We give new proofs of the evaluation of the connection relation for the Askey–Wilson polynomials and for expressing the Askey–Wilson basis in those polynomials using $q$-Taylor series. This led to some inverse relations. We also evaluate the coefficients in the expansions of $(x + b)^n$ in various $q$-orthogonal polynomials, including the Askey–Wilson polynomials, which leads to explicit expressions for the moments of the Askey–Wilson weight function. We generalize the $q$-plane wave expansion by expanding $\mathcal{E}_q(x; \alpha)$ in Askey–Wilson polynomials. Further, we prove a bibasic extension of the Nassrallah–Rahman integral and establish a recently conjectured identity of George Andrews.

1. Introduction

Richard Askey and James Wilson introduced the polynomials that bear their names in their memoir [1985], where they derived, among other properties, the connection relation between Askey–Wilson polynomials with different parameters. One fundamental result of theirs is the evaluation of the Askey–Wilson $q$-beta integral,

\begin{equation}
\int_{0}^{\pi} \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{\prod_{j=1}^{4} (t_j e^{i\theta}, t_j e^{-i\theta}; q)_\infty} \, d\theta = \frac{2\pi (t_1 t_2 t_3 t_4; q)_\infty}{(q; q)_\infty \prod_{1 \leq j < k \leq 4} (t_j t_k; q)_\infty}.
\end{equation}

All this work was done in the late 1970s and the results were made available to researchers in the area, but the writing took a long time. In the mean time, Nassrallah and Rahman [1985] generalized the Askey–Wilson integral to

\begin{equation}
\int_{0}^{\pi} \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty (t_6 e^{i\theta}, t_6 e^{-i\theta}; q)_\infty}{\prod_{j=1}^{5} (t_j e^{i\theta}, t_j e^{-i\theta}; q)_\infty} \, d\theta = \frac{2\pi (t_1 t_2 t_3 t_4 t_5 / t_6; q)_\infty \prod_{j=1}^{5} (t_j t_6; q)_\infty}{(q, t_6^2; q)_\infty \prod_{1 \leq j < k \leq 5} (t_j t_k; q)_\infty} \times 8 W_7(t_6^2 / q; t_6 / t_1, t_6 / t_2, t_6 / t_3, t_6 / t_4, t_6 / t_5; q, t_1 t_2 t_3 t_4 t_5 / t_6).
\end{equation}

**MSC2000:** primary 05A19, 33D15; secondary 33D70.

**Keywords:** connection relations, bibasic integrals, moments of the Askey–Wilson and $q$-ultraspherical distributions, $q$-plane wave expansions, bibasic integrals, Andrews conjecture.
**Remark.** The preceding equation is (6.3.9) in [Gasper and Rahman 2004]. As in that reference and in [Ismail 2009], we follow the notation of [Andrews et al. 1999] for $q$-shifted factorials and basic hypergeometric series, and that of [Koekoek and Swarttouw 1998] for orthogonal polynomials.

The Askey–Wilson and Nassrallah–Rahman integrals play a fundamental role in the derivation of the results of this article, which is laid out as follows. **Section 2** contains many of the formulas needed, other than (1-1) and (1-2). In particular, the Askey–Wilson polynomials are defined in (2-14).

In **Section 3**, we first solve the connection-coefficient problem of expanding an Askey–Wilson basis element

$$(ae^{i\theta}, ae^{-i\theta}; q)_n$$

in Askey–Wilson polynomials. The proof utilizes the $q$-integration by parts technique of [Brown et al. 1996]. One application of this expansion is to give a new derivation of a $q$-analogue of the plane wave expansion [Ismail 2009, (4.8.3)]

$$(1-3) \quad e^{ixy} = (2/y)^v \Gamma(v) \sum_{n=0}^{\infty} (n+v) i^n J_{n+v}(y) C_n^v(x),$$

a result first proved in [Ismail and Zhang 1994]. More importantly, we generalize the $q$-plane wave expansion to expand the Ismail–Zhang $q$-exponential function $\mathcal{E}_q(x; \alpha)$ in Askey–Wilson polynomials, which is a new result. The aforementioned connection-coefficient problem is also used to give a new proof of the connection relation of the Askey–Wilson polynomials. Each connection relation may be used to discover an inverse relation of the form $y_n = \sum_{k=0}^{n} Y_{n,k} x_k$ if and only if $x_n = \sum_{k=1}^{n} X_{n,k} y_k$. Inverse relations play a fundamental role in combinatorial-enumeration problems, as discussed in Riordan’s classic [1968]. In the 1970s, interpretations of inverse relations involving $q$-shifted factorials and $q$-binomial coefficients were shown to be instances of Möbius inversion [Rota 1964] and of counting problems involving vector spaces over a finite field [Goldman and Rota 1970]. More recently, very general inverse relations were derived in [Krattenthaler 1989, 1996; Krattenthaler and Schlosser 1999].

**Section 4** contains expansions of $x^n$ and $(1\pm x)^\rho$ in $q$-ultraspherical polynomials.

**Section 5** contains the evaluation of two bibasic integrals which extend the Nassrallah–Rahman integral. They are stated as Theorems 5.1 and 5.2; the latter contains as a special case the evaluation of the moments of the Askey–Wilson weight function. [Corteel and Williams 2007] recently found a beautiful combinatorial expression for the $n$-th moment of the Askey–Wilson measure; this is also part of the results announced in [Corteel and Williams 2010]. Our analytic expression of the moments of the Askey–Wilson weight function is a double sum.
George Andrews [2011] studied identities involving the Catalan numbers he introduced in [Andrews 1987]. One of his identities was motivated by earlier work of L. Shapiro. Andrews’ investigations led him to two summation theorems. One summation theorem is

\[ (1-4) \quad \phi_3 (q^{-2n}, a/b, q^{-2n}/ab, q^2, q^2) = \frac{q^{-n}(a, b, -q; q)_n (a; q^2)_n}{(ab; q)_n (a, b; q^2)_n}, \]

which he proved. He conjectured the validity of the other summation theorem,

\[ (1-5) \quad \phi_3 (q^{-2n}, a/b, q^{-2n}/ab, q^2, q^2) = \frac{q^{-n}(a, b/q, -q; q)_n (q - ab)}{(1 - b/q)(q - a^2)(1 - abq^{2n-1})} \times \frac{(ab/q^2; q^2)_n}{(ab; q)_n (a, b/q^2; q^2)_n} \]

Andrews verified (1-5) for \(1 \leq n \leq 6\). In Section 6, we give basic hypergeometric-series proofs of both (1-4) and the conjectured identity (1-5). We show that both (1-4) and (1-5) follow from a limiting case of the \(5\phi_4\) to \(12\phi_{11}\) transformation stated in [Gasper and Rahman 2004, (2.8.4)].

