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We give new proofs of the evaluation of the connection relation for the Askey–Wilson polynomials and for expressing the Askey–Wilson basis in those polynomials using q-Taylor series. This led to some inverse relations. We also evaluate the coefficients in the expansions of $(x+b)^n$ in various q-orthogonal polynomials, including the Askey–Wilson polynomials, which leads to explicit expressions for the moments of the Askey–Wilson weight function. We generalize the q-plane wave expansion by expanding $\mathcal{E}_q(x;\alpha)$ in Askey–Wilson polynomials. Further, we prove a bibasic extension of the Nassrallah–Rahman integral and establish a recently conjectured identity of George Andrews.

1. Introduction

Richard Askey and James Wilson introduced the polynomials that bear their names in their memoir [1985], where they derived, among other properties, the connection relation between Askey–Wilson polynomials with different parameters. One fundamental result of theirs is the evaluation of the Askey–Wilson q-beta integral,

$$\int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{\prod_{j=1}^4 (t_j e^{i\theta}, t_j e^{-i\theta}; q)_\infty}\, d\theta = \frac{2\pi (t_1t_2t_3t_4; q)_\infty}{(q;q)_\infty \prod_{1\le j<k\le 4}(t_jt_k; q)_\infty}. \tag{1-1}$$

All this work was done in the late 1970s and the results were made available to researchers in the area, but the writing took a long time. In the mean time, Nassrallah and Rahman [1985] generalized the Askey–Wilson integral to

$$\begin{aligned}\int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty\,(t_6e^{i\theta}, t_6e^{-i\theta}; q)_\infty}{\prod_{j=1}^5 (t_j e^{i\theta}, t_j e^{-i\theta}; q)_\infty}\, d\theta \\ = \frac{2\pi (t_1t_2t_3t_4t_5/t_6; q)_\infty \prod_{j=1}^5 (t_jt_6; q)_\infty}{(q, t_6^2; q)_\infty \prod_{1\le j<k\le 5}(t_jt_k; q)_\infty} \\ \times {}_8W_7(t_6^2/q; t_6/t_1, t_6/t_2, t_6/t_3, t_6/t_4, t_6/t_5; q, t_1t_2t_3t_4t_5/t_6).\end{aligned} \tag{1-2}$$

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Remark. The preceding equation is (6.3.9) in [Gasper and Rahman 2004]. As in that reference and in [Ismail 2009], we follow the notation of [Andrews et al. 1999] for q-shifted factorials and basic hypergeometric series, and that of [Koekoek and Swarttouw 1998] for orthogonal polynomials.

The Askey–Wilson and Nassrallah–Rahman integrals play a fundamental role in the derivation of the results of this article, which is laid out as follows. Section 2 contains many of the formulas needed, other than (1-1) and (1-2). In particular, the Askey–Wilson polynomials are defined in (2-14).

In Section 3, we first solve the connection-coefficient problem of expanding an Askey–Wilson basis element

$$(ae^{i\theta}, ae^{-i\theta}; q)_n$$

in Askey–Wilson polynomials. The proof utilizes the q-integration by parts technique of [Brown et al. 1996]. One application of this expansion is to give a new derivation of a q-analogue of the plane wave expansion [Ismail 2009, (4.8.3)]

$$e^{ixy} = (2/y)^{\nu}\,\Gamma(\nu)\sum_{n=0}^{\infty}(n+\nu)\,i^n J_{n+\nu}(y)\,C_n^{\nu}(x), \tag{1-3}$$

a result first proved in [Ismail and Zhang 1994]. More importantly, we generalize the q-plane wave expansion to expand the Ismail–Zhang q-exponential function $\mathcal{E}_q(x;\alpha)$ in Askey–Wilson polynomials, which is a new result. The aforementioned connection-coefficient problem is also used to give a new proof of the connection relation of the Askey–Wilson polynomials. Each connection relation may be used to discover an inverse relation of the form $y_n = \sum_{k=0}^{n} Y_{n,k}x_k$ if and only if $x_n = \sum_{k=1}^{n} X_{n,k}y_k$. Inverse relations play a fundamental role in combinatorial-enumeration problems, as discussed in Riordan's classic [1968]. In the 1970s, interpretations of inverse relations involving q-shifted factorials and q-binomial coefficients were shown to be instances of Möbius inversion [Rota 1964] and of counting problems involving vector spaces over a finite field [Goldman and Rota 1970]. More recently, very general inverse relations were derived in [Krattenthaler 1989, 1996; Krattenthaler and Schlosser 1999].

Section 4 contains expansions of x^n and $(1\pm x)^{\rho}$ in q-ultraspherical polynomials.

Section 5 contains the evaluation of two bibasic integrals which extend the Nassrallah–Rahman integral. They are stated as Theorems 5.1 and 5.2; the latter contains as a special case the evaluation of the moments of the Askey–Wilson weight function. [Corteel and Williams 2007] recently found a beautiful combinatorial expression for the n-th moment of the Askey–Wilson measure; this is also part of the results announced in [Corteel and Williams 2010]. Our analytic expression of the moments of the Askey–Wilson weight function is a double sum.

George Andrews [2011] studied identities involving the Catalan numbers he introduced in [Andrews 1987]. One of his identities was motivated by earlier work of L. Shapiro. Andrews' investigations led him to two summation theorems. One summation theorem is

$$(1\text{-}4)\qquad {}_4\phi_3\left(\begin{matrix} q^{-2n}, a, b, q^{1-2n}/ab \\ q^{2-2n}/a, q^{2-2n}/b, qab \end{matrix}\,\middle|\, q^2, q^2\right) = \frac{q^{-n}(a, b, -q; q)_n\,(ab; q^2)_n}{(ab; q)_n\,(a, b; q^2)_n},$$

which he proved. He conjectured the validity of the other summation theorem,

$$(1\text{-}5)\qquad {}_4\phi_3\left(\begin{matrix} q^{-2n}, a, b, q^{3-2n}/ab \\ q^{2-2n}/a, q^{2-2n}/b, qab \end{matrix}\,\middle|\, q^2, q^2\right)$$
$$= \frac{q^{-n}(a, b/q, -q; q)_n\,(q-ab)}{(1-b/q)\,(ab-q^2)\,(1-abq^{2n-1})}$$
$$\times\frac{(ab/q^2; q^2)_n}{(ab; q)_n\,(a, b/q^2; q^2)_n}\big(abq^{2n-2}(q^2-b)+abq^{n-1}(1-q)+b-q\big).$$

Andrews verified (1-5) for $1 \le n \le 6$. In Section 6, we give basic hypergeometric-series proofs of both (1-4) and the conjectured identity (1-5). We show that both (1-4) and (1-5) follow from a limiting case of the ${}_5\phi_4$ to ${}_{12}\phi_{11}$ transformation stated in [Gasper and Rahman 2004, (2.8.4)].

2. Preliminaries

The expansions of x^n and $(1-x)^\rho$ in ultraspherical polynomials are

$$(2\text{-}1)\qquad \frac{(2x)^n}{n!} = \sum_{k=0}^{\lfloor n/2\rfloor} \frac{\nu+n-2k}{k!\,(\nu)_{n+1-k}} C^{\nu}_{n-2k}(x)$$

[Rainville 1960, (36), p. 283], and

$$(2\text{-}2)\qquad (1-x)^\rho = \Gamma(\nu)\,\Gamma(\nu+\rho+1/2)\,\frac{2^{2\nu+\rho}}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(k+\nu)\,(-\rho)_k}{\Gamma(k+2\nu+\rho+1)}\,C^{\nu}_k(x),$$

valid for $-1 < x < 1$, $-\rho < \frac{1}{2}(\nu+1)$ if $\nu \ge 0$, and $-\rho < \nu + \frac{1}{2}$ if $-\frac{1}{2} < \nu \le 0$ [Erdélyi et al. 1953, (10.20.6)]. The Chebyshev polynomials are the special cases

$$(2\text{-}3)\qquad T_n(x) = \lim_{\nu\to 0}\frac{n+2\nu}{2\nu}\,C^{\nu}_n(x) \quad\text{and}\quad U_n(x) = C^1_n(x).$$

The Chebyshev polynomials are also special cases of the continuous q-ultraspherical polynomials, since

$$(2\text{-}4)\qquad T_n(x) = \lim_{\beta\to 1}\frac{1-\beta q^n}{1-\beta^2}C^{\nu}_n(x; \beta \mid q) \quad\text{and}\quad U_n(x) = C_n(x; q \mid q).$$

The Rogers connection relation for the q-ultraspherical polynomials is

$$C_n(x;\gamma \mid q) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{\beta^k(\gamma/\beta;q)_k(\gamma;q)_{n-k}}{(q;q)_k(q\beta;q)_{n-k}} \frac{1-\beta q^{n-2k}}{1-\beta} C_{n-2k}(x;\beta \mid q) \tag{2-5}$$

[Ismail 2009, (13.3.1)]. The Ismail–Zhang q-exponential function is

$$\begin{aligned}\mathcal{E}_q(\cos\theta;\alpha) &\\ &= \frac{(\alpha^2;q^2)_\infty}{(q\alpha^2;q^2)_\infty} \sum_{n=0}^{\infty} (-ie^{i\theta}q^{(1-n)/2}, -ie^{-i\theta}q^{(1-n)/2};q)_n \frac{(-i\alpha)^n}{(q;q)_n} q^{n^2/4}\end{aligned} \tag{2-6}$$

[Ismail 2009, §14.1].

