CHARACTERIZING ALMOST PRÜFER v-MULTIPLICATION DOMAINS IN PULLBACKS

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Let $I$ be an ideal of an integral domain $T$, let $\varphi : T \to T/I$ be the projection, let $D$ be an integral domain contained in $T/I$, and let $R = \varphi^{-1}(D)$. We characterize when $R$ is an almost Prüfer $v$-multiplication domain, an almost valuation domain, and an almost Prüfer domain, in the context of pullbacks.

1. Introduction

Let $I$ be an ideal of an integral domain $T$, let $\varphi : T \to T/I$ be the natural projection, let $D$ be an integral domain contained in $T/I$, and let $k = qf(D)$ be the quotient field of $D$. Let $R = \varphi^{-1}(D)$ be the integral domain arising from the following pullback of canonical homomorphisms:

$$
\begin{array}{ccc}
R & \longrightarrow & D \\
\downarrow & & \downarrow \\
T & \longrightarrow & T/I
\end{array}
$$

It is well-known that $D = R/I$ and that $I$ is a prime ideal of $R$. Notice that $I$ is a common ideal of $R$ and $T$, and hence $T$ is an overring of $R$. We assume that $R$ is properly contained in $T$, and we refer to this as a pullback diagram of type $(\triangle)$. For the diagram $(\triangle)$, if $qf(D) \subseteq T/I$, then we refer to this as a diagram of type $(\triangle')$. For the diagram $(\triangle)$, if $I$ is a prime ideal of $T$ and $qf(D) = qf(T/I)$, then we refer to this as a diagram of type $(\triangle^*)$. Here $qf(T/I)$ denotes the quotient field of $T/I$. For the diagram $(\triangle)$, if $I = M$ is a maximal ideal of $T$, we refer to this as a diagram of type $(\triangle_M)$. For the diagram $(\triangle_M)$, if $qf(D) = T/M$, then we refer to this as a diagram of type $(\triangle_M^*)$.

Pullbacks are an important tool in constructing interesting examples and counter-examples. They have become so important that in recent years there have been many papers devoted to ring- and ideal-theoretic properties in pullback domains.

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For more details on pullbacks, see [Mimouni 2004; Houston and Taylor 2007; Fontana and Gabelli 1996; Gilmer 1972; Gabelli and Houston 1997].

Zafrullah [1985] began a general theory of almost factoriality and introduced the notion of an almost GCD-domain. Zafrullah defined $R$ to be an almost GCD-domain (AGCD-domain for short) if for each $a, b \in D \setminus \{0\}$, there is a positive integer $n = n(a, b)$ such that $a^n D \cap b^n D$ is principal (or equivalently, $(a^n, b^n)_v$ is principal). Anderson and Zafrullah [AZ 1991] introduced several classes of integral domains related to almost GCD-domains, including almost Bézout domains (AB-domains), almost Prüfer domains (AP-domains), and almost valuation domains (AV-domains). As in [AZ 1991], an integral domain $R$ is an AB-domain if for each $a, b \in D \setminus \{0\}$, there is a positive integer $n = n(a, b)$ such that $(a^n, b^n)$ is principal; while $R$ is an AP-domain if for each $a, b \in D \setminus \{0\}$, there is a positive integer $n = n(a, b)$ such that $(a^n, b^n)$ is invertible. Following [AZ 1991], an integral domain $R$ is said to be an AV-domain if for each $a, b \in D \setminus \{0\}$, there is a positive integer $n = n(a, b)$ such that $a^n b^n$ or $b^n a^n$. Similarly, in [Li 2012] we defined an integral domain $R$ to be an almost Prüfer $v$-multiplication domain (APVMD) if for each $a, b \in R \setminus \{0\}$, there is a positive integer $n = n(a, b)$ such that $a^n D \cap b^n D$ is $t$-invertible, or equivalently, $(a^n, b^n)$ is $t$-invertible. Recall that an integral domain $R$ is said to be a Prüfer $v$-multiplication domain (PVMD) if each $a \in R \setminus \{0\}$, $(a, b)$ is $t$-invertible. The class of APVMDs includes a lot of important rings, such as AV-domains, AB-domains, AGCD-domains, AP-domains, PVMDs, and so on.

Anderson and Zafrullah [1991, Theorem 4.9] proved that $D$ is an AB-domain (respectively, AP-domain) if and only if $R = D + Xk[X]$ is an AB-domain (respectively, AP-domain). However, we notice that the $(D + Xk[X])$-construction is a special case of the pullback of type $(\triangle_M)$. Mimouni [2004, Theorem 2.2] generalized these results and proved that for the diagram $(\triangle_M)$, $R$ is an AP-domain if and only if $T$ and $D$ are AP-domains and the extension $k \subseteq T/M$ is a root extension. He also gave a similar characterization for AV-domains. Mimouni [2004, Corollary 2.6] continued to show that for the diagram $(\triangle_M)$, assuming that $D = k$ is a field, then $R$ is an AB-domain if and only if $T$ is an AB-domain and the extension $k \subseteq T/M$ is a root extension. In [Li 2012, Theorem 3.10], we proved that $D$ is an APVMD if and only if $R = D + Xk[X]$ is an APVMD.

