ON INTRINSICALLY KNOTTED OR COMPLETELY 3-LINKED GRAPHS

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We say that a graph is intrinsically knotted or completely 3-linked if every embedding of the graph into the 3-sphere contains a nontrivial knot or a 3-component link each of whose 2-component sublinks is nonsplittable. We show that a graph obtained from the complete graph on seven vertices by a finite sequence of \( \Delta Y \)-exchanges and \( Y \Delta \)-exchanges is a minor-minimal intrinsically knotted or completely 3-linked graph.

1. Introduction

Throughout this paper we work in the piecewise linear category. Let \( f \) be an embedding of a finite graph \( G \) into the 3-sphere. Then \( f \) is called a spatial embedding of \( G \) and \( f(G) \) is called a spatial graph. We denote the set of all spatial embeddings of \( G \) by \( SE(G) \). We call a subgraph \( \gamma \) of \( G \) that is homeomorphic to the circle a cycle of \( G \). For a positive integer \( n \), let \( \Gamma^{(n)}(G) \) denote the set of all cycles of \( G \) if \( n = 1 \) and the set of all unions of \( n \) mutually disjoint cycles of \( G \) if \( n \geq 2 \). For simplicity, we also write \( \Gamma(G) \) for \( \Gamma^{(1)}(G) \). For an element \( \lambda \) in \( \Gamma^{(n)}(G) \) and a spatial embedding \( f \) of \( G \), \( f(\lambda) \) is a knot if \( n = 1 \) and an \( n \)-component link if \( n \geq 2 \).

A graph \( G \) is said to be intrinsically linked (IL) if for every spatial embedding \( f \) of \( G \), \( f(G) \) contains a nonsplittable 2-component link. Conway and Gordon [1983] and Sachs [1984] showed that \( K_6 \) is IL, where \( K_m \) denotes the complete graph on \( m \) vertices. Also, IL graphs have been completely characterized as follows. For a graph \( G \) and an edge \( e \) of \( G \), we denote the subgraph \( G \setminus \text{int } e \) by \( G - e \). Let \( e = uv \) be an edge of \( G \) that is not a loop. We call the graph obtained from \( G - e \) by identifying the end vertices \( u \) and \( v \) the edge contraction of \( G \) along \( e \), and denote it by \( G/e \). A graph \( H \) is called a minor of a graph \( G \) if there exists a subgraph \( G' \) of \( G \) and edges \( e_1, e_2, \ldots, e_m \) of \( G' \) such that \( H \) is obtained from \( G' \) by a

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sequence of edge contractions along $e_1, e_2, \ldots, e_m$. A minor $H$ of $G$ is called a proper minor if $H$ does not equal $G$. Let $\mathcal{P}$ be a property for graphs that is closed under minor reductions; that is, for any graph $G$ that does not have $\mathcal{P}$, all minors of $G$ also do not have $\mathcal{P}$. A graph $G$ is said to be minor-minimal with respect to $\mathcal{P}$ if $G$ has $\mathcal{P}$ but all proper minors of $G$ do not have $\mathcal{P}$. Note that $G$ has $\mathcal{P}$ if and only if $G$ has a minor-minimal graph with respect to $\mathcal{P}$ as a minor. By the famous theorem of Robertson and Seymour [2004], there are finitely many minor-minimal graphs with respect to $\mathcal{P}$. Nešetřil and Thomas [1985] showed that IL is closed under minor reductions, and Robertson, Seymour and Thomas [Robertson et al. 1995] showed that the set of all minor-minimal graphs with respect to IL equals the Petersen family, which is the set of all graphs obtained from $K_6$ by a finite sequence of $\triangle Y$-exchanges and $Y\triangle$-exchanges. A $\triangle Y$-exchange is the left-to-right operation shown here:

That is, a graph $G_\triangle$ containing a three-edge cycle $\triangle$ is changed into a new graph $G_Y$ by removing the edges of the cycle and adding a new vertex $x$ connected to each of the vertices of the deleted cycle, thus forming a $Y$. A $Y\triangle$-exchange is the reverse of this operation. $\triangle Y$- and $Y\triangle$-exchanges preserve IL: if $G_\triangle$ is IL, so is $G_Y$ [Motwani et al. 1988], and if $G_Y$ is IL, so is $G_\triangle$ [Robertson et al. 1995].

The Petersen family contains seven graphs, including the Petersen graph $P_{10}$:

(An arrow between two graphs indicates the application of a single $\triangle Y$-exchange.)

A graph $G$ is said to be intrinsically knotted (IK) if for every spatial embedding $f$ of $G$, $f(G)$ contains a nontrivial knot. Conway and Gordon [1983] showed that $K_7$ is IK. Fellows and Langston [1988] showed that IK is closed under minor
reductions. Motwani, Raghunathan, and Saran [Motwani et al. 1988] showed that $K_7$ is a minor-minimal IK graph, and additional minor-minimal IK graphs were given in [Kohara and Suzuki 1992] and [Foisy 2002; 2003].

IK graphs have not been completely characterized yet. If $G_\Delta$ is IK then $G_\nabla$ is also IK [Motwani et al. 1988], but if $G_\nabla$ is IK then $G_\Delta$ may not always be IK. That is, the $\nabla \Delta$-exchange does not preserve IK in general. Flapan and Naimi [2008] showed that there exists a graph $G_{FN}$ obtained from $K_7$ by five $\nabla Y$-exchanges and two $Y \Delta$-exchanges that is not IK. We call the set of all graphs obtained from $K_7$ by a finite sequence of $\nabla Y$ and $Y \Delta$-exchanges the Heawood family.\(^1\) This family contains exactly twenty graphs, as illustrated in Figure 1; of these, $C_{14}$ is the Heawood graph (Remark 4.7).

Kohara and Suzuki [1992] showed that a graph $G$ in the Heawood family is a minor-minimal IK graph if $G$ is obtained from $K_7$ by a finite sequence of $\Delta Y$-exchanges, that is, if $G$ is one of fourteen graphs $K_7$, $H_8$, $H_9$, $\ldots$, $H_{12}$, $F_9$, $F_{10}$, $E_{10}$, $E_{11}$ and $C_{11}$, $C_{12}$, $\ldots$, $C_{14}$.\(^2\) On the other hand, $N_{10}'$ is isomorphic to $G_{FN}$, that is, $N_{10}'$ is not IK. Our first purpose in this paper is to determine completely when a graph in the Heawood family is IK.

**Theorem 1.1.** For a graph $G$ in the Heawood family, the following are equivalent:

1. $G$ is IK.
2. $G$ is obtained from $K_7$ by a finite sequence of $\Delta Y$-exchanges.
3. $\Gamma^{(3)}(G)$ is the empty set.

Hence the members $N_9$, $N_{10}$, $N_{11}$, $N_{10}'$, $N_{11}'$ and $N_{12}'$ of the Heawood family are not IK, and only they contain a union of three mutually disjoint cycles.

Our second purpose is to show that any of the graphs in the Heawood family is a minor-minimal graph with respect to a certain kind of intrinsic nontriviality even if it is not IK. We say that a graph $G$ is *intrinsically knotted or completely 3-linked*—I(K or C3L) for short—if for every spatial embedding $f$ of $G$, $f(G)$ contains a nontrivial knot or a 3-component link all of whose 2-component sublinks are nonsplittable. An IK graph is I(K or C3L). As we show in Proposition 2.2, I(K or C3L) is closed under minor reductions.

**Theorem 1.2.** All graphs in the Heawood family are minor-minimal I(K or C3L) graphs.

As we have seen, $N_9$, $N_{10}$, $N_{11}$, $N_{10}'$, $N_{11}'$ and $N_{12}'$ are not IK, but they are but I(K or C3L) and are minor-minimal with respect to I(K or C3L).

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\(^1\)Van der Holst [2006] calls the set of all graphs obtained from $K_7$ or $K_{3,3,1,1}$ by a finite sequence of $\Delta Y$-exchanges and $Y \Delta$-exchanges the Heawood family, where $K_{3,3,1,1}$ is the complete 4-partite graph on $3+3+1+1$ vertices.

\(^2\)One edge of $F_{10}$ in [Kohara and Suzuki 1992, Figure 5] is wanting.
Figure 1. The Heawood family. An arrow between two graphs indicates the application of a single $\Delta Y$-exchange.