We shall always use the notation

$$x = \cos\theta, \qquad z = e^{i\theta}, \qquad f(x) = \breve{f}(z). \tag{2-7}$$

The set of polynomials $\{(ae^{i\theta}, ae^{-i\theta};q)_n : n = 0, 1, \dots\}$ is a basis for the space of all polynomials, and is called the Askey–Wilson basis. The connection formula for the Askey–Wilson basis is

$$\frac{(be^{i\theta}, be^{-i\theta};q)_n}{(q, ab;q)_n} = \sum_{k=0}^{n} \frac{(ae^{i\theta}, ae^{-i\theta};q)_k}{(q,ab;q)_k} \frac{(b/a;q)_{n-k}}{(q;q)_{n-k}} \left(\frac{b}{a}\right)^k \tag{2-8}$$

[Ismail 1995]; see also the proof of Theorem 12.2.3 in [Ismail 2009].

We recall the definition of the Askey–Wilson operator,

$$(\mathcal{D}_q f)(x) = \frac{\breve{f}(q^{1/2}z) - \breve{f}(q^{-1/2}z)}{(q^{1/2}-q^{-1/2})(z-1/z)/2}. \tag{2-9}$$

It is easy to see that

$$\mathcal{D}_q(ae^{i\theta}, ae^{-i\theta};q)_n = -\frac{2a(1-q^n)}{1-q}(aq^{1/2}e^{i\theta}, aq^{1/2}e^{-i\theta};q)_{n-1} \tag{2-10}$$

[Ismail 2009, (12.2.2)]. We shall use the inner product

$$\langle f, g\rangle := \int_{-1}^{1} f(x)\,\overline{g(x)}\,\frac{dx}{\sqrt{1-x^2}}. \tag{2-11}$$

Let

$$H_\nu := \left\{ f : f((z+1/z)/2) \text{ is analytic for } q^\nu \le |z| \le q^{-\nu} \right\}. \tag{2-12}$$

The following theorem — an analogue of integration by parts — is due to Brown, Evans and Ismail [Brown et al. 1996]; see also [Ismail 2009, §16.1].

Theorem 2.1. *The Askey–Wilson operator $\mathcal{D}_q$ satisfies, for $f, g \in H_{1/2}$,*

$$\langle \mathcal{D}_q f, g\rangle = \frac{\pi\sqrt{q}}{1-q}\left[f\Big(\frac{q^{1/2}+q^{-1/2}}{2}\Big)\overline{g(1)} - f\Big(-\frac{q^{1/2}+q^{-1/2}}{2}\Big)\overline{g(-1)}\right] - \Big\langle f, \sqrt{1-x^2}\,\mathcal{D}_q\big(g(x)(1-x^2)^{-1/2}\big)\Big\rangle. \tag{2-13}$$

The Askey–Wilson polynomials have the basic hypergeometric representation

$$\begin{aligned} &p_n(x;\mathbf{t}\mid q) \\ &\quad = t_1^{-n}(t_1t_2, t_1t_3, t_1t_4; q)_n\, {}_4\phi_3\left(\left.\begin{matrix} q^{-n}, t_1t_2t_3t_4q^{n-1}, t_1e^{i\theta}, t_1e^{-i\theta} \\ t_1t_2,\ t_1t_3,\ t_1t_4 \end{matrix}\,\right|\, q, q\right), \end{aligned} \tag{2-14}$$

where $\mathbf{t}$ stands for the ordered quadruple (t_1, t_2, t_3, t_4). Their weight function is

$$w(x,\mathbf{t}\mid q) = \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{\prod_{j=1}^4 (t_je^{i\theta}, t_je^{-i\theta}; q)_\infty}\frac{1}{\sqrt{1-x^2}}, \quad x=\cos\theta\in(-1,1), \tag{2-15}$$

The Askey–Wilson polynomials satisfy the orthogonality relation

$$\begin{aligned} \int_{-1}^{1} p_m(x;\mathbf{t}\mid q)\, p_n(x;\mathbf{t}\mid q)\, w(x;\mathbf{t}\mid q)\, dx &= h_n(\mathbf{t})\,\delta_{m,n} \\ &= \frac{2\pi(t_1t_2t_3t_4q^{2n};q)_\infty\,(t_1t_2t_3t_4q^{n-1};q)_n}{(q^{n+1};q)_\infty \prod_{1\le j<k\le 4}(t_jt_kq^n;q)_\infty}\,\delta_{m,n}, \end{aligned} \tag{2-16}$$

for $\max\big\{|t_1|, |t_2|, |t_3|, |t_4|\big\} < 1$. The Askey–Wilson polynomials also satisfy the Rodrigues-type formula

$$w(x;\mathbf{t}\mid q)\, p_n(x;\mathbf{t}\mid q) = \Big(\frac{q-1}{2}\Big)^n q^{n(n-1)/4}\,\mathcal{D}_q^n w(x; q^{n/2}\mathbf{t}\mid q). \tag{2-17}$$

The Chebyshev polynomials are also special Askey–Wilson polynomials; indeed,

$$\begin{aligned} &p_n(x; q, -q, \sqrt{q}, -\sqrt{q}\mid q) = (q^{n+2};q)_n\, U_n(x), \\ &p_0(x;\mathbf{t}\mid q) = T_0(x) = 1, \\ &p_n(x; 1, -1, \sqrt{q}, -\sqrt{q}\mid q) = 2(q^n;q)_n\, T_n(x) \qquad \text{for } n>0. \end{aligned} \tag{2-18}$$

We shall also use the q-Taylor expansion stated next.

Theorem 2.2 [Ismail 1995]. *Let*

$$x_n = (aq^{n/2} + q^{-n/2}/a)/2 \quad \textit{for } 0<q<1,\ 0<a<1, \tag{2-19}$$

If $f(x)$ is a polynomial, then

$$f(x)=\sum_{k=0}^{\infty} f_k\,(ae^{i\theta},ae^{-i\theta};q)_k,$$

with

$$f_k=\frac{(q-1)^k}{(2a)^k\,(q;q)_k}q^{-k(k-1)/4}\,(\mathcal{D}_q^k f)(x_k).$$

For a proof and details, see [Ismail 2009, Theorem 12.2.2].

3. Connection formulas and expansions

Lemma 3.1. *We have the integral evaluation*

$$\begin{aligned}\text{(3-1)}\quad \int_{-1}^{1}(ae^{i\theta},ae^{-i\theta};q)_n\,w(x;\mathbf{t}\mid q)\,dx &= \\ &=\frac{2\pi\,(t_1a,a/t_1;q)_n\,(t_1t_2t_3t_4;q)_\infty}{(q;q)_\infty\prod_{1\le j<m\le 4}(t_jt_m;q)_\infty}\,{}_4\phi_3\left(\begin{matrix}q^{-n},\ t_1t_2,\ t_1t_3,\ t_1t_4\\ t_1a,\,t_1t_2t_3t_4,\,q^{1-n}t_1/a\end{matrix}\,\middle|\,q,q\right).\end{aligned}$$

This integral can be evaluated by writing

$$(ae^{i\theta},ae^{-i\theta};q)_n=\frac{(ae^{i\theta},ae^{-i\theta};q)_\infty}{(aq^ne^{i\theta},aq^ne^{-i\theta};q)_\infty},$$

then using the Nassrallah–Rahman integral (1-2) and the Watson transformation [Gasper and Rahman 2004, (III.18)]. It also follows by expanding $(ae^{i\theta},ae^{-i\theta};q)_n$ in $\{(t_1e^{i\theta},t_1e^{-i\theta};q)_k:0\le k\le n\}$ by using (2-8), and then applying the Askey–Wilson integral (1-1); see also [Ismail and Stanton 1998, Thm. 3].

Our first result is the next expansion of $(be^{i\theta},be^{-i\theta};q)_n$ in Askey–Wilson polynomials.

