CHARACTERIZING ALMOST PRÜFER $v$-MULTIPLICATION DOMAINS IN PULLBACKS

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Let $I$ be an ideal of an integral domain $T$, let $\varphi : T \to T/I$ be the projection, let $D$ be an integral domain contained in $T/I$, and let $R = \varphi^{-1}(D)$. We characterize when $R$ is an almost Prüfer $v$-multiplication domain, an almost valuation domain, and an almost Prüfer domain, in the context of pullbacks.

1. Introduction

Let $I$ be an ideal of an integral domain $T$, let $\varphi : T \to T/I$ be the natural projection, let $D$ be an integral domain contained in $T/I$, and let $k = qf(D)$ be the quotient field of $D$. Let $R = \varphi^{-1}(D)$ be the integral domain arising from the following pullback of canonical homomorphisms:

\[
\begin{array}{ccc}
R & R \\
\downarrow & \downarrow \\
D & D \\
\text{T} & \text{T}/I
\end{array}
\]

It is well-known that $D = R/I$ and that $I$ is a prime ideal of $R$. Notice that $I$ is a common ideal of $R$ and $T$, and hence $T$ is an overring of $R$. We assume that $R$ is properly contained in $T$, and we refer to this as a pullback diagram of type $(\triangle)$. For the diagram $(\triangle)$, if $qf(D) \subseteq T/I$, then we refer to this as a diagram of type $(\triangle')$. For the diagram $(\triangle)$, if $I$ is a prime ideal of $T$ and $qf(D) = qf(T/I)$, then we refer to this as a diagram of type $(\triangle^*)$. Here $qf(T/I)$ denotes the quotient field of $T/I$. For the diagram $(\triangle)$, if $I = M$ is a maximal ideal of $T$, we refer to this as a diagram of type $(\triangle_M)$. For the diagram $(\triangle_M)$, if $qf(D) = T/M$, then we refer to this as a diagram of type $(\triangle_{M}^*)$.

Pullbacks are an important tool in constructing interesting examples and counter-examples. They have become so important that in recent years there have been many papers devoted to ring- and ideal-theoretic properties in pullback domains.

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For more details on pullbacks, see [Mimouni 2004; Houston and Taylor 2007; Fontana and Gabelli 1996; Gilmer 1972; Gabelli and Houston 1997].

Zafrullah [1985] began a general theory of almost factoriality and introduced the notion of an almost GCD-domain. Zafrullah defined $R$ to be an almost GCD-domain (AGCD-domain for short) if for each $a, b \in D \setminus \{0\}$, there is a positive integer $n = n(a, b)$ such that $a^n D \cap b^n D$ is principal (or equivalently, $(a^n, b^n)_v$ is principal). Anderson and Zafrullah [AZ 1991] introduced several classes of integral domains related to almost GCD-domains, including almost Bézout domains (AB-domains), almost Prüfer domains (AP-domains), and almost valuation domains (AV-domains). As in [AZ 1991], an integral domain $R$ is an AB-domain if for each $a, b \in D \setminus \{0\}$, there is a positive integer $n = n(a, b)$ such that $(a^n, b^n)$ is principal; while $R$ is an AP-domain if for each $a, b \in D \setminus \{0\}$, there is a positive integer $n = n(a, b)$ such that $(a^n, b^n)$ is invertible. Following [AZ 1991], an integral domain $R$ is said to be an AV-domain if for each $a, b \in D \setminus \{0\}$, there is a positive integer $n = n(a, b)$ such that $a^n | b^n$ or $b^n | a^n$. Similarly, in [Li 2012] we defined an integral domain $R$ to be an almost Prüfer $v$-multiplication domain (APVMD) if for each $a, b \in R \setminus \{0\}$, there is a positive integer $n = n(a, b)$ such that $a^n D \cap b^n D$ is $t$-invertible, or equivalently, $(a^n, b^n)$ is $t$-invertible. Recall that an integral domain $R$ is said to be a Prüfer $v$-multiplication domain (PVMD) if each $a, b \in R \setminus \{0\}$, $(a, b)$ is $t$-invertible. The class of APVMDs includes a lot of important rings, such as AV-domains, AB-domains, AGCD-domains, AP-domains, PVMDs, and so on.

Anderson and Zafrullah [1991, Theorem 4.9] proved that $D$ is an AB-domain (respectively, AP-domain) if and only if $R = D + Xk[X]$ is an AB-domain (respectively, AP-domain). However, we notice that the $(D + Xk[X])$-construction is a special case of the pullback of type $(\triangle_M)$. Mimouni [2004, Theorem 2.2] generalized these results and proved that for the diagram $(\triangle_M)$, $R$ is an AP-domain if and only if $T$ and $D$ are AP-domains and the extension $k \subseteq T/M$ is a root extension. He also gave a similar characterization for AV-domains. Mimouni [2004, Corollary 2.6] continued to show that for the diagram $(\triangle_M)$, assuming that $D = k$ is a field, then $R$ is an AB-domain if and only if $T$ is an AB-domain and the extension $k \subseteq T/M$ is a root extension. In [Li 2012, Theorem 3.10], we proved that $D$ is an APVMD if and only if $R = D + Xk[X]$ is an APVMD.

