Pacific Journal of Mathematics

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Volume 252 No. 2 August 2011

PACIFIC JOURNAL OF MATHEMATICS
Vol. 252, No. 2, 2011

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Let I be an ideal of an integral domain T, let $\varphi : T \to T/I$ be the projection, let D be an integral domain contained in T/I, and let $R = \varphi^{-1}(D)$. We characterize when R is an almost Prüfer v-multiplication domain, an almost valuation domain, and an almost Prüfer domain, in the context of pullbacks.

1. Introduction

Let I be an ideal of an integral domain T, let $\varphi : T \to T/I$ be the natural projection, let D be an integral domain contained in T/I, and let $k = qf(D)$ be the quotient field of D. Let $R = \varphi^{-1}(D)$ be the integral domain arising from the following pullback of canonical homomorphisms:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/I \end{array}$$

It is well-known that $D = R/I$ and that I is a prime ideal of R. Notice that I is a common ideal of R and T, and hence T is an overring of R. We assume that R is properly contained in T, and we refer to this as a pullback diagram of type $(\triangle)$. For the diagram $(\triangle)$, if $qf(D) \subseteq T/I$, then we refer to this as a diagram of type $(\triangle')$. For the diagram $(\triangle)$, if I is a prime ideal of T and $qf(D) = qf(T/I)$, then we refer to this as a diagram of type $(\triangle^*)$. Here $qf(T/I)$ denotes the quotient field of T/I. For the diagram $(\triangle)$, if $I = M$ is a maximal ideal of T, we refer to this as a diagram of type $(\triangle_M)$. For the diagram $(\triangle_M)$, if $qf(D) = T/M$, then we refer to this as a diagram of type $(\triangle_M^*)$.

Pullbacks are an important tool in constructing interesting examples and counterexamples. They have become so important that in recent years there have been many papers devoted to ring- and ideal-theoretic properties in pullback domains.

The author is supported by the National Natural Science Foundation of China (Grant No. 11171240) and Fundamental Research Funds for the Central Universities, Southwest University for Nationalities (Grant No. 11NZYQN24).
MSC2000: 13G05, 13A15.
Keywords: pullback, almost Prüfer v-multiplication domain, almost valuation domain, almost Prüfer domain.

For more details on pullbacks, see [Mimouni 2004; Houston and Taylor 2007; Fontana and Gabelli 1996; Gilmer 1972; Gabelli and Houston 1997].

Zafrullah [1985] began a general theory of almost factoriality and introduced the notion of an almost GCD-domain. Zafrullah defined R to be an almost GCD-domain (AGCD-domain for short) if for each $a, b \in D \setminus \{0\}$, there is a positive integer $n = n(a, b)$ such that $a^n D \cap b^n D$ is principal (or equivalently, $(a^n, b^n)_v$ is principal). Anderson and Zafrullah [AZ 1991] introduced several classes of integral domains related to almost GCD-domains, including almost Bézout domains (AB-domains), almost Prüfer domains (AP-domains), and almost valuation domains (AV-domains). As in [AZ 1991], an integral domain R is an AB-domain if for each $a, b \in D \setminus \{0\}$, there is a positive integer $n = n(a, b)$ such that (a^n, b^n) is principal; while R is an AP-domain if for each $a, b \in D \setminus \{0\}$, there is a positive integer $n = n(a, b)$ such that (a^n, b^n) is invertible. Following [AZ 1991], an integral domain R is said to be an AV-domain if for each $a, b \in D \setminus \{0\}$, there is a positive integer $n = n(a, b)$ such that $a^n | b^n$ or $b^n | a^n$. Similarly, in [Li 2012] we defined an integral domain R to be an almost Prüfer v-multiplication domain (APVMD) if for each $a, b \in R \setminus \{0\}$, there is a positive integer $n = n(a, b)$ such that $a^n D \cap b^n D$ is t-invertible, or equivalently, (a^n, b^n) is t-invertible. Recall that an integral domain R is said to be a Prüfer v-multiplication domain (PVMD) if each $a, b \in R \setminus \{0\}$, (a, b) is t-invertible. The class of APVMDs includes a lot of important rings, such as AV-domains, AB-domains, AGCD-domains, AP-domains, PVMDs, and so on.