Theorem 3.2.

$$\text{(3-2)}\qquad (be^{i\theta},be^{-i\theta};q)_n=\sum_{k=0}^{n} f_{n,k}(b,\mathbf{t})\,p_k(x;\mathbf{t}\mid q),$$

where

$$\begin{aligned}\text{(3-3)}\quad f_{n,k}(b,\mathbf{t})&=\frac{(-b)^kq^{\binom{k}{2}}\,(q;q)_n\,(b/t_4,bt_4q^k;q)_{n-k}}{(q,t_1t_2t_3t_4q^{k-1};q)_k\,(q;q)_{n-k}}\\ &\qquad\times{}_4\phi_3\left(\begin{matrix}q^{k-n},\ t_1t_4q^k,\ t_2t_4q^k,\ t_3t_4q^k\\ bt_4q^k,\,t_1t_2t_3t_4q^{2k},\,q^{1-n+k}t_4/b\end{matrix}\,\middle|\,q,q\right).\end{aligned}$$

Proof. It is clear that

$$f_{n,k}h_k(\mathbf{t})=\left\langle p_k(x;\mathbf{t}\mid q)\,w(x;\mathbf{t}\mid q),\ \sqrt{1-x^2}(be^{i\theta},be^{-i\theta};q)_n\right\rangle$$

$$= \Big(\frac{q-1}{2}\Big)^k q^{k(k-1)/4} \big\langle \mathcal{D}_q^k w(x; q^{k/2}\mathbf{t} \mid q),\ \sqrt{1-x^2}(be^{i\theta}, be^{-i\theta}; q)_n \big\rangle$$
$$= \Big(\frac{1-q}{2}\Big)^k q^{k(k-1)/4} \int_{-1}^{1} w(x; q^{k/2}\mathbf{t} \mid q)\, \mathcal{D}_q^k (be^{i\theta}, be^{-i\theta}; q)_n \, dx$$
$$= \frac{(-b)^k (q;q)_n}{(q;q)_{n-k}} q^{\binom{k}{2}} \int_{-1}^{1} (bq^{k/2}e^{i\theta}, bq^{k/2}e^{-i\theta}; q)_{n-k}\, w(x; q^{k/2}\mathbf{t} \mid q)\, dx.$$

In these steps we used the Rodrigues formula (2-17), as well as (2-13) and (2-10). The result follows from a slight variation of Lemma 3.1. □

Our first application of Theorem 3.2 is the connection relation for the Askey–Wilson polynomials.

Corollary 3.3. *We have the connection relation*

$$p_n(x; \mathbf{b}) = \sum_{k=o}^{n} c_{n,k}(\mathbf{b}, \mathbf{a})\, p_k(x; \mathbf{a}), \tag{3-4}$$

where

$$c_{n,k}(\mathbf{b}, \mathbf{a}) = \frac{b_4^{k-n} (b_1b_2b_3b_4q^{n-1}; q)_k\, (q, b_1b_4, b_2b_4, b_3b_4; q)_n}{(q;q)_{n-k}\, (q, a_1a_2a_3a_4q^{k-1}; q)_k\, (b_1b_4, b_2b_4, b_3b_4; q)_k} \tag{3-5}$$
$$\times\, q^{k(k-n)} \sum_{j,l\ge 0} \frac{(q^{k-n}, b_1b_2b_3b_4q^{n+k-1}, a_4b_4q^k; q)_{j+l}\, q^{j+l}}{(b_1b_4q^k, b_2b_4q^k, b_3b_4q^k; q)_{j+l}\, (q;q)_j\, (q;q)_l}$$
$$\times\, \frac{(a_1a_4q^k, a_2a_4q^k, a_3a_4q^k; q)_l\, (b_4/a_4; q)_j}{(a_4b_4q^k, a_1a_2a_3a_4q^{2k}; q)_l} \Big(\frac{b_4}{a_4}\Big)^l.$$

Proof. The follows by expanding the left-hand side of (3-4) in the Askey–Wilson basis $\{(a_1e^{i\theta}, a_1e^{-i\theta}; q)_k\}$, then applying Theorem 3.2. □

Corollary 3.3 is Theorem 14.4.2 in [Ismail 2009]. When $a_4 = b_4$, the double series in (3-4) reduces to a ${}_5\phi_4$ and we get a result of [Askey and Wilson 1985]. See also [Gasper and Rahman 2004, (7.6.2)–(7.6.3)]. For another proof, see [Ismail and Zhang 2005], which also uses (2-13). Note that, in view of the orthogonality relation (2-16), Corollary 3.3 is equivalent to Theorem 3.2.

The special case $b = t_3$ of Theorem 3.2 is interesting. The result, after interchanging t_1 and t_3, is

$$(t_1e^{i\theta}, t_1e^{-i\theta}; q)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-t_1)^k q^{\binom{k}{2}} \frac{(t_1t_2, t_1t_3, t_1t_4; q)_n}{(t_1t_2, t_1t_3, t_1t_4; q)_k} \frac{1 - t_1t_2t_3t_4q^{2k-1}}{1 - t_1t_2t_3t_4/q} \times \frac{(t_1t_2t_3t_4/q; q)_k}{(t_1t_2t_3t_4; q)_{n+k}}\, p_k(x; \mathbf{t} \mid q). \tag{3-6}$$

Theorem 3.4. *The following relations are equivalent*:

$$B_n = \frac{(t_1t_2, t_1t_3, t_1t_4; q)_n}{t_1^n} \sum_{k=0}^{n} \frac{(q^{-n}, t_1t_2t_3t_4q^{n-1}; q)_k}{(q; q)_k \prod_{j=2}^{4}(t_1t_j; q)_k} q^k A_k \tag{3-7}$$

$$A_n = \sum_{k=0}^{n} \frac{t_1^k q^{\binom{k}{2}} (t_1/t_4, t_1t_2q^k, t_1t_3q^k, t_1t_4q^k; q)_{n-k}}{(q, t_1t_2t_3t_4q^{k-1}; q)_k (q, t_1t_2t_3t_4q^{2k}; q)_{n-k}} B_k \tag{3-8}$$

Proof. We set $b = t_1$ in (3-2) and take (2-14) into account. The ${}_4\phi_3$ in (3-3) becomes a ${}_3\phi_2$, and can be summed by the q-analogue of the Pfaff–Saalschütz theorem. □

Theorem 3.4 is known [Krattenthaler 1989; 1996]. An interesting question is to explore where such inverse pair lives from the point of view of the Möbius function on lattices [Rota 1964], because the lattices which will lead to such a deep result will be very interesting. It is also interesting to explore the concept of Bailey pairs [Andrews 1986] from the Möbius-inversion point of view.

The q-ultraspherical polynomials are special Askey–Wilson polynomials, since

$$p_n\big(x; \sqrt{\beta}, -\sqrt{\beta}, \sqrt{q\beta}, -\sqrt{q\beta} \mid q\big) = \frac{(q, \beta^2q^n; q)_n}{(\beta; q)_n} C_n(x; \beta \mid q). \tag{3-9}$$

The q-plane wave expansion in q-ultraspherical polynomials is

$$\begin{aligned}\mathcal{E}_q(x; i\alpha) = \frac{(\alpha)^{-\nu} (q; q)_\infty}{(-q\alpha^2; q^2)_\infty (q^{\nu+1}; q)_\infty} \sum_{n=0}^{\infty} \frac{(1-q^{n+\nu})}{(1-q^\nu)} q^{n^2/4} i^n \\ \times J_{\nu+n}^{(2)}(2\alpha; q)\, C_n(x; q^\nu \mid q);\end{aligned} \tag{3-10}$$

see [Ismail and Zhang 1994].

Another application of Theorem 3.2 is this generalization of (3-10):

Theorem 3.5. *We have the following generalization of the q-plane wave expansion function*:

$$\begin{aligned}\mathcal{E}_q(x; \alpha) &= \frac{(\alpha^2; q^2)_\infty}{(q\alpha^2; q)_\infty} \sum_{n=0}^{\infty} \frac{\alpha^n q^{n^2/4}\, p_n(x; \mathbf{t})}{(q, t_1t_2t_3t_4q^{n-1}; q)_n} \\ &\quad \times \sum_{k=0}^{\infty} \frac{(-\alpha/t_4)^k}{(q; q)_k} (-q^{1+n-k}t_4^2; q^2)_k\, q^{k(k-2n)/4} \\ &\quad \times {}_4\phi_3\left(\left. \begin{matrix} q^{-k},\ t_1t_4q^n,\ t_2t_4q^n,\ t_3t_4q^n \\ -it_4q^{(1-k+n)/2}, it_4q^{(1-k+n)/2}, t_1t_2t_3t_4q^{2n} \end{matrix} \right| q, q \right).\end{aligned} \tag{3-11}$$

Proof. Expand the $\mathcal{E}_q$ in the Askey–Wilson basis via (2-6), then apply (3-2). □