From this we notice that the characterization of AV-domains and AP-domains is known only in the context of the special pullback of type $(\triangle_M)$, and that the study of APVMDs is only in the $(D + Xk[X])$-construction, a special case of type $(\triangle_M)$. So the main purpose of this paper is to characterize APVMDs in pullbacks in greater generality and to generalize the characterization of AV-domains and AP-domains for the pullback of type $(\triangle_M)$ to that for the pullback of type $(\triangle')$.

In Section 2, we mainly prove that in the pullback of type $(\triangle_M)$, $R$ is an APVMD if and only if $D$ and $T$ are APVMDs, $T_M$ is an AV-domain, and the extension
$qf(D) \subseteq T/M$ is a root extension. Using this fact, we give Example 2.2 to show that an APVMD is not necessarily a PVMD. We also show that for the diagram $(\triangle_M^*)$, $R$ is an APVMD if and only if $D$ and $T$ are APVMDs and $T_M$ is an AV-domain. Using this result, we prove that $D$ is an APVMD if and only if $R = D + Xk\llbracket X \rrbracket$ is an APVMD.

In Section 3, we mainly indicate that for the diagram $(\triangle')$, if $T$ is an AV-domain, then $R$ is an APVMD if and only if $D$ and $T$ are APVMDs and $T_M$ is an AV-domain. Using this result, we prove that $D$ is an APVMD if and only if $R = D + Xk\llbracket X \rrbracket$ is integrally closed, but $R$ is not an APVMD. Consider the following pullback:

$$
\begin{array}{c}
R = D + XD_S[X] \\
\downarrow \\
T = D_S[X] \\
\downarrow \\
D_S \cong T/I
\end{array}
$$

Here $I$ denotes $XD_S[X]$. The example indicates that $qf(D) = qf(T/I)$, $D$ and $T$ are APVMDs, $I$ is principal in $T$, and $T = D_S[X]$ is a PVMD. It follows that $T_I$ is an AV-domain by [Li 2012, Theorem 2.3]. However, $R$ is not an APVMD. The pullback above belongs to the pullback of type $(\triangle^*)$. Therefore, for the diagram $(\triangle^*)$, without some other assumption on $T$, $D$ or $T/I$, there is no hope of proving that $R$ is an APVMD even when $T$ and $D$ are APVMDs and $T_I$ is an AV-domain. So in Section 4, we prove that in a pullback of type $(\triangle^*)$, if $T = (I_v : I_v)$, then $R$ is an APVMD if and only if $T$ is an APVMD, $T_I$ is an AV-domain, and for each nonzero prime ideal $\bar{P}$ of $D$, either (1) $D_{\bar{P}}$ and $T_{\varphi^{-1}(D_{\bar{P}})}$ are AV-domains, or (2) there exists a finitely generated ideal $A$ of $D$ such that $A \subseteq \bar{P}$, $A^{-1} \cap E = D$, and $(\varphi^{-1}(\bar{P})T)_I = T$.

For details on star operations, see [Gilmer 1972, Sections 32 and 34].

2. Pullbacks of type $(\triangle_M)$

We begin with the characterization of APVMDs in a pullback of type $(\triangle_M)$.

**Theorem 2.1.** For the diagram $(\triangle_M)$, $R$ is an APVMD if and only if $D$ and $T$ are APVMDs, $T_M$ is an AV-domain, and the extension $qf(D) \subseteq T/M$ is a root extension.
Proof. ($\Rightarrow$) Assume that $R$ is an APVMD. Let $x, y \in D \setminus \{0\}$; then $\varphi(a) = x$ and $\varphi(b) = y$ for some $a, b \in R \setminus M$. Because $R$ is an APVMD, there is a positive integer $n = n(a, b)$ such that $(a^n, b^n)$ is $t$-invertible in $R$. By [Wang 2006, Theorem 10.3.11], $(\varphi(a^n), \varphi(b^n))$ is $t$-invertible in $D$. Because $(x^n, y^n) = (\varphi(a)^n, \varphi(b)^n) = (\varphi(a^n), \varphi(b^n))$, it follows that $(x^n, y^n)$ is $t$-invertible in $D$. Thus $D$ is an APVMD. Let $c, d \in T \setminus \{0\}$. Because $T$ and $R$ have the same quotient field, there is an element $r \in R \setminus \{0\}$ with $rc, rd \in R$. Then $((rc)^n, (rd)^n)R$ is a $t$-invertible ideal of $R$ for some positive integer $n$. According to [Wang 2006, Theorem 10.3.11], $((rc)^n, (rd)^n)T$ is $t$-invertible in $T$. It is well-known that $((rc)^n, (rd)^n)T = r^n(c^n, d^n)T$, so $(c^n, d^n)T$ is $t$-invertible in $T$. Therefore $T$ is an APVMD. As we know, $M$ is a $v$-ideal of $R$. Then $R_M$ is an AV-domain by [Li 2012, Theorem 2.3]. By [Wang 2006, Theorem 10.2.2], we have the pullback \[
abla R_M \longrightarrow D_{R \setminus M} \hfill
abla \]
\[
abla T_M \longrightarrow T / M \hfill
abla \]
By [Mimouni 2004, Theorem 2.2], $T_M$ and $D_{R \setminus M}$ are AV-domains and the extension $\mathit{qf}(D) \subseteq T / M$ is a root extension.