Remark 1.3. A graph $G$ is said to be intrinsically $n$-linked (InL) if for every spatial embedding $f$ of $G$, $f(G)$ contains a nonsplittable $n$-component link [Flapan et al. 2001a; 2001b]. I2L coincides with IL. Let $G$ be a graph in the Heawood family.
that is not IK. Then we show in Example 4.6 that there exists a spatial embedding $f$ of $G$ such that $f(G)$ does not contain a nonsplittable 3-component link. That is, $G$ is neither IK nor I3L.

**Remark 1.4.** A graph $G$ is called *intrinsically knotted* or *3-linked*—I(K or 3L) for short—if for every spatial embedding $f$ of $G$, $f(G)$ contains a nontrivial knot or a nonsplittable 3-component link. Clearly I(K or C3L) implies I(K or 3L), but the converse is not true: [Foisy 2006] exhibits an I(K or 3L) graph $G$ and a spatial embedding $f$ of $G$ such that $f(G)$ contains no nontrivial knot and all nonsplittable 3-component links contained in $f(G)$ have split 2-component sublinks.

The rest of this paper is organized as follows. Section 2 contains general results about graph minors, $\Delta Y$-exchanges and spatial graphs. We prove Theorem 1.1 in Section 3 and Theorem 1.2 in Section 4.

## 2. Graph minors, $\Delta Y$-exchanges and spatial graphs

Let $H$ be a minor of a graph $G$. Then there exists a natural injection

$$
\Psi(n) = \Psi_{H,G}^{(n)} : \Gamma(n)(H) \rightarrow \Gamma(n)(G)
$$

for any positive integer $n$. We write $\Psi$ for $\Psi(1)$. Let $f$ be a spatial embedding of $G$ and $e$ an edge of $G$ that is not a loop. Then by contracting $f(e)$ into one point, we obtain a spatial embedding $\psi(f)$ of $G/e$. Similarly, we can also obtain a spatial embedding $\psi(f)$ of $H$ from $f$. Thus we obtain a map

$$
\psi = \psi_{G,H} : \text{SE}(G) \rightarrow \text{SE}(H).
$$

Then we immediately have:

**Proposition 2.1.** For a spatial embedding $f$ of $G$ and an element $\lambda$ in $\Gamma(n)(H)$, $\psi(f)(\lambda)$ is ambient isotopic to $f(\Psi(n)(\lambda))$. \hfill \Box

**Proposition 2.2.** I(K or C3L) is closed under minor reductions.

*Proof.* Let $G$ be a graph that is not I(K or C3L), and $H$ be a minor of $G$. Let $f$ be a spatial embedding of $G$ that contains neither a nontrivial knot nor a 3-component link all of whose 2-component sublinks are nonsplittable. Then by Proposition 2.1, $\psi(f)$ has the same property. This implies that $H$ is not I(K or C3L). \hfill \Box

**Remark 2.3.** Proposition 2.1 also implies that IK, InL and I(K or 3L) are closed under minor reductions.

Let $G_\Delta$ and $G_Y$ be two graphs such that $G_Y$ is obtained from $G_\Delta$ by a single $\Delta Y$-exchange, as in the previous section. Let $\lambda$ be an element in $\Gamma(n)(G_\Delta)$ that does not contain $\Delta$. Then there exists an element $\Phi(n)(\lambda)$ in $\Gamma(n)(G_Y)$ such that
\( \lambda \setminus \Delta = \Phi^{(n)}(\lambda) \setminus Y \). Thus we obtain a map

\[
\Phi^{(n)} = \Phi^{(n)}_{G_\Delta, G_Y} : \{ \lambda \in \Gamma^{(n)}(G_\Delta) \mid \lambda \not\supset \Delta \} \to \Gamma^{(n)}(G_Y),
\]

for any positive integer \( n \). We denote \( \Phi^{(1)} \) by \( \Phi \). Note that \( \Phi^{(n)} \) is surjective and the inverse image of \( \lambda \) by \( \Phi^{(n)} \) contains at most two elements in \( \Gamma^{(n)}(G_\Delta) \) for any element \( \lambda \) in \( \Gamma^{(n)}(G_Y) \). The surjectivity of \( \Phi^{(n)} \) implies Proposition 2.4.

**Proposition 2.4.** For \( n \geq 2 \), if \( \Gamma^{(n)}(G_\Delta) = \emptyset \), then \( \Gamma^{(n)}(G_Y) = \emptyset \). \( \square \)

Let \( f \) be a spatial embedding of \( G_Y \), and let \( D \) be a 2-disk in the 3-sphere such that \( D \cap f(G_\Delta) = f(Y) \) and \( \partial D \cap f(G_\Delta) = \{ f(u), f(v), f(w) \} \). (Throughout the paper we use \( u, v, w, x \) for the vertices of the Y of interest, as in the first figure on page 408.) Let \( \varphi(f) \) be a spatial embedding of \( G_\Delta \) such that \( \varphi(f)(x) = f(x) \) for \( x \in G_Y \setminus Y \) and \( \varphi(f)(G_\Delta) = (f(G_Y) \setminus f(Y)) \cup \partial D \). Then we obtain a map

\[ \varphi = \varphi_{G_Y, G_\Delta} : \text{SE}(G_Y) \to \text{SE}(G_\Delta), \]

and we immediately have Proposition 2.5.

**Proposition 2.5.** For a spatial embedding \( f \) of \( G_Y \) and an element \( \lambda \) in \( \Gamma^{(n)}(G_Y) \), \( f(\lambda) \) is ambient isotopic to \( \varphi(f)(\lambda') \) for each element \( \lambda' \) in the inverse image of \( \lambda \) by \( \Phi^{(n)}. \) \( \square \)

**Lemma 2.6.** If \( G_\Delta \) is \( I(K \text{ or } C3L) \), then \( G_Y \) is also \( I(K \text{ or } C3L) \).

**Proof.** Assume that \( G_Y \) is not \( I(K \text{ or } C3L) \), that is, there exists a spatial embedding \( f \) of \( G_Y \) that contains neither a nontrivial knot nor a 3-component link all of whose 2-component sublinks are nonsplittable. We show that \( \varphi(f)(G_\Delta) \) also has the same property.

Let \( \gamma \) be an element in \( \Gamma(G_\Delta) \). If \( \gamma \) is not \( \Delta \), then \( \varphi(f)(\gamma) \) is ambient isotopic to \( f(\Phi(\gamma)) \) by Proposition 2.5, and \( f(\Phi(\gamma)) \) is a trivial knot by the assumption. Since \( \varphi(f)(\Delta) \) is also a trivial knot, it follows that \( \varphi(f)(G_\Delta) \) does not contain a nontrivial knot. Let \( \lambda \) be an element in \( \Gamma^{(3)}(G_\Delta) \). If \( \lambda \) does not contain \( \Delta \), then \( \varphi(f)(\lambda) \) is ambient isotopic to \( f(\Phi^{(3)}(\lambda)) \) by Proposition 2.5, and \( f(\Phi^{(3)}(\lambda)) \) is a 3-component link that contains a split 2-component sublink by the assumption. If \( \lambda \) contains \( \Delta \), then \( \varphi(f)(\lambda) \) is a split 3-component link. Thus we see that \( \varphi(f)(G_\Delta) \) does not contain a 3-component link with a nonsplittable 2-component sublink. \( \square \)

**Lemma 2.7.** If \( G_Y \) is minor-minimal for \( I(K \text{ or } C3L) \), then \( G_\Delta \) is also minor-minimal for \( I(K \text{ or } C3L) \).