2. Preliminaries

The expansions of \(x^n\) and \((1 - x)^\rho\) in ultraspherical polynomials are

\[ (2-1) \quad \frac{(2x)^n}{n!} = \sum_{k=0}^{[n/2]} \frac{v+n-2k}{k!(v+1-k)} C_{n-2k}^v(x) \]

[Rainville 1960, (36), p. 283], and

\[ (2-2) \quad (1 - x)^\rho = \Gamma(v) \Gamma(v + \rho + 1/2) \frac{2^{2v+\rho}}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(k + v)(-\rho)_k}{\Gamma(k + 2v + \rho + 1)} C_k^v(x), \]

valid for \(-1 < x < 1\), \(-\rho < \frac{1}{2}(v+1)\) if \(v \geq 0\), and \(-\rho < v + \frac{1}{2}\) if \(-\frac{1}{2} < v \leq 0\) [Erdélyi et al. 1953, (10.20.6)]. The Chebyshev polynomials are the special cases

\[ (2-3) \quad T_n(x) = \lim_{v \to 0} \frac{n + 2v}{2v} C_n^v(x) \quad \text{and} \quad U_n(x) = C_n^1(x). \]

The Chebyshev polynomials are also special cases of the continuous \(q\)-ultrasphe-rical polynomials, since

\[ (2-4) \quad T_n(x) = \lim_{\beta \to 1} \frac{1 - \beta q^n}{1 - \beta^2} C_n^v(x; \beta | q) \quad \text{and} \quad U_n(x) = C_n(x; q | q). \]
The Rogers connection relation for the $q$-ultraspherical polynomials is

\[(2-5) \quad C_n(x; \gamma \mid q) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{\beta^k (\gamma / \beta; q)_k (\gamma ; q)_{n-k}}{(q; q)_k (q\beta; q)_{n-k}} \frac{1 - \beta q^{n-2k}}{1 - \beta} C_{n-2k}(x; \beta \mid q)\]

[Ismail 2009, (13.3.1)]. The Ismail–Zhang $q$-exponential function is

\[(2-6) \quad \mathcal{E}_q(\cos \theta; \alpha) = \frac{(\alpha^2; q^2)_{\infty}}{(q\alpha^2; q^2)_{\infty}} \sum_{n=0}^{\infty} (-ie^{i\theta} q^{(1-n)/2}, -ie^{-i\theta} q^{(1-n)/2}; q)_n \frac{(-i\alpha)^n}{(q; q)_n} q^{n^2/4}\]

[Ismail 2009, §14.1].

We shall always use the notation

\[(2-7) \quad x = \cos \theta, \quad z = e^{i\theta}, \quad f(x) = \tilde{f}(z)\]

The set of polynomials $\{(ae^{i\theta}, ae^{-i\theta}; q)_n : n = 0, 1, \ldots\}$ is a basis for the space of all polynomials, and is called the Askey–Wilson basis. The connection formula for the Askey–Wilson basis is

\[(2-8) \quad \frac{(be^{i\theta}, be^{-i\theta}; q)_n}{(q, ab; q)_n} = \sum_{k=0}^{n} \frac{(ae^{i\theta}, ae^{-i\theta}; q)_k (b/a; q)_{n-k}}{(q, ab; q)_k} \left(\frac{b}{a}\right)^k\]

[Ismail 1995]; see also the proof of Theorem 12.2.3 in [Ismail 2009].

We recall the definition of the Askey–Wilson operator,

\[(2-9) \quad (\mathcal{D}_q f)(x) = \frac{\tilde{f}(q^{1/2}z) - \tilde{f}(q^{-1/2}z)}{(q^{1/2} - q^{-1/2})(z - 1/z)/2}\]

It is easy to see that

\[(2-10) \quad \mathcal{D}_q (ae^{i\theta}, ae^{-i\theta}; q)_n = -\frac{2a(1 - q^n)}{1 - q} (aq^{1/2} e^{i\theta}, aq^{1/2} e^{-i\theta}; q)_{n-1}\]

[Ismail 2009, (12.2.2)]. We shall use the inner product

\[(2-11) \quad \langle f, g \rangle := \int_{-1}^{1} f(x) g(x) \frac{dx}{\sqrt{1 - x^2}}\]

Let

\[(2-12) \quad H_v := \{ f : f((z + 1/z)/2) \text{ is analytic for } q^v \leq |z| \leq q^{-v} \}\]

The following theorem — an analogue of integration by parts — is due to Brown, Evans and Ismail [Brown et al. 1996]; see also [Ismail 2009, §16.1].
Theorem 2.1. The Askey–Wilson operator $\mathcal{D}_q$ satisfies, for $f, g \in H_{1/2}$,

$$\langle \mathcal{D}_q f, g \rangle = \frac{\pi \sqrt{q}}{1 - q} \left[ f \left( \frac{q^{1/2} + q^{-1/2}}{2} \right) g(1) - f \left( -\frac{q^{1/2} + q^{-1/2}}{2} \right) g(-1) \right]$$

$$- \left( f, \sqrt{1 - x^2} \mathcal{D}_q (g(x)(1 - x^2)^{-1/2}) \right).$$

The Askey–Wilson polynomials have the basic hypergeometric representation

$$p_n(x; t | q) = t_1^{-n} (t_1 t_2, t_1 t_3, t_1 t_4; q)_n \Phi_3 \left( q^{-n}, t_1 t_2 t_3 t_4 q^{n-1}; t_1 e^{i\theta}, t_1 e^{-i\theta} \middle| q, q \right),$$

where $t$ stands for the ordered quadruple $(t_1, t_2, t_3, t_4)$. Their weight function is

$$w(x, t | q) = \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{\prod_{j=1}^4 (t_j e^{i\theta}, t_j e^{-i\theta}; q)_\infty} \frac{1}{\sqrt[2]{1 - x^2}}, \quad x = \cos \theta \in (-1, 1),$$

The Askey–Wilson polynomials satisfy the orthogonality relation

$$\int_{-1}^1 p_m(x; t | q) p_n(x; t | q) w(x; t | q) \, dx = h_n(t) \delta_{m,n}$$

$$= \frac{2\pi (t_1 t_2 t_3 t_4 q^{2n}; q)_\infty (t_1 t_2 t_3 t_4 q^{n-1}; q)_n}{(q^{n+1}; q)_\infty \prod_{1 \leq j < k \leq 4} (t_j t_k q^n; q)_\infty} \delta_{m,n},$$

for $\max\{|t_1|, |t_2|, |t_3|, |t_4|\} < 1$. The Askey–Wilson polynomials also satisfy the Rodrigues-type formula

$$w(x; t | q) p_n(x; t | q) = \left( \frac{q - 1}{2} \right)^n q^{n(n-1)/4} \mathcal{D}_q^n w(x; q^{n/2} t | q).$$

The Chebyshev polynomials are also special Askey–Wilson polynomials; indeed,

$$p_n(x; q, -q, \sqrt{q}, -\sqrt{q} | q) = (q^{n+2}; q)_n U_n(x),$$

$$p_0(x; t | q) = T_0(x) = 1,$$

$$p_n(x; 1, -1, \sqrt{q}, -\sqrt{q} | q) = 2 (q^n; q)_n T_n(x) \quad \text{for } n > 0.$$

We shall also use the $q$-Taylor expansion stated next.

Theorem 2.2 [Ismail 1995]. Let

$$x_n = (aq^{n/2} + q^{-n/2}/a)/2 \quad \text{for } 0 < q < 1, \; 0 < a < 1,$$
If \( f(x) \) is a polynomial, then
\[
f(x) = \sum_{k=0}^{\infty} f_k (ae^{i\theta}, ae^{-i\theta}; q)_k,
\]
with
\[
f_k = \frac{(q - 1)^k}{(2a)^k (q; q)_k} q^{-k(k-1)/4} (\overline{\varphi}_k^q f)(x_k).
\]
For a proof and details, see [Ismail 2009, Theorem 12.2.2].