From this we notice that the characterization of AV-domains and AP-domains is known only in the context of the special pullback of type $(\triangle_M)$, and that the study of APVMDs is only in the $(D + Xk[X])$-construction, a special case of type $(\triangle_M)$. So the main purpose of this paper is to characterize APVMDs in pullbacks in greater generality and to generalize the characterization of AV-domains and AP-domains for the pullback of type $(\triangle_M)$ to that for the pullback of type $(\triangle)$.

In Section 2, we mainly prove that in the pullback of type $(\triangle_M)$, $R$ is an APVMD if and only if $D$ and $T$ are APVMDs, $T_M$ is an AV-domain, and the extension
qf(D) ⊆ T/M is a root extension. Using this fact, we give Example 2.2 to show that an APVMD is not necessarily a PVMD. We also show that for the diagram (Δ_M^*), R is an APVMD if and only if D and T are APVMDs and TM is an AV-domain. Using this result, we prove that D is an APVMD if and only if R = D + Xk[[X]] is an APVMD.

In Section 3, we mainly indicate that for the diagram (Δ'), if T is an AV-domain, then R is an APVMD if and only if D is an APVMD and the extension qf(D) ⊆ T/I is a root extension. We prove that for the diagram (Δ'), R is an AV-domain if and only if D and T are APVMDs and TM is an AV-domain. Using this result, we prove that D is an APVMD if and only if R = D + Xk[[X]] is an APVMD.

Following [Zafrullah 1988, p. 95], assume that D is the ring of entire functions and S is the multiplicative set generated by the principal primes of D; then D is integrally closed, and hence R = D + XD_S[X] is integrally closed, but R = D + XD_S[X] is not a PVMD. Because an integrally closed APVMD is a PVMD by [Li 2012, Theorem 2.4], R is not an APVMD. Consider the following pullback:

\[
\begin{array}{ccc}
R = D + XD_S[X] & \longrightarrow & D \\
\downarrow & & \downarrow \\
T = D_S[X] & \longrightarrow & D_S \cong T/I
\end{array}
\]

Here I denotes XD_S[X]. The example indicates that qf(D) = qf(T/I), D and T are APVMDs, I is principal in T, and T = D_S[X] is a PVMD. It follows that T_I is an AV-domain by [Li 2012, Theorem 2.3]. However, R is not an APVMD. The pullback above belongs to the pullback of type (Δ^*). Therefore, for the diagram (Δ^*), without some other assumption on T, D or T/I, there is no hope of proving that R is an APVMD even when T and D are APVMDs and T_I is an AV-domain. So in Section 4, we prove that in a pullback of type (Δ^*), if T = (I_v : I_v), then R is an APVMD if and only if T is an APVMD, T_I is an AV-domain, and for each nonzero prime ideal \( \overline{P} \) of D, either (1) \( D_{\overline{P}} \) and \( T_{\varphi^{-1}(D \setminus \overline{P})} \) are AV-domains, or (2) there exists a finitely generated ideal A of D such that A \subseteq \overline{P}, A^{-1} \cap E = D, and \( (\varphi^{-1}(\overline{P}) T)_I = T \).

For details on star operations, see [Gilmer 1972, Sections 32 and 34].

2. Pullbacks of type (Δ_M)

We begin with the characterization of APVMDs in a pullback of type (Δ_M).

**Theorem 2.1.** For the diagram (Δ_M), R is an APVMD if and only if D and T are APVMDs, T_M is an AV-domain, and the extension qf(D) ⊆ T/M is a root extension.
Proof. (⇒) Assume that $R$ is an APVMD. Let $x, y \in D \setminus \{0\}$; then $\varphi(a) = x$ and $\varphi(b) = y$ for some $a, b \in R \setminus M$. Because $R$ is an APVMD, there is a positive integer $n = n(a, b)$ such that $(a^n, b^n)$ is $t$-invertible in $R$. By [Wang 2006, Theorem 10.3.11], $(\varphi(a^n), \varphi(b^n))$ is $t$-invertible in $D$. Because $(x^n, y^n) = (\varphi(a)^n, \varphi(b)^n) = (\varphi(a^n), \varphi(b^n))$, it follows that $(x^n, y^n)$ is $t$-invertible in $D$. Thus $D$ is an APVMD. Let $c, d \in T \setminus \{0\}$. Because $T$ and $R$ have the same quotient field, there is an element $r \in R \setminus \{0\}$ with $rc, rd \in R$. Then $((rc)^n, (rd)^n)R$ is a $t$-invertible ideal of $R$ for some positive integer $n$. According to [Wang 2006, Theorem 10.3.11], $((rc)^n, (rd)^n)T$ is $t$-invertible in $T$. It is well-known that $((rc)^n, (rd)^n)T = r^n(c^n, d^n)T$, so $(c^n, d^n)T$ is $t$-invertible in $T$. Therefore $T$ is an APVMD. As we know, $M$ is a $v$-ideal of $R$. Then $R_M$ is an AV-domain by [Li 2012, Theorem 2.3]. By [Wang 2006, Theorem 10.2.2], we have the pullback

$$
\begin{array}{c}
R_M \longrightarrow D_{R \setminus M} \\
\downarrow \hspace{1cm} \downarrow \\
T_M \longrightarrow T/M
\end{array}
$$