**Proof.** (This lemma has already been proven in more general form [Ozawa and Tsutsumi 2007, Lemma 3.1, Exercise 3.2], but we prove it here for convenience.)
We show that for any edge \( e \) of \( G_\Delta \) that is not a loop, there exist a spatial embedding \( f \) of \( G_\Delta - e \) and a spatial embedding \( g \) of \( G_\Delta/e \) such that each of \( f(G_\Delta - e) \) and
\[ g(G_\Delta/e) \text{ contains neither a nontrivial knot nor a 3-component link all of whose 2-component sublinks are nonsplittable. If } e \text{ is not one of the edges } \overline{uv}, \overline{vw} \text{ or } \overline{wu} \text{ of the } \Delta \text{ then there exist a spatial embedding } f' \text{ of } G_Y - e \text{ and a spatial embedding } g' \text{ of } G_Y/e \text{ such that both } f'(G_Y - e) \text{ and } g'(G_Y/e) \text{ contain neither a nontrivial knot nor a 3-component link all of whose 2-component sublinks are nonsplittable.} \]

The graph \( G_Y - e \) is obtained from \( G_\Delta - e \), and likewise \( G_Y/e \) from \( G_\Delta/e \), by a single \( \Delta Y \)-exchange at the same \( \Delta \). Then we see that each of \( \varphi(f')(G_\Delta - e) \) and \( \varphi(g')(G_\Delta/e) \) contains neither a nontrivial knot nor a 3-component link having only nonsplittable 2-component sublinks, in a way similar to the proof of Lemma 2.6. If \( e \) is one of \( uv, vw \) and \( wu \), we may assume that \( e = uv \) without loss of generality. Now there exists a spatial embedding \( g' \) of \( G_Y/\overline{uv} \) such that \( g'(G_Y/\overline{uw}) \) contains neither a nontrivial knot nor a 3-component link having only nonsplittable 2-component sublinks. Then we can see that \( G_\Delta - uv = G_Y/\overline{uv} \).

On the other hand, there exists a spatial embedding \( g' \) of \( G_Y/\overline{uv}/\overline{wu} \) such that \( g'(G_Y/\overline{uw}) \) contains neither a nontrivial knot nor a 3-component link having only nonsplittable 2-component sublink. Take a 2-disk \( D' \) in the 3-sphere such that \( D' \cap g'(G_Y/\overline{uv}/\overline{wu}) = g'(\overline{uw}) \) and \( \partial D' \cap g'(G_Y/\overline{uw}/\overline{wu}) = \{g'(u), g'(w))\}. \)

Then \( (g'(G_Y/\overline{uw}) \setminus \text{int } g'(\overline{uw})) \cup \partial D' \) may be regarded as the image of a spatial embedding of \( G_\Delta/\overline{uv} \), denoted by \( g \). Clearly \( g(G_\Delta/\overline{uv}) \) contains neither a nontrivial knot nor a 3-component link having only nonsplittable 2-component sublink. \( \square \)

### 3. Proof of Theorem 1.1

**Lemma 3.1.** Each of the graphs \( N_9, N_{10}, N_{11}, N'_ {10}, N'_{11} \) and \( N'_{12} \) in the Heawood family is not IK.

**Proof.** For \( N'_{10} \), see [Flapan and Naimi 2008]. We show that \( N_9, N_{10}, N_{11}, N'_{11} \) and \( N'_{12} \) are not IK. Let \( f_9 \) be the spatial embedding of \( N_9 \) illustrated in Figure 2. It can be checked directly that \( f_9(N_9) \) does not contain a nontrivial knot. Thus \( N_9 \) is
not IK. Let $f_{10}$ be the spatial embedding of $N_{10}$ illustrated in Figure 2. Let $\varphi_{N_{10},N_9}$ be the map from $\text{SE}(N_{10})$ to $\text{SE}(N_9)$ induced by the $Y\Delta$-exchange from $N_{10}$ to $N_9$ at the Y-fork marked * in Figure 2. Then clearly $\varphi(f_{10}) = f_9$. Since $f_9(N_9)$ does not contain a nontrivial knot, by Proposition 2.5 it follows that $f_{10}(N_{10})$ also does not contain a nontrivial knot. Thus, $N_{10}$ is not IK. By repeating this argument, we can see that each of the graphs $N_{11}$, $N'_{11}$ and $N'_{12}$ is also not IK; see Figure 2. \[ \square \]

Proof of Theorem 1.1. First we show that (1) and (2) are equivalent. Since we already know that (2) implies (1), we show that (1) implies (2). If $G$ is IK, then by Lemma 3.1 we see that $G$ is not one of $N_9$, $N_{10}$, $N_{11}$, $N'_{10}$, $N'_{11}$ or $N'_{12}$. Thus $G$ is obtained from $K_7$ by a finite sequence of $\Delta Y$-exchanges. Next we show that (2) and (3) are equivalent. Assume that $G$ is obtained from $K_7$ by a finite sequence of $\Delta Y$-exchanges. $\Gamma^{(3)}(K_7)$ is the empty set. Thus, by Proposition 2.4, we see that $\Gamma^{(3)}(G)$ is the empty set. Conversely, if $G$ is one of $N_9$, $N_{10}$, $N_{11}$, $N'_{10}$, $N'_{11}$, and $N'_{12}$, then $\Gamma^{(3)}(G)$ is not the empty set. This completes the proof. \[ \square \]

Remark 3.2. Let $f'_{11}$ be the spatial embedding of $N'_{11}$ illustrated in Figure 2, and let $f'_{10}$ be the spatial embedding of $N'_{10}$ illustrated in the figure below. Let $\varphi_{N'_{11},N'_{10}}$ be the map from $\text{SE}(N'_{11})$ to $\text{SE}(N'_{10})$ induced by the $Y\Delta$-exchange from $N'_{11}$ to $N'_{10}$ at the Y-fork marked **. Then clearly $\varphi(f'_{11}) = f'_{10}$. Also, we can see that $f'_{10}$ coincides with Flapan and Naimi’s example [2008] of a spatial embedding of $N'_{10}$ whose image does not contain a nontrivial knot, as illustrated in the leftmost diagram:

\[ \begin{align*}
\text{\includegraphics[width=0.5\textwidth]{diagram1.png}}
\end{align*} \]

4. Proof of Theorem 1.2

Lemma 4.1 [Conway and Gordon 1983; Taniyama and Yasuhara 2001]. Let $G$ be a graph in the Petersen family and $f$ a spatial embedding of $G$. Then there exists an element $\lambda$ in $\Gamma^{(2)}(G)$ such that $\text{lk}(f(\lambda))$ is odd, where $\text{lk}$ denotes the linking number in the 3-sphere.

Let $D_4$ be the graph illustrated on the right. We denote the set of all cycles of $D_4$ with exactly four edges by $\Gamma_4(D_4)$. For a spatial embedding $f$ of $D_4$, we define

\[ \alpha(f) \equiv \sum_{\gamma \in \Gamma_4(D_4)} a_2(f(\gamma)) \pmod{2}, \]
where $a_2$ denotes the second coefficient of the Conway polynomial. Note that $a_2(K)$ of a knot $K$ is congruent to the Arf invariant modulo 2 [Kauffman 1983].

**Lemma 4.2** [Taniyama and Yasuhara 2001]. Let $f$ be a spatial embedding of $D_4$ and $\lambda, \lambda'$ all elements in $\Gamma^{(2)}(D_4)$. If both $\text{lk}(f(\lambda))$ and $\text{lk}(f(\lambda'))$ are odd, then $\alpha(f) = 1$.

Let $G$ be a graph that contains $D_4$ as a minor and $f$ a spatial embedding of $G$. Then we define

$$\alpha(f) \equiv \sum_{\gamma \in \Gamma_4(D_4)} a_2(f(\Psi_{D_4,G}(\gamma))) \pmod{2}.$$  

**Lemma 4.3.** Let $G$ be a graph that contains $D_4$ as a minor and let $f$ be a spatial embedding of $G$. For two elements $\mu$ and $\mu'$ in $\Psi^{(2)}_{D_4,G}(\Gamma^{(2)}(D_4))$, if both $\text{lk}(f(\mu))$ and $\text{lk}(f(\mu'))$ are odd, then $\alpha(f) = 1$.

**Proof.** For two elements $\lambda$ and $\lambda'$ in $\Gamma^{(2)}(D_4)$, we see that both $\text{lk}(f(\Psi^{(2)}_{D_4,G}(\lambda)))$ and $\text{lk}(f(\Psi^{(2)}_{D_4,G}(\lambda')))$ are odd by the assumption. Then by Proposition 2.1, it follows that $\text{lk}(\psi_{G,D_4}(f)(\lambda))$ and $\text{lk}(\psi_{G,D_4}(f)(\lambda'))$ are also odd. Therefore, by Lemma 4.2, we have that

$$\alpha(f) \equiv \sum_{\gamma \in \Gamma_4(D_4)} a_2(f(\Psi_{D_4,G}(\gamma))) \equiv \sum_{\gamma \in \Gamma_4(D_4)} a_2(\psi_{G,D_4}(f)(\gamma)) \equiv 1 \pmod{2}. \quad \square$$

The next theorem is the most important part of the proof of Theorem 1.2.