($\Leftarrow$) Let $P$ be a maximal $t$-ideal of $R$.

Case 1. Suppose that $M \nsubseteq P$. By [Wang 2006, Theorem 10.2.4(3)], there is a prime ideal $Q$ of $T$ with $P = Q \cap R$. Clearly, $M \nsubseteq Q$. In fact $P \nsubseteq M$. Because $M$ is a $v$-ideal of $R$, $M$ is a $t$-ideal of $R$. As the maximality, $P \nsubseteq M$. So $Q \nsubseteq M$. Hence $Q$ is incomparable to $M$. According to [Fontana et al. 1998, Lemma 3.3], $Q$ is a maximal $t$-ideal of $T$. Since $T$ is an APVMD, $T_Q$ is an AV-domain. By [Wang 2006, Theorem 10.2.1(6)], $R_P = T_Q$. Hence $R_P$ is an AV-domain.

Case 2. Suppose that $M \subseteq P$. There exists a prime ideal $p$ of $D$ such that $P = \varphi^{-1}(p)$. Because $P$ is a $t$-ideal of $R$, $P = P_t$. Then $\varphi^{-1}(p) = (\varphi^{-1}(p))_t = \varphi^{-1}(p_t)$ by [Wang 2006, Theorem 10.3.5(3)]. So $p = p_t$. Thus $p$ is a $t$-ideal of $D$. Since $D$ is an APVMD, $D_p$ is an AV-domain. In this case, consider the following pullback:

\[
abla R_P \longrightarrow D_p \hfill
abla \]
\[
abla T_M \longrightarrow T / M \hfill
abla \]
Since $T_M$ and $D_p$ are AV-domains and the extension $\mathit{qf}(D) \subseteq T / M$ is a root extension, $R_P$ is an AV-domain by [Mimouni 2004, Theorem 2.2]. Therefore $R$ is an APVMD. \qed
Gabelli and Houston [1997, Theorem 4.13] showed that for the diagram $(\triangle_M)$, $R$ is a PVMD if and only if $T$ and $D$ are PVMDs, $k = T/M$, and $T_M$ is a valuation domain. Using this result and Theorem 2.1, we can easily get the following result.

**Example 2.2.** Let $R = K + XL[X]$, where $K$ and $L$ are fields, $K \subseteq L$, and for some prime $p$, $L^p \subseteq K$. Consider the pullback

$$
\begin{array}{c}
K + XL[X] \longrightarrow K \\
\downarrow \\
L[X] \longrightarrow L
\end{array}
$$

Then $R$ is an APVMD but not a PVMD. Thus an APVMD need not be a PVMD.

**Corollary 2.3.** For the diagram $(\triangle^*_M)$, $R$ is an APVMD if and only if $D$ and $T$ are APVMDs and $T_M$ is an AV-domain.

*Proof.* It easily follows from Theorem 2.1 and [Mimouni 2004, Lemma 2.3].

**Corollary 2.4.** For the diagram $(\triangle^*_M)$, $R$ is an AP-domain if and only if $D$ and $T$ are AP-domains.

*Proof.* $(\Rightarrow)$ It follows from [Mimouni 2004, Theorem 2.2].

$(\Leftarrow)$ Let $P$ be a maximal ideal of $R$.

**Case 1.** Suppose that $M \not\subseteq P$. By [Wang 2006, Theorems 10.2.4(3) and 10.2.1(6)], there is a prime ideal $Q$ of $T$ with $P = Q \cap R$ and $R_P = T_Q$. Since $T$ is an AP-domain, $T_Q$ is an AV-domain by [AZ 1991, Theorem 5.8]. Hence $R_P$ is an AV-domain.