Anderson and Zafrullah [1991, Theorem 4.9] proved that D is an AB-domain (respectively, AP-domain) if and only if $R = D + Xk[X]$ is an AB-domain (respectively, AP-domain). However, we notice that the $(D + Xk[X])$-construction is a special case of the pullback of type $(\triangle_M)$. Mimouni [2004, Theorem 2.2] generalized these results and proved that for the diagram $(\triangle_M)$, R is an AP-domain if and only if T and D are AP-domains and the extension $k \subseteq T/M$ is a root extension. He also gave a similar characterization for AV-domains. Mimouni [2004, Corollary 2.6] continued to show that for the diagram $(\triangle_M)$, assuming that $D = k$ is a field, then R is an AB-domain if and only if T is an AB-domain and the extension $k \subseteq T/M$ is a root extension. In [Li 2012, Theorem 3.10], we proved that D is an APVMD if and only if $R = D + Xk[X]$ is an APVMD.

From this we notice that the characterization of AV-domains and AP-domains is known only in the context of the special pullback of type $(\triangle_M)$, and that the study of APVMDs is only in the $(D + Xk[X])$-construction, a special case of type $(\triangle_M)$. So the main purpose of this paper is to characterize APVMDs in pullbacks in greater generality and to generalize the characterization of AV-domains and AP-domains for the pullback of type $(\triangle_M)$ to that for the pullback of type $(\triangle')$.

In Section 2, we mainly prove that in the pullback of type $(\triangle_M)$, R is an APVMD if and only if D and T are APVMDs, T_M is an AV-domain, and the extension

$qf(D) \subseteq T/M$ is a root extension. Using this fact, we give Example 2.2 to show that an APVMD is not necessarily a PVMD. We also show that for the diagram $(\triangle_M^*)$, R is an APVMD if and only if D and T are APVMDs and T_M is an AV-domain. Using this result, we prove that D is an APVMD if and only if $R = D + Xk[\![X]\!]$ is an APVMD.

In Section 3, we mainly indicate that for the diagram $(\triangle')$, if T is an AV-domain, then R is an APVMD if and only if D is an APVMD and the extension $qf(D) \subseteq T/I$ is a root extension. We prove that for the diagram $(\triangle')$, R is an AV-domain if and only if T and D are AV-domains and the extension $k = qf(D) \subseteq T/I$ is a root extension. We also show that for the diagram $(\triangle')$, assuming that T is an AV-domain, then R is an AP-domain if and only if D is an AP-domain and the extension $k = qf(D) \subseteq T/I$ is a root extension.

Following [Zafrullah 1988, p. 95], assume that D is the ring of entire functions and S is the multiplicative set generated by the principal primes of D; then D is integrally closed, and hence $R = D + XD_S[X]$ is integrally closed, but $R = D + XD_S[X]$ is not a PVMD. Because an integrally closed APVMD is a PVMD by [Li 2012, Theorem 2.4], R is not an APVMD. Consider the following pullback:

$$\begin{array}{ccc} R = D + XD_S[X] & \longrightarrow & D \\ \downarrow & & \downarrow \\ T = D_S[X] & \longrightarrow & D_S \cong T/I \end{array}$$

Here I denotes $XD_S[X]$. The example indicates that $qf(D) = qf(T/I)$, D and T are APVMDs, I is principal in T, and $T = D_S[X]$ is a PVMD. It follows that T_I is an AV-domain by [Li 2012, Theorem 2.3]. However, R is not an APVMD. The pullback above belongs to the pullback of type $(\triangle^*)$. Therefore, for the diagram $(\triangle^*)$, without some other assumption on T, D or T/I, there is no hope of proving that R is an APVMD even when T and D are APVMDs and T_I is an AV-domain. So in Section 4, we prove that in a pullback of type $(\triangle^*)$, if $T = (I_v : I_v)$, then R is an APVMD if and only if T is an APVMD, T_I is an AV-domain, and for each nonzero prime ideal $\bar{P}$ of D, either (1) $D_{\bar{P}}$ and $T_{\varphi^{-1}(D\setminus\bar{P})}$ are AV-domains, or (2) there exists a finitely generated ideal A of D such that $A \subseteq \bar{P}$, $A^{-1} \cap E = D$, and $(\varphi^{-1}(\bar{P})T)_t = T$.

For details on star operations, see [Gilmer 1972, Sections 32 and 34].

2. Pullbacks of type $(\triangle_M)$

We begin with the characterization of APVMDs in a pullback of type $(\triangle_M)$.