3. Connection formulas and expansions

Lemma 3.1. We have the integral evaluation
\[
(3-1) \quad \int_{-1}^{1} (ae^{i\theta}, ae^{-i\theta}; q)_n w(x; t \mid q) \, dx = \frac{2\pi (t_1 a, a/t_1; q)_n (t_1 t_2 t_3 t_4; q)_\infty}{(q; q)_\infty \prod_{1 \leq j < m \leq 4} (t_j t_m; q)_\infty} \, 4\varphi_3 \left( \begin{array}{c} q^{-n}, t_1 t_2, t_1 t_3, t_1 t_4 \\ t_1 a, t_1 t_2 t_3 t_4, q^{1-n} t_1 / a \end{array} \bigg| q, q \right).
\]

This integral can be evaluated by writing
\[
(ae^{i\theta}, ae^{-i\theta}; q)_n = \frac{(ae^{i\theta}, ae^{-i\theta}; q)_\infty}{(aq^n e^{i\theta}, aq^n e^{-i\theta}; q)_\infty},
\]
then using the Nassrallah–Rahman integral (1-2) and the Watson transformation [Gasper and Rahman 2004, (III.18)]. It also follows by expanding \((ae^{i\theta}, ae^{-i\theta}; q)_n\) in \(\{t_1 e^{i\theta}, t_1 e^{-i\theta}; q\}_k : 0 \leq k \leq n\) by using (2-8), and then applying the Askey–Wilson integral (1-1); see also [Ismail and Stanton 1998, Thm. 3].

Our first result is the next expansion of \((be^{i\theta}, be^{-i\theta}; q)_n\) in Askey–Wilson polynomials.

Theorem 3.2.
\[
(3-2) \quad (be^{i\theta}, be^{-i\theta}; q)_n = \sum_{k=0}^{n} f_{n,k}(b, t) p_k(x; t \mid q),
\]
where
\[
(3-3) \quad f_{n,k}(b, t) = \frac{(-b)^k \varphi^{(k)}_n (q; q)_n (b/t_4, bt_4 q^k; q)_{n-k}}{(q, t_1 t_2 t_3 t_4 q^{k-1}; q)_k (q; q)_{n-k}} \times 4\varphi_3 \left( \begin{array}{c} q^{k-n}, t_1 t_4 q^k, t_2 t_4 q^k, t_3 t_4 q^k \\ bt_4 q^k, t_1 t_2 t_3 t_4 q^{2k}, q^{1-n+k} t_4 / b \end{array} \bigg| q, q \right).
\]

Proof. It is clear that
\[
f_{n,k} h_k(t) = \langle p_k(x; t \mid q) w(x; t \mid q), \sqrt{1 - x^2 (be^{i\theta}, be^{-i\theta}; q)_n} \rangle
\]
\[ \frac{1-\theta}{2} = \left( \frac{q-1}{2} \right)^k q^{k(k-1)/4} \left( \sum_q^k w(x; q^{k/2} t \mid q) \right) \]

\[ = \left( \frac{1-q}{2} \right)^k q^{k(k-1)/4} \int_{-1}^1 w(x; q^{k/2} t \mid q) \sum_q^k (be^{i\theta}, be^{-i\theta}; q)_n dx \]

\[ = \frac{(-b)^k (q; q)_n}{(q; q)_{n-k}} q^{(k-3)/2} \int_{-1}^1 (bq^{k/2} e^{i\theta}, bq^{k/2} e^{-i\theta}; q)_{n-k} w(x; q^{k/2} t \mid q) dx. \]

In these steps we used the Rodrigues formula (2-17), as well as (2-13) and (2-10). The result follows from a slight variation of Lemma 3.1.

Our first application of Theorem 3.2 is the connection relation for the Askey–Wilson polynomials.

**Corollary 3.3.** We have the connection relation

\[ p_n(x; b) = \sum_{k=0}^n c_{n,k}(b, a) p_k(x; a), \]

where

\[ c_{n,k}(b, a) = \frac{b_4^{k-n} (b_1 b_2 b_3 b_4 q^{n-1}; q)_k (q, b_1 b_4, b_2 b_4, b_3 b_4; q)_n}{(q; q)_{n-k} (q, a_1 a_2 a_3 a_4 q^{k-1}; q)_k (b_1 b_4, b_2 b_4, b_3 b_4; q)_n} \]

\[ \times q^{k(k-n)} \sum_{j,l\geq0} \frac{(q^{k-n}, b_1 b_2 b_3 b_4 q^{n+k-1}, a_4 b_4 q^k; q)_{j+l} (q; q)_j (q; q)_l}{(b_1 b_4 q^k, b_2 b_4 q^k, b_3 b_4 q^k; q)_{j+l} (q; q)_j (q; q)_l} \]

\[ \times \frac{(a_1 a_4 q^k, a_2 a_4 q^k, a_3 a_4 q^k; q)_l (b_4/a_4; q)_j (b_4/a_4)\, _1} {(a_4 b_4 q^k, a_1 a_2 a_3 a_4 q^{2k}; q)_l} \]

**Proof.** The follows by expanding the left-hand side of (3-4) in the Askey–Wilson basis \{(a_1 e^{i\theta}, a_1 e^{-i\theta}; q)_k\}, then applying Theorem 3.2.

Corollary 3.3 is Theorem 14.4.2 in [Ismail 2009]. When \(a_4 = b_4\), the double series in (3-4) reduces to a \(_5\phi_4\) and we get a result of [Askey and Wilson 1985]. See also [Gasper and Rahman 2004, (7.6.2)–(7.6.3)]. For another proof, see [Ismail and Zhang 2005], which also uses (2-13). Note that, in view of the orthogonality relation (2-16), Corollary 3.3 is equivalent to Theorem 3.2.

The special case \(b = t_3\) of Theorem 3.2 is interesting. The result, after interchanging \(t_1\) and \(t_3\), is

\[ (t_1 e^{i\theta}, t_1 e^{-i\theta}; q)_n = \sum_{k=0}^n \left( \begin{array}{c} n \\ k \end{array} \right) (-1)^k q^{\frac{k(k-1)}{2}} \frac{(t_1 t_2, t_1 t_3, t_1 t_4; q)_n 1 - t_1 t_2 t_3 t_4 q^{2k-1}}{(t_1 t_2, t_1 t_3, t_1 t_4; q)_k 1 - t_1 t_2 t_3 t_4} \]

\[ \times \frac{(t_1 t_2 t_3 t_4; q)_k}{(t_1 t_2 t_3 t_4; q)_{n+k}} p_k(x; t \mid q). \]
Theorem 3.4. The following relations are equivalent:

\begin{equation}
B_n = \frac{(t_1 t_2, t_1 t_3, t_1 t_4; q)_n}{t_1^n} \sum_{k=0}^{n} \frac{(q^{-n}, t_1 t_2 t_3 t_4 q^{n-1}; q)_k}{(q; q)_k \prod_{j=2}^{4} (t_1 t_j; q)_k} q^k A_k
\end{equation}

\begin{equation}
A_n = \sum_{k=0}^{n} t_1^k q^{\binom{k}{2}} (t_1 t_2 q^k, t_1 t_3 q^k, t_1 t_4 q^k; q)_{n-k} B_k \tag{3-8}
\end{equation}

Proof. We set $b = t_1$ in (3-2) and take (2-14) into account. The $4\phi_3$ in (3-3) becomes a $3\phi_2$, and can be summed by the $q$-analogue of the Pfaff–Saalschütz theorem. \qed

Theorem 3.4 is known [Krattenthaler 1989; 1996]. An interesting question is to explore where such inverse pair lives from the point of view of the Möbius function on lattices [Rota 1964], because the lattices which will lead to such a deep result will be very interesting. It is also interesting to explore the concept of Bailey pairs [Andrews 1986] from the Möbius-inversion point of view.