**Case 2.** Suppose that $M \subseteq P$. There exists a prime ideal $p$ of $D$ such that $P = \varphi^{-1}(p)$. Since $D$ is an AP-domain, $D_p$ is an AV-domain. In this case, consider the pullback

$$
\begin{array}{c}
R_P \longrightarrow D_p \\
\downarrow \\
T_M \longrightarrow T/M
\end{array}
$$

Since $T_M$ and $D_p$ are AV-domains and $qf(D) = qf(D_p) = T/M$, $R_P$ is an AV-domain by [Mimouni 2004, Lemma 2.3]. Therefore $R$ is an AP-domain.

**Proposition 2.5.** For the diagram $(\triangle_M)$, suppose that $(T, M)$ is a quasilocal domain and $D = k$ is a proper field of $T/M$. Then $R$ is an APVMD if and only if $R$ is an AV-domain.

*Proof.* $(\Leftarrow)$ It easily follows from their definitions.

$(\Rightarrow)$ Assume that $D$ is a field. Since $D = R/M$, $M$ is a maximal ideal of $R$. Because $T$ is quasilocal, $R$ is quasilocal by [Wang 2006, Corollary 10.2.1]. Also
\[M = (R : T)\] is a \(v\)-ideal of \(R\). Hence \(M\) is the unique maximal \(t\)-ideal of \(R\). Therefore \(R = R_M\) is an AV-domain. \[\square\]

In [Li 2012, Theorem 3.10], we considered the polynomial ring case and proved that \(D\) is an APVM if and only if \(R = D + Xk[X]\) is an APVM. Similarly, we consider the power series ring case and get the following result.

**Corollary 2.6.** Let \(D\) be an integral domain with quotient field \(k\). Then \(D\) is an APVM if and only if \(R = D + Xk[[X]]\) is an APVM.

**Proof.** Consider the pullback

\[
\begin{align*}
R &= D + Xk[[X]] \\
\downarrow &\quad \downarrow \\
T &= k[[X]] \\
\quad \downarrow &\quad \downarrow \\
k &= k[[X]]/Xk[[X]]
\end{align*}
\]

\(T = k[[X]]\) is a UFD, so \(T\) is an APVM. The rest follows from Corollary 2.3. \[\square\]

### 3. Pullbacks of type \((\Delta')\)

Mimouni [2004] considered the pullbacks of type \((\Delta_M)\) in AP-domains and AV-domains. He proved that for the diagram \((\Delta_M)\), \(R\) is an AV-domain (respectively AP-domain) if and only if \(T\) and \(D\) are AV-domains (respectively AP-domains) and the extension \(k \subseteq T/M\) is a root extension. We generalize these results for the special pullback of type \((\Delta_M)\) to those for the pullback of type \((\Delta')\).

**Lemma 3.1.** For the diagram \((\Delta')\), if \(R\) is an AP-domain (resp. AGCD-domain), then the extension \(k = qf(D) \subseteq T/I\) is a root extension.

**Proof.** Assume that \(R\) is an AP-domain (resp. AGCD-domain). By way of contradiction, suppose that the extension \(k \subseteq T/I\) is not a root extension. So there is \(\lambda \in T/I\) such that \(\lambda^n\) is not in \(k\) for each positive integer \(n\). Set \(\lambda = \varphi(a)\) for some \(a \in T \setminus I\). Let \(b\) be a nonzero fixed element of \(I\). Since \(R\) is an AP-domain (resp. AGCD-domain), \(((ab)^n, b^n)\) is invertible (resp. \(((ab)^n, b^n)\) is principal) for some positive integer \(n\). Let \(J\) denote \(((ab)^n, b^n)\). Then \(JJ^{-1} = R\) (resp. \(J_v = cR\) for some \(c \in R\)). By [Wang 2006, Example 8.1.10(1)], \(J^{-1} = (ab)^{-n}R \cap b^{-n}R\). Let \(f \in J^{-1}\); then \(f = (ab)^{-n}f_1 = b^{-n}f_2\) for some \(f_1, f_2 \in R\). Thus \(a^{-n}f_1 = f_2\) and so \(f_1 = a^n f_2\). If \(f_2\) is not in \(I\), then \(\varphi(f_2) \in D \setminus \{0\}\). Hence \(\varphi(f_1) = \varphi(a^n f_2) = \varphi(a)^n \varphi(f_2) = \lambda^n \varphi(f_2)\). So \(\lambda^n \in qf(D) = k\), a contradiction. Therefore \(f_2 \in I\). So \(J^{-1} \subseteq b^{-n}I\). We claim \(b^{-n}I \subseteq J^{-1}\). Let \(z \in I\) and \(x \in J\) and write \(x = \alpha(ab)^n + \beta b^n\) for some \(\alpha, \beta \in R\). Then \((b^{-n}z)x = (b^{-n}z)(\alpha(ab)^n + \beta b^n) = z\alpha a^n + z\beta \subseteq I \subseteq R\), so \(b^{-n}z \in J^{-1}\). Then \(b^{-n}I \subseteq J^{-1}\). Therefore \(b^{-n}I = J^{-1}\). So \(J_v = b^n I^{-1}\). Since \(JJ^{-1} = R\) (resp. \(J_v = cR\)), we have \(1 = g_1 h_1 + \cdots + g_m h_m\) for \(g_1, \ldots, g_m \in J\), \(h_1, \ldots, h_m \in J^{-1}\) (resp. \(b^n I^{-1} = cR\)). For each \(i \in \{1, 2, \ldots, m\}\), write \(g_i = \)
\[ \alpha_i(ab)^n + \beta_i b^n \] and \( h_i = b^{-n} f_i \), where \( \alpha_i, \beta_i \in R, f_i \in I \). Then we have \( 1 = g_1 h_1 + \cdots + g_m h_m = (\alpha_1(ab)^n + \beta_1 b^n)(b^{-n} f_1) + \cdots + (\alpha_m(ab)^n + \beta_m b^n)(b^{-n} f_m) = (\alpha_1 a^n + \beta_1) f_1 + \cdots + (\alpha_m a^n + \beta_m) f_m \in I \), which is absurd. (Respectively, for each \( y \in I^{-1} \), \( TyI \subseteq yI \subseteq R \), so \( Ty \in I^{-1} \), hence \( T \subseteq (I^{-1} : I^{-1}) \). Then \( R \subseteq T \subseteq (I^{-1} : I^{-1}) = (b^n I^{-1} : b^n I^{-1}) = (J^{-1} : J^{-1}) = (cR : cR) = R \), which is absurd.) Therefore the extension \( k \subseteq T/I \) is a root extension. \( \square \)

