectively, then $D' \otimes D''$ is also admissible by Lemma 2.3. But not all the submodules of $D_1 \otimes D_2$ are of this form.

However, we saw in Theorem 6.1 that for $D = D_\tau \otimes D_{\mathrm{Sp}(n)}$, all the (φ, N, F, E)-submodules of D are of the form $D_\tau \otimes D_{\mathrm{Sp}(r)}$ for some $1 \le r \le n$. This fact allows us to study the crystal D and its submodules, once we introduce the Hodge filtration.

Filtration in general position. Assume that the Hodge filtration on D is in general position with respect to the Newton filtration (Assumption 3.6). Let m be the rank of D_τ. Let $\{\beta_{i,j}\}_{i=1,j=1}^{i=n,j=m}$ be the jumps in the Hodge filtration with $\beta_{i_1,j_1} > \beta_{i_2,j_2}$, if $i_1 > i_2$, or if $i_1 = i_2$ and $j_1 > j_2$. Thus

$$\beta_{n,m} > \beta_{n,m-1} > \cdots > \beta_{n,1} > \beta_{n-1,m} > \cdots > \beta_{1,m} > \cdots > \beta_{1,1}.$$

Define, for every $1 \le k \le n$,

$$b_k = \sum_{j=1}^{j=m} \beta_{k,j},$$

and

$$a_k = t_N(D_\tau \otimes D_{\mathrm{Sp}(k)}) - t_N(D_\tau \otimes D_{\mathrm{Sp}(k-1)}) = t_N(D_\tau) + m(k-1),$$

where the last equality follows from Lemma 2.3. Clearly,

$$b_n > b_{n-1} > \cdots > b_2 > b_1,$$
$$a_n > a_{n-1} > \cdots > a_2 > a_1.$$

Observe that $b_{i+1} - b_i \ge m^2$ and $a_{i+1} - a_i = m$ for every $1 \le i \le n$. Since D is admissible, the submodule $D_\tau \otimes D_{\mathrm{Sp}(i)}$ of D is admissible if and only if $\sum_{k=1}^{i} b_k = \sum_{k=1}^{i} a_k$.

The arguments below are similar to the ones used when analyzing the Steinberg case. We start with an analog of Lemma 5.2.

Lemma 6.5. *Let $\{a_i\}_{i=1}^n$ be an increasing sequence of integers such that $a_{i+1} - a_i = m$ for every i and for some fixed natural number m. Let $\{b_i\}_{i=1}^n$ be an increasing sequence of integers such that $b_{i+1} - b_i \ge m^2$ for every i. Suppose that $\sum_i a_i = \sum_i b_i$. If $a_n = b_n$ or $a_1 = b_1$, then $m = 1$ and hence $a_i = b_i$ for all i.*

Proof. We prove the lemma when $a_n = b_n$; the case of $a_1 = b_1$ is similar. Write

$$m^2(n-1+n-2+\cdots+1) \le \sum_{i=1}^{n}(b_n-b_i) = \sum_{i=1}^{n}(a_n-a_i) = m(n-1+n-2+\cdots+1),$$

where the first equality follows from $a_n = b_n$. From the inequality we see that $m = 1$. Now, the rest of the proof follows from Lemma 5.2. □

Theorem 6.6. *If $D_\tau \otimes D_{\mathrm{Sp}(i)}$ and $D_\tau \otimes D_{\mathrm{Sp}(i+1)}$ are admissible submodules of D, then $m = 1$, in which case all the $D_\tau \otimes D_{\mathrm{Sp}(i)}$, for $1 \le i \le n$, are admissible.*

Proof. Since $D_\tau \otimes D_{\mathrm{Sp}(i)}$ and $D_\tau \otimes D_{\mathrm{Sp}(i+1)}$ are admissible, we have

$$b_1 + b_2 + \cdots + b_i = \sum_{r=1}^{i} a_r \quad \text{and} \quad b_1 + b_2 + \cdots + b_{i+1} = \sum_{r=1}^{i+1} a_r. \tag{6-2}$$

From these expressions, $b_{i+1} = a_{i+1}$. As recalled above:

$$b_n > \cdots > b_{i+2} > b_{i+1} > b_i > \cdots > b_1,$$
$$a_n > \cdots > a_{i+2} > a_{i+1} > a_i > \cdots > a_1.$$

Since $a_{i+1} = b_{i+1}$ and (6-2) holds, by Lemma 6.5 we have $m = 1$ and $a_i = b_i$ for all $1 \le i \le n$. This shows that all the $D_\tau \otimes D_{\mathrm{Sp}(i)}$ are admissible. □

Theorem 6.7. *Let $D = D_\tau \otimes D_{\mathrm{Sp}(n)}$ and assume that the Hodge filtration on D is in general position (Assumption 3.6). Then either D is irreducible or D is reducible, in which case $m = 1$ and the (φ, N, F, E)-submodules $D_\tau \otimes D_{\mathrm{Sp}(i)}$, for $1 \le i \le n$, are all admissible.*