The $q$-ultraspherical polynomials are special Askey–Wilson polynomials, since

\begin{equation}
p_n(x; \sqrt{\beta}, -\sqrt{\beta}, \sqrt{q}\beta, -\sqrt{q}\beta \mid q) = \frac{(q, \beta^2 q^n; q)_n}{(\beta; q)_n} C_n(x; \beta \mid q).
\end{equation}

The $q$-plane wave expansion in $q$-ultraspherical polynomials is

\begin{equation}
\mathcal{E}_q(x; i\alpha) = \frac{(\alpha)^{-v} (q; q)_\infty}{(-q \alpha^2; q^2)_\infty (q^{v+1}; q)_\infty} \sum_{n=0}^{\infty} \frac{(1 - q^{n+v})}{(1 - q^v)} q^{n^2/4} i^n \times J_{v+n}^{(2)}(2\alpha; q) C_n(x; q^v \mid q);
\end{equation}

see [Ismail and Zhang 1994].

Another application of Theorem 3.2 is this generalization of (3-10):

Theorem 3.5. We have the following generalization of the $q$-plane wave expansion function:

\begin{equation}
\mathcal{E}_q(x; \alpha) = \frac{(\alpha^2; q^2)_\infty}{(q \alpha^2; q)_\infty} \sum_{n=0}^{\infty} \frac{\alpha^n q^{n^2/4} p_n(x; t)}{(q, t_1 t_2 t_3 t_4 q^{n-1}; q)_n} \times \sum_{k=0}^{\infty} \frac{(-\alpha/t_4)^k}{(q; q)_k} (-q^{1+n-k} t_4^2, q^2)_k q^{k(k-2n)/4} \times 4\phi_3\left(\begin{array}{c}
q^{-k}, t_1 t_4 q^n, t_2 t_4 q^n, t_3 t_4 q^n
\end{array}; q, q \right).
\end{equation}

Proof. Expand the $\mathcal{E}_q$ in the Askey–Wilson basis via (2-6), then apply (3-2). \qed
Another proof of Theorem 3.5. Since \( \mathcal{E}_q(x; \alpha) \in L_2[-1, 1, w(x; t)] \), we set
\[
\mathcal{E}_q(x; \alpha) = \sum_{n=0}^{\infty} c_n p_n(x; t).
\]
Using (2-17), the divided-difference relation \( \mathcal{D}_q \mathcal{E}_q(x; \alpha) = 2\alpha q^{1/4}/(1-q) \mathcal{E}_q(x; \alpha) \) and the \( q \)-integration by parts (2-13), we find that
\[
c_n h_n(t) = \int_{-1}^{1} \mathcal{E}_q(x; \alpha) p_n(x; t) w(x; t) \, dx
= (\frac{q-1}{2})^n q^{(\frac{3}{2})/2} \int_{-1}^{1} \mathcal{E}_q(x; \alpha) \mathcal{D}_q^n w(x; q^{n/2} t) \, dx
= \alpha^n q^{n^2/4} \int_{-1}^{1} \mathcal{E}_q(x; \alpha) w(x; q^{n/2} t) \, dx
= \frac{(\alpha^2; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \alpha^n q^{n^2/4} \sum_{k=0}^{\infty} (-i\alpha)^k (q; q)_k
\times \int_{0}^{\pi} w(\cos \theta; q^{n/2} t) \left( -i q^{(1-k)/2} e^{i\theta}, -i q^{(1-k)/2} e^{-i\theta}; q \right)_k \sin \theta \, d\theta.
\]
The integral above is
\[
2\pi (-i t_4 q^{(1+n-k)/2}, -i t_4 q^{(1-n-k)/2} / t_4; q)_k (t_1 t_2 t_3 t_4 q^{2n}; q)_{\infty} (q; q)_{\infty} \prod_{1 \leq j < m \leq 4} (t_j t_m q^n; q)_{\infty}
\times _4 \Phi_3 \left( \begin{array}{c}
q^{-k}, -i t_4 q^{n}, t_2 t_4 q^n, t_3 t_4 q^n
\end{array} \right| q, q).
\]
The result now follows from (2-16). \( \square \)

In the case of \( q \)-ultraspherical polynomials, the \( _4 \Phi_3 \) in (3-11) can be summed by Andrews’ \( q \)-analogue of Watson’s \( _3 F_2 \) sum [Gasper and Rahman 2004, (II.17)]. Thus, the \( _4 \Phi_3 \) is zero for \( k \) odd and, when \( k \) is replaced by \( 2k \), the \( _4 \Phi_3 \) is
\[
\beta^{2k} q^{2nk+k} \frac{(q, q^{1-n-2k} / \beta; q^2)_k}{(q^{n+2-2k}, \beta^2 q^{2n+2}; q^2)_k}.
\]
Thus, the \( k \)-sum in (3-11) is \( _2 \Phi_1 (-\beta q^{n+2}, -\beta q^{n+1}; \beta^2 q^{2n+2}; q^2, \alpha^2) \). Therefore,
\[
(3-12) \quad \mathcal{E}_q(x; \alpha) = \frac{(\alpha^2; q^2)_{\infty}}{(q \alpha^2; q^2)_{\infty}} \sum_{n=0}^{\infty} \alpha^n q^{n^2/4}
\times _2 \Phi_1 \left( \begin{array}{c}
-\beta q^{n+2}, -\beta q^{n+1}
\end{array} \right| q^2, \alpha^2 C_n(x; \beta \mid q).
\]
By equating the left sides of (3-12) and (3-10), we establish the identity

\[
J_\nu^{(2)}(2\alpha; q) = \frac{\alpha^\nu(-\alpha^2; q^2)_\infty}{(q^\nu+1; q)_\infty} 2\phi_1\left(\begin{array}{c|c}
-q^{\nu+2}, -q^{\nu+1} \\
q^{2\nu+2}
\end{array}\mid q^2, -\alpha^2\right)
\]

\[
= \frac{\alpha^\nu(q^{\nu+1}\alpha^2; q^2)_\infty}{(q^{\nu+1}; q)_\infty} 2\phi_2\left(\begin{array}{c|c}
-q^{\nu+2}, -q^{\nu+1} \\
q^{2\nu+2}, q^{\nu+2}\alpha^2
\end{array}\mid q^2, q^{\nu+1}\alpha^2\right),
\]

after applying the \(2\phi_1\) to \(2\phi_2\) transformation [Gasper and Rahman 2004, (III.4)]. The representation of \(J\) after applying the \(\phi\)-continuous \(q\)-Jacobi polynomials, \(t_2 = t_1q^{1/2}\) and \(t_4 = t_3q^{1/2}\), yielding a result in [Ismail et al. 1996]. The details however are not lengthy and will be omitted.