Theorem 2.1. *For the diagram $(\triangle_M)$, R is an APVMD if and only if D and T are APVMDs, T_M is an AV-domain, and the extension $qf(D) \subseteq T/M$ is a root extension.*

Proof. (⇒) Assume that R is an APVMD. Let $x, y \in D \setminus \{0\}$; then $\varphi(a) = x$ and $\varphi(b) = y$ for some $a, b \in R \setminus M$. Because R is an APVMD, there is a positive integer $n = n(a, b)$ such that (a^n, b^n) is t-invertible in R. By [Wang 2006, Theorem 10.3.11], $(\varphi(a^n), \varphi(b^n))$ is t-invertible in D. Because $(x^n, y^n) = (\varphi(a)^n, \varphi(b)^n) = (\varphi(a^n), \varphi(b^n))$, it follows that (x^n, y^n) is t-invertible in D. Thus D is an APVMD. Let $c, d \in T \setminus \{0\}$. Because T and R have the same quotient field, there is an element $r \in R \setminus \{0\}$ with $rc, rd \in R$. Then $((rc)^n, (rd)^n)R$ is a t-invertible ideal of R for some positive integer n. According to [Wang 2006, Theorem 10.3.11], $((rc)^n, (rd)^n)T$ is t-invertible in T. It is well-known that $((rc)^n, (rd)^n)T = r^n(c^n, d^n)T$, so $(c^n, d^n)T$ is t-invertible in T. Therefore T is an APVMD. As we know, M is a v-ideal of R. Then R_M is an AV-domain by [Li 2012, Theorem 2.3]. By [Wang 2006, Theorem 10.2.2], we have the pullback

$$\begin{array}{ccc} R_M & \longrightarrow & D_{R\setminus M} \\ \downarrow & & \downarrow \\ T_M & \longrightarrow & T/M \end{array}$$

By [Mimouni 2004, Theorem 2.2], T_M and $D_{R\setminus M}$ are AV-domains and the extension $qf(D) = qf(D_{R\setminus M}) \subseteq T/M$ is a root extension.

(⇐) Let P be a maximal t-ideal of R.

Case 1. Suppose that $M \nsubseteq P$. By [Wang 2006, Theorem 10.2.4(3)], there is a prime ideal Q of T with $P = Q \cap R$. Clearly, $M \nsubseteq Q$. In fact $P \nsubseteq M$. Because M is a v-ideal of R, M is a t-ideal of R. As the maximality, $P \nsubseteq M$. So $Q \nsubseteq M$. Hence Q is incomparable to M. According to [Fontana et al. 1998, Lemma 3.3], Q is a maximal t-ideal of T. Since T is an APVMD, T_Q is an AV-domain. By [Wang 2006, Theorem 10.2.1(6)], $R_P = T_Q$. Hence R_P is an AV-domain.

Case 2. Suppose that $M \subseteq P$. There exists a prime ideal p of D such that $P = \varphi^{-1}(p)$. Because P is a t-ideal of R, $P = P_t$. Then $\varphi^{-1}(p) = (\varphi^{-1}(p))_t = \varphi^{-1}(p_t)$ by [Wang 2006, Theorem 10.3.5(3)]. So $p = p_t$. Thus p is a t-ideal of D. Since D is an APVMD, D_p is an AV-domain. In this case, consider the following pullback:

$$\begin{array}{ccc} R_P & \longrightarrow & D_p \\ \downarrow & & \downarrow \\ T_M & \longrightarrow & T/M \end{array}$$

Since T_M and D_p are AV-domains and the extension $qf(D) \subseteq T/M$ is a root extension, R_P is an AV-domain by [Mimouni 2004, Theorem 2.2]. Therefore R is an APVMD. □

Gabelli and Houston [1997, Theorem 4.13] showed that for the diagram $(\triangle_M)$, R is a PVMD if and only if T and D are PVMDs, $k = T/M$, and T_M is a valuation domain. Using this result and Theorem 2.1, we can easily get the following result.

Example 2.2. Let $R = K + XL[X]$, where K and L are fields, $K \subseteq L$, and for some prime p, $L^p \subseteq K$. Consider the pullback

$$\begin{array}{ccc} K + XL[X] & \longrightarrow & K \\ \downarrow & & \downarrow \\ L[X] & \longrightarrow & L \end{array}$$

Then R is an APVMD but not a PVMD. Thus an APVMD need not be a PVMD.

Corollary 2.3. *For the diagram $(\triangle^*_M)$, R is an APVMD if and only if D and T are APVMDs and T_M is an AV-domain.*

Proof. It easily follows from Theorem 2.1 and [Mimouni 2004, Lemma 2.3]. □

Corollary 2.4. *For the diagram $(\triangle^*_M)$, R is an AP-domain if and only if D and T are AP-domains.*

Proof. (⇒) It follows from [Mimouni 2004, Theorem 2.2].

(⇐) Let P be a maximal ideal of R.