**Lemma 3.2.** For the diagram \((\triangle')\), assume that \( D = k \) is a field. Then \( R \) is an AV-domain if and only if \( T \) is an AV-domain and the extension \( k \subseteq T/I \) is a root extension.

**Proof.** \((\Rightarrow)\) It follows from Lemma 3.1 and the fact that \( T \) is an overring of \( R \).

\((\Leftarrow)\) Let \( x \in qf(R) \); then \( x \in qf(T) \). Since \( T \) is an AV-domain, there is a positive integer \( n = n(x) \) such that \( x^n \in T \) or \( x^{-n} \in T \). Assume that, for example, \( x^n \in T \). If \( x^n \in I \), then \( x^n \in R \). If \( x^n \in T \setminus I \), then \( \phi(x)^n = \phi(x^n) \in T/I \setminus \{0\} \). Since the extension \( k \subseteq T/I \) is a root extension, there is a positive integer \( m \) such that \( \phi(x^{nm}) = \phi(x)^{nm} \in k \). Hence \( x^{nm} \in \phi^{-1}(k) = R \). It follows that \( R \) is an AV-domain. \( \square \)

**Theorem 3.3.** For the diagram \((\triangle')\), \( R \) is an AV-domain if and only if \( T \) and \( D \) are AV-domains and the extension \( k = qf(D) \subseteq T/I \) is a root extension.

**Proof.** \((\Rightarrow)\) By [AZ 1991, Lemma 4.5], \( T \) is an AV-domain as an overring of \( R \); and by [AZ 1991, Theorem 4.10], \( D = R/I \) is an AV-domain. Also by Lemma 3.1, the extension \( k = qf(D) \subseteq T/I \) is a root extension.

\((\Leftarrow)\) We use the fact that the diagram \((\triangle')\) splits into two parts as follows:

\[
\begin{array}{ccc}
R & \longrightarrow & D \\
\downarrow & & \downarrow \\
R_0 = \phi^{-1}(k) & \longrightarrow & k = R_0/I \\
\downarrow & & \downarrow \\
T & \longrightarrow & T/I \\
\end{array}
\]

Consider the second part of this diagram:

\[
\begin{array}{ccc}
R_0 & \longrightarrow & k \\
\downarrow & & \downarrow \\
T & \longrightarrow & T/I \\
\end{array}
\]
Since $T$ is an AV-domain and the extension $k \subseteq T/I$ is a root extension, by Lemma 3.2 $R_0$ is an AV-domain. The first part of the diagram—

\[
\begin{array}{ccc}
R & \longrightarrow & D \\
\downarrow & & \downarrow \\
R_0 & \longrightarrow & k
\end{array}
\]

—is a pullback diagram of type $(\Delta^*_M)$. Since $D$ and $R_0$ are AV-domains, $R$ is an AV-domain by [Mimouni 2004, Lemma 2.3]. □

**Lemma 3.4.** For the diagram $(\triangle)$, let $Q(A) = \{x \in T \mid x I \subseteq A\}$ for an ideal $A$ of $R$. Then if $P$ is a prime ideal of $R$ and $I \nsubseteq P$, then $Q(P)$ is a prime ideal of $T$, $P = Q(P) \cap R$ and $R_P = T_{Q(P)}$.