Another proof of Theorem 3.5. Since $\mathscr{E}_q(x;\alpha) \in L_2[-1,1,w(x;\mathbf{t})]$, we set

$$\mathscr{E}_q(x;\alpha) = \sum_{n=0}^{\infty} c_n\, p_n(x;\mathbf{t}).$$

Using (2-17), the divided-difference relation $\mathscr{D}_q\mathscr{E}_q(x;\alpha) = 2\alpha q^{1/4}/(1-q)\mathscr{E}_q(x;\alpha)$ and the q-integration by parts (2-13), we find that

$$\begin{aligned}
c_n h_n(\mathbf{t}) &= \int_{-1}^{1} \mathscr{E}_q(x;\alpha)\, p_n(x;\mathbf{t})\, w(x;\mathbf{t})\, dx \\
&= \Big(\frac{q-1}{2}\Big)^n q^{\binom{n}{2}/2} \int_{-1}^{1} \mathscr{E}_q(x;\alpha)\, \mathscr{D}_q^n w(x; q^{n/2}\mathbf{t})\, dx \\
&= \alpha^n q^{n^2/4} \int_{-1}^{1} \mathscr{E}_q(x;\alpha)\, w(x; q^{n/2}\mathbf{t})\, dx \\
&= \frac{(\alpha^2;q^2)_\infty}{(q\alpha^2;q^2)_\infty} \alpha^n q^{n^2/4} \sum_{k=0}^{\infty} \frac{(-i\alpha)^k}{(q;q)_k} q^{k^2/4} \\
&\qquad \times \int_0^{\pi} w(\cos\theta; q^{n/2}\mathbf{t})\, (-iq^{(1-k)/2}e^{i\theta}, -iq^{(1-k)/2}e^{-i\theta}; q)_k \sin\theta\, d\theta.
\end{aligned}$$

The integral above is

$$\frac{2\pi(-it_4q^{(1+n-k)/2}, -it_4q^{(1-n-k)/2}/t_4; q)_k\, (t_1t_2t_3t_4q^{2n}; q)_\infty}{(q;q)_\infty \prod_{1\le j<m\le 4}(t_jt_mq^n;q)_\infty} \times {}_4\phi_3\left(\begin{matrix} q^{-k},\ t_1t_4q^n,\ t_2t_4q^n,\ t_3t_4q^n \\ -it_4q^{(1-k+n)/2}, it_4q^{(1-k+n)/2}, t_1t_2t_3t_4q^{2n} \end{matrix}\,\middle|\, q, q\right).$$

The result now follows from (2-16). □

In the case of q-ultraspherical polynomials, the ${}_4\phi_3$ in (3-11) can be summed by Andrews' q-analogue of Watson's ${}_3F_2$ sum [Gasper and Rahman 2004, (II.17)]. Thus, the ${}_4\phi_3$ is zero for k odd and, when k is replaced by $2k$, the ${}_4\phi_3$ is

$$\beta^{2k} q^{2nk+k} \frac{(q, -q^{1-n-2k}/\beta; q^2)_k}{(-\beta q^{n+2-2k}, \beta^2 q^{2n+2}; q^2)_k}.$$

Thus, the k-sum in (3-11) is ${}_2\phi_1(-\beta q^{n+2}, -\beta q^{n+1}; \beta^2 q^{2n+2}; q^2, \alpha^2)$. Therefore,

$$\mathscr{E}_q(x;\alpha) = \frac{(\alpha^2;q^2)_\infty}{(q\alpha^2;q)_\infty} \sum_{n=0}^{\infty} \frac{\alpha^n q^{n^2/4}}{(\beta;q)_n} \times {}_2\phi_1\left(\begin{matrix} -\beta q^{n+2}, -\beta q^{n+1} \\ \beta^2 q^{2n+2} \end{matrix}\,\middle|\, q^2, \alpha^2\right) C_n(x;\beta \mid q). \tag{3-12}$$

By equating the left sides of (3-12) and (3-10), we establish the identity

$$J_\nu^{(2)}(2\alpha;q) = \frac{\alpha^\nu(-\alpha^2;q^2)_\infty}{(q^{\nu+1};q)_\infty}\,{}_2\phi_1\left(\begin{matrix}-q^{\nu+2},\,-q^{\nu+1}\\ q^{2\nu+2}\end{matrix}\,\middle|\,q^2,-\alpha^2\right)$$
$$= \frac{\alpha^\nu(q^{\nu+1}\alpha^2;q^2)_\infty}{(q^{\nu+1};q)_\infty}\,{}_2\phi_2\left(\begin{matrix}-q^{\nu+2},\,-q^{\nu+1}\\ q^{2\nu+2},\,q^{\nu+2}\alpha^2\end{matrix}\,\middle|\,q^2,q^{\nu+1}\alpha^2\right),$$

after applying the ${}_2\phi_1$ to ${}_2\phi_2$ transformation [Gasper and Rahman 2004, (III.4)]. The representation of $J_\nu^{(2)}$ as a ${}_2\phi_2$ is due to [Rahman 1987].

The double series in (3-11) also reduces to a single series in the case of continuous q-Jacobi polynomials, $t_2 = t_1q^{1/2}$ and $t_4 = t_3q^{1/2}$, yielding a result in [Ismail et al. 1996]. The details however are not lengthy and will be omitted.

4. Expansions of x^n and $(1 \pm x)^\rho$

Theorem 4.1. *The expansion*

$$(1-x)^\rho = \frac{4}{\sqrt{\pi}}\,2^\rho\,\Gamma(\rho+3/2) \times \sum_{k=0}^\infty \frac{1-\beta q^k}{1-\beta}\left(\sum_{j=0}^\infty \frac{(k+2j+1)(-\rho)_{k+2j}\,\beta^j\,(q/\beta;q)_j\,(q;q)_{k+j}}{(q,q)_j\,(q\beta;q)_{k+j}\,\Gamma(k+2j+\rho+3)}\right)C_k(x;\beta\mid q) \tag{4-1}$$

holds for $-1<x<1$, $\rho>-1$ *and* $\beta\in(0,1)$. *The expansion for* $(1+x)^\rho$ *is similar, since* $C_n(-x;\beta\mid q) = (-1)^n C_n(x;\beta\mid q)$.

Proof. Apply (2-2) with $\nu=1$, then expand $C_k^1(x) = U_k(x) = C_k(x;q\mid q)$ in $C_j(x;\beta\mid q)$ by using (2-5), then rearrange the series. The expansion (2-2) holds for $\rho>-1$. The rearrangement is valid because the double series in the theorem converges absolutely for $\rho>-1$, in view of the asymptotic formula [Ismail 2009, (13.4.5)] and the well-known fact that $n^{b-a}\Gamma(n+a)/\Gamma(n+b)\to 1$ as $n\to+\infty$. □

It is interesting to note that, as $q\to 1$, the expansion (4-1) should reduce to (2-2). Indeed with $\beta = q^\nu$ the $q\to 1$ limit of the quantity in square brackets is a well-poised ${}_5F_4$ at $x=1$, which can be summed, see Slater [Slater 1966, (III.12)]. So we could have discovered the abovementioned sum if it was not already known.

Theorem 4.2. *For nonnegative integers n we have the q-ultraspherical expansion*

$$x^n = \frac{n!}{2^n}\sum_{m=0}^{\lfloor n/2\rfloor}\frac{1-\beta q^{n-2m}}{1-\beta}C_{n-2m}(x;\beta\mid q) \times \sum_{k=0}^m \frac{n+1-2k}{k!\,(n+1-k)!}\,\frac{\beta^{m-k}(q/\beta;q)_{m-k}(q;q)_{n-m-k}}{(q;q)_{m-k}(q\beta;q)_{n-m-k}}. \tag{4-2}$$

Proof. The expansion (4-2) follows immediately from letting $\nu = 1$ in (2-1) then use (2-5) with $\gamma = 1$. □

Note that
$$\frac{n!\,(n+1-2k)}{k!\,(n+1-k)!} = \binom{n}{k} - \binom{n}{k-1}.$$
With $\beta = q^{\nu}$, the limit of the k-sum in (4-2) as $q \to 1$ is a very well-poised ${}_4F_3$ at $x = -1$, which can be summed [Slater 1966, (III.11)].

5. Two bibasic integrals

In this section we give evaluations of the integral (5-2) and the more general integral (5-3). The proof uses the bibasic expansion

$$\tag{5-1} \frac{(q, qa^2; q)_\infty}{(qae^{i\theta}, qae^{-i\theta}; q)_\infty}\,(be^{i\theta}, be^{-i\theta}; p)_\infty = \sum_{k=0}^{\infty} \frac{1-a^2q^{2k}}{1-a^2}\,\frac{(a^2, ae^{i\theta}, ae^{-i\theta}; q)_k}{(q, qae^{i\theta}, aqe^{-i\theta}; q)_k}(-1)^k q^{\binom{k+1}{2}}(abq^k, bq^{-k}/a; p)_\infty,$$

which is valid for $0 < p < q$, or $p = q$ and $|b| < |a|$ [Ismail and Stanton 2003].