Case 1. Suppose that $M \nsubseteq P$. By [Wang 2006, Theorems 10.2.4(3) and 10.2.1(6)], there is a prime ideal Q of T with $P = Q \cap R$ and $R_P = T_Q$. Since T is an AP-domain, T_Q is an AV-domain by [AZ 1991, Theorem 5.8]. Hence R_P is an AV-domain.

Case 2. Suppose that $M \subseteq P$. There exists a prime ideal p of D such that $P = \varphi^{-1}(p)$. Since D is an AP-domain, D_p is an AV-domain. In this case, consider the pullback

$$\begin{array}{ccc} R_P & \longrightarrow & D_p \\ \downarrow & & \downarrow \\ T_M & \longrightarrow & T/M \end{array}$$

Since T_M and D_p are AV-domains and $qf(D) = qf(D_p) = T/M$, R_P is an AV-domain by [Mimouni 2004, Lemma 2.3]. Therefore R is an AP-domain. □

Proposition 2.5. *For the diagram $(\triangle_M)$, suppose that (T, M) is a quasilocal domain and $D = k$ is a proper field of T/M. Then R is an APVMD if and only if R is an AV-domain.*

Proof. (⇐) It easily follows from their definitions.

(⇒) Assume that D is a field. Since $D = R/M$, M is a maximal ideal of R. Because T is quasilocal, R is quasilocal by [Wang 2006, Corollary 10.2.1]. Also

$M = (R : T)$ is a v-ideal of R. Hence M is the unique maximal t-ideal of R. Therefore $R = R_M$ is an AV-domain. □

In [Li 2012, Theorem 3.10], we considered the polynomial ring case and proved that D is an APVMD if and only if $R = D + Xk[X]$ is an APVMD. Similarly, we consider the power series ring case and get the following result.

Corollary 2.6. *Let D be an integral domain with quotient field k. Then D is an APVMD if and only if $R = D + Xk[\![X]\!]$ is an APVMD.*

Proof. Consider the pullback

$$\begin{array}{ccc} R = D + Xk[\![X]\!] & \longrightarrow & D \\ \downarrow & & \downarrow \\ T = k[\![X]\!] & \longrightarrow & k = k[\![X]\!]/Xk[\![X]\!] \end{array}$$

$T = k[\![X]\!]$ is a UFD, so T is an APVMD. The rest follows from Corollary 2.3. □

3. Pullbacks of type (Δ')

Mimouni [2004] considered the pullbacks of type $(\triangle_M)$ in AP-domains and AV-domains. He proved that for the diagram $(\triangle_M)$, R is an AV-domain (respectively AP-domain) if and only if T and D are AV-domains (respectively AP-domains) and the extension $k \subseteq T/M$ is a root extension. We generalize these results for the special pullback of type $(\triangle_M)$ to those for the pullback of type $(\triangle')$.

Lemma 3.1. *For the diagram $(\triangle')$, if R is an AP-domain (resp. AGCD-domain), then the extension $k = qf(D) \subseteq T/I$ is a root extension.*

Proof. Assume that R is an AP-domain (resp. AGCD-domain). By way of contradiction, suppose that the extension $k \subseteq T/I$ is not a root extension. So there is $\lambda \in T/I$ such that λ^n is not in k for each positive integer n. Set $\lambda = \varphi(a)$ for some $a \in T \setminus I$. Let b be a nonzero fixed element of I. Since R is an AP-domain (resp. AGCD-domain), $((ab)^n, b^n)$ is invertible (resp. $((ab)^n, b^n)_v$ is principal) for some positive integer n. Let J denote $((ab)^n, b^n)$. Then $JJ^{-1} = R$ (resp. $J_v = cR$ for some $c \in R$). By [Wang 2006, Example 8.1.10(1)], $J^{-1} = (ab)^{-n}R \cap b^{-n}R$. Let $f \in J^{-1}$; then $f = (ab)^{-n} f_1 = b^{-n} f_2$ for some $f_1, f_2 \in R$. Thus $a^{-n} f_1 = f_2$ and so $f_1 = a^n f_2$. If f_2 is not in I, then $\varphi(f_2) \in D \setminus \{0\}$. Hence $\varphi(f_1) = \varphi(a^n f_2) = \varphi(a)^n \varphi(f_2) = \lambda^n \varphi(f_2)$. So $\lambda^n \in qf(D) = k$, a contradiction. Therefore $f_2 \in I$. So $J^{-1} \subseteq b^{-n} I$. We claim $b^{-n} I \subseteq J^{-1}$. Let $z \in I$ and $x \in J$ and write $x = \alpha(ab)^n + \beta b^n$ for some $\alpha, \beta \in R$. Then $(b^{-n}z)x = (b^{-n}z)(\alpha(ab)^n + \beta b^n) = z\alpha a^n + z\beta \subseteq I \subseteq R$, so $b^{-n}z \in J^{-1}$. Then $b^{-n} I \subseteq J^{-1}$. Therefore $b^{-n} I = J^{-1}$. So $J_v = b^n I^{-1}$. Since $JJ^{-1} = R$ (resp. $J_v = cR$), we have $1 = g_1 h_1 + \cdots + g_m h_m$ for $g_1, \ldots, g_m \in J$, $h_1, \ldots, h_m \in J^{-1}$ (resp. $b^n I^{-1} = cR$). For each $i \in \{1, 2, \ldots, m\}$, write $g_i =$