**Theorem 4.4.** Let $G$ be $N_9$ or $N'_9$. For every spatial embedding $f$ of $G$, there exists an element $\gamma$ in $\Gamma(G)$ such that $a_2(f(\gamma))$ is odd, or there exists an element $\lambda$ in $\Gamma^{(3)}(G)$ such that each 2-component sublink of $f(\lambda)$ has an odd linking number.

**Proof.** We will denote by $[i_1 i_2 \ldots i_k]$ the cycle $i_1 i_2 \cup i_2 i_3 \cup \ldots \cup i_{k-1} i_k \cup i_k i_1$ of $G$. We label each vertex of $G$ as follows:

![Graphs $N_9$ and $N'_9$](image)

First we show the case of $G = N_9$. Let $f$ be a spatial embedding of $N_9$. Note that $N_9$ contains $K_6$ as the proper minor

$$(((N_9 - 78) - 89) - 97)/47/58/69.$$  

By Lemma 4.1, there is thus an element $v$ in $\Gamma^{(2)}(K_6)$ such that $\text{lk}(\psi_{N_9,K_6}(f)(v))$ is odd. Hence, by Proposition 2.1, there exists an element $\mu$ in $\Psi^{(2)}_{K_6,N_9}(\Gamma^{(2)}(K_6))$ such that $\text{lk}(f(\mu))$ is odd. $\Psi^{(2)}_{K_6,N_9}(\Gamma^{(2)}(K_6))$ consists of ten elements, and by the
symmetry of $N_9$, we may assume that $\mu = [1 7 4 3] \cup [2 6 5 8]$ or $[1 2 3] \cup [4 5 6]$ without loss of generality.

**Case 1.** Let $\mu = [1 7 4 3] \cup [2 6 5 8]$. Note that $N_9$ contains $P_7$ as the proper minor

$$(((N_9 - \overline{61}) - \overline{62}) - \overline{64}) - \overline{65} - \overline{69})/39.$$

Thus, by Lemma 4.1, there is an element $\nu'$ in $\Gamma (2)(P_7)$ such that $\text{lk}(\psi_{N_9,P_7}(f)(\nu'))$ is odd. Hence, by Proposition 2.1, there exists an element $\mu'$ in $\Psi_{P_7,N_9}(\Gamma (2)(P_7))$ such that $\text{lk}(f(\mu'))$ is odd. $\Psi_{P_7,N_9}(\Gamma (2)(P_7))$ consists of the nine elements

$$\begin{align*}
\mu'_1 &= [3 4 5] \cup [1 2 8 7], & \mu'_2 &= [1 5 4 7] \cup [2 3 9 8], & \mu'_3 &= [2 8 5 4] \cup [3 1 7 9], \\
\mu'_4 &= [1 2 4 7] \cup [3 5 8 9], & \mu'_5 &= [1 2 3] \cup [4 7 8 5], & \mu'_6 &= [1 2 8 5] \cup [3 4 7 9], \\
\mu'_7 &= [2 3 4] \cup [1 5 8 7], & \mu'_8 &= [7 8 9] \cup [1 2 4 5], & \mu'_9 &= [1 5 3] \cup [2 8 7 4].
\end{align*}$$

For $i = 1, 2, \ldots, 9$, let $J^i$ be the subgraph of $N_9$ that is $\mu \cup \mu'_i \cup \overline{69}$ if $i = 3, 6$ and $\mu \cup \mu'_i$ if $i \neq 3, 6$. Assume that $\text{lk}(f(\mu'_i))$ is odd for some $i \neq 8$. Then it can be easily seen that $J^i$ contains a graph $D^i$ as a minor, such that $D^i$ is isomorphic to $D_4$ and $\{\mu, \mu'_i\} = \Psi_{D^i,\mu}((\Gamma (2)(D^i))$. Since both $\text{lk}(f(\mu))$ and $\text{lk}(f(\mu'_i))$ are odd, by Lemma 4.3 there exists an element $\gamma$ in $\Gamma (J^i)$ such that $a_2(f(\gamma))$ is odd. Next assume that $\text{lk}(f(\mu'_8))$ is odd. We denote two elements $[7 8 9] \cup [1 2 6 5]$ and $[7 8 9] \cup [4 2 6 5]$ in $\Gamma (2)(J^8)$ by $\mu'_{8,1}$ and $\mu'_{8,2}$, respectively. We denote the subgraph $\mu \cup \mu'_{8,j}$ of $J^8$ by $J^{8,j}$ $(j = 1, 2)$. Then it can be easily seen that $J^{8,j}$ contains a graph $D^{8,j}$ as a minor, such that $D^{8,j}$ is isomorphic to $D_4$ and $\{\mu, \mu'_{8,j}\} = \Psi_{D^{8,j},\mu}(\Gamma (2)(D^{8,j})) (j = 1, 2)$. Note that

$$[1 2 4 5] = [1 2 6 5] + [4 2 6 5]$$

in $H_1(J^8; \mathbb{Z}_2)$, where $H_n(\cdot ; \mathbb{Z}_2)$ denotes the homology group with $\mathbb{Z}_2$-coefficients. Then, by the homological property of the linking number, we have that

$$1 \equiv \text{lk}(f(\mu'_8)) \equiv \text{lk}(f(\mu'_{8,1})) + \text{lk}(f(\mu'_{8,2})) \pmod{2}.$$

Thus we see that $\text{lk}(f(\mu'_{8,j}))$ is odd or $\text{lk}(f(\mu'_{8,2}))$ is odd. In either case, by Lemma 4.3 there exists an element $\gamma$ in $\Gamma (J^{8,j})$ such that $a_2(f(\gamma))$ is odd.

**Case 2.** Let $\mu = [1 2 3] \cup [4 5 6]$. Note that $N_9$ contains $P_9$ as the proper minor

$$(((N_9 - \overline{12}) - \overline{23}) - \overline{45}) - \overline{56} - 64.$$

Thus, by Lemma 4.1, there is an element $\nu'$ in $\Gamma (2)(P_9)$ such that $\text{lk}(\psi_{N_9,P_9}(f)(\nu'))$ is odd. Hence by Proposition 2.1, there exists an element $\mu'$ in $\Psi_{P_9,N_9}(\Gamma (2)(P_9))$ such that $\text{lk}(f(\mu'))$ is odd. $\Psi_{P_9,N_9}(\Gamma (2)(P_9))$ consists of seven elements, and by the symmetry of $N_9$, we may assume, without loss of generality, that $\mu' = [1 5 8 7] \cup [2 6 9 3 4]$ or $[7 8 9] \cup [1 5 3 4 2 6]$. Denote by $J$ the subgraph $\mu \cup \mu'$ of $N_9$. Assume
that \( \mu' = [1 5 8 7] \cup [2 6 9 3 4] \). We denote the two elements \([1 5 8 7] \cup [4 3 2]\) and \([1 5 8 7] \cup [6 9 3 2]\) in \( \Gamma^{(2)}(J) \) by \( \mu'_1 \) and \( \mu'_2 \), respectively. We denote the subgraph \( \mu \cup \mu'_i \) of \( J \) by \( J^i \) (\( i = 1, 2 \)). Then \( J^i \) contains a graph \( D^i \) as a minor such that \( D^i \) is isomorphic to \( D_4 \) and

\[
\{\mu, \mu'_i\} = \Psi^{(2)}_{D^i, J^i}(\Gamma^{(2)}(D^i)) \quad (i = 1, 2).
\]

Since \([2 6 9 3 4] = [4 3 2] \cup [6 9 3 2] \) in \( H_1(J; Z_2) \), it follows that

\[
1 \equiv \text{lk}(f(\mu')) \equiv \text{lk}(f(\mu'_1)) + \text{lk}(f(\mu'_2)) \pmod{2}.
\]