By [Mimouni 2004, Theorem 2.2], $T_M$ and $D_{R \setminus M}$ are AV-domains and the extension $qf(D) = qf(D_{R \setminus M}) \subseteq T/M$ is a root extension.

(⇐) Let $P$ be a maximal $t$-ideal of $R$.

**Case 1.** Suppose that $M \nsubseteq P$. By [Wang 2006, Theorem 10.2.4(3)], there is a prime ideal $Q$ of $T$ with $P = Q \cap R$. Clearly, $M \nsubseteq Q$. In fact $P \nsubseteq M$. Because $M$ is a $v$-ideal of $R$, $M$ is a $t$-ideal of $R$. As the maximality, $P \nsubseteq M$. So $Q \nsubseteq M$. Hence $Q$ is incomparable to $M$. According to [Fontana et al. 1998, Lemma 3.3], $Q$ is a maximal $t$-ideal of $T$. Since $T$ is an APVMD, $T_Q$ is an AV-domain. By [Wang 2006, Theorem 10.2.1(6)], $R_P = T_Q$. Hence $R_P$ is an AV-domain.

**Case 2.** Suppose that $M \subseteq P$. There exists a prime ideal $p$ of $D$ such that $P = \varphi^{-1}(p)$. Because $P$ is a $t$-ideal of $R$, $P = P_I$. Then $\varphi^{-1}(p) = (\varphi^{-1}(p))_I = \varphi^{-1}(p_I)$ by [Wang 2006, Theorem 10.3.5(3)]. So $p = p_I$. Thus $p$ is a $t$-ideal of $D$. Since $D$ is an APVMD, $D_p$ is an AV-domain. In this case, consider the following pullback:

$$
\begin{array}{c}
R_P \longrightarrow D_p \\
\downarrow \hspace{1cm} \downarrow \\
T_M \longrightarrow T/M
\end{array}
$$

Since $T_M$ and $D_p$ are AV-domains and the extension $qf(D) \subseteq T/M$ is a root extension, $R_P$ is an AV-domain by [Mimouni 2004, Theorem 2.2]. Therefore $R$ is an APVMD.

\qed
Gabelli and Houston [1997, Theorem 4.13] showed that for the diagram \((\triangle_M)\), \(R\) is a PVMD if and only if \(T\) and \(D\) are PVMDs, \(k = T/M\), and \(T_M\) is a valuation domain. Using this result and Theorem 2.1, we can easily get the following result.

**Example 2.2.** Let \(R = K + XL[X]\), where \(K\) and \(L\) are fields, \(K \subseteq L\), and for some prime \(p\), \(L^p \subseteq K\). Consider the pullback

\[
\begin{array}{ccc}
K + XL[X] & \longrightarrow & K \\
\downarrow & & \downarrow \\
L[X] & \longrightarrow & L
\end{array}
\]

Then \(R\) is an APVMD but not a PVMD. Thus an APVMD need not be a PVMD.

**Corollary 2.3.** For the diagram \((\triangle^*_M)\), \(R\) is an APVMD if and only if \(D\) and \(T\) are APVMDs and \(T_M\) is an AV-domain.

*Proof.* It easily follows from Theorem 2.1 and [Mimouni 2004, Lemma 2.3].

**Corollary 2.4.** For the diagram \((\triangle^*_M)\), \(R\) is an AP-domain if and only if \(D\) and \(T\) are AP-domains.

*Proof.* \((\Rightarrow)\) It follows from [Mimouni 2004, Theorem 2.2].

\((\Leftarrow)\) Let \(P\) be a maximal ideal of \(R\).

**Case 1.** Suppose that \(M \not\subseteq P\). By [Wang 2006, Theorems 10.2.4(3) and 10.2.1(6)], there is a prime ideal \(Q\) of \(T\) with \(P = Q \cap R\) and \(R_P = T_Q\). Since \(T\) is an AP-domain, \(T_Q\) is an AV-domain by [AZ 1991, Theorem 5.8]. Hence \(R_P\) is an AV-domain.