$\alpha_i(ab)^n + \beta_i b^n$ and $h_i = b^{-n} f_i$, where $\alpha_i, \beta_i \in R$, $f_i \in I$. Then we have $1 = g_1h_1+\cdots+g_mh_m = (\alpha_1(ab)^n+\beta_1 b^n)(b^{-n} f_1)+\cdots+(\alpha_m(ab)^n+\beta_m b^n)(b^{-n} f_m) = (\alpha_1 a^n + \beta_1) f_1 + \cdots + (\alpha_m a^n + \beta_m) f_m \in I$, which is absurd. (Respectively, for each $y \in I^{-1}$, $TyI \subseteq yI \subseteq R$, so $Ty \in I^{-1}$, hence $T \subseteq (I^{-1} : I^{-1})$. Then $R \subset T \subseteq (I^{-1} : I^{-1}) = (b^n I^{-1} : b^n I^{-1}) = (J^{-1} : J^{-1}) = (cR : cR) = R$, which is absurd.) Therefore the extension $k \subseteq T/I$ is a root extension. □

Lemma 3.2. *For the diagram* ($\triangle'$), *assume that* $D = k$ *is a field. Then* R *is an AV-domain if and only if* T *is an AV-domain and the extension* $k \subseteq T/I$ *is a root extension.*

Proof. ($\Rightarrow$) It follows from Lemma 3.1 and the fact that T is an overring of R.

($\Leftarrow$) Let $x \in qf(R)$; then $x \in qf(T)$. Since T is an AV-domain, there is a positive integer $n = n(x)$ such that $x^n \in T$ or $x^{-n} \in T$. Assume that, for example, $x^n \in T$. If $x^n \in I$, then $x^n \in R$. If $x^n \in T \setminus I$, then $\varphi(x)^n = \varphi(x^n) \in T/I \setminus \{0\}$. Since the extension $k \subseteq T/I$ is a root extension, there is a positive integer m such that $\varphi(x^{nm}) = \varphi(x)^{nm} \in k$. Hence $x^{nm} \in \varphi^{-1}(k) = R$. It follows that R is an AV-domain. □

Theorem 3.3. *For the diagram* ($\triangle'$), *R is an AV-domain if and only if T and D are AV-domains and the extension $k = qf(D) \subseteq T/I$ is a root extension.*

Proof. ($\Rightarrow$) By [AZ 1991, Lemma 4.5], T is an AV-domain as an overring of R; and by [AZ 1991, Theorem 4.10], $D = R/I$ is an AV-domain. Also by Lemma 3.1, the extension $k = qf(D) \subseteq T/I$ is a root extension.

($\Leftarrow$) We use the fact that the diagram ($\triangle'$) splits into two parts as follows:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ R_0 = \varphi^{-1}(k) & \longrightarrow & k = R_0/I \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/I \end{array}$$

Consider the second part of this diagram:

$$\begin{array}{ccc} R_0 & \longrightarrow & k \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/I \end{array}$$

Since T is an AV-domain and the extension $k \subseteq T/I$ is a root extension, by Lemma 3.2 R_0 is an AV-domain. The first part of the diagram —

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ R_0 & \longrightarrow & k \end{array}$$

— is a pullback diagram of type $(\triangle_M^*)$. Since D and R_0 are AV-domains, R is an AV-domain by [Mimouni 2004, Lemma 2.3]. □

Lemma 3.4. *For the diagram* $(\triangle)$, *let* $Q(A) = \{x \in T \mid xI \subseteq A\}$ *for an ideal* A *of* R. *Then if* P *is a prime ideal of* R *and* $I \nsubseteq P$, *then* $Q(P)$ *is a prime ideal of* T, $P = Q(P) \cap R$ *and* $R_P = T_{Q(P)}$.