This implies that \( \text{lk}(f(\mu'_1)) \) is odd or \( \text{lk}(f(\mu'_2)) \) is odd. In both cases, by Lemma 4.3 there exists an element \( \gamma \) in \( \Gamma(J^i) \) such that \( a_2(f(\gamma)) \) is odd. Next assume that \( \mu' = [7 8 9] \cup [1 5 3 4 2 6] \). We denote four elements \([7 8 9] \cup [3 4 5], [7 8 9] \cup [4 5 6], [7 8 9] \cup [1 5 6] \) and \([7 8 9] \cup [2 4 6] \) in \( \Gamma^{(2)}(J) \) by \( \mu'_1, \mu'_2, \mu'_3 \) and \( \mu'_4 \), respectively. Since \([1 5 3 4 2 6] = [3 4 5] + [4 5 6] + [1 5 6] + [2 4 6] \) in \( H_1(J; Z_2) \), we get

\[
1 \equiv \text{lk}(\mu') \equiv \text{lk}(\mu'_1) + \text{lk}(\mu'_2) + \text{lk}(\mu'_3) + \text{lk}(\mu'_4) \pmod{2}.
\]

This implies that \( \text{lk}(\mu'_i) \) is odd for some \( i = 1, 2, 3 \) or 4. Moreover, by the symmetry of \( J \), we may assume that \( \text{lk}(\mu'_i) \) is odd or \( \text{lk}(\mu'_2) \) is odd without loss of generality. Assume that \( \text{lk}(\mu'_i) \) is odd. We denote the subgraph \( \mu \cup \mu'_1 \cup 1 7 \cup 6 9 \) of \( N_9 \) by \( J^1 \). Then \( J^1 \) contains a graph \( D^1 \) as a minor such that \( D^1 \) is isomorphic to \( D_4 \) and \( \{\mu, \mu'_1\} = \Psi^{(2)}_{D^1, J^1}(\Gamma^{(2)}(D^1)) \). Since both \( \text{lk}(f(\mu)) \) and \( \text{lk}(f(\mu'_1)) \) are odd, by Lemma 4.3 there exists an element \( \gamma \) in \( \Gamma(J^1) \) such that \( a_2(f(\gamma)) \) is odd. Next assume that \( \text{lk}(\mu'_2) \) is odd. We denote four elements \([7 8 9] \cup [1 2 6], [7 8 9] \cup [1 2 3], [7 8 9] \cup [2 3 4] \) and \([7 8 9] \cup [1 3 5] \) in \( \Gamma^{(2)}(J) \) by \( \mu'_5, \mu'_6, \mu'_7 \) and \( \mu'_8 \), respectively. Since \([1 5 3 4 2 6] = [1 2 6] + [1 2 3] + [2 3 4] + [1 3 5] \) in \( H_1(J; Z_2) \), we have

\[
1 \equiv \text{lk}(\mu') \equiv \text{lk}(\mu'_5) + \text{lk}(\mu'_6) + \text{lk}(\mu'_7) + \text{lk}(\mu'_8) \pmod{2}.
\]

Thus we see that \( \text{lk}(\mu'_i) \) is odd for some \( i = 5, 6, 7 \) or 8. Moreover, by the symmetry of \( J \), we may assume that \( \text{lk}(\mu'_5) \) is odd or \( \text{lk}(\mu'_6) \) is odd without loss of generality. Assume that \( \text{lk}(\mu'_5) \) is odd. We denote the subgraph \( \mu \cup \mu'_5 \cup 4 7 \cup 3 9 \) of \( N_9 \) by \( J^5 \). Then \( J^5 \) contains a graph \( D^5 \) as a minor such that \( D^5 \) is isomorphic to \( D_4 \) and \( \{\mu, \mu'_5\} = \Psi^{(2)}_{D^5, J^5}(\Gamma^{(2)}(D^5)) \). Since both \( \text{lk}(f(\mu)) \) and \( \text{lk}(f(\mu'_5)) \) are odd, by Lemma 4.3 there exists an element \( \gamma \) in \( \Gamma(J^5) \) such that \( a_2(f(\gamma)) \) is odd. Finally, assume that \( \text{lk}(\mu'_6) \) is odd. Let us consider the 3-component link \( L = f([1 2 3] \cup [4 5 6] \cup [7 8 9]) \). Since all 2-component sublinks of \( L \) are \( f(\mu), f(\mu'_5) \) and \( f(\mu'_6) \), each of the 2-component sublinks of \( L \) has an odd linking number.

Now we show the case of \( G = N'_{10} \). Let \( f \) be a spatial embedding of \( N'_{10} \). Note that \( N'_{10} \) contains \( P_7 \) as the proper minor

\[
(((N'_{10} - 78) - 89) - 97)/47/58/69.
\]
Thus by Lemma 4.1, there is an element $v$ in $\Gamma^{(2)}(P_7)$ such that $\text{lk}(\psi_{N'_{10}, P_7}(f)(v))$ is odd. Hence by Proposition 2.1, there exists an element $\mu$ in $\Psi_{P_7, N'_{10}}^{(2)}(\Gamma^{(2)}(P_7))$ such that $\text{lk}(f(\mu))$ is odd. $\Psi_{P_7, N'_{10}}^{(2)}(\Gamma^{(2)}(P_7))$ consists of nine elements, and by the symmetry of $N'_{10}$, we may assume that $\mu = [1\ 7\ 4\ 5\ 1\ 2\ 10\ 3\ 9\ 6\ 2\ 4\ 5\ 8\ 1\ 10\ 3\ 9\ 6\ 3\ 1\ 0\ 8\ 5\ 1\ 6\ 2\ 4\ 7\ 3\ 4\ 5\ 1\ 10\ 2\ 6\ 0\ 2\ 4\ 1\ 8\ 9\ 3\ 4\ 7\ 3\ 0\ 1\ 6\ 9\ 3\ 4\ 7]$ without loss of generality.

**Case 1.** Let $\mu = [1\ 7\ 4\ 5\ 1\ 2\ 10\ 3\ 9\ 6]$. Note that $N'_{10}$ contains $P_9$ as the proper minor

$$(((N'_{10} - 5\ 1) - 5\ 3) - 5\ 4) - 5\ 6) - 5\ 8) - 7\ 9).$$

Thus by Lemma 4.1, there is an element $v'$ in $\Gamma^{(2)}(P_9)$ such that $\text{lk}(\psi_{N'_{10}, P_9}(f)(v'))$ is odd. Hence by Proposition 2.1, there exists an element $\mu'$ in $\Psi_{P_9, N'_{10}}^{(2)}(\Gamma^{(2)}(P_9))$ such that $\text{lk}(f(\mu'))$ is odd. $\Psi_{P_9, N'_{10}}^{(2)}(\Gamma^{(2)}(P_9))$ consists of seven elements

$$\mu_1' = [3\ 1\ 0\ 8\ 9\ 1\ 6\ 2\ 4\ 7], \quad \mu_2' = [1\ 7\ 8\ 10\ 1\ 2\ 4\ 3\ 9\ 6],$$

$$\mu_3' = [1\ 1\ 0\ 2\ 6\ 1\ 3\ 4\ 7\ 8\ 9], \quad \mu_4' = [2\ 4\ 3\ 1\ 0\ 1\ 7\ 8\ 9\ 6],$$

$$\mu_5' = [2\ 4\ 7\ 8\ 1\ 1\ 0\ 3\ 9\ 6], \quad \mu_6' = [2\ 8\ 9\ 6\ 1\ 1\ 0\ 3\ 4\ 7],$$

$$\mu_7' = [2\ 8\ 10\ 1\ 1\ 6\ 9\ 3\ 4\ 7].$$

For $i = 1, 2, \ldots, 7$, let $J^i$ be the subgraph of $N'_{10}$ that is $\mu \cup \mu_i' \cup 5\ 8$ if $i = 1, 6, 7$ and $\mu \cup \mu_i'$ if $i = 2, 3, 4, 5$. Assume that $\text{lk}(f(\mu_i'))$ is odd for some $i$. Then $J^i$ contains a graph $D^i$ as a minor such that $D^i$ is isomorphic to $D_4$ and $\{\mu, \mu_i'\} = \Psi_{D_i, J}(\Gamma^{(2)}(D^i))$. Because both $\text{lk}(f(\mu))$ and $\text{lk}(f(\mu_i'))$ are odd, by Lemma 4.3 there exists an element $\gamma$ in $\Gamma(J^i)$ such that $a_2(f(\gamma))$ is odd.