**Case 2.** Suppose that \(M \subseteq P\). There exists a prime ideal \(p\) of \(D\) such that \(P = \varphi^{-1}(p)\). Since \(D\) is an AP-domain, \(D_p\) is an AV-domain. In this case, consider the pullback

\[
\begin{array}{ccc}
R_P & \longrightarrow & D_P \\
\downarrow & & \downarrow \\
T_M & \longrightarrow & T/M
\end{array}
\]

Since \(T_M\) and \(D_p\) are AV-domains and \(qf(D) = qf(D_P) = T/M\), \(R_P\) is an AV-domain by [Mimouni 2004, Lemma 2.3]. Therefore \(R\) is an AP-domain.

**Proposition 2.5.** For the diagram \((\triangle_M)\), suppose that \((T, M)\) is a quasilocal domain and \(D = k\) is a proper field of \(T/M\). Then \(R\) is an APVMD if and only if \(R\) is an AV-domain.

*Proof.* \((\Leftarrow)\) It easily follows from their definitions.

\((\Rightarrow)\) Assume that \(D\) is a field. Since \(D = R/M\), \(M\) is a maximal ideal of \(R\). Because \(T\) is quasilocal, \(R\) is quasilocal by [Wang 2006, Corollary 10.2.1]. Also
$M = (R : T)$ is a $v$-ideal of $R$. Hence $M$ is the unique maximal $t$-ideal of $R$. Therefore $R = R_M$ is an AV-domain. 

In [Li 2012, Theorem 3.10], we considered the polynomial ring case and proved that $D$ is an APVMD if and only if $R = D + Xk[X]$ is an APVMD. Similarly, we consider the power series ring case and get the following result.

**Corollary 2.6.** Let $D$ be an integral domain with quotient field $k$. Then $D$ is an APVMD if and only if $R = D + Xk[[X]]$ is an APVMD.

**Proof.** Consider the pullback

\[
\begin{array}{ccc}
R = D + Xk[[X]] & \longrightarrow & D \\
\downarrow & & \downarrow \\
T = k[[X]] & \longrightarrow & k = k[[X]]/Xk[[X]]
\end{array}
\]

$T = k[[X]]$ is a UFD, so $T$ is an APVMD. The rest follows from Corollary 2.3. □

3. **Pullbacks of type ($\Delta'$)**

Mimouni [2004] considered the pullbacks of type ($\Delta_M$) in AP-domains and AV-domains. He proved that for the diagram ($\Delta_M$), $R$ is an AV-domain (respectively AP-domain) if and only if $T$ and $D$ are AV-domains (respectively AP-domains) and the extension $k \subseteq T/M$ is a root extension. We generalize these results for the special pullback of type ($\Delta_M$) to those for the pullback of type ($\Delta'$).

**Lemma 3.1.** For the diagram ($\Delta'$), if $R$ is an AP-domain (resp. AGCD-domain), then the extension $k = qf(D) \subseteq T/I$ is a root extension.

**Proof.** Assume that $R$ is an AP-domain (resp. AGCD-domain). By way of contradiction, suppose that the extension $k \subseteq T/I$ is not a root extension. So there is $\lambda \in T/I$ such that $\lambda^n$ is not in $k$ for each positive integer $n$. Let $\lambda = \varphi(a)$ for some $a \in T \setminus I$. Let $b$ be a nonzero fixed element of $I$. Since $R$ is an AP-domain (resp. AGCD-domain), $((ab)^n, b^n)$ is invertible (resp. $((ab)^n, b^n)_v$ is principal) for some positive integer $n$. Let $J$ denote $((ab)^n, b^n)$. Then $JJ^{-1} = R$ (resp. $J_v = cR$ for some $c \in R$). By [Wang 2006, Example 8.1.10(1)], $J^{-1} = (ab)^{-n}R \cap b^{-n}R$. Let $f \in J^{-1}$; then $f = (ab)^{-n}f_1 = b^{-n}f_2$ for some $f_1, f_2 \in R$. Thus $a^{-n}f_1 = f_2$ and so $f_1 = a^n f_2$. If $f_2$ is not in $I$, then $\varphi(f_2) \in D \setminus \{0\}$. Hence $\varphi(f_1) = \varphi(a^n f_2) = \varphi(a)^n \varphi(f_2) = \lambda^n \varphi(f_2)$. So $\lambda^n \in qf(D) = k$, a contradiction. Therefore $f_2 \in I$. So $J^{-1} \subseteq b^{-n}I$. We claim $b^{-n}I \subseteq J^{-1}$. Let $z \in I$ and $x \in J$ and write $x = \alpha(ab)^n + \beta b^n$ for some $\alpha, \beta \in R$. Then $(b^{-n}z)x = (b^{-n}z)(\alpha(ab)^n + \beta b^n) = z\alpha a^n + z\beta \subseteq I \subseteq R$, so $b^{-n}z \in J^{-1}$. Then $b^{-n}I \subseteq J^{-1}$. Therefore $b^{-n}I = J^{-1}$. So $J_v = b^n I^{-1}$. Since $JJ^{-1} = R$ (resp. $J_v = cR$), we have $1 = g_1 h_1 + \cdots + g_m h_m$ for $g_1, \ldots, g_m \in J$, $h_1, \ldots, h_m \in J^{-1}$ (resp. $b^n I^{-1} = cR$). For each $i \in \{1, 2, \ldots, m\}$, write $g_i =$
\[
\alpha_i(ab)^n + \beta_i b^n \quad \text{and} \quad h_i = b^{-n} f_i,
\]
where \( \alpha_i, \beta_i \in R, f_i \in I \). Then we have
\[
1 = g_1 h_1 + \cdots + g_m h_m = (\alpha_1(ab)^n + \beta_1 b^n)(b^{-n} f_1) + \cdots + (\alpha_m(ab)^n + \beta_m b^n)(b^{-n} f_m) = (\alpha_1 a^n + \beta_1) f_1 + \cdots + (\alpha_m a^n + \beta_m) f_m \in I,
\]
which is absurd. (Respectively, for each \( y \in I^{-1} \), \( Ty \subseteq y I \subseteq R \), so \( T \subseteq (I^{-1} : I^{-1}) \). Then
\[
R \subseteq T \subseteq (I^{-1} : I^{-1}) = (b^n I^{-1} : b^n I^{-1}) = (J^{-1} : J^{-1}) = (cR : cR) = R,
\]
which is absurd.) Therefore the extension \( k \subseteq T/I \) is a root extension. □