**Proof.** Let $I \nsubseteq P$, let $x, y \in T$, and let $xy \in Q(P)$. Then $xyI^2 \subseteq xyI \subseteq P$. Since $xI, yI \subseteq I \subseteq R$ and $P$ is a prime ideal of $R$, we have $xI \subseteq P$ or $yI \subseteq P$. So $x \in Q(P)$ or $y \in Q(P)$. Thus $Q(P)$ is a prime ideal of $T$. We claim $P = Q(P) \cap R$. Because $PI \subseteq P$, we have $P \subseteq Q(P) \cap R$. Let $x \in Q(P) \cap R$; then $xI \subseteq P$. Since $I \nsubseteq P$, we have $x \in P$. Hence $Q(P) \cap R \subseteq P$. Thus $P = Q(P) \cap R$. Next we show that $R_P = T_{Q(P)}$. It easily follows that $R_P \subseteq T_{Q(P)}$. For the reverse inclusion, let $x \in T_{Q(P)}$. Then $x = z_1/z_2$ for some $z_1 \in T, z_2 \in T \setminus Q(P)$. Since $I \nsubseteq P$, there exists $u \in I \setminus P$. Of course $u \in I \setminus Q(P)$. Then $uz_1 \in I \subseteq R, uz_2 \in I \setminus Q(P) \subseteq R \setminus Q(P)$. Thus $uz_2 \in R \setminus P$. So $x = uz_1/uz_2 \in R_P$. Thus $T_{Q(P)} \subseteq R_P$, so $R_P = T_{Q(P)}$. □

**Theorem 3.5.** For the diagram $(\triangle')$, assume that $T$ is an AV-domain. Then $R$ is an APVMD if and only if $D$ is an APVMD and the extension $k = qf(D) \subseteq T/I$ is a root extension.

**Proof.** As in Theorem 3.3, we consider the diagram

\[
\begin{array}{ccc}
R & \longrightarrow & D \\
\downarrow & & \downarrow \\
R_0 = \varphi^{-1}(k) & \longrightarrow & k = R_0/I \\
\downarrow & & \downarrow \\
T & \longrightarrow & T/I
\end{array}
\]

$(\Leftarrow)$ Since $T$ is an AV-domain, $R_0$ is an AV-domain by Lemma 3.2. Because $D$ is an APVMD, by Corollary 2.3 $R$ is an APVMD.

$(\Rightarrow)$ Assume that $R$ is an APVMD; by Corollary 2.3 $D$ and $R_0$ are APVMDs and $(R_0)_I$ is an AV-domain. Set $S = R \setminus I$. Then $R_S = R_I$ and $(R_0)_I = (R_0)_S$. By
[Houston and Taylor 2007, Lemma 1.2], consider the pullback

\[
\begin{array}{ccc}
(R_0)_S & \rightarrow & k = k_{\varphi(S)} \\
\downarrow & & \downarrow \\
T_S & \rightarrow & (T/I)_{\varphi(S)}
\end{array}
\]

As \((R_0)_S = (R_0)_I\) is an AV-domain, the extension \(k \subseteq (T/I)_{\varphi(S)}\) is a root extension by Lemma 3.2. So the extension \(k \subseteq T/I\) is a root extension. □

**Theorem 3.6.** For the diagram \((\triangle')\), assume that \(T\) is an AV-domain. Then \(R\) is an AP-domain if and only if \(D\) is an AP-domain and the extension \(k = qf(D) \subseteq T/I\) is a root extension.

**Proof.** \((\Leftarrow)\) As in Theorem 3.3, we consider the diagram

\[
\begin{array}{ccc}
R & \rightarrow & D \\
\downarrow & & \downarrow \\
R_0 = \varphi^{-1}(k) & \rightarrow & k = R_0/I \\
\downarrow & & \downarrow \\
T & \rightarrow & T/I
\end{array}
\]

Since \(T\) is an AV-domain, \(R_0\) is an AV-domain by Lemma 3.2. Then \(R\) is an AP-domain by Corollary 2.4.

\((\Rightarrow)\) Assume that \(R\) is an AP-domain; then \(D = R/I\) is an AP-domain by [AZ 1991, Theorem 4.10]. Also by Lemma 3.1, the extension \(k \subseteq T/I\) is a root extension. □

### 4. Pullbacks of type \((\triangle^*)\)

**Lemma 4.1.** For a diagram \((\triangle^*)\), \(R\) is an AV-domain if and only if \(T\) and \(D\) are AV-domains.

**Proof.** The proof is similar to that of Lemma 3.2.

\((\Rightarrow)\) If \(R\) is an AV-domain, so are its homomorphic image of \(D\) and its overring \(T\).