Proof. Let $I \nsubseteq P$, let $x, y \in T$, and let $xy \in Q(P)$. Then $xyI^2 \subseteq xyI \subseteq P$. Since $xI, yI \subseteq I \subseteq R$ and P is a prime ideal of R, we have $xI \subseteq P$ or $yI \subseteq P$. So $x \in Q(P)$ or $y \in Q(P)$. Thus $Q(P)$ is a prime ideal of T. We claim $P = Q(P) \cap R$. Because $PI \subseteq P$, we have $P \subseteq Q(P) \cap R$. Let $x \in Q(P) \cap R$; then $xI \subseteq P$. Since $I \nsubseteq P$, we have $x \in P$. Hence $Q(P) \cap R \subseteq P$. Thus $P = Q(P) \cap R$. Next we show that $R_P = T_{Q(P)}$. It easily follows that $R_P \subseteq T_{Q(P)}$. For the reverse inclusion, let $x \in T_{Q(P)}$. Then $x = z_1/z_2$ for some $z_1 \in T$, $z_2 \in T \setminus Q(P)$. Since $I \nsubseteq P$, there exists $u \in I \setminus P$. Of course $u \in I \setminus Q(P)$. Then $uz_1 \in I \subseteq R$, $uz_2 \in I \setminus Q(P) \subseteq R \setminus Q(P)$. Thus $uz_2 \in R \setminus P$. So $x = uz_1/uz_2 \in R_P$. Thus $T_{Q(P)} \subseteq R_P$, so $R_P = T_{Q(P)}$. □

Theorem 3.5. *For the diagram* $(\triangle')$, *assume that* T *is an AV-domain. Then* R *is an APVMD if and only if* D *is an APVMD and the extension* $k = qf(D) \subseteq T/I$ *is a root extension.*

Proof. As in Theorem 3.3, we consider the diagram

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ R_0 = \varphi^{-1}(k) & \longrightarrow & k = R_0/I \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/I \end{array}$$

($\Leftarrow$) Since T is an AV-domain, R_0 is an AV-domain by Lemma 3.2. Because D is an APVMD, by Corollary 2.3 R is an APVMD.

($\Rightarrow$) Assume that R is an APVMD; by Corollary 2.3 D and R_0 are APVMDs and $(R_0)_I$ is an AV-domain. Set $S = R \setminus I$. Then $R_S = R_I$ and $(R_0)_I = (R_0)_S$. By

[Houston and Taylor 2007, Lemma 1.2], consider the pullback

$$\begin{array}{ccc} (R_0)_S & \longrightarrow & k = k_{\varphi(S)} \\ \downarrow & & \downarrow \\ T_S & \longrightarrow & (T/I)_{\varphi(S)} \end{array}$$

As $(R_0)_S = (R_0)_I$ is an AV-domain, the extension $k \subseteq (T/I)_{\varphi(S)}$ is a root extension by Lemma 3.2. So the extension $k \subseteq T/I$ is a root extension. □

Theorem 3.6. *For the diagram* $(\triangle')$, *assume that T is an AV-domain. Then R is an AP-domain if and only if D is an AP-domain and the extension $k = qf(D) \subseteq T/I$ is a root extension.*

Proof. ($\Leftarrow$) As in Theorem 3.3, we consider the diagram

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ R_0 = \varphi^{-1}(k) & \longrightarrow & k = R_0/I \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/I \end{array}$$

Since T is an AV-domain, R_0 is an AV-domain by Lemma 3.2. Then R is an AP-domain by Corollary 2.4.

($\Rightarrow$) Assume that R is an AP-domain; then $D = R/I$ is an AP-domain by [AZ 1991, Theorem 4.10]. Also by Lemma 3.1, the extension $k \subseteq T/I$ is a root extension. □

4. Pullbacks of type (Δ^*)

Lemma 4.1. *For a diagram* $(\triangle^*)$, *R is an AV-domain if and only if T and D are AV-domains.*

Proof. The proof is similar to that of Lemma 3.2.

($\Rightarrow$) If R is an AV-domain, so are its homomorphic image of D and its overring T.