**Lemma 3.2.** For the diagram \((\triangle')\), assume that \( D = k \) is a field. Then \( R \) is an AV-domain if and only if \( T \) is an AV-domain and the extension \( k \subseteq T/I \) is a root extension.

**Proof.** \((\Rightarrow)\) It follows from Lemma 3.1 and the fact that \( T \) is an overring of \( R \).

\((\Leftarrow)\) Let \( x \in qf(R) \); then \( x \in qf(T) \). Since \( T \) is an AV-domain, there is a positive integer \( n = n(x) \) such that \( x^n \in T \) or \( x^{-n} \in T \). Assume that, for example, \( x^n \in T \). If \( x^n \in I \), then \( x^n \in R \). If \( x^n \in T \setminus I \), then \( \varphi(x)^n = \varphi(x^n) \in T/I \setminus \{0\} \). Since the extension \( k \subseteq T/I \) is a root extension, there is a positive integer \( m \) such that \( \varphi(x)^n = \varphi(x)^{nm} \in k \). Hence \( x^{nm} \in \varphi^{-1}(k) = R \). It follows that \( R \) is an AV-domain. □

**Theorem 3.3.** For the diagram \((\triangle')\), \( R \) is an AV-domain if and only if \( T \) and \( D \) are AV-domains and the extension \( k = qf(D) \subseteq T/I \) is a root extension.

**Proof.** \((\Rightarrow)\) By [AZ 1991, Lemma 4.5], \( T \) is an AV-domain as an overring of \( R \); and by [AZ 1991, Theorem 4.10], \( D = R/I \) is an AV-domain. Also by Lemma 3.1, the extension \( k = qf(D) \subseteq T/I \) is a root extension.

\((\Leftarrow)\) We use the fact that the diagram \((\triangle')\) splits into two parts as follows:

\[
\begin{array}{ccc}
R & \rightarrow & D \\
\downarrow & & \downarrow \\
R_0 = \varphi^{-1}(k) & \rightarrow & k = R_0/I \\
\downarrow & & \downarrow \\
T & \rightarrow & T/I
\end{array}
\]

Consider the second part of this diagram:

\[
\begin{array}{ccc}
R_0 & \rightarrow & k \\
\downarrow & & \downarrow \\
T & \rightarrow & T/I
\end{array}
\]
Since \( T \) is an AV-domain and the extension \( k \subseteq T/I \) is a root extension, by Lemma 3.2 \( R_0 \) is an AV-domain. The first part of the diagram—

\[
\begin{array}{ccc}
R & \longrightarrow & D \\
\downarrow & & \downarrow \\
R_0 & \longrightarrow & k
\end{array}
\]

—is a pullback diagram of type \((\Delta^*_M)\). Since \( D \) and \( R_0 \) are AV-domains, \( R \) is an AV-domain by [Mimouni 2004, Lemma 2.3]. □

**Lemma 3.4.** For the diagram \((\Delta)\), let \( Q(A) = \{x \in T \mid xI \subseteq A\} \) for an ideal \( A \) of \( R \). Then if \( P \) is a prime ideal of \( R \) and \( I \nsubseteq P \), then \( Q(P) \) is a prime ideal of \( T \), 

\[ P = Q(P) \cap R \text{ and } R_P = T_{Q(P)}. \]