**Case 2.** Let $\mu = [2\ 4\ 5\ 8\ 1\ 1\ 0\ 3\ 9\ 6]$. Note that $N'_{10}$ contains another $P_9$ as the proper minor

$$(((N'_{10} - 8\ 2) - 8\ 5) - 8\ 7) - 8\ 9) - 8\ 10) - 3\ 4.$$

Thus by Lemma 4.1, there is an element $v'$ in $\Gamma^{(2)}(P_9)$ such that $\text{lk}(\psi_{N'_{10}, P_9}(f)(v'))$ is odd. Hence by Proposition 2.1, there exists an element $\mu'$ in $\Psi_{P_9, N'_{10}}^{(2)}(\Gamma^{(2)}(P_9))$ such that $\text{lk}(f(\mu'))$ is odd. $\Psi_{P_9, N'_{10}}^{(2)}(\Gamma^{(2)}(P_9))$ consists of the seven elements

$$\mu_1' = [1\ 6\ 9\ 7\ 1\ 2\ 4\ 5\ 3\ 1\ 0], \quad \mu_2' = [1\ 7\ 4\ 5\ 1\ 2\ 10\ 3\ 9\ 6],$$

$$\mu_3' = [3\ 5\ 6\ 9\ 1\ 1\ 0\ 2\ 4\ 7], \quad \mu_4' = [1\ 5\ 3\ 1\ 0\ 1\ 2\ 4\ 7\ 9\ 6],$$

$$\mu_5' = [1\ 1\ 0\ 2\ 6\ 1\ 3\ 9\ 7\ 4\ 5], \quad \mu_6' = [1\ 5\ 6\ 1\ 2\ 4\ 7\ 9\ 3\ 10],$$

$$\mu_7' = [2\ 4\ 5\ 6\ 1\ 1\ 0\ 3\ 9\ 7].$$

For $i = 1, 2, \ldots, 7$, let $J^i$ be the subgraph of $N'_{10}$ that is $\mu \cup \mu_i' \cup 7\ 8$ if $i = 1, 7$ and $\mu \cup \mu_i'$ if $i \neq 1, 7$. Assume that $\text{lk}(f(\mu_i'))$ is odd for some $i$. Then $J^i$ contains
a graph $D^i$ as a minor such that $D^i$ is isomorphic to $D_4$ and

$$\{\mu, \mu'_i\} = \Psi^{(2)}_{D_i, J_i}(\Gamma^{(2)}(D^i)).$$

Since both $\text{lk}(f(\mu))$ and $\text{lk}(f(\mu'_i))$ are odd, by Lemma 4.3 there exists an element $\gamma$ in $\Gamma(J^i)$ such that $a_2(f(\gamma))$ is odd.

**Case 3.** Let $\mu = [3 \ 10 \ 8 \ 5] \cup [1 \ 6 \ 2 \ 4 \ 7]$. Let $P_9$ be the proper minor of $N'_{10}$ and $\mu'_i$ the element in

$$\Psi^{(2)}_{P_9, N'_{10}}(\Gamma^{(2)}(P_9)) \quad (i = 1, 2, \ldots, 7)$$

as in Case 2. For $i = 1, 2, \ldots, 7$, let $J^i$ be the subgraph of $N'_{10}$ that is $\mu \cup \mu'_i \cup \overline{\mathbf{9}}$ if $i = 1, 4$ and $\mu \cup \mu'_i$ if $i \neq 1, 4$. Assume that $\text{lk}(f(\mu'_i))$ is odd for some $i$. Then $J^i$ contains a graph $D^i$ as a minor such that $D^i$ is isomorphic to $D_4$ and $\{\mu, \mu'_i\} = \Psi^{(2)}_{D_i, J_i}(\Gamma^{(2)}(D^i))$. Because both $\text{lk}(f(\mu))$ and $\text{lk}(f(\mu'_i))$ are odd, by Lemma 4.3 there exists an element $\gamma$ in $\Gamma(J^i)$ such that $a_2(f(\gamma))$ is odd.

**Case 4.** Let $\mu = [3 \ 4 \ 5] \cup [1 \ 10 \ 2 \ 6]$. Note that $N'_{10}$ contains another $P_7$ as the proper minor

$$(((N'_{10} - 3 4) - 4 5) - 5 3) / 39 / 47 / 58.$$  

Thus by Lemma 4.1, there is an element $\nu'$ in $\Gamma^{(2)}(P_7)$ such that $\text{lk}(\Psi^{(2)}_{N'_{10}, P_7}(f)(\nu'))$ is odd. Hence by Proposition 2.1, there exists an element $\mu'$ in $\Psi^{(2)}_{P_7, N'_{10}}(\Gamma^{(2)}(P_7))$ such that $\text{lk}(f(\mu'))$ is odd. $\Psi^{(2)}_{P_7, N'_{10}}(\Gamma^{(2)}(P_7))$ consists of the nine elements

- $\mu'_1 = [5 \ 6 \ 9 \ 8] \cup [1 \ 10 \ 2 \ 4 \ 7], \quad \mu'_2 = [3 \ 10 \ 8 \ 9] \cup [1 \ 6 \ 2 \ 4 \ 7],$
- $\mu'_3 = [1 \ 5 \ 8 \ 10] \cup [2 \ 4 \ 7 \ 9 \ 6], \quad \mu'_4 = [7 \ 8 \ 9] \cup [1 \ 10 \ 2 \ 6],$
- $\mu'_5 = [2 \ 8 \ 10] \cup [1 \ 6 \ 9 \ 7], \quad \mu'_6 = [2 \ 8 \ 5 \ 6] \cup [1 \ 10 \ 3 \ 9 \ 7],$
- $\mu'_7 = [1 \ 7 \ 8 \ 5] \cup [2 \ 1 \ 0 \ 3 \ 9 \ 6], \quad \mu'_8 = [1 \ 5 \ 6] \cup [2 \ 4 \ 7 \ 9 \ 3 \ 10],$
- $\mu'_9 = [2 \ 4 \ 7 \ 8] \cup [1 \ 10 \ 3 \ 9 \ 6].$

For $i = 1, 2, \ldots, 9$, let $J^i$ be the subgraph of $N'_{10}$ that is $\mu \cup \mu'_i \cup 47 \cup 58$ if $i = 5$ and $\mu \cup \mu'_i$ if $i \neq 5$. Assume that $\text{lk}(f(\mu'_i))$ is odd for some $i \neq 4, 8$. Then $J^i$ contains a graph $D^i$ as a minor such that $D^i$ is isomorphic to $D_4$ and $\{\mu, \mu'_i\} = \Psi^{(2)}_{D_i, J_i}(\Gamma^{(2)}(D^i))$. Since both $\text{lk}(f(\mu))$ and $\text{lk}(f(\mu'_i))$ are odd, by Lemma 4.3 there exists an element $\gamma$ in $\Gamma(J^i)$ such that $a_2(f(\gamma))$ is odd. Next assume that $\text{lk}(f(\mu'_8))$ is odd. We denote two elements $[1 \ 5 \ 6] \cup [2 \ 4 \ 3 \ 10]$ and $[1 \ 5 \ 6] \cup [3 \ 4 \ 7 \ 9]$ in $\Gamma^{(2)}(J^8)$ by $\mu'_{8,1}$ and $\mu'_{8,2}$, respectively. We denote the subgraph $\mu \cup \mu'_{8,1}$ of $J^8$ by $J^{8,1}$ and the subgraph $\mu \cup \mu'_{8,2} \cup 89 \cup 10$ of $N'_{10}$ by $J^{8,2}$. Then $J^{8,j}$ contains a graph $D^{8,j}$ as a minor such that $D^{8,j}$ is isomorphic to $D_4$ and $\{\mu, \mu'_{8,j}\} = \Psi^{(2)}_{D^{8,j}, J^{8,j}}(\Gamma^{(2)}(D^{8,j}))$ ($j = 1, 2$). Since $[2 \ 4 \ 7 \ 9 \ 3 \ 10] = [2 \ 4 \ 3 \ 10] + [3 \ 4 \ 7 \ 9]$ in $H_1(J^{8}; \mathbb{Z}_2)$, it follows that

$$1 \equiv \text{lk}(f(\mu'_8)) \equiv \text{lk}(f(\mu'_{8,1})) + \text{lk}(f(\mu'_{8,2})) \pmod{2}.$$
This implies that \( \text{lk}(f(\mu_{8,1}')) \) is odd or \( \text{lk}(f(\mu_{8,2}')) \) is odd. In either case, by Lemma 4.3 there exists an element \( \gamma \) in \( \Gamma(\mathcal{P}^8, f) \) such that \( a_2(f(\gamma)) \) is odd. Finally assume that \( \text{lk}(f(\mu_{4}')) \) is odd. Note that \( N'_{10} \) contains another \( P_9 \) as the proper minor

\[
(((N'_{10} - 24) - 26) - 28) - 210) - 5T) - 53.
\]