\((\Leftarrow)\) Let \(x \in qf(R)\); then \(x \in qf(T)\). Since \(T\) is an AV-domain, there is a positive integer \(n = n(x)\) such that \(x^n \in T\) or \(x^{-n} \in T\). Assume that, for example, \(x^n \in T\). If \(x^n \in I\), then \(x^n \in R\). If \(x^n \in T \setminus I\), then \(\varphi(x)^n = \varphi(x^n) \in T/I \setminus \{0\} \subseteq qf(T/I) = qf(D)\). Since \(D\) is an AV-domain, there is a positive integer \(m\) such that \(\varphi(x)^{nm} \in D\). Hence \(x^{nm} \in \varphi^{-1}(D) = R\). It follows that \(R\) is an AV-domain. □

**Proposition 4.2.** Let \(R\) be an integral domain and \(I\) a nonzero ideal of \(R\). If \(R\) is an APVMD, then \((I_v : I_v)\) is an APVMD.
Proof. Set $T = (I_v : I_v)$. Assume that $x, y \in T = (I_v : I_v)$. Choose a fixed element $a \in I_v$. Then $ax, ay \in I_v \subseteq R$. Since $R$ is an APVMD, there is a positive integer $n = n(ax, ay)$ such that $((ax)^n, (ay)^n)$ is $t$-invertible in $R$. Let $J$ denote $((ax)^n, (ay)^n)$. So $(JJ^{-1})_t = R$. There is a finitely generated ideal $H \subseteq JJ^{-1} \subseteq R$ such that $H_v = R$. By [Houston and Taylor 2007, Lemma 2.3], $(I_v : I_v)$ is $t$-linked over $R$. Then $(HT)_v = T$. So $(JJ^{-1}T)_t = T$. Thus $(a^n(ax^n, ay^n)J^{-1}T)_t = (((ax)^n, (ay)^n)J^{-1}T)_t = T$. So $(x^n, y^n)$ is $t$-invertible in $T$. Therefore $T = (I_v : I_v)$ is an APVMD.

**Proposition 4.3.** For a diagram $(\triangle^*)$, if $R$ is an APVMD, then $I$ is a prime $t$-ideal of both $R$ and $T$.

Proof. We claim $R_I$ is an AV-domain, and thus $I$ is a $t$-ideal of $R$. Let $x, y \in R \setminus \{0\}$. If $(x^n, y^n)(x^n, y^n)^{-1} \subseteq I$ for each positive integer $n$, then $((x^n, y^n)(x^n, y^n)^{-1})^{-1} \supseteq I^{-1} \supseteq T \supseteq R$, which contradicts that $R$ is an APVMD. Hence there exists a positive integer $n$ such that $(x^n, y^n)(x^n, y^n)^{-1} \not\subseteq I$. Thus $((x^n, y^n)(x^n, y^n)^{-1})R_I = R_I$. So $(x^n, y^n)R_I$ is invertible in $R_I$. Since $R_I$ is quasilocal, $(x^n, y^n)R_I$ is principal. Then $R_I$ is an AV-domain. So $IR_I$ is a maximal $t$-ideal of $R_I$. By [Kang 1989, Lemma 3.17], $I = IR_I \cap R$ is a $t$-ideal of $R$. Since $qf(D) = qf(T/I)$, we have $R_I = T_I$ by [Houston and Taylor 2007, Lemma 1.2]. So $T_I$ is an AV-domain. Then $IT_I$ is a maximal $t$-ideal of $T$. Therefore $I$ is a prime $t$-ideal of $T$.

Houston and Taylor [2007, Theorem 2.8] characterized the PVMD-property in a pullback of type $(\triangle^*)$. Similarly, we are ready to study the APVMD-property in a pullback of type $(\triangle^*)$. For convenience, let $E$ denote $T/I$.

**Theorem 4.4.** For a diagram $(\triangle^*)$, assume that $T = (I_v : I_v)$. Then $R$ is an APVMD if and only if $T$ is an APVMD and $T_I$ is an AV-domain, and for each nonzero prime ideal $\bar{P}$ of $D$, either

1. $D_{\bar{P}}$ and $T_{qf^{-1}(D \setminus \bar{P})}$ are AV-domains, or
2. there is a finitely generated ideal $A$ of $D$ such that $A \subseteq \bar{P}$, $A^{-1} \cap E = D$, and $(\varphi^{-1}(\bar{P})T)_t = T$.

**Proof.** ($\Rightarrow$) Assume that $R$ is an APVMD. By Proposition 4.2, $T = (I_v : I_v)$ is an APVMD. Also, $T_I$ is an AV-domain by Proposition 4.3. Let $\bar{P}$ be a prime ideal of $D$, and let $P = \varphi^{-1}(\bar{P})$.