Theorem 5.1. *We have the bibasic integral evaluation*

$$\tag{5-2} \begin{aligned} &\int_0^{\pi} \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty\,(be^{i\theta}, be^{-i\theta}; p)_\infty}{\prod_{j=1}^{5}(a_je^{i\theta}, a_je^{-i\theta}; q)_\infty}\,d\theta \\ &= \frac{2\pi\,(a_2a_3a_4a_5/q; q)_\infty}{(q; q)_\infty \prod_{2\le r<s\le 5}(a_ra_s; q)_\infty}\,\frac{1}{(q, qa_1^2; q)_\infty} \\ &\quad\times \sum_{k=0}^{\infty} \frac{1-a_1^2q^{2k}}{1-a_1^2}\,\frac{(a_1^2; q)_k}{(q; q)_k}\,\frac{1-a_1^2q^{2k+1}}{\prod_{s=2}^{5}(1-a_1a_sq^k)}(-1)^k q^{\binom{k+1}{2}}\Big(a_1bq^k, \frac{bq^{-k}}{a_1}; p\Big)_\infty \\ &\quad\times {}_8W_7\Big(a_1^2q^{2k+1}; q, q^{k+1}\frac{a_1}{a_2}, q^{k+1}\frac{a_1}{a_3}, q^{k+1}\frac{a_1}{a_4}, q^{k+1}\frac{a_1}{a_5}; q, \frac{a_2a_3a_4a_5}{q}\Big) \\ &= \frac{2\pi \prod_{j=2}^{5}(a_1a_2a_3a_4a_5/a_j; q)_\infty}{(q, a_1^2a_2a_3a_4a_5; q)_\infty \prod_{1\le r<s\le 5}(a_ra_s; q)_\infty} \\ &\quad\times \sum_{k=0}^{\infty} \frac{(a_1^2; q)_k\,(a_1^2a_2a_3a_4a_5; q)_{2k}}{(q; q)_k\,(a_1^2; q)_{2k}} \\ &\quad\times \prod_{j=2}^{5} \frac{(a_1a_j; q)_k}{(a_1a_2a_3a_4a_5/a_j; q)_k}(-1)^k q^{\binom{k+1}{2}}\Big(a_1bq^k, \frac{bq^{-k}}{a_1}; p\Big)_\infty \\ &\quad\times {}_8W_7\Big(a_1^2a_2a_3a_4a_5q^{2k-1}; a_1a_2q^k, a_1a_3q^k, a_1a_4q^k, a_1a_5q^k, \frac{a_2a_3a_4a_5}{q}; q, q\Big). \end{aligned}$$

Proof. In view of (5-1), the left-hand side of (5-2) is

$$\frac{1}{(q, qa_1^2; q)_\infty} \sum_{k=0}^{\infty} \frac{1-a_1^2 q^{2k}}{1-a_1^2} \frac{(a_1^2; q)_k}{(q; q)_k} (-1)^k q^{\binom{k+1}{2}} (a_1 b q^k, b q^{-k}/a_1; p)_\infty$$
$$\times \int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty \, (a_1 q^{k+1} e^{i\theta}, a_1 q^{k+1} e^{-i\theta}; q)_\infty}{(a_1 q^k e^{i\theta}, a_1 q^k e^{-i\theta}; q)_\infty \prod_{j=2}^5 (a_j e^{i\theta}, a_j e^{-i\theta}; q)_\infty} \, d\theta.$$

The first equality in (5-2) follows from (1-2). The second equality follows from the form of the Nassrallah–Rahman integral stated in [Gasper and Rahman 2004, (6.3.7)] with

$$f = a_1 q^k. \qquad \square$$

When $p = q$, Theorem 5.1 should reduce to the Nassrallah–Rahman integral (1-2). This is not obvious, so we will indicate how it works. When $p = q$,

$$(-1)^k q^{\binom{k+1}{2}} (a_1 b q^k, b q^{-k}/a_1; p)_\infty = (a_1 b, b/a_1; q)_\infty \frac{b^k (q a_1/b; q)_k}{a^k (a_1 b; q)_k}.$$

We use the second equation in (5-2) and write the ${}_8W_7$ as a sum over j. With $\ell = j + k$, the left-hand side of (5-2) becomes

$$\frac{2\pi (a_1 b, b/a_1; q)_\infty \prod_{s=2}^5 (a_1 a_2 a_3 a_4 a_5/a_s; q)_\infty}{(q, a_1^2 a_2 a_3 a_4 a_5; q)_\infty \prod_{1 \le r < s \le 5} (a_r a_s; q)_\infty}$$
$$\times \sum_{\ell=0}^{\infty} \frac{1 - a_1^2 a_2 a_3 a_4 a_5 q^{2\ell-1}}{1 - a_1^2 a_2 a_3 a_4 a_5/q} \frac{(a_2 a_3 a_4 a_5/q, a_1^2 a_2 a_3 a_4 a_5/q; q)_\ell}{(q, q a_1^2; q)_\ell} q^\ell$$
$$\times \prod_{r=2}^5 \frac{(a_1 a_r; q)_\ell}{(a_1 a_2 a_3 a_4 a_5/a_r; q)_\ell}$$
$$\times {}_6W_5(a_1^2; q a_1/b, a_1^2 a_2 a_3 a_4 a_5 q^{\ell-1}, q^{-\ell}; q, q b/a_1^2 a_2 a_3 a_4 a_5).$$

The ${}_6W_5$ can be summed by [Gasper and Rahman 2004, (II.20)], and the expression above reduces to the integral evaluation [Gasper and Rahman 2004, (6.3.7)].

The next theorem generalizes the evaluation of the moments of the Askey–Wilson weight function.

Theorem 5.2. *We have the integral evaluation*

$$\text{(5-3)} \qquad \int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty \, (b e^{i\theta}, b e^{-i\theta}; p)_n}{\prod_{j=1}^4 (a_j e^{i\theta}, a_j e^{-i\theta}; q)_\infty} \, d\theta$$

$$= \frac{2\pi (a_1a_2a_3a_4;\, q)_\infty}{(q;\, q)_\infty \prod_{1\le j<k\le 4}(a_ja_k;\, q)_\infty} \frac{(a_1a_2, qa_1a_3, qa_1a_4;\, q)_n}{(q, qa_1^2, a_1a_2a_3a_4;\, q)_n}$$

$$\times \sum_{k=0}^{n} \frac{1-a_1^2q^{2k}}{1-a_1^2} \frac{(a_1^2, q^{-n};\, q)_k}{(q, a_1^2q^{n+1};\, q)_k} (a_1bq^k, bq^{-k}/a_1;\, p)_n$$

$$\times q^{k(n+1)} \frac{(1-a_1a_3)(1-a_1a_4)}{(1-a_1a_3q^k)(1-a_1a_4q^k)}$$

$$\times {}_4\phi_3\left(\begin{matrix} q^{k-n}, q, a_3a_4, a_1q^{k+1}/a_2 \\ a_1a_3q^{k+1}, a_1a_4q^{k+1}, q^{1-n}/a_1a_2 \end{matrix} \middle| \, q, q \right).$$

Proof. Observe that

$$\begin{aligned}
&(abq^k, bq^{-k}/a;\, p)_n \\
&\quad = (ab;\, p)_n \prod_{j=0}^{n-1}(1-ap^{-j}/b)q^{-kn}\Big(-\frac{b}{a}\Big)^n \prod_{j=0}^{n-1} \frac{(abp^j;\, q)_k\, (aqp^{-j}/b;\, q)_k}{(abp^j;\, q)_k\, (ap^{-j}/b;\, q)_k} \\
&\quad = (ab, b/a;\, p)_n q^{-kn} \prod_{j=0}^{n-1} \frac{(abp^j;\, q)_k\, (aqp^{-j}/b;\, q)_k}{(ap^{-j}/b, abp^j;\, q)_k}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{k=0}^{n} \frac{1-a_1^2q^{2k}}{1-a_1^2} \frac{(a_1^2, a_1e^{i\theta}, a_1e^{-i\theta}, q^{-n};\, q)_k}{(q, qa_1e^{-i\theta}, qa_1e^{i\theta}, a_1^2q^{n+1};\, q)_k} q^{k(n+1)} \Big(a_1bq^k, \frac{bq^{-k}}{a_1};\, p\Big)_n \\
&\qquad = (a_1b, b/a_1;\, p)_n \sum_{k=0}^{n} \frac{1-a_1^2q^{2k}}{1-a_1^2} \frac{(a_1^2, a_1e^{i\theta}, a_1e^{-i\theta}, q^{-n};\, q)_k}{(q, qa_1e^{i\theta}, qa_1e^{-i\theta}, a_1^2q^{n+1};\, q)_k} q^k \\
&\qquad\qquad \times \prod_{j=0}^{n-1} \frac{(qa_1p^{-j}/b, qa_1bp^j;\, q)_k}{(a_1bp^j, a_1p^{-j}/b;\, q)_k} \\
&\qquad = (a_1b, b/a_1;\, p)_n \frac{(qa_1^2, q;\, q)_n\, (be^{i\theta}, be^{-i\theta};\, p)_n}{(qa_1e^{i\theta}, qa_1e^{-i\theta};\, q)_n\, (a_1b, b/a_1;\, p)_n} \\
&\qquad = \frac{(q, qa_1^2;\, q)_n\, (be^{i\theta}, be^{-i\theta};\, p)_n}{(qa_1e^{i\theta}, qa_1e^{-i\theta};\, q)_n}.
\end{aligned}$$