Thus by Lemma 4.1, there is an element \( \nu'' \) in \( \Gamma(2)(P_9) \) such that \( \text{lk}(\Psi_{P_9,N'_{10}}(\Gamma(2)(P_9))) \) is odd. Hence by Proposition 2.1, there exists an element \( \mu'' \) in \( \Psi_{P_9,N'_{10}}(\Gamma(2)(P_9)) \) such that \( \text{lk}(f(\mu'')) \) is odd. \( \Psi_{P_9,N'_{10}}(\Gamma(2)(P_9)) \) consists of the seven elements

\[
\mu'_{1} = [5698] \cup [110347], \quad \mu'_{2} = [4587] \cup [110396],
\mu'_{3} = [17810] \cup [34569], \quad \mu'_{4} = [31089] \cup [17456],
\mu'_{5} = [1697] \cup [345810], \quad \mu'_{6} = [3974] \cup [110856],
\mu'_{7} = [789] \cup [1103456].
\]

For \( j = 1, 2, \ldots, 7 \), let \( J^{4,j} \) be the subgraph of \( N'_{10} \) which is \( \mu'_{4} \cup \mu'_{j} \cup 24 \) if \( j = 2, 6 \) and \( \mu'_{4} \cup \mu'_{j} \) if \( j \neq 2, 6 \). Assume that \( \text{lk}(f(\mu'_{j})) \) is odd for some \( j \neq 7 \). Then \( J^{4,j} \) contains a graph \( D^{4,j} \) as a minor such that \( D^{4,j} \) is isomorphic to \( D_{4} \) and \( \{\mu'_{4}, \mu''_{j}\} = \Psi_{D^{4,j}, f}(\Gamma(2)(D^{4,j})) \). Since both \( \text{lk}(f(\mu'_{4})) \) and \( \text{lk}(f(\mu''_{j})) \) are odd, by Lemma 4.3 there exists an element \( \gamma \) in \( \Gamma(4, J) \) such that \( a_2(f(\gamma)) \) is odd. Next assume that \( \text{lk}(f(\mu'_{j})) \) is odd. We denote three elements \( [789] \cup [15310], [789] \cup [156] \) and \( [789] \cup [345] \) in \( \Gamma(2)(N'_{10}) \) by \( \mu''_{1}, \mu''_{2}, \mu''_{3} \). We denote the subgraph \( \mu \cup \mu''_{k} \cup 24 \cup 28 \) of \( N'_{10} \) by \( J^{4,k} \) \( (k = 1, 2) \). Then \( J^{4,k} \) contains a graph \( D^{4,k} \) as a minor such that \( D^{4,k} \) is isomorphic to \( D_{4} \) and \( \{\mu, \mu''_{k}\} = \Psi_{D^{4,k}, f}(\Gamma(2)(D^{4,k})) \) \( (k = 1, 2) \). Since \( [1103456] = [15310] + [156] + [345] \) in \( H_{1}(N'_{10}; \mathbb{Z}_2) \), it follows that

\[
1 \equiv \text{lk}(f(\mu_{7,1})) \equiv \text{lk}(f(\mu_{7,1}')) + \text{lk}(f(\mu_{7,2}')) + \text{lk}(f(\mu_{7,3}')) \pmod{2}.
\]

This implies that \( \text{lk}(f(\mu_{7,k})) \) is odd for some \( k \). If \( \text{lk}(f(\mu_{7,1})) \) is odd or \( \text{lk}(f(\mu_{7,2})) \) is odd, then by Lemma 4.3 there exists an element \( \gamma \) in \( \Gamma(4, J) \) such that \( a_2(f(\gamma)) \) is odd. If \( \text{lk}(f(\mu_{7,3})) \) is odd, let us consider the 3-component link

\[
L = f([345] \cup [789] \cup [11026]).
\]

Since all 2-component sublinks of \( L \) are \( f(\mu), f(\mu'_{4}) \) and \( f(\mu'_{7,3}) \), each of the 2-component sublinks of \( L \) has an odd linking number.

**Case 5.** Let \( \mu = [2810] \cup [169347] \). We denote two elements \([2810] \cup [1697] \) and \([2810] \cup [3974] \) in \( \Gamma(2)(N'_{0}) \) by \( \mu_{1} \) and \( \mu_{2} \), respectively. Since \([169347] = [1697] + [3974] \) in \( H_{1}(N'_{0}; \mathbb{Z}_2) \), it follows that

\[
1 \equiv \text{lk}(f(\mu)) \equiv \text{lk}(f(\mu_{1})) + \text{lk}(f(\mu_{2})) \pmod{2}.
\]
This implies that \( \text{lk}(f(\mu_1)) = \text{odd} \) or \( \text{lk}(f(\mu_2)) = \text{odd} \). By the symmetry of \( N'_{10} \),
we may assume that \( \text{lk}(f(\mu_1)) = \text{odd} \). Note that \( N'_{10} \) contains another \( P_7 \) as the proper minor

\(((N'_{10} - 28) - 810) - 102)/260/310/58.\)

Thus by Lemma 4.1, there is an element \( v' \) in \( \Gamma^{(2)}(P_7) \) such that \( \text{lk}(\psi_{N'_{10}, P_7}(f)(v')) = \text{odd} \). Hence by Proposition 2.1, there exists an element \( \mu' \) in \( \Psi^{(2)}_{P_7, N'_{10}}(\Gamma^{(2)}(P_7)) \) such that \( \text{lk}(f(\mu')) = \text{odd} \). \( \Psi_{P_7, N'_{10}}^{(2)}(\Gamma^{(2)}(P_7)) \) consists of the nine elements

\[
\begin{align*}
\mu'_1 &= [3589] \cup [16247], & \mu'_2 &= [1785] \cup [24396], \\
\mu'_3 &= [156] \cup [3974], & \mu'_4 &= [345] \cup [1697], \\
\mu'_5 &= [5698] \cup [110347], & \mu'_6 &= [4587] \cup [110396], \\
\mu'_7 &= [15310] \cup [24796], & \mu'_8 &= [2456] \cup [110397], \\
\mu'_9 &= [789] \cup [1103426].
\end{align*}
\]

For \( i = 1, 2, \ldots, 9 \), let \( J^i \) be the subgraph of \( N'_{10} \) that is \( \mu_1 \cup \mu'_3 \cup 310 \cup 58 \) if \( i = 3 \) and \( \mu_1 \cup \mu'_i \) if \( i \neq 3 \). Assume that \( \text{lk}(f(\mu'_i)) = \text{odd} \) for some \( i \neq 4, 9 \). Then \( J^i \) contains a graph \( D^i \) as a minor such that \( D^i \) is isomorphic to \( D_4 \) and \( \{ \mu_1, \mu'_i \} = \Psi_{D^i, J^i}^{(2)}(\Gamma^{(2)}(D^i)) \). Since both \( \text{lk}(f(\mu_1)) \) and \( \text{lk}(f(\mu'_i)) \) are odd, by Lemma 4.3 there exists an element \( \gamma \) in \( \Gamma(J^i) \) such that \( a_2(\gamma) = \text{odd} \). Next assume that \( \text{lk}(f(\mu'_9)) = \text{odd} \). We denote two elements \([789] \cup [16210]\) and \([789] \cup [24310]\) in \( \Gamma^{(2)}(J^9) \) by \( \mu'_{9,1} \) and \( \mu'_{9,2} \), respectively. We denote the subgraph \( \mu_1 \cup \mu'_{9,1} \) of \( J^9 \) by \( J'^{9,1} \) and the subgraph \( \mu_1 \cup \mu'_{9,2} \cup 33 \cup 51 \) of \( N'_{10} \) by \( J'^{9,2} \). Then \( J'^{9,j} \) contains a graph \( D^{9,j} \) as a minor such that \( D^{9,j} \) is isomorphic to \( D_4 \) and

\[
\{ \mu_1, \mu'_{9,j} \} = \Psi_{D^{9,j}, J'^{9,j}}^{(2)}(\Gamma^{(2)}(D^{9,j})) \quad (j = 1, 2).
\]