($\Leftarrow$) Let $x \in qf(R)$; then $x \in qf(T)$. Since T is an AV-domain, there is a positive integer $n = n(x)$ such that $x^n \in T$ or $x^{-n} \in T$. Assume that, for example, $x^n \in T$. If $x^n \in I$, then $x^n \in R$. If $x^n \in T \setminus I$, then $\varphi(x)^n = \varphi(x^n) \in T/I \setminus \{0\} \subseteq qf(T/I) = qf(D)$. Since D is an AV-domain, there is a positive integer m such that $\varphi(x)^{nm} \in D$. Hence $x^{nm} \in \varphi^{-1}(D) = R$. It follows that R is an AV-domain. □

Proposition 4.2. *Let R be an integral domain and I a nonzero ideal of R. If R is an APVMD, then $(I_v : I_v)$ is an APVMD.*

Proof. Set $T = (I_v : I_v)$. Assume that $x, y \in T = (I_v : I_v)$. Choose a fixed element $a \in I_v$. Then $ax, ay \in I_v \subseteq R$. Since R is an APVMD, there is a positive integer $n = n(ax, ay)$ such that $((ax)^n, (ay)^n)$ is t-invertible in R. Let J denote $((ax)^n, (ay)^n)$. So $(JJ^{-1})_t = R$. There is a finitely generated ideal $H \subseteq JJ^{-1} \subseteq R$ such that $H_v = R$. By [Houston and Taylor 2007, Lemma 2.3], $(I_v : I_v)$ is t-linked over R. Then $(HT)_v = T$. So $(JJ^{-1}T)_t = T$. Thus $(a^n(x^n, y^n)J^{-1}T)_t = \big(((ax)^n, (ay)^n)J^{-1}T\big)_t = T$. So (x^n, y^n) is t-invertible in T. Therefore $T = (I_v : I_v)$ is an APVMD. □

Proposition 4.3. *For a diagram* $(\triangle^*)$, *if* R *is an APVMD*, *then* I *is a prime* t*-ideal of both* R *and* T.

Proof. We claim R_I is an AV-domain, and thus I is a t-ideal of R. Let $x, y \in R\setminus\{0\}$. If $(x^n, y^n)(x^n, y^n)^{-1} \subseteq I$ for each positive integer n, then $((x^n, y^n)(x^n, y^n)^{-1})^{-1} \supseteq I^{-1} \supseteq T \supsetneq R$, which contradicts that R is an APVMD. Hence there exists a positive integer n such that $(x^n, y^n)(x^n, y^n)^{-1} \nsubseteq I$. Thus $((x^n, y^n)(x^n, y^n)^{-1})R_I = R_I$. So $(x^n, y^n)R_I$ is invertible in R_I. Since R_I is quasilocal, $(x^n, y^n)R_I$ is principal. Then R_I is an AV-domain. So IR_I is a maximal t-ideal of R_I. By [Kang 1989, Lemma 3.17], $I = IR_I \cap R$ is a t-ideal of R. Since $qf(D) = qf(T/I)$, we have $R_I = T_I$ by [Houston and Taylor 2007, Lemma 1.2]. So T_I is an AV-domain. Then IT_I is a maximal t-ideal of T. Therefore I is a prime t-ideal of T. □

Houston and Taylor [2007, Theorem 2.8] characterized the PVMD-property in a pullback of type $(\triangle^*)$. Similarly, we are ready to study the APVMD-property in a pullback of type $(\triangle^*)$. For convenience, let E denote T/I.

Theorem 4.4. *For a diagram* $(\triangle^*)$, *assume that* $T = (I_v : I_v)$. *Then* R *is an APVMD if and only if* T *is an APVMD and* T_I *is an AV-domain*, *and for each nonzero prime ideal* $\bar{P}$ *of* D, *either*

(1) $D_{\bar{P}}$ *and* $T_{\varphi^{-1}(D\setminus\bar{P})}$ *are AV-domains*, *or*

(2) *there is a finitely generated ideal* A *of* D *such that* $A \subseteq \bar{P}$, $A^{-1} \cap E = D$, *and* $(\varphi^{-1}(\bar{P})T)_t = T$.

Proof. (⇒) Assume that R is an APVMD. By Proposition 4.2, $T = (I_v : I_v)$ is an APVMD. Also, T_I is an AV-domain by Proposition 4.3. Let $\bar{P}$ be a prime ideal of D, and let $P = \varphi^{-1}(\bar{P})$.

Case 1. If P is a t-ideal of R, then R_P is an AV-domain. By [Houston and Taylor 2007, Lemma 1.2], we have the pullback

$$\begin{array}{ccc} R_P & \longrightarrow & D_{\varphi(R\setminus P)} = D_{\bar{P}} \\ \downarrow & & \downarrow \\ T_{R\setminus P} = T_{\varphi^{-1}(D\setminus\bar{P})} & \longrightarrow & E_{\varphi(S)} = E_{D\setminus\bar{P}} \end{array}$$

By Lemma 4.1, $D_{\bar{P}}$ and $T_{R\setminus P} = T_{\varphi^{-1}(D\setminus\bar{P})}$ are AV-domains.