**Proof.** Let \( I \nsubseteq P \), let \( x, y \in T \), and let \( xy \in Q(P) \). Then \( xyI^2 \subseteq xyI \subseteq P \). Since \( xI, yI \subseteq I \subseteq R \) and \( P \) is a prime ideal of \( R \), we have \( xI \subseteq P \) or \( yI \subseteq P \). So \( x \in Q(P) \) or \( y \in Q(P) \). Thus \( Q(P) \) is a prime ideal of \( T \). We claim \( P = Q(P) \cap R \). Because \( P \subseteq R \), we have \( P \subseteq Q(P) \cap R \). Let \( x \in Q(P) \cap R \); then \( xI \subseteq P \). Since \( I \nsubseteq P \), we have \( x \in P \). Hence \( Q(P) \cap R \subseteq P \). Thus \( P = Q(P) \cap R \). Next we show that \( R_P = T_{Q(P)} \). It easily follows that \( R_P \subseteq T_{Q(P)} \). For the reverse inclusion, let \( x \in T_{Q(P)} \). Then \( x = z_1/z_2 \) for some \( z_1 \in T, z_2 \in T \setminus Q(P) \). Since \( I \nsubseteq P \), there exists \( u \in I \setminus P \). Of course \( u \in I \setminus Q(P) \). Then \( uz_1 \in I \subseteq R, uz_2 \in I \setminus Q(P) \subseteq R \setminus Q(P) \). Thus \( uz_2 \in R \setminus P \). So \( x = uz_1/uz_2 \in R_P \). Hence \( T_{Q(P)} \subseteq R_P \), so \( R_P = T_{Q(P)} \). □

**Theorem 3.5.** For the diagram \((\Delta')\), assume that \( T \) is an AV-domain. Then \( R \) is an APVMD if and only if \( D \) is an APVMD and the extension \( k = qf(D) \subseteq T/I \) is a root extension.

**Proof.** As in Theorem 3.3, we consider the diagram

\[
\begin{array}{ccc}
R & \longrightarrow & D \\
\downarrow & & \downarrow \\
R_0 = \varphi^{-1}(k) & \longrightarrow & k = R_0/I \\
\downarrow & & \downarrow \\
T & \longrightarrow & T/I
\end{array}
\]

\((\Leftarrow)\) Since \( T \) is an AV-domain, \( R_0 \) is an AV-domain by Lemma 3.2. Because \( D \) is an APVMD, by Corollary 2.3 \( R \) is an APVMD.

\((\Rightarrow)\) Assume that \( R \) is an APVMD; by Corollary 2.3 \( D \) and \( R_0 \) are APVMDs and \((R_0)_T \) is an AV-domain. Set \( S = R \setminus I \). Then \( R_S = R_I \) and \((R_0)_I = (R_0)_S \). By
[Houston and Taylor 2007, Lemma 1.2], consider the pullback

\[
\begin{array}{c}
(R_0)_S \\
\downarrow
\end{array}
\begin{array}{c}
k = k_{\varphi(S)}
\end{array}
\begin{array}{c}
T_S \\
\downarrow
\end{array}
\begin{array}{c}
(T/I)_{\varphi(S)}
\end{array}
\]

As \((R_0)_S = (R_0)_I\) is an AV-domain, the extension \(k \subseteq (T/I)_{\varphi(S)}\) is a root extension by Lemma 3.2. So the extension \(k \subseteq T/I\) is a root extension. \(\square\)

**Theorem 3.6.** For the diagram \((\triangle')\), assume that \(T\) is an AV-domain. Then \(R\) is an AP-domain if and only if \(D\) is an AP-domain and the extension \(k = qf(D) \subseteq T/I\) is a root extension.

**Proof.** \((\Leftarrow)\) As in Theorem 3.3, we consider the diagram

\[
\begin{array}{c}
R \\
\downarrow
\end{array}
\begin{array}{c}
D \\
\downarrow
\end{array}
\begin{array}{c}
R_0 = \varphi^{-1}(k) \\
\downarrow
\end{array}
\begin{array}{c}
k = R_0/I \\
\downarrow
\end{array}
\begin{array}{c}
T \\
\downarrow
\end{array}
\begin{array}{c}
T/I
\end{array}
\]

Since \(T\) is an AV-domain, \(R_0\) is an AV-domain by Lemma 3.2. Then \(R\) is an AP-domain by Corollary 2.4.

\((\Rightarrow)\) Assume that \(R\) is an AP-domain; then \(D = R/I\) is an AP-domain by [AZ 1991, Theorem 4.10]. Also by Lemma 3.1, the extension \(k \subseteq T/I\) is a root extension. \(\square\)

### 4. Pullbacks of type \((\triangle^*)\)

**Lemma 4.1.** For a diagram \((\triangle^*)\), \(R\) is an AV-domain if and only if \(T\) and \(D\) are AV-domains.

**Proof.** The proof is similar to that of Lemma 3.2.

\((\Rightarrow)\) If \(R\) is an AV-domain, so are its homomorphic image of \(D\) and its overring \(T\).