Proof. Let $D_i = D_\tau \otimes D_{\mathrm{Sp}(i)}$ for $1 \le i \le n$. If D is irreducible, we are done. If not, by Theorem 6.1, there exists an $1 \le i \le n$ such that D_i is admissible. If D_{i-1} or D_{i+1} is also admissible, then by the theorem above, $m = 1$ and hence all the (φ, N, F, E)-submodules of D are admissible. So, assume D_{i-1} and D_{i+1} are not admissible, but D_i is admissible. We shall show that this is not possible. Indeed, we have

$$b_1 + b_2 + \cdots + b_{i-1} < \sum_{r=1}^{r=i-1} a_r, \tag{6-3a}$$

$$b_1 + b_2 + \cdots + b_i = \sum_{r=1}^{r=i} a_r, \tag{6-3b}$$

$$b_1 + b_2 + \cdots + b_{i+1} < \sum_{r=1}^{r=i+1} a_r. \tag{6-3c}$$

Subtracting (6-3b) from (6-3a), we get $-b_i < -a_i$. Subtracting (6-3b) from (6-3c), we get $b_{i+1} < a_{i+1}$. Adding these two inequalities, we get $b_{i+1} - b_i < a_{i+1} - a_i = m$. But this is a contradiction, since $b_{i+1} - b_i \geq m$. $\square$

For emphasis we state separately:

Corollary 6.8. *With assumptions as above, the crystal $D = D_\tau \otimes D_{\mathrm{Sp}(n)}$ is irreducible if $m \geq 2$.*

Definition 6.9. Say π is *ordinary* at p if $a_1 = b_1$, that is, $t_N(D_\tau) = \sum_{j=1}^{m} \beta_{1,j}$.

This condition implies $m = 1$; the definition then coincides with Definition 5.6.

Applying the discussion above to the local (p, p)-Galois representation in a strictly compatible system, we obtain:

Theorem 6.10 (indecomposable case). *Say π is a cuspidal automorphic representation with infinitesimal character consisting of distinct integers. Suppose that*

$$\mathrm{WD}(\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}) \sim \tau_m \otimes \mathrm{Sp}(n),$$

where τ_m is an irreducible representation of $W_{\mathbb{Q}_p}$ of dimension $m \geq 1$, and $n \geq 1$. Assume that Assumption 3.6 holds.

- *If π is ordinary at p, then $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ is reducible, in which case $m = 1$, τ_1 is a character, and $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ is (quasi)ordinary as in Theorems 5.7 and 5.8.*
- *If π is not ordinary at p, then $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ is irreducible.*

Tensor product filtration. One might wonder what happens if the filtration on D is not necessarily in general position. As an example, we consider here just one case arising from the so-called tensor product filtration.

Assume that D_τ and $D_{\mathrm{Sp}(n)}$ are the usual filtered (φ, N, F, E)-modules, and equip $D_\tau \otimes D_{\mathrm{Sp}(n)}$ with the tensor product filtration. By the formulas in Lemma 2.3, one can prove:

Lemma 6.11. *Suppose that $D = D_\tau \otimes D_{\mathrm{Sp}(n)}$ has the tensor product filtration. Fix $1 \leq r \leq n$. Then $D_\tau \otimes D_{\mathrm{Sp}(r)}$ is an admissible submodule of D if and only if $D_{\mathrm{Sp}(r)}$ is an admissible submodule of $D_{\mathrm{Sp}(n)}$.*

We recall that if the filtration on $D_{\mathrm{Sp}(n)}$ is in general position (as in Assumption 3.6), then we have shown that furthermore $D_{\mathrm{Sp}(r)}$ is an admissible submodule of $D_{\mathrm{Sp}(n)}$ if and only if $D_{\mathrm{Sp}(1)}$ is an admissible submodule.

The lemma can be used to give an example where the tensor product filtration on D is not in general position (i.e., does not satisfy Assumption 3.6). Suppose that τ is an irreducible representation of dimension $m = 2$ and $D_{\mathrm{Sp}(2)}$ has weight 2 (as in [Breuil 2001] or [Ghate and Mézard 2009, §3.1]). Note that $D_{\mathrm{Sp}(1)}$ is an admissible submodule of $D_{\mathrm{Sp}(2)}$. Hence, by the lemma, $D_\tau \otimes D_{\mathrm{Sp}(1)}$ is an admissible submodule of $D_\tau \otimes D_{\mathrm{Sp}(2)}$. If the tensor product filtration satisfies Assumption 3.6, then the admissibility of $D_\tau \otimes D_{\mathrm{Sp}(1)}$ would contradict Theorem 6.7, since $m = 2$.

In any case, we have the following application to local Galois representations.

Proposition 6.12. *Suppose that $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}} \sim \rho_\tau \otimes \rho_{\mathrm{Sp}(n)}$ is a tensor product of two (p, p)-Galois representations, with underlying Weil–Deligne representations τ and $\mathrm{Sp}(n)$, respectively. If $\rho_{\mathrm{Sp}(n)}$ is irreducible, so is $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$.*

Errata

We end this paper by correcting some errors in [Ghate and Mézard 2009]:

- p. 2254, lines 6 and 7: $\mathbf{Q}$ should be $\mathbf{Q}_p$.
- p. 2257, first two lines should be $\varphi(e_1) = (1/\sqrt{t})\, e_1$ and $\varphi(e_2) = (1/\sqrt{t})\, e_2$.
- p. 2260: first three lines in the middle display should be $\varphi(e_1) = (1/\sqrt{t})\, e_1$, $\varphi(e_2) = (1/\sqrt{t})\, e_2$, and $t \in \mathbb{O}_E$, $\mathrm{val}_p(t) = k - 1$. Moreover, t is to be chosen in §3.4.3 satisfying $t^2 = 1/c$ (we may take $c = d$, since ι commutes with φ, and s is no longer required).

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