Case 2. Suppose that P is not a t-ideal of R. Since R is an APVMD, it is a UMT-domain by [Li 2012, Theorem 3.8]. By [Fontana et al. 1998, Corollary 1.6], $P_t = R$. Hence there is a finitely generated ideal $J \subseteq P$ such that $J^{-1} = R$. Since T is t-linked over R by [Houston and Taylor 2007, Lemma 2.3], we have $(JT)^{-1} = T$. So $(\varphi^{-1}(\bar{P})T)_t = (PT)_t = T$. Now let $A = \varphi(J)$ and $e \in A^{-1} \cap E$. Then $\varphi(t) = e$ for some $t \in T$ and $eA \subseteq D$. Hence $\varphi^{-1}(eA) \subseteq \varphi^{-1}(D) = R$. Also, $\varphi^{-1}(eA) = \varphi^{-1}(e)\varphi^{-1}(A) = \varphi^{-1}(\varphi(t))\varphi^{-1}(\varphi(J)) \supseteq tJ$. So $tJ \subseteq R$. Then $t \in J^{-1} = R$. Thus $e = \varphi(t) \in D$. Therefore $A^{-1} \cap E = D$.

($\Leftarrow$) Let P be a maximal t-ideal of R. It suffices to show that R_P is an AV-domain.

Case 1. Assume that $I \nsubseteq P$. By Lemma 3.4, there is a prime ideal Q of T such that $P = Q \cap R$ and $R_P = T_Q$. By Proposition 4.3, we know that I is a prime t-ideal of R. Then $(PT)_t \neq T$ by [Houston and Taylor 2007, Lemma 2.6]. Hence $PT \subseteq Q_1$ for some prime t-ideal Q_1 of T. Since $T = (I_v : I_v)$ is t-linked over R by [Houston and Taylor 2007, Lemma 2.3], it follows that $(Q_1 \cap R)_t \neq R$. However, $P \subseteq Q_1 \cap R$ and P is a maximal t-ideal of R. It follows that $Q = Q_1$. Then Q is t-ideal of T. Therefore $R_P = T_Q$ is an AV-domain.

Case 2. Assume that $I \subseteq P$. Let $\bar{P}$ denote $\varphi(P)$. By way of contradiction, suppose that condition (2) of the hypothesis holds: there is a finitely generated ideal A of D such that $A \subseteq \bar{P}$, $A^{-1} \cap E = D$, and $(\varphi^{-1}(\bar{P})T)_t = (PT)_t = T$. Then $A = \varphi(J_1)$ and $(J_2T)^{-1} = T$ for some finitely generated ideals J_1, J_2 of R. Also $J_1 + J_2 \subseteq P$. Set $J = J_1 + J_2$. Then $J^{-1} \subseteq J_2^{-1}$. Let $x \in J_2^{-1}$; then $xJ_2 \subseteq R$, and hence $xJ_2T \subseteq T$. So $x \in (J_2T)^{-1} = T$. So $J^{-1} \subseteq J_2^{-1} \subseteq T$. Since $J \subseteq P$ and P is a prime t-ideal of R, then $J^{-1} \neq R$. Otherwise, if $J^{-1} = R$, then $R = J_v \subseteq P_t = P$, a contradiction. So $R \subsetneq J^{-1}$. Therefore, there is an element $t \in J^{-1} \setminus R$ with $tJ \subseteq R$. So $\varphi(t)A \subseteq \varphi(t)\varphi(J_1) \subseteq \varphi(t)\varphi(J) = \varphi(tJ) \subseteq D$. Then $\varphi(t) \in A^{-1} \cap E = D$. So $t \in R$, a contradiction. Hence condition (1) must hold. Localize the diagram at P and consider the pullback

$$\begin{array}{ccc} R_P & \longrightarrow & D_{\varphi(R\setminus P)} = D_{\bar{P}} \\ \downarrow & & \downarrow \\ T_{R\setminus P} = T_{\varphi^{-1}(D\setminus\bar{P})} & \longrightarrow & E_{\varphi(S)} = E_{D\setminus\bar{P}} \end{array}$$

By Lemma 4.1, R_P is an AV-domain. Therefore, R is an APVMD. □

Acknowledgments

The author would like to thank the referee for valuable comments and suggestions. Also, the author is very grateful to Muhammad Zafrullah for his valuable advice.

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Received April 16, 2010. Revised July 23, 2010.

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E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

PACIFIC JOURNAL OF MATHEMATICS

Volume 252 No. 2 August 2011

0030-8730(201108)252:2;1-9