\((\Leftarrow)\) Let \(x \in qf(R)\); then \(x \in qf(T)\). Since \(T\) is an AV-domain, there is a positive integer \(n = n(x)\) such that \(x^n \in T\) or \(x^{-n} \in T\). Assume that, for example, \(x^n \in T\). If \(x^n \in I\), then \(x^n \in R\). If \(x^n \in T \setminus I\), then \(\varphi(x)x^n = \varphi(x^n) \in T/I \setminus \{0\} \subseteq qf(T/I) = qf(D)\). Since \(D\) is an AV-domain, there is a positive integer \(m\) such that \(\varphi(x)^{nm} \in D\). Hence \(x^{nm} \in \varphi^{-1}(D) = R\). It follows that \(R\) is an AV-domain. \(\square\)

**Proposition 4.2.** Let \(R\) be an integral domain and \(I\) a nonzero ideal of \(R\). If \(R\) is an APVMD, then \((I_v : I_v)\) is an APVMD.
Proof. Set $T = (I_v : I_v)$. Assume that $x, y \in T = (I_v : I_v)$. Choose a fixed element $a \in I_v$. Then $ax, ay \in I_v \subseteq R$. Since $R$ is an APVMD, there is a positive integer $n = n(ax, ay)$ such that $((ax)^n, (ay)^n)$ is $t$-invertible in $R$. Let $J$ denote $((ax)^n, (ay)^n)$. So $(JJ^{-1})_t = R$. There is a finitely generated ideal $H \subseteq JJ^{-1} \subseteq R$ such that $HV = R$. By [Houston and Taylor 2007, Lemma 2.3], $(I_v : I_v)$ is $t$-linked over $R$. Then $(HT)_v = T$. So $(JJ^{-1})_t = T$. Thus $(a^n(x^n, y^n)J^{-1}T)_t = (((ax)^n, (ay)^n)J^{-1}T)_t = T$. So $(x^n, y^n)$ is $t$-invertible in $T$. Therefore $T = (I_v : I_v)$ is an APVMD.

Proposition 4.3. For a diagram $(\Delta^*)$, if $R$ is an APVMD, then $I$ is a prime $t$-ideal of both $R$ and $T$.

Proof. We claim $R_I$ is an AV-domain, and thus $I$ is a $t$-ideal of $R$. Let $x, y \in R \setminus \{0\}$. If $(x^n, y^n)(x^n, y^n)^{-1} \subseteq I$ for each positive integer $n$, then $((x^n, y^n)(x^n, y^n)^{-1})^{-1} \supseteq I^{-1} \supseteq T \supseteq R$, which contradicts that $R$ is an APVMD. Hence there exists a positive integer $n$ such that $(x^n, y^n)(x^n, y^n)^{-1} \not\subseteq I$. Thus $((x^n, y^n)(x^n, y^n)^{-1})R_I = R_I$. So $(x^n, y^n)R_I$ is invertible in $R_I$. Since $R_I$ is quasilocal, $(x^n, y^n)R_I$ is principal. Then $R_I$ is an AV-domain. So $IR_I$ is a maximal $t$-ideal of $R_I$. By [Kang 1989, Lemma 3.17], $I = IR_I \cap R$ is a $t$-ideal of $R$. Since $qf(D) = qf(T/I)$, we have $R_I = T_I$ by [Houston and Taylor 2007, Lemma 1.2]. So $T_I$ is an AV-domain. Then $IT_I$ is a maximal $t$-ideal of $T$. Therefore $I$ is a prime $t$-ideal of $T$.

Houston and Taylor [2007, Theorem 2.8] characterized the PVMD-property in a pullback of type $(\Delta^*)$. Similarly, we are ready to study the APVMD-property in a pullback of type $(\Delta^*)$. For convenience, let $E$ denote $T/I$.

Theorem 4.4. For a diagram $(\Delta^*)$, assume that $T = (I_v : I_v)$. Then $R$ is an APVMD if and only if $T$ is an APVMD and $T_I$ is an AV-domain, and for each nonzero prime ideal $P$ of $D$, either

1. $D_P$ and $T_{q^{-1}(D \setminus P)}$ are AV-domains, or
2. there is a finitely generated ideal $A$ of $D$ such that $A \subseteq P$, $A^{-1} \cap E = D$, and $(q^{-1}(P)T)_t = T$.

Proof. ($\Rightarrow$) Assume that $R$ is an APVMD. By Proposition 4.2, $T = (I_v : I_v)$ is an APVMD. Also, $T_I$ is an AV-domain by Proposition 4.3. Let $P$ be a prime ideal of $D$, and let $P = 

\varphi^{-1}(\tilde{P})$.

Case 1. If $P$ is a $t$-ideal of $R$, then $R_P$ is an AV-domain. By [Houston and Taylor 2007, Lemma 1.2], we have the pullback

\[
\begin{array}{ccc}
R_P & \longrightarrow & D_{\varphi(R \setminus P)} = D_{\tilde{P}} \\
\downarrow & & \downarrow \\
T_{R \setminus P} = T_{q^{-1}(D \setminus \tilde{P})} & \longrightarrow & E_{\varphi(S)} = E_{D \setminus \tilde{P}}
\end{array}
\]
By Lemma 4.1, $D_{\tilde{p}}$ and $T_{R \setminus P} = T_{\varphi^{-1}(D \setminus \tilde{p})}$ are AV-domains.