So, we have

$$\begin{aligned}
&\frac{(q, qa_1^2;\, q)_n\, (be^{i\theta}, be^{-i\theta};\, p)_n}{(qa_1e^{i\theta}, qa_1e^{-i\theta};\, q)_n} \\
&\quad = \sum_{k=0}^{n} \frac{1-a_1^2q^{2k}}{1-a_1^2} \frac{(a_1^2, a_1e^{i\theta}, a_1e^{-i\theta}, q^{-n};\, q)_k}{(q, qa_1e^{-i\theta}, qa_1e^{i\theta}, a_1^2q^{n+1};\, q)_k} q^{k(n+1)} \Big(a_1bq^k, \frac{bq^{-k}}{a_1};\, p\Big)_n.
\end{aligned}$$

Therefore,

$$\int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty\, (be^{i\theta}, be^{-i\theta}; p)_n}{\prod_{j=1}^4 (a_j e^{i\theta}, a_j e^{-i\theta}; q)_\infty}\, d\theta$$

$$= \frac{1}{(q, qa_1^2; q)_n} \sum_{k=0}^n \frac{1-a_1^2 q^{2k}}{1-a_1^2} \frac{(a_1^2, q^{-n}; q)_k}{(q, a_1^2 q^{n+1}; q)_k} q^{k(n+1)} \Big(a_1 b q^k, \frac{bq^{-k}}{a_1}; p\Big)_n$$

$$\times \int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(a_1 q^{n+1} e^{i\theta}, a_1 q^{n+1} e^{-i\theta}, a_1 q^k e^{i\theta}, a_1 q^k e^{-i\theta}; q)_\infty}$$

$$\times \frac{(a_1 q^{k+1} e^{i\theta}, a_1 q^{k+1} e^{-i\theta}; q)_\infty}{\prod_{j=2}^4 (a_j e^{i\theta}, a_j e^{-i\theta}; q)_\infty}\, d\theta$$

Using [Gasper and Rahman 2004, (6.3.8)] and Watson's formula [Gasper and Rahman 2004, (III.18)], the integral in the equation above becomes

$$\frac{2\pi (a_1 a_2 a_3 a_4; q)_\infty}{(q; q)_\infty \prod_{1\le j<k\le 4} (a_j a_k; q)_\infty} \frac{(a_1 a_2; q)_n (a_1 a_3, a_1 a_4; q)_{n+1}}{(a_1 a_2 a_3 a_4; q)_n}$$

$$\times \frac{1}{(1-a_1 a_3 q^k)(1-a_1 a_4 q^k)}\, {}_4\phi_3\left(\begin{matrix} q^{k-n}, q, a_3 a_4, a_1 q^{k+1}/a_2 \\ a_1 a_3 q^{k+1}, a_1 a_4 q^{k+1}, q^{1-n}/a_1 a_2 \end{matrix} \,\middle|\, q, q \right).$$

This completes the proof. □

We give a second proof of (5-3) because it has an idea which may be useful in other cases. The second proof uses the following recent result of [Ismail and Stanton 2010]:

$$\frac{(q, qa^2; q)_n}{(qae^{i\theta}, qae^{-i\theta}; q)_n} (be^{i\theta}, be^{-i\theta}; p)_n \tag{5-4}$$

$$= \sum_{k=0}^n \frac{1-a^2 q^{2k}}{1-a^2} \frac{(q^{-n}, a^2, ae^{i\theta}, ae^{-i\theta}; q)_k}{(q, a^2 q^{n+1}, aqe^{i\theta}, aqe^{-i\theta}; q)_k} q^{k(1+n)} (abq^k, bq^{-k}/a; p)_n.$$

Second proof of Theorem 5.2. In view of (5-4), the left-hand side of (5-3) is

$$\frac{1}{(q, qa_1^2; q)_n} \sum_{k=0}^n \frac{1-a_1^2 q^{2k}}{1-a_1^2} \frac{(q^{-n}, a_1^2; q)_k}{(q, a_1^2 q^{n+1}; q)_k} q^{k(1+n)} (a_1 b q^k, bq^{-k}/a_1; p)_n$$

$$\times \int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty\, (a_1 q^{k+1} e^{i\theta}, a_1 q^{k+1} e^{-i\theta})_{n-k}}{(a_1 q^k e^{i\theta}, a_1 q^k e^{-i\theta}; q)_\infty \prod_{j=2}^4 (a_j e^{i\theta}, a_j e^{-i\theta}; q)_\infty}\, d\theta.$$

This integral can be evaluated by (3-1) and equals

$$\frac{2\pi (a_1^2 q^{2k+1}, q; q)_{n-k}\, (q^k a_1a_2a_3a_4; q)_\infty}{(q;q)_\infty \prod_{j=2}^4 (q^k a_1 a_j; q)_\infty \prod_{2\le r<s\le 4}(a_r a_s; q)_\infty} \times {}_4\phi_3\left(\left.\begin{matrix} q^{k-n}, a_1a_2q^k, a_1a_3q^k, a_1a_4q^k \\ a_1^2q^{2k+1}, a_1a_2a_3a_4q^k, q^{-n}\end{matrix}\,\right|\, q, q\right)$$

The application of the iterated Sears transformation [Gasper and Rahman 2004, (III.16)] reduces ${}_4\phi_3$ to

$$\frac{(a_1a_2q^k, a_1a_3q^{k+1}, a_1a_4q^{k+1}; q)_{n-k}}{(a_1^2q^{2k+1}, a_1a_2a_3a_4q^k, q; q)_{n-k}}$$

times the ${}_4\phi_3$ in (5-3). Simple manipulations now establish (5-3). □

Let $p = 1$ and $\zeta = \frac{1}{2}(b + 1/b)$. Then,

$$\text{(5-5)}\quad \int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{\prod_{j=1}^4 (a_j e^{i\theta}, a_j e^{-i\theta}; q)_\infty} (\cos\theta - \zeta)^n d\theta = \frac{2\pi (a_1a_2a_3a_4; q)_\infty}{(q;q)_\infty \prod_{1\le j<k\le 4}(a_ja_k;q)_\infty} \frac{(a_1a_2, qa_1a_3, qa_1a_4; q)_n}{(q, qa_1^2, a_1a_2a_3a_4; q)_n} \times \sum_{k=0}^n \frac{1-a_1^2q^{2k}}{1-a_1^2} \frac{(a_1^2, q^{-n}; q)_k}{(q, a_1^2q^{n+1}; q)_k} \left(\tfrac{1}{2}(a_1q^k + q^{-k}/a_1) - \zeta\right)^n \times q^k \frac{(1-a_1a_3)(1-a_1a_4)}{(1-a_1a_3q^k)(1-a_1a_4q^k)} \times {}_4\phi_3\left(\left.\begin{matrix} q^{k-n}, q, a_3a_4, a_1q^{k+1}/a_2 \\ a_1a_3q^{k+1}, a_1a_4q^{k+1}, q^{1-n}/a_1a_2\end{matrix}\,\right|\, q, q\right).$$

The special case $\zeta = 0$ gives the Askey–Wilson moments

$$\text{(5-6)}\quad \int_{-1}^1 W(x; \mathbf{a})\, x^n\, dx = \frac{(a_1a_2, qa_1a_3, qa_1a_4; q)_n}{(2a_1)^n (q, qa_1^2, a_1a_2a_3a_4; q)_n} \times \sum_{k=0}^n \frac{1-a_1^2q^{2k}}{1-a_1^2} \frac{(a_1^2, q^{-n}; q)_k}{(q, a_1^2q^{n+1}; q)_k} (1 + a_1^2q^{2k})^n \times q^{k(n+1)} \frac{(1-a_1a_3)(1-a_1a_4)}{(1-a_1a_3q^k)(1-a_1a_4q^k)} \times {}_4\phi_3\left(\left.\begin{matrix} q^{k-n}, q, a_3a_4, a_1q^{k+1}/a_2 \\ a_1a_3q^{k+1}, a_1a_4q^{k+1}, q^{1-n}/a_1a_2\end{matrix}\,\right|\, q, q\right),$$

where W is the normalized weight function

$$W(x;\mathbf{a}) := \frac{(q;q)_\infty \prod_{1\le r<s\le 4}(a_r a_s;q)_\infty}{2\pi(a_1a_2a_3a_4;q)_\infty}\, w(x;\mathbf{a}). \tag{5-7}$$

The moments of the Askey–Wilson weight functions were first computed in the very interesting paper [Corteel and Williams 2007]. Corteel and Williams used purely combinatorial techniques and showed that the moments of the Askey–Wilson weight is a generating function for purely combinatorial objects. The Corteel-Williams formula is very different in nature from our (5-6), and a very interesting but difficult exercise is to show the equivalence of the two results.