4. Expansions of \(x^n\) and \((1 \pm x)^\rho\)

**Theorem 4.1.** The expansion

\[
(4-1) \quad (1 - x)^\rho = \frac{4}{\sqrt{\pi}} 2^\rho \Gamma(\rho + 3/2)
\times \sum_{k=0}^{\infty} \frac{1 - \beta q^k}{1 - \beta} \left(\sum_{j=0}^{\infty} \frac{(k+2j+1)(-\rho)_{k+2j} \beta^j(q/\beta; q)_{j} (q; q)_{k+j}}{(q, q)_j (q^2, q)_{k+j} \Gamma(k+2j+\rho+3)}\right) C_k(x; \beta \mid q)
\]

holds for \(-1 < x < 1\), \(\rho > -1\) and \(\beta \in (0, 1)\). The expansion for \((1 + x)^\rho\) is similar, since \(C_n(-x; \beta \mid q) = (-1)^n C_n(x; \beta \mid q)\).

**Proof.** Apply (2-2) with \(\nu = 1\), then expand \(C_k^1(x) = U_k(x) = C_k(x; q \mid q)\) in \(C_j(x; \beta \mid q)\) by using (2-5), then rearrange the series. The expansion (2-2) holds for \(\rho > -1\). The rearrangement is valid because the double series in the theorem converges absolutely for \(\rho > -1\), in view of the asymptotic formula [Ismail 2009, (13.4.5)] and the well-known fact that \(n^{b-a} \Gamma(n+a)/\Gamma(n+b) \to 1\) as \(n \to +\infty\). \(\Box\)

It is interesting to note that, as \(q \to 1\), the expansion (4-1) should reduce to (2-2). Indeed with \(\beta = q^\nu\) the \(q \to 1\) limit of the quantity in square brackets is a well-poised \(5F_4\) at \(x = 1\), which can be summed, see Slater [Slater 1966, (III.12)]. So we could have discovered the abovementioned sum if it was not already known.

**Theorem 4.2.** For nonnegative integers \(n\) we have the \(q\)-ultraspherical expansion

\[
(4-2) \quad x^n = \frac{n!}{2^n} \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{1 - \beta q^{n-2m}}{1 - \beta} C_{n-2m}(x; \beta \mid q)
\times \sum_{k=0}^{m} \frac{n + 1 - 2k}{k!(n + 1 - k)!} \frac{\beta^{m-k}(q/\beta; q)_{m-k}(q; q)_{n-m-k}}{(q; q)_{m-k}(q^2; q)_{n-m-k}}.
\]
Proof. The expansion (4-2) follows immediately from letting \( \nu = 1 \) in (2-1) then use (2-5) with \( \gamma = 1 \).

Note that
\[
\frac{n!(n+1-2k)}{k!(n+1-k)!} = \binom{n}{k} - \binom{n}{k-1}.
\]

With \( \beta = q^\nu \), the limit of the \( k \)-sum in (4-2) as \( q \to 1 \) is a very well-poised \( {}_4F_3 \) at \( x = -1 \), which can be summed [Slater 1966, (III.11)].

5. Two bibasic integrals

In this section we give evaluations of the integral (5-2) and the more general integral (5-3). The proof uses the bibasic expansion

(5-1) \[
\frac{(q, qa^2; q)_\infty}{(qae^{i\theta}, qae^{-i\theta}; q)_\infty} (be^{i\theta}, be^{-i\theta}; p)_\infty \quad = \quad \sum_{k=0}^{\infty} \frac{1-a^2 q^{2k}}{1-a^2} \frac{(a^2, a^2 e^{i\theta}, ae^{-i\theta}; q)_k}{(q, qae^{i\theta}, aqe^{-i\theta}; q)_k} (-1)^k q^{(k+1)\frac{1}{2}} (abq^k, bq^{-k}/a; p)_\infty,
\]

which is valid for \( 0 < p < q \), or \( p = q \) and \( |b| < |a| \) [Ismail and Stanton 2003].

Theorem 5.1. We have the bibasic integral evaluation

(5-2) \[
\int_0^\pi \frac{e^{2i\theta} - e^{-2i\theta}}{\prod_{j=1}^{\infty} (a_j e^{i\theta}, a_j e^{-i\theta}; q)_\infty} d\theta \quad = \quad \frac{2\pi (a_2 a_3 a_4 a_5/q; q)_\infty}{(q; q)_\infty \prod_{2 \leq r < s \leq 5} (a_r a_s; q)_\infty} \frac{1}{(q, qa^2_1; q)_\infty} \times \\
\times 8 W_7 \left( a_1^2 q^{2k+1}; q, q^{k+1} a_1/a_2, q^{k+1} a_1/a_3, q^{k+1} a_1/a_4, q^{k+1} a_1/a_5; q, a_2 a_3 a_4 a_5/q \right) \quad = \quad \frac{2\pi \prod_{j=2}^{\infty} (a_1 a_2 a_3 a_4 a_5/a_j; q)_\infty}{(q, a_1^2 a_2 a_3 a_4 a_5; q)_\infty \prod_{1 \leq r < s \leq 5} (a_r a_s; q)_\infty} \times \\
\times \sum_{k=0}^{\infty} \frac{(a^2_1; q)_k (a_1^2 a_2 a_3 a_4 a_5; q)_{2k}}{(q; q)_k (a^2_1; q)_{2k}} \times \\
\times \prod_{j=2}^{5} \frac{(a_1 a_j; q)_k}{(a_1 a_2 a_3 a_4 a_5/a_j; q)_k} (-1)^k q^{(k+1)\frac{1}{2}} \left( abq^k, bq^{-k}/a; p \right)_\infty \times \\
\times 8 W_7 \left( a_1^2 a_2 a_3 a_4 a_5 q^{-2k-1}; a_1 a_2 q^k, a_1 a_3 q^k, a_1 a_4 q^k, a_1 a_5 q^k, a_2 a_3 a_4 a_5; q, q \right).
\]
Proof. In view of (5-1), the left-hand side of (5-2) is

\[
\frac{1}{(q, qa_1^2; q)\infty} \sum_{k=0}^{\infty} \frac{1 - a_1^2 q^{2k}}{1 - a_1^2} \frac{(a_1^2; q)_k}{(q; q)_k} (-1)^k q^{\ell+1} (a_1 b q^k, bq^{-k}/a_1; p)\infty
\]

\[
\times \int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q)\infty (a_1 q^{k+1} e^{i\theta}, a_1 q^{k+1} e^{-i\theta}; q)\infty}{(a_1 q^k e^{i\theta}, a_1 q^k e^{-i\theta}; q)\infty \prod_{j=2}^5 (a_j e^{i\theta}, a_j e^{-i\theta}; q)\infty} d\theta.
\]

The first equality in (5-2) follows from (1-2). The second equality follows from the form of the Nassrallah–Rahman integral stated in [Gasper and Rahman 2004, (6.3.7)] with

\[f = a_1 q^k.\]

When \(p = q\), Theorem 5.1 should reduce to the Nassrallah–Rahman integral (1-2). This is not obvious, so we will indicate how it works. When \(p = q\),

\[(-1)^k q^{\ell+1} (a_1 b q^k, bq^{-k}/a_1; p)\infty = (a_1 b, b/a_1; q)\infty \frac{b^k (qa_1/b; q)_k}{a^k (a_1 b; q)_k}.
\]

We use the second equation in (5-2) and write the \(8W_7\) as a sum over \(j\). With \(\ell = j + k\), the left-hand side of (5-2) becomes

\[
\frac{2\pi (a_1 b, b/a_1; q)\infty \prod_{s=2}^5 (a_1 a_2 a_3 a_4 a_5/a_s; q)\infty}{(q, a_1^2 a_2 a_3 a_4 a_5; q)\infty \prod_{1 \leq r < s \leq 5} (a_r a_s; q)\infty}
\]

\[
\times \sum_{\ell=0}^{\infty} \frac{1 - a_1^2 a_2 a_3 a_4 a_5 q^{2\ell-1}}{1 - a_1^2 a_2 a_3 a_4 a_5/q} \frac{(a_2 a_3 a_4 a_5/q, a_1^2 a_2 a_3 a_4 a_5/q; q)_\ell}{(q, qa_1^2; q)_\ell}
\]

\[
\times \prod_{r=2}^5 \frac{(a_1 a_r; q)_\ell}{(a_1 a_2 a_3 a_4 a_5/a_r; q)_\ell}
\]

\[
\times 6W_5(q^2; qa_1/b, a_1^2 a_2 a_3 a_4 a_5 q^{\ell-1}, q^{-\ell}; q, qb/a_1^2 a_2 a_3 a_4 a_5).
\]

The \(6W_5\) can be summed by [Gasper and Rahman 2004, (II.20)], and the expression above reduces to the integral evaluation [Gasper and Rahman 2004, (6.3.7)].