Since \([1103426] = [16210] + [24310] \) in \( H_1(J^9; \mathbb{Z}_2) \), it follows that

\[
1 \equiv \text{lk}(f(\mu'_9)) \equiv \text{lk}(f(\mu'_{9,1})) + \text{lk}(f(\mu'_{9,2})) \pmod{2}.
\]

This implies that \( \text{lk}(f(\mu'_{9,1})) = \text{odd} \) or \( \text{lk}(f(\mu'_{9,2})) = \text{odd} \). In either case, by
Lemma 4.3 there exists an element \( \gamma \) in \( \Gamma(J'^{9,j}) \) such that \( a_2(\gamma) = \text{odd} \). Finally assume that \( \text{lk}(f(\mu'_j)) = \text{odd} \). \( N'_{10} \) contains another \( P_9 \) as the proper minor

\(((N'_{10} - 61) - 62) - 65) - 69) - 87) - 810.\)

Thus, by Lemma 4.1, there is \( v'' \) in \( \Gamma^{(2)}(P_9) \) such that \( \text{lk}(\psi_{N'_{10}, P_9}(f)(v'')) = \text{odd} \). Hence by Proposition 2.1, there exists \( \mu'' \in \Psi_{P_9, N'_{10}}^{(2)}(\Gamma^{(2)}(P_9)) \) such that \( \text{lk}(f(\mu'')) \)
is odd. The set \( \Psi_{P_9, N_9'}(\Gamma(2)(P_9)) \) consists of the seven elements
\[
\begin{align*}
\mu_1'' &= [3589] \cup [10247], \\
\mu_2'' &= [3974] \cup [158210], \\
\mu_3'' &= [1745] \cup [289310], \\
\mu_4'' &= [2458] \cup [10397], \\
\mu_5'' &= [24310] \cup [15897], \\
\mu_6'' &= [15310] \cup [24798], \\
\mu_7'' &= [345] \cup [1102897].
\end{align*}
\]
For \( j = 1, 2, \ldots, 7 \), let \( J^{4,j} \) be the subgraph of \( N_9' \) that is \( \mu_4' \cup \mu_7'' \cup 26 \) if \( j = 4, 5 \) and \( \mu_4' \cup \mu_7'' \) if \( j \neq 4, 5 \). Assume that \( \text{lk}(f(\mu_7'')) \) is odd for some \( j \neq 7 \). Then \( J^{4,j} \) contains a graph \( D^{4,j} \) as a minor such that \( D^{4,j} \) is isomorphic to \( D_4 \) and \( \{ \mu_4', \mu_7'' \} = \Psi_{D_4, J^{4,j}}(\Gamma(2)(D^{4,j})) \). Since both \( \text{lk}(f(\mu_4')) \) and \( \text{lk}(f(\mu_7'')) \) are odd, by Lemma 4.3 there exists an element \( \gamma \) in \( \Gamma(J^{4,j}) \) such that \( \alpha_2(f(\gamma)) \) is odd. Next assume that \( \text{lk}(f(\mu_7'')) \) is odd. We denote two elements \( [345] \cup [110897] \) and \( [345] \cup [2810] \) in \( \Gamma(2)(N_9') \) by \( \mu_{7,1}'' \) and \( \mu_{7,2}'' \), respectively. We denote the subgraph \( \mu_1 \cup \mu_{7,1}'' \cup 24 \cup 56 \) of \( N_7 \) by \( J^{4,7} \). Then \( J^{4,7} \) contains a graph \( D^{4,7} \) as a minor such that \( D^{4,7} \) is isomorphic to \( D_4 \) and
\[
\{ \mu_1, \mu_{7,1}'' \} = \Psi_{D_4, J^{4,7}}(\Gamma(2)(D^{4,7})).
\]
Since \([1102897] = [110897] \cup [2810] \) in \( H_1(N_9'; \mathbb{Z}_2) \), it follows that
\[
1 \equiv \text{lk}(f(\mu_{7,1}'')) \equiv \text{lk}(f(\mu_{7,1}'')) + \text{lk}(f(\mu_{7,2}'')) \pmod{2}.
\]
This implies that \( \text{lk}(f(\mu_{7,1}'')) \) is odd or \( \text{lk}(f(\mu_{7,2}'')) \) is odd. If \( \text{lk}(f(\mu_{7,1}'')) \) is odd, then by Lemma 4.3 there exists an element \( \gamma \) in \( \Gamma(J^{4,7}) \) such that \( \alpha_2(f(\gamma)) \) is odd. If \( \text{lk}(f(\mu_{7,2}'')) \) is odd, let us consider the 3-component link
\[
L = f([345] \cup [2810] \cup [1697]).
\]
Since all 2-component sublinks of \( L \) are \( f(\mu_1), f(\mu_4') \) and \( f(\mu_{7,2}'') \), each of the 2-component sublinks of \( L \) has an odd linking number. This completes the proof.

**Proof of Theorem 1.2.** A graph in the Heawood family is obtained from one of \( K_7, N_9 \) and \( N_9' \) by a finite sequence of \( \Delta Y \)-exchanges. Thus by Lemma 2.6, Theorem 4.4, and the fact that \( K_7 \) is IK — and thus I(K or C3L) — it follows that every graph in the Heawood family is I(K or C3L). On the other hand, a graph in the Heawood family is obtained from one of \( H_{12} \) and \( C_{14} \) by a finite sequence of \( \gamma^\Delta \)-exchanges. Since each of \( H_{12} \) and \( C_{14} \) is a minor-minimal IK graph and \( \Gamma(3)(H_{12}) \) and \( \Gamma(3)(C_{14}) \) are the empty sets, it follows that \( H_{12} \) and \( C_{14} \) are minor-minimal I(K or C3L) graphs. By Lemma 2.7, we have the desired conclusion.

**Remark 4.5.** A graph is said to be 2-apex if it can be embedded in the 2-sphere after the deletion of at most two vertices and all of their incidental edges. It is not hard to see that any 2-apex graph may have a spatial embedding whose image
contains neither a nontrivial knot nor a 3-component link all of whose 2-component sublinks are nonsplittable. Thus any 2-apex graph is not I(K or C3L). It is known that every graph of at most twenty edges is 2-apex [Mattman 2011] (see also [Johnson et al. 2010]). Since the number of all edges of every graph in the Heawood family is twenty-one, we see that any proper minor of a graph in the Heawood family is 2-apex, and thus not I(K or C3L). This also implies that any graph in the Heawood family is minor-minimal for I(K or C3L).

**Example 4.6.** Let $g_9$ be the spatial embedding of $N_9$ and $g_{10}'$ the spatial embedding of $N_{10}'$ illustrated here:

Then it can be checked directly that both $g_9(N_9)$ and $g_{10}'(N_{10}')$ do not contain a nonsplittable 3-component link. Thus neither $N_9$ nor $N_{10}'$ is I3L. Also, we can see that $N_{10}, N_{11}, N_{11}', N_{12}'$ are not I3L in a similar way as the proof of Lemma 3.1 (see figure above).

**Remark 4.7.** The Heawood graph is IK. The Heawood graph $H$ is the dual graph of $K_7$, which is embedded in a torus. It is known that there exists a unique graph $C_{14}$ obtained from $K_7$ by seven applications of △Y-exchanges [Kohara and Suzuki 1992]. The seven triangles correspond to the black triangles of a black-and-white coloring of the torus by $K_7$. Then $C_{14}$ and $H$ are mapped to each other by a translation of the torus:

Thus they are isomorphic. Since $C_{14}$ is IK, we have the result.
**Remark 4.8.** It is known that all twenty-six graphs obtained from the complete four-partite graph $K_{3,3,1,1}$ by a finite sequence of $\Delta Y$-exchanges are minor-minimal IK graphs [Kohara and Suzuki 1992; Foisy 2002]. There exist thirty-two graphs that are obtained from $K_{3,3,1,1}$ by a finite sequence of $\Delta Y$-exchanges and $Y \Delta$-exchanges but that cannot be obtained from $K_{3,3,1,1}$ by a finite sequence of $\Delta Y$-exchanges. Recently, Goldberg, Mattman, and Naimi [2011] announced that these thirty-two graphs are also minor-minimal IK graphs.

**References**


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