6. The Andrews identities

We now prove both (1-4) and (1-5) using the ${}_5\phi_4$ to ${}_{12}\phi_{11}$ transformation [Gasper and Rahman 2004, (2.8.4)].

Proof of (1-4). The limiting case $e \to 0$ of the ${}_5\phi_4$ to ${}_{12}\phi_{11}$ transformation (2.8.4) of [Gasper and Rahman 2004] is

$$
{}_4\phi_3\left(\left.\begin{matrix} q^{-n}, & b, & c, & d \\ \dfrac{q^{1-n}}{b}, & \dfrac{q^{1-n}}{c}, & \dfrac{q^{1-n}}{d} & \end{matrix}\,\right|\, q, q\right) = \frac{(\lambda^2 q^{n+1};q)_n}{(q\lambda;q)_n}(\lambda q^n)^{-n}
$$
$$
\times\, {}_{10}\phi_9\left(\left.\begin{matrix} \lambda, q\sqrt{\lambda}, -q\sqrt{\lambda}, \lambda b q^n, \lambda c q^n, \lambda d q^n, q^{-\frac{n}{2}}, -q^{-\frac{n}{2}}, q^{\frac{1-n}{2}}, -q^{\frac{1-n}{2}}, \\ \sqrt{\lambda}, -\sqrt{\lambda}, \dfrac{q^{1-n}}{b}, \dfrac{q^{1-n}}{c}, \dfrac{q^{1-n}}{d}, \lambda q^{1+\frac{n}{2}}, -\lambda q^{1+\frac{n}{2}}, \lambda q^{\frac{1+n}{2}}, -\lambda q^{\frac{1+n}{2}} \end{matrix}\,\right|\, q, \lambda q^{n+1}\right),
$$

where $bcd\lambda = q^{1-2n}$. Thus, the ${}_4\phi_3$ above is

$$
\frac{(\lambda q^n)^{-n}}{(\lambda q;q)_n}\sum_{k=0}^{\lfloor n/2\rfloor}\frac{1-\lambda q^{2k}}{1-\lambda}\,\frac{(q^{-n};q)_{2k}\,(\lambda.\lambda b q^n, \lambda c q^n, \lambda d q^n;q)_k}{(q, q^{1-n}/b, q^{1-n}/c, q^{1-n}/d;q)_k}
\times \frac{(\lambda^2 q^{n+1};q)_n}{(\lambda^2 q^{n+1};q)_{2k}}\,(\lambda q^{n+1})^k
$$
$$
= \frac{(\lambda q^n)^{-n}}{(\lambda q;q)_n}\sum_{k=0}^{\lfloor n/2\rfloor}\frac{1-\lambda q^{2k}}{1-\lambda}\,\frac{(q^{-n};q)_{2k}\,(\lambda.\lambda b q^n, \lambda c q^n, \lambda d q^n;q)_k}{(q, q^{1-n}/b, q^{1-n}/c, q^{1-n}/d;q)_k}
\times (\lambda^2 q^{n+1+2k};q)_{n-2k}\,(\lambda q^{n+1})^k,
$$

since $(a;q)_n/(a;q)_j = (aq^j;q)_{n-j}$. In the case of (1-4), we replace q by q^2, then replace b, c and d by a, b and q^{1-2n}/ab, respectively. These choices make $\lambda = q^{1-2n}$. Hence, the ${}_4\phi_3$ in (1-4) transforms to

$$\text{(6-1)}\quad \frac{q^{-n}}{(q^{3-2n};q^2)_n}\sum_{k=0}^{\lfloor n/2\rfloor}\frac{1-q^{1-2n+4k}}{1-q^{1-2n}}\frac{(q^{1-2n},qa,qb,q^{2-2n}/ab;q^2)_k}{(q^2,q^{2-2n}/a,q^{2-2n}/b,qab;q^2)_k}\times q^{3k}(q^{4-2n+4k};q^2)_{n-2k}\,(q^{-2n};q^2)_{2k}.$$

Since $(q^{4-2n+2k};q^2)_{n-2k}=q^{-2\binom{n-2k}{2}}(-q^2)^{n-2k}(q^{-2};q^2)_{n-2k}$ by [Gasper and Rahman 2004, (I.8)], we find that the summand of the series above vanishes, unless $0\le n-2k\le 1$, which implies that the only nonvanishing term is when $k=\lfloor n/2\rfloor$. Computing and simplifying this last term gives the right-hand side of (1-4). □

Proof of (1-5). The use of the easily verifiable identity

$$\frac{1-abq^{-1}}{1-ab^2q^{2n-3}}\frac{(b,q^{3-2n}/ab;q^2)_k}{(q^{4-2n}/b,ab/q;q^2)_k}-ab^2q^{2n-3}\frac{(b,q^{3-2n}/ab;q^2)_k}{(q^{2-2n}/b,qab;q^2)_k}=\frac{(b,q^{2-2n}/ab;q^2)_k}{(q^{4-2n}/b,qab;q^2)_k},$$

gives

$$\text{(6-2)}\quad {}_4\phi_3\left(\begin{matrix}q^{-2n},a,b,q^{3-2n}/ab\\ q^{2-2n}/a,q^{4-2n}/b,qab\end{matrix}\,\middle|\,q^2,q^2\right)=\frac{1-abq^{-1}}{1-ab^2q^{2n-3}}\,{}_4\phi_3\left(\begin{matrix}q^{-2n},a,b,q^{3-2n}/ab\\ q^{2-2n}/a,q^{4-2n}/b,ab/q\end{matrix}\,\middle|\,q^2,q^2\right)-ab^2q^{2n-3}{}_4\phi_3\left(\begin{matrix}q^{-2n},a,b,q^{3-2n}/ab\\ q^{2-2n}/a,q^{2-2n}/b,qab\end{matrix}\,\middle|\,q^2,q^2\right),$$

yielding two balanced and nearly-poised series of the second kind on the right-hand side. Now we use the Watson transformation formula [Gasper and Rahman 2004, (III.18)] to transform the right-hand side of into ${}_8\phi_7$ series. Thus

$${}_4\phi_3\left(\begin{matrix}q^{-2n},a,b,\frac{q^{3-2n}}{ab}\\ \frac{q^{2-2n}}{a},\frac{q^{4-2n}}{b},qab\end{matrix}\,\middle|\,q^2,q^2\right)=\frac{1-abq^{-1}}{1-ab^2q^{2n-3}}\frac{(a/q,b/q;q^2)_n}{(1/q,ab/q;q^2)_n}\times{}_8\phi_7\left(\begin{matrix}q^{1-2n},q^{-n+5/2},-q^{-n+5/2},q^{-2n},a,qa,\frac{b}{q},b\\ q^{-n+1/2},-q^{-n+1/2},q^3,\frac{q^{3-2n}}{a},\frac{q^{2-2n}}{a},\frac{q^{4-2n}}{b},\frac{q^{3-2n}}{b}\end{matrix}\,\middle|\,q^2,\frac{q^{6-2n}}{a^2b^2}\right)$$
$$-ab^2q^{2n-3}\frac{(b/q,ab;q^2)_n}{(1/q,a;q^2)_n}{}_8\phi_7\left(\begin{matrix}q^{1-2n},q^{-n+5/2},-q^{-n+5/2},q^{-2n},b,qb,\frac{q^{2-2n}}{ab},\frac{q^{3-2n}}{ab}\\ q^{-n+1/2},-q^{-n+1/2},q^3,\frac{q^{3-2n}}{b},\frac{q^{2-2n}}{b},qab,ab\end{matrix}\,\middle|\,q^2,a^2\right).$$