The next theorem generalizes the evaluation of the moments of the Askey–Wilson weight function.

**Theorem 5.2.** We have the integral evaluation

\[
(5-3) \int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q)\infty (be^{i\theta}, be^{-i\theta}; q)_n}{\prod_{j=1}^4 (a_j e^{i\theta}, a_j e^{-i\theta}; q)\infty} d\theta
\]
So, we have

\[
\frac{2\pi (a_1a_2a_3_4; q) \infty}{(q; q) \infty \prod_{1 \leq j < k \leq 4} (a_j a_k; q) \infty (q, qa_1^2, a_1a_2a_3a_4; q)_n}
\times \sum_{k=0}^{n} \frac{1 - a_1^2 q^{2k}}{1 - a_1^2} \frac{(a_1^2, q^{-n}; q)_k}{(q, a_1^2 q^{n+1}; q)_k} \frac{(a_1 b q^k, b q^{-k}/a_1; p)_n}{(a_1 b q^k, b q^{-k}/a_1; p)_n}
\times q^{k(n+1)} \frac{(1 - a_1 a_3)(1 - a_1 a_4)}{(1 - a_1 a_3 q^k)(1 - a_1 a_4 q^k)}
\times 4\phi_3 \left( \begin{array}{l}
q^{-k-n}, a_3 a_4, a_1 q^{k+1}/a_2 \\
\end{array} \right)
\left( \begin{array}{l}
a_1 a_3 q^{k+1}, a_1 a_4 q^{k+1}, q^{-1-n}/a_1 a_2 \\
\end{array} \right) q, q \right).
\]

**Proof.** Observe that

\[
(ab q^k, b q^{-k}/a; p)_n
= (ab; p)_n \prod_{j=0}^{n-1} (1 - ap^{-j}/b) q^{-kn}( -b/a) \prod_{j=0}^{n-1} \frac{(ab p^j; q)_{k} (a q p^{-j}/b; q)_{k}}{(ab p^j; q)_{k} (a q p^{-j}/b; q)_{k}}
= (ab, b/a; p)_n q^{-kn} \prod_{j=0}^{n-1} \frac{(ab p^j; q)_{k} (a q p^{-j}/b; q)_{k}}{(a q p^{-j}/b, ab p^j; q)_{k}}.
\]

Hence,

\[
\sum_{k=0}^{n} \frac{1 - a_1^2 q^{2k}}{1 - a_1^2} \frac{(a_1^2, a_1 e^{i \theta}, a_1 e^{-i \theta}, q^{-n}; q)_k}{(q, qa_1 e^{-i \theta}, qa_1 e^{i \theta}, a_1^2 q^{n+1}; q)_k} q^{k(n+1)} \frac{(a_1 b q^k, b q^{-k}/a_1; p)_n}{(a_1 b q^k, b q^{-k}/a_1; p)_n}
\times \prod_{j=0}^{n-1} \frac{(a_1 p^{-j}/b, qa_1 b p^j; q)_k}{(a_1 b p^j, a_1 p^{-j}/b; q)_k}
= (a_1 b, b/a_1; p)_n \frac{(qa_1^2, q; q)_n (be^{i \theta}, be^{-i \theta}; p)_n}{(qa_1 e^{i \theta}, qa_1 e^{-i \theta}; q)_n (a_1 b, b/a_1; p)_n}
= \frac{(q, qa_1^2; q)_n (be^{i \theta}, be^{-i \theta}; p)_n}{(qa_1 e^{i \theta}, qa_1 e^{-i \theta}; q)_n}.
\]

So, we have

\[
\frac{(q, qa_1^2; q)_n (be^{i \theta}, be^{-i \theta}; p)_n}{(qa_1 e^{i \theta}, qa_1 e^{-i \theta}; q)_n}
= \sum_{k=0}^{n} \frac{1 - a_1^2 q^{2k}}{1 - a_1^2} \frac{(a_1^2, a_1 e^{i \theta}, a_1 e^{-i \theta}, q^{-n}; q)_k}{(q, qa_1 e^{-i \theta}, qa_1 e^{i \theta}, a_1^2 q^{n+1}; q)_k} q^{k(n+1)} \frac{(a_1 b q^k, b q^{-k}/a_1; p)_n}{(a_1 b q^k, b q^{-k}/a_1; p)_n}.
\]
Therefore,
\[
\int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty (be^{i\theta}, be^{-i\theta}; p)_n}{\prod_{j=1}^4(a_j e^{i\theta}, a_j e^{-i\theta}; q)_\infty} d\theta
\]
\[
= \frac{1}{(q, qa^2_1; q)_n} \sum_{k=0}^{n} \frac{1 - a^2_1 q^{2k}}{1 - a^2_1} \frac{(a^2_1, q^{-n}; q)_k}{(q, a^2_1 q^{n+1}; q)_k} q^{k(n+1)} (a_1 b q^k, b q^{-k}/a_1; p)_n
\]
\[
\times \int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(a_1 q^{n+1} e^{i\theta}, a_1 q^{n+1} e^{-i\theta}, a_1 q^k e^{i\theta}, a_1 q^k e^{-i\theta}; q)_\infty} \times \frac{(a_1 q^{k+1} e^{i\theta}, a_1 q^{k+1} e^{-i\theta}; q)_\infty}{\prod_{j=2}^4(a_j e^{i\theta}, a_j e^{-i\theta}; q)_\infty} d\theta
\]

Using [Gasper and Rahman 2004, (6.3.8)] and Watson’s formula [Gasper and Rahman 2004, (III.18)], the integral in the equation above becomes
\[
\frac{2\pi (a_1 a_2 a_3 a_4; q)_\infty}{(q; q)_\infty \prod_{1 \leq j < k \leq 4}(a_j a_k; q)_\infty} \frac{(a_1 a_2; q)_n (a_1 a_3, a_1 a_4; q)_{n+1}}{(a_1 a_2 a_3 a_4; q)_n} \times \frac{1}{(1 - a_1 a_3 q^k)(1 - a_1 a_4 q^k)} \phi_3 \left( q^{k-n}; q, a_3 a_4, a_1 q^{k+1}/a_2 \left| q, q \right. \right).
\]

This completes the proof.