**Case 2.** Suppose that $P$ is not a $t$-ideal of $R$. Since $R$ is an APVMD, it is a UMT-domain by [Li 2012, Theorem 3.8]. By [Fontana et al. 1998, Corollary 1.6], $P_t = R$. Hence there is a finitely generated ideal $J \subseteq P$ such that $J^{-1} = R$. Since $T$ is $t$-linked over $R$ by [Houston and Taylor 2007, Lemma 2.3], we have $(JT)^{-1} = T$. So $(\varphi^{-1}(\tilde{p})T)_t = (PT)_t = T$. Now let $A = \varphi(J)$ and $e \in A^{-1} \cap E$. Then $\varphi(t) = e$ for some $t \in T$ and $eA \subseteq D$. Hence $\varphi^{-1}(eA) \subseteq \varphi^{-1}(D) = R$. Also, $\varphi^{-1}(eA) = \varphi^{-1}(e)\varphi^{-1}(A) = \varphi^{-1}(\varphi(t))\varphi^{-1}(\varphi(J)) \supseteq tJ$. So $tJ \subseteq R$. Then $t \in J^{-1} = R$. Thus $e = \varphi(t) \in D$. Therefore $A^{-1} \cap E = D$.

$(\Leftarrow)$ Let $P$ be a maximal $t$-ideal of $R$. It suffices to show that $R_P$ is an AV-domain.

**Case 1.** Assume that $I \not\subseteq P$. By Lemma 3.4, there is a prime ideal $Q$ of $T$ such that $P = Q \cap R$ and $R_P = T_Q$. By Proposition 4.3, we know that $I$ is a prime $t$-ideal of $R$. Then $(PT)_t \neq T$ by [Houston and Taylor 2007, Lemma 2.6]. Hence $PT \subseteq Q_1$ for some prime $t$-ideal $Q_1$ of $T$. Since $T = (I_v : I_v)$ is $t$-linked over $R$ by [Houston and Taylor 2007, Lemma 2.3], it follows that $(Q_1 \cap R)_t \neq R$. However, $P \subseteq Q_1 \cap R$ and $P$ is a maximal $t$-ideal of $R$. It follows that $Q = Q_1$. Then $Q$ is a $t$-ideal of $T$. Therefore $R_P = T_Q$ is an AV-domain.

**Case 2.** Assume that $I \subseteq P$. Let $\tilde{P}$ denote $\varphi(P)$. By way of contradiction, suppose that condition (2) of the hypothesis holds: there is a finitely generated ideal $A$ of $D$ such that $A \subseteq \tilde{P}$, $A^{-1} \cap E = D$, and $(\varphi^{-1}(\tilde{p})T)_t = (PT)_t = T$. Then $A = \varphi(J_1)$ and $(J_2T)^{-1} = T$ for some finitely generated ideals $J_1$, $J_2$ of $R$. Also $J_1 + J_2 \subseteq P$. Set $J = J_1 + J_2$. Then $J^{-1} \subseteq J_2^{-1}$. Let $x \in J_2^{-1}$; then $xJ_2 \subseteq R$, and hence $xJ_2T \subseteq T$. So $x \in (J_2T)^{-1} = T$. So $J^{-1} \subseteq J_2^{-1} \subseteq T$. Since $J \subseteq P$ and $P$ is a prime $t$-ideal of $R$, then $J^{-1} \neq R$. Otherwise, if $J^{-1} = R$, then $R = J_v \subseteq P_t = P$, a contradiction. So $R \not\subseteq J^{-1}$. Therefore, there is an element $t \in J^{-1} \setminus R$ with $tJ \subseteq R$. So $\varphi(t)A \subseteq \varphi(t)\varphi(J_1) \subseteq \varphi(t)\varphi(J) = \varphi(tJ) \subseteq D$. Then $\varphi(t) \in A^{-1} \cap E = D$. So $t \in R$, a contradiction. Hence condition (1) must hold. Localize the diagram at $P$ and consider the pullback

\[
\begin{array}{ccc}
R_P & \longrightarrow & D_{\varphi(R \setminus P)} = D_{\tilde{p}} \\
\downarrow & & \downarrow \\
T_{R \setminus P} = T_{\varphi^{-1}(D \setminus \tilde{p})} & \longrightarrow & E_{\varphi(S)} = E_{D \setminus \tilde{p}}
\end{array}
\]

By Lemma 4.1, $R_P$ is an AV-domain. Therefore, $R$ is an APVMD.

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References


QING LI

COLLEGE OF COMPUTER SCIENCE AND TECHNOLOGY

SOUTHWEST UNIVERSITY FOR NATIONALITIES

CHENGDU, 610041

CHINA

lqop80@163.com
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