The crucial formula to use now is the quadratic transformation formula [Gasper and Rahman 2004, (3.5.10)], that after some simplification, gives

$$
\begin{aligned}
&{}_4\phi_3\left(\begin{matrix} q^{-2n}, a, b, q^{3-2n}/ab \\ q^{2-2n}/a, q^{4-2n}/b, qab \end{matrix}\,\middle|\, q^2, q^2\right) \\
&= \frac{(q^{2-2n};q)_{2n}}{(q^{2-2n}/a;q)_{2n}} \frac{(abq^{n-2}, -abq^{n-2};q)_n}{(bq^{n-2}, -bq^{n-2};q)_n} a^{-2n} \frac{1-abq^{-1}}{1-ab^2q^{2n-3}} \\
&\quad\times \sum_{k=0}^{n} \frac{1+q^{2-2n+2k}/b}{1+q^{2-2n}/b} \frac{(-q^{2-2n}/b, q^{2-n}/b, -q^{2-n}/b, a;q)_k}{(q, q^{3-n}/b, -q^{3-n}/b, -q^{3-2n}/a;q)_k} \frac{q^{2k}(q^{-2n};q^2)_k}{a^k(q^{2-2n};q^2)_k} \\
&\quad - abq^{3n-3} \frac{(b/q, ab, a^2;q^2)_n}{(1/q, a, a^2b^2q^2;q^2)_n} \frac{(q^{2-2n};q)_{2n}}{(q^{2-2n}/b;q)_{2n}} \\
&\quad\times \sum_{k=0}^{n} \frac{1+abq^{2k}}{1+ab} \frac{(-ab, b, abq^{n-1}, -abq^{n-1};q)_k}{(q, -qa, b, abq^{n+1}, -abq^{n+1};q)_k} \frac{q^{2k}(q^{-2n};q^2)_k}{b^k(q^{2-2n};q^2)_k}
\end{aligned}
$$

However, in each of the two series above there is the common factor

$$
\frac{(q^{2-2n};q)_{2n}(q^{-2n};q^2)_k}{(q^{2-2n};q^2)_k} = (q^{2-2n+2k};q^2)_{n-k}(q^{-2n};q^2)_k(q^{3-2n};q^2)_n,
$$

which vanishes unless $k = n$. So, the only term that survives in each is the one term with $k = n$. Combining the two terms after a lot of messy but straightforward simplifications, we obtain (1-5). □

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References

[Andrews 1986] G. E. Andrews, *q-series: their development and application in analysis, number theory, combinatorics, physics, and computer algebra*, CBMS Regional Conference Series in Mathematics **66**, American Mathematical Society, Providence, RI, 1986. MR 88b:11063 Zbl 0594.33001

[Andrews 1987] G. E. Andrews, "Catalan numbers, q-Catalan numbers and hypergeometric series", *J. Combin. Theory Ser. A* **44**:2 (1987), 267–273. MR 88f:05015 Zbl 0607.05006

[Andrews 2011] G. E. Andrews, "On Shapiro's Catalan convolution", *Adv. in Appl. Math.* **46**:1-4 (2011), 15–24. MR 2794010 Zbl 05895437

[Andrews et al. 1999] G. E. Andrews, R. Askey, and R. Roy, *Special functions*, Encyclopedia of Mathematics and its Applications **71**, Cambridge University Press, Cambridge, 1999. MR 2000g:33001 Zbl 0920.33001

[Askey and Wilson 1985] R. Askey and J. Wilson, *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*, Mem. Amer. Math. Soc. **319**, American Mathematical Society, Providence, RI, 1985. MR 87a:05023

[Brown et al. 1996] B. M. Brown, W. D. Evans, and M. E. H. Ismail, “The Askey-Wilson polynomials and q-Sturm–Liouville problems”, *Math. Proc. Cambridge Philos. Soc.* **119**:1 (1996), 1–16. MR 96j:33012 Zbl 0860.33012

[Corteel and Williams 2007] S. Corteel and L. K. Williams, “Tableaux combinatorics for the asymmetric exclusion process”, *Adv. in Appl. Math.* **39**:3 (2007), 293–310. MR 2008g:05220 Zbl 1129.05057

[Corteel and Williams 2010] S. Corteel and L. K. Williams, “Staircase tableaux, the asymmetric exclusion process, and Askey–Wilson polynomials”, *Proc. Natl. Acad. Sci. USA* **107**:15 (2010), 6726–6730. MR 2011k:05268 Zbl 1205.05243

[Erdélyi et al. 1953] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher transcendental functions*, vol. 2, McGraw-Hill Book Company, New York-Toronto-London, 1953. Based, in part, on notes left by Harry Bateman. MR 15,419i Zbl 0052.29502

[Gasper and Rahman 2004] G. Gasper and M. Rahman, *Basic hypergeometric series*, 2nd ed., Encycl. of Mathematics and its Applications **96**, Cambridge University Press, 2004. MR 2006d:33028 Zbl 1129.33005

[Goldman and Rota 1970] J. Goldman and G.-C. Rota, “On the foundations of combinatorial theory. IV. Finite vector spaces and Eulerian generating functions”, *Studies in Appl. Math.* **49** (1970), 239–258. MR 42 #93

[Ismail 1995] M. E. H. Ismail, “The Askey–Wilson operator and summation theorems”, pp. 171–178 in *Mathematical analysis, wavelets, and signal processing* (Cairo, 1994), edited by M. E. H. Ismail et al., Contemp. Math. **190**, Amer. Math. Soc., Providence, RI, 1995. MR 96j:33011 Zbl 0839.33010

[Ismail 2009] M. E. H. Ismail, *Classical and quantum orthogonal polynomials in one variable*, Encycl. of Mathematics and its Applications **98**, Cambridge University Press, 2009. Reprint of the 2005 original. MR 2010i:33001 Zbl 1172.42008

[Ismail and Stanton 1998] M. E. H. Ismail and D. Stanton, “More orthogonal polynomials as moments”, pp. 377–396 in *Mathematical essays in honor of Gian-Carlo Rota* (Cambridge, MA, 1996), edited by B. E. Sagan and R. P. Stanley, Progr. Math. **161**, Birkhäuser, Boston, MA, 1998. MR 99f:33011 Zbl 0905.05083

[Ismail and Stanton 2003] M. E. H. Ismail and D. Stanton, “q-Taylor theorems, polynomial expansions, and interpolation of entire functions”, *J. Approx. Theory* **123**:1 (2003), 125–146. MR 2004g:30040 Zbl 1035.30025

[Ismail and Stanton 2010] M. E. H. Ismail and D. Stanton, “Some combinatorial and analytical identities”, preprint, 2010, available at www.math.umn.edu/~stanton/PAPERS/FuLascoux.pdf.

[Ismail and Zhang 1994] M. E. H. Ismail and R. Zhang, “Diagonalization of certain integral operators”, *Adv. Math.* **109**:1 (1994), 1–33. MR 96d:39005 Zbl 0838.33012

[Ismail and Zhang 2005] M. E. H. Ismail and R. Zhang, “New proofs of some q-series results”, pp. 285–299 in *Theory and applications of special functions*, edited by M. E. H. Ismail and E. Koelink, Dev. Math. **13**, Springer, New York, 2005. MR 2006b:33029 Zbl 1219.33019

[Ismail et al. 1996] M. E. H. Ismail, M. Rahman, and R. Zhang, “Diagonalization of certain integral operators. II”, *J. Comput. Appl. Math.* **68**:1-2 (1996), 163–196. MR 98d:33011 Zbl 0868.33015

[Koekoek and Swarttouw 1998] R. Koekoek and R. F. Swarttouw, “The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogues”, technical report 98-17, Faculty of Technical Mathematics and Informatics, TU Delft, 1998, available at http://homepage.tudelft.nl/11r49/documents/as98.pdf.

[Krattenthaler 1989] C. Krattenthaler, "q-analogue of a two variable inverse pair of series with applications to basic double hypergeometric series", *Canad. J. Math.* **41**:4 (1989), 743–768. MR 90g:33004 Zbl 0667.33005

[Krattenthaler 1996] C. Krattenthaler, "A new matrix inverse", *Proc. Amer. Math. Soc.* **124**:1 (1996), 47–59. MR 96d:15004 Zbl 0843.15005

[Krattenthaler and Schlosser 1999] C. Krattenthaler and M. Schlosser, "A new multidimensional matrix inverse with applications to multiple q-series", *Discrete Math.* **204**:1-3 (1999), 249–279. MR 2000j:33011 Zbl 0936.33011

[Nassrallah and Rahman 1985] B. Nassrallah and M. Rahman, "Projection formulas, a reproducing kernel and a generating function for q-Wilson polynomials", *SIAM J. Math. Anal.* **16**:1 (1985), 186–197. MR 87b:33009 Zbl 0564.33009

[Rahman 1987] M. Rahman, "An integral representation and some transformation properties of q-Bessel functions", *J. Math. Anal. Appl.* **125**:1 (1987), 58–71. MR 88h:33020 Zbl 0634.33013

[Rainville 1960] E. D. Rainville, *Special functions*, Macmillan, New York, 1960. MR 21 #6447 Zbl 0092.06503

[Riordan 1968] J. Riordan, *Combinatorial identities*, Wiley, New York, 1968. MR 38 #53 Zbl 0194.00502

[Rota 1964] G.-C. Rota, "On the foundations of combinatorial theory. I. Theory of Möbius functions", *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **2** (1964), 340–368. MR 30 #4688

[Slater 1966] L. J. Slater, *Generalized hypergeometric functions*, Cambridge University Press, 1966. MR 34 #1570 Zbl 0135.28101

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