We give a second proof of (5-3) because it has an idea which may be useful in other cases. The second proof uses the following recent result of [Ismail and Stanton 2010]:

(5-4)
\[
\frac{(q, qa^2; q)_n}{(q a e^{i\theta}, q a e^{-i\theta}; q)_n} (be^{i\theta}, be^{-i\theta}; p)_n
\]
\[
= \sum_{k=0}^{n} \frac{1 - a^2 q^{2k}}{1 - a^2} \frac{(q^{-n}, a^2, a e^{i\theta}, a e^{-i\theta}; q)_k}{(q, a^2 q^{n+1}, a q e^{i\theta}, a q e^{-i\theta}; q)_k} q^{k(1+n)} (ab q^k, b q^{-k}/a; p)_n.
\]

Second proof of Theorem 5.2. In view of (5-4), the left-hand side of (5-3) is
\[
\frac{1}{(q, qa^2_1; q)_n} \sum_{k=0}^{n} \frac{1 - a^2 q^{2k}}{1 - a^2} \frac{(q^{-n}, a^2_1; q)_k}{(q, a^2_1 q^{n+1}; q)_k} q^{k(1+n)} (a_1 b q^k, b q^{-k}/a_1; p)_n
\]
\[
\times \int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(a_1 q^{k+1} e^{i\theta}, a_1 q^{k+1} e^{-i\theta}; q)_\infty \prod_{j=2}^4(a_j e^{i\theta}, a_j e^{-i\theta}; q)_\infty} d\theta.
\]
This integral can be evaluated by (3-1) and equals

\[
\frac{2\pi (a_1^2 q^{2k+1}, q; q)_{n-k} (q^k a_1 a_2 a_3 a_4; q)_{\infty}}{(q; q)_{\infty} \prod_{j=2}^{4} (q^k a_1 a_j; q)_{\infty} \prod_{2 \leq r < s \leq 4} (a_r a_s; q)_{\infty}}
\times 4\phi_3 \left( \begin{array}{c} q^{k-n}, a_1 a_2 q^k, a_1 a_3 q^k, a_1 a_4 q^k \\ a_1^2 q^{2k+1}, a_1 a_2 a_3 a_4 q^k, q^{-n} \end{array} \right| q, q \right)
\]

The application of the iterated Sears transformation [Gasper and Rahman 2004, (III.16)] reduces \(4\phi_3\) to

\[
\frac{(a_1 a_2 q^k, a_1 a_3 q^{k+1}, a_1 a_4 q^{k+1}; q)_{n-k}}{(a_1^2 q^{2k+1}, a_1 a_2 a_3 a_4 q^k, q; q)_{n-k}}
\]
times the \(4\phi_3\) in (5-3). Simple manipulations now establish (5-3). \(\square\)

Let \(p = 1\) and \(\zeta = \frac{1}{2} (b + 1/b)\). Then,

\[
\int_{0}^{\pi} \frac{(e^{2i\theta}, e^{-2i\theta}; q)_{\infty}}{\prod_{j=1}^{4} (a_j e^{i\theta}, a_j e^{-i\theta}; q)_{\infty}} (\cos \theta - \zeta)^n d\theta
\]

\[
= \frac{2\pi (a_1 a_2 a_3 a_4; q)_{\infty}}{(q; q)_{\infty} \prod_{1 \leq j < k \leq 4} (a_j a_k; q)_{\infty}} \frac{(a_1 a_2, qa_1 a_3, qa_1 a_4; q)_n}{(q, qa_1^2, a_1 a_2 a_3 a_4; q)_n}
\times \sum_{k=0}^{n} \frac{1 - a_1^2 q^{2k}}{1 - a_1^2} \frac{(a_1^2, q^{-n}; q)_k}{(q, a_1^2 q^{n+1}; q)_k} \left( \frac{1}{2} (a_1 q^k + q^{-k} / a_1) - \zeta \right)^n
\times q^k \frac{(1 - a_1 a_3)(1 - a_1 a_4)}{(1 - a_1 a_3 q^k)(1 - a_1 a_4 q^k)}
\times 4\phi_3 \left( \begin{array}{c} q^{k-n}, q, a_3 a_4, a_1 q^{k+1} / a_2 \\ a_1 a_3 q^{k+1}, a_1 a_4 q^{k+1}, q^{1-n} / a_1 a_2 \end{array} \right| q, q \right).
\]

The special case \(\zeta = 0\) gives the Askey–Wilson moments

\[
\int_{-1}^{1} W(x; a) x^n dx = \frac{(a_1 a_2, qa_1 a_3, qa_1 a_4; q)_n}{(2a_1)^n (q, qa_1^2, a_1 a_2 a_3 a_4; q)_n}
\times \sum_{k=0}^{n} \frac{1 - a_1^2 q^{2k}}{1 - a_1^2} \frac{(a_1^2, q^{-n}; q)_k}{(q, a_1^2 q^{n+1}; q)_k} \left( 1 + a_1^2 q^{2k} \right)^n
\times q^{k(n+1)} \frac{(1 - a_1 a_3)(1 - a_1 a_4)}{(1 - a_1 a_3 q^k)(1 - a_1 a_4 q^k)}
\times 4\phi_3 \left( \begin{array}{c} q^{k-n}, q, a_3 a_4, a_1 q^{k+1} / a_2 \\ a_1 a_3 q^{k+1}, a_1 a_4 q^{k+1}, q^{1-n} / a_1 a_2 \end{array} \right| q, q \right),
\]
where \( W \) is the normalized weight function

\[
W(x; \mathbf{a}) := \frac{(q; q)_\infty \prod_{1 \leq r < s \leq 4} (a_r a_s; q)_\infty}{2\pi (a_1 a_2 a_3 a_4; q)_\infty} w(x; \mathbf{a}).
\]

The moments of the Askey–Wilson weight functions were first computed in the very interesting paper [Corteel and Williams 2007]. Corteel and Williams used purely combinatorial techniques and showed that the moments of the Askey–Wilson weight is a generating function for purely combinatorial objects. The Corteel-Williams formula is very different in nature from our (5-6), and a very interesting but difficult exercise is to show the equivalence of the two results.

6. The Andrews identities

We now prove both (1-4) and (1-5) using the \( 5\phi_4 \) to \( 12\phi_{11} \) transformation [Gasper and Rahman 2004, (2.8.4)].

**Proof of (1-4).** The limiting case \( e \to 0 \) of the \( 5\phi_4 \) to \( 12\phi_{11} \) transformation (2.8.4) of [Gasper and Rahman 2004] is

\[
4\phi_3 \left( \begin{array}{l}
q^{-n}, b, c, d \\
q^{1-n} q^{-n} q^{1-n}
\end{array} \right) \Bigg| q, q \Bigg) = \frac{(\lambda^2 q^{n+1}; q)_n}{(q; q)_n} \frac{(\lambda q^n)^{-n}}{(\lambda q)^n}
\]

\[
\times 10\phi_9 \left( \begin{array}{l}
\lambda, q \sqrt{\lambda}, -q \sqrt{\lambda}, \lambda b q^n, \lambda c q^n, \lambda d q^n, q^{-\frac{n}{2}}, -q^{-\frac{n}{2}}, q^{\frac{1-n}{2}}, -q^{\frac{1-n}{2}} \\
q^{-\frac{n}{2}}, q^{\frac{1-n}{2}}, q^{1-n}, q^{-1-n}, q^{\frac{1+n}{2}}, -q^{\frac{1+n}{2}}
\end{array} \right) \left| q, \lambda q^{n+1} \right),
\]