**Case 1.** If $P$ is a $t$-ideal of $R$, then $R_P$ is an AV-domain. By [Houston and Taylor 2007, Lemma 1.2], we have the pullback

\[
\begin{array}{cccc}
R_P & \rightarrow & D_{\varphi^{-1}(P \setminus \bar{P})} = D_{\bar{P}} \\
\downarrow & & \downarrow \\
T_{R \setminus P} = T_{\varphi^{-1}(D \setminus \bar{P})} & \rightarrow & E_{\varphi(S)} = E_{D \setminus \bar{P}}
\end{array}
\]
By Lemma 4.1, $D\bar{p}$ and $T_{R\setminus P} = T_{\varphi^{-1}(D\setminus \bar{p})}$ are AV-domains.

**Case 2.** Suppose that $P$ is not a $t$-ideal of $R$. Since $R$ is an APVMD, it is a UMT-domain by [Li 2012, Theorem 3.8]. By [Fontana et al. 1998, Corollary 1.6], $P_t = R$. Hence there is a finitely generated ideal $J \subseteq P$ such that $J^{-1} = R$. Since $T$ is $t$-linked over $R$ by [Houston and Taylor 2007, Lemma 2.3], we have $(JT)^{-1} = T$. So $(\varphi^{-1}(\bar{P})T)_t = (PT)_t = T$. Now let $A = \varphi(J)$ and $e \in A^{-1} \cap E$. Then $\varphi(t) = e$ for some $t \in T$ and $eA \subseteq D$. Hence $\varphi^{-1}(eA) \subseteq \varphi^{-1}(D) = R$. Also, $\varphi^{-1}(eA) = \varphi^{-1}(e)\varphi^{-1}(A) = \varphi^{-1}(\varphi(t))\varphi^{-1}(\varphi(J)) \supseteq tJ$. So $tJ \subseteq R$. Then $t \in J^{-1} = R$. Thus $e = \varphi(t) \in D$. Therefore $A^{-1} \cap E = D$.

$(\Leftarrow)$ Let $P$ be a maximal $t$-ideal of $R$. It suffices to show that $R_P$ is an AV-domain.

**Case 1.** Assume that $I \not\subseteq P$. By Lemma 3.4, there is a prime ideal $Q$ of $T$ such that $P = Q \cap R$ and $R_P = T_Q$. By Proposition 4.3, we know that $I$ is a prime $t$-ideal of $R$. Then $(PT)_t \neq T$ by [Houston and Taylor 2007, Lemma 2.6]. Hence $PT \subseteq Q_1$ for some prime $t$-ideal $Q_1$ of $T$. Since $T = (I_v : I_v)$ is $t$-linked over $R$ by [Houston and Taylor 2007, Lemma 2.3], it follows that $(Q_1 \cap R)_t \neq R$. However, $P \subseteq Q_1 \cap R$ and $P$ is a maximal $t$-ideal of $R$. It follows that $Q = Q_1$. Then $Q$ is $t$-ideal of $T$. Therefore $R_P = T_Q$ is an AV-domain.

**Case 2.** Assume that $I \subseteq P$. Let $\bar{P}$ denote $\varphi(P)$. By way of contradiction, suppose that condition (2) of the hypothesis holds: there is a finitely generated ideal $A$ of $D$ such that $A \subseteq \bar{P}$, $A^{-1} \cap E = D$, and $(\varphi^{-1}(\bar{P})T)_t = (PT)_t = T$. Then $A = \varphi(J_1)$ and $(J_2T)^{-1} = T$ for some finitely generated ideals $J_1$, $J_2$ of $R$. Also $J_1 + J_2 \subseteq P$. Set $J = J_1 + J_2$. Then $J^{-1} \subseteq J_2^{-1}$. Let $x \in J_2^{-1}$; then $xJ_2 \subseteq R$, and hence $xJ_2T \subseteq T$. So $x \in (J_2T)^{-1} = T$. So $J^{-1} \subseteq J_2^{-1} \subseteq T$. Since $J \subseteq P$ and $P$ is a prime $t$-ideal of $R$, then $J^{-1} \neq R$. Otherwise, if $J^{-1} = R$, then $R = J_v \subseteq P_t = P$, a contradiction. So $R \not\subseteq J^{-1}$. Therefore, there is an element $t \in J^{-1} \setminus R$ with $tJ \subseteq R$. So $\varphi(t)A \subseteq \varphi(t)\varphi(J_1) \subseteq \varphi(t)\varphi(J) = \varphi(tJ) \subseteq D$. Then $\varphi(t) \in A^{-1} \cap E = D$. So $t \in R$, a contradiction. Hence condition (1) must hold. Localize the diagram at $P$ and consider the pullback

$$
\begin{array}{ccc}
R_P & \to & D_{\varphi(R \setminus P)} = D\bar{p} \\
\downarrow & & \downarrow \\
T_{R\setminus P} = T_{\varphi^{-1}(D\setminus \bar{p})} & \to & E_{\varphi(S)} = E_{D \setminus \bar{p}}
\end{array}
$$

By Lemma 4.1, $R_P$ is an AV-domain. Therefore, $R$ is an APVMD.

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