where \( bcd \lambda = q^{1-2n} \). Thus, the \( 4\phi_3 \) above is

\[
\frac{(\lambda q^n)^{-n}}{(\lambda q; q)_n} \sum_{k=0}^{[n/2]} \frac{1 - \lambda q^{2k}}{1 - \lambda} \frac{(q^{-n}; q)_2^{2k} (\lambda, \lambda b q^n, \lambda c q^n, \lambda d q^n; q)_k}{(q, q^{1-n}/b, q^{1-n}/c, q^{1-n}/d; q)_k}
\]

\[
\times \frac{(\lambda^2 q^{n+1}; q)_n}{(\lambda^2 q^{n+1}; q)_2^{2k}} \frac{(\lambda q^{n+1})^k}{(\lambda q^{n+1})^k}
\]

\[
= \frac{(\lambda q^n)^{-n}}{(\lambda q; q)_n} \sum_{k=0}^{[n/2]} \frac{1 - \lambda q^{2k}}{1 - \lambda} \frac{(q^{-n}; q)_2^{2k} (\lambda, \lambda b q^n, \lambda c q^n, \lambda d q^n; q)_k}{(q, q^{1-n}/b, q^{1-n}/c, q^{1-n}/d; q)_k}
\]

\[
\times (\lambda^2 q^{n+1+2k}; q)_{n-2k} (\lambda q^{n+1})^k,
\]

since \((a; q)_n/(a; q)_j = (aq^j; q)_{n-j}\). In the case of (1-4), we replace \( q \) by \( q^2 \), then replace \( b, c \) and \( d \) by \( a, b \) and \( q^{1-2n}/ab \), respectively. These choices make \( \lambda = q^{1-2n} \). Hence, the \( 4\phi_3 \) in (1-4) transforms to
\[(6-1) \quad \frac{q^n}{(q^{3-2n}; q^2)_n} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1 - q^{1-2n+4k}}{1 - q^{1-2n}} (q^{1-2n}, qa, qb, q^{2-2n}/ab; q^2)_k \]

\[
\times q^k (q^{4-2n+4k}; q^2)_{n-2k} (q^{2-2n}; q^2)_{2k}.
\]

Since \((q^{4-2n+2k}; q^2)_{n-2k} = q^{-2(n-2k)} (-q^2)^{n-2k} (q^{2-2k}; q^2)_{n-2k}\) by [Gasper and Rahman 2004, (I.8)], we find that the summand of the series above vanishes, unless \(0 \leq n - 2k \leq 1\), which implies that the only nonvanishing term is when \(k = \lfloor n/2 \rfloor\). Computing and simplifying this last term gives the right-hand side of (1-4). □

**Proof of (1-5).** The use of the easily verifiable identity

\[
\frac{1 - abq^{-1}}{1 - ab^2q^{2n-3}} (b, q^{3-2n}/ab; q^2)_k - ab^2q^{2n-3} \frac{(b, q^{3-2n}/ab; q^2)_k}{(q^{2-2n}/b, qab; q^2)_k} = \frac{(b, q^{2-2n}/ab; q^2)_k}{(q^{4-2n}/b, qab; q^2)_k},
\]

gives

\[
(6-2) \quad 4\phi_3\left( q^{2n}, a, b, q^{3-2n}/ab \left| q^2, q^2 \right. \right) = \frac{1 - abq^{-1}}{1 - ab^2q^{2n-3}} 4\phi_3\left( q^{2n}/a, q^{4-2n}/b, qab \left| q^2, q^2 \right. \right) - ab^2q^{2n-3} 4\phi_3\left( q^{2n}/a, q^{2-2n}/b, qab \left| q^2, q^2 \right. \right),
\]

yielding two balanced and nearly-poised series of the second kind on the right-hand side. Now we use the Watson transformation formula [Gasper and Rahman 2004, (III.18)] to transform the right-hand side of into \(8\phi_7\) series. Thus

\[
4\phi_3\left( q^{2n}, a, b, q^{3-2n}/ab \left| q^2, q^2 \right. \right) = \frac{1 - abq^{-1}}{1 - ab^2q^{2n-3}} \frac{(a/q, b/q; q^2)_n}{(1/q, ab/q; q^2)_n} \times 8\phi_7\left( \begin{array}{c}
q^{1-2n}, q^{-n+5/2}, -q^{-n+5/2}, q^{-2n}, qa, -b, \frac{b}{q}, q \\
q^{-n+1/2}, q^{3-2n}, q^{2-2n}, q^{4-2n}, q^{3-2n}/a, q^{2-2n}/b, q^{3-2n}/b, b
\end{array} \right) \left( q^2, q^{6-2n}/a^2b^2 \right)
\]

\[
- ab^2q^{2n-3} \frac{(b/q, ab; q^2)_n}{(1/q, a; q^2)_n} 8\phi_7\left( \begin{array}{c}
q^{1-2n}, q^{-n+5/2}, -q^{-n+5/2}, q^{-2n}, b, q, \frac{q^{2-2n}}{ab}, \frac{q^{3-2n}}{ab} \left| q^2, q^2 \right. \right) \left( q^{-n+1/2}, q^{3-2n}/b, qab, ab \right) \left( q^{2}, a^2 \right).
\]
The crucial formula to use now is the quadratic transformation formula [Gasper and Rahman 2004, (3.5.10)], that after some simplification, gives

\[
4\phi_3 \left( \frac{q^{-2n}, a, b, q^{3-2n}/ab}{q^{2-2n}/a, q^{4-2n}/b, qab} \right) = \frac{(q^{2-2n}; q)_{2n}}{(q^{2-2n}/a; q)_{2n}} \frac{(abq^{n-2}, -abq^{n-2}; q)_n}{(bq^{n-2}, -bq^{n-2}; q)_n} \frac{1 - abq^{-1}}{1 - ab^2q^{2n-3}} \\
\times \sum_{k=0}^{n} \frac{1 + q^{2-2n+2k}/b}{1 + q^{2-2n}/b} \frac{(-q^{2-2n}/b, q^{2-n}/b, -q^{2-n}/b, a; q)_k}{(q, q^{3-n}/b, -q^{3-n}/b, -q^{3-2n}/a; q)_k} \frac{q^{2k}(q^{2n}; q^2)_k}{a^k(q^{2-2n}; q^2)_k} \frac{abq^{3n-3}}{(1/q, a, a^2b^2q^2; q^2)_n (q^{2-2n}/b; q)_{2n}}
\]

However, in each of the two series above there is the common factor

\[
\frac{(q^{2-2n}; q)_{2n}(q^{-2n}; q^2)_k}{(q^{2-2n}; q^2)_k} = (q^{2-2n+2k}; q^2)_{n-k} (q^{-2n}; q^2)_k (q^{3-2n}; q^2)_n,
\]

which vanishes unless \( k = n \). So, the only term that survives in each is the one term with \( k = n \). Combining the two terms after a lot of messy but straightforward simplifications, we obtain (1-5).

\[\square\]

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MOURAD E. H. ISMAIL
DEPARTMENT OF MATHEMATICS
CITY UNIVERSITY OF HONG KONG
TAT CHEE AVENUE
KOWLOON
HONG KONG

and

DEPARTMENT OF MATHEMATICS
KING SAUD UNIVERSITY
RIYADH
SAUDI ARABIA

ismail@math.ucf.edu

MIZAN RAHMAN
DEPARTMENT OF MATHEMATICS
CARLETON UNIVERSITY
OTTAWA, ONTARIO K1S5B6
CANADA

mrahman@math.carleton.ca
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