NOTE ON THE RELATIONS IN THE TAUETOLOGICAL RING OF $M_g$

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We give some nontrivial relations in the tautological ring of $\mathcal{M}_g$. These are derived from some new geometric relations obtained by localization on the moduli of stable quotients, which was recently introduced by A. Marian, D. Oprea and R. Pandharipande.

1. Introduction

We denote by $\mathcal{M}_g$ the moduli space of smooth curves of genus $g \geq 2$ over an algebraically closed field. Let $\pi : \mathcal{C}_g \to \mathcal{M}_g$ be its tautological family and $\omega_\pi$ be the dualizing sheaf. We denote by $\mathcal{E} = \pi_* \omega_\pi$ the Hodge bundle. Define $\kappa_i = \pi_*(c_1(\omega_\pi)^{i+1}) \in A^i(\mathcal{M}_g)$, $\lambda_i = c_i(\mathcal{E})$, and in particular, $k_0 = 2g - 2$, $k_{-1} = 0$. The tautological ring $R^*(\mathcal{M}_g)$ is defined to be the subring generated by $\lambda$-classes and $\kappa$-classes. By Mumford’s formula [1983], the tautological ring is in fact generated by the $\kappa$-classes $\kappa_1, \ldots, \kappa_{g-2}$.

Faber [1999] proposed a series of remarkable conjectures about the structure of $R^*(\mathcal{M}_g)$:

(a) The tautological ring $R^*(\mathcal{M}_g)$ is Gorenstein with socle in degree $g - 2$, and when an isomorphism $R^{g-2}(\mathcal{M}_g) = \mathbb{Q}$ is fixed, the natural pairing

$$R^i(\mathcal{M}_g) \times R^{g-2-i}(\mathcal{M}_g) \to R^{g-2}(\mathcal{M}_g) = \mathbb{Q}$$

is perfect.

(b) The $[g/3]$ classes $\kappa_1, \ldots, \kappa_{[g/3]}$ generate the ring $R^*(\mathcal{M}_g)$, with no relations in degrees $\leq [g/3]$.

(c) Let $\sum_{j=1}^n d_j = g - 2$ and $d_j \geq 0$. Then

$$\sum_{\sigma \in S_n} \kappa_\sigma = \frac{(2g - 3 + n)!}{(2g - 2)!! \prod_{j=1}^n (2d_j + 1)!!} \kappa_{g-2},$$

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where $\kappa_\sigma$ is defined as follows: write the permutation $\sigma = \beta_1 \ldots \beta_{v(\sigma)}$, where we think of the symmetric group $S_n$ as acting on the $n$-tuple $(d_1, \ldots, d_n)$. Denote by $|\beta|$ the sum of the elements of a cycle $\beta$. Then $\kappa_\sigma = \kappa_{|\beta_1|} \kappa_{|\beta_2|} \cdots \kappa_{|\beta_{v(\sigma)}|}$.

By now there are many works related to Faber’s conjecture. Looijenga [1995] illustrated that
\[
\dim R^k(\mathcal{M}_g) = 0, \quad k > g - 2, \quad \text{and} \quad \dim R^{g-2}(\mathcal{M}_g) \leq 1.
\]
Faber [1999] proved that actually $\dim R^{g-2}(\mathcal{M}_g) = 1$ and thus $R^*(\mathcal{M}_g)$ has the Gorenstein property. But the perfect pairing conjecture is still open.

Part (b) of the conjecture was proved independently by Morita [2003] and Ionel [2005] with different methods.

Part (c) of the conjecture (that is, Faber’s intersection number conjecture) is equivalent to a closed formula of the $\lambda_g \lambda_{g-1}$ Hodge integral, itself a consequence of the degree-zero Virasoro conjecture for surfaces [Getzler and Pandharipande 1998]. A short and direct proof of that integral formula can be found in [Liu and Xu 2009]. Recently, Buryak and Shadrin [2009] gave another combinatoric approach to this problem.

Thus, only the perfect pairing conjecture is open in Faber’s original conjecture. Liu and Xu [2010] proved some effective recursive relations in the top-degree tautological ring $R^{g-2}(\mathcal{M}_g)$ based on Faber’s intersection number conjecture. We know that it is important to find explicit relations in the tautological ring independent of genus. Faber [1999] also proposed a conjecture that all the tautological relations can be generated by the Brill–Noether method. Recently, Faber and Pandharipande have found some counterexamples when $g \geq 24$, and thus Faber’s approach may not produce all tautological relations starting from $g = 24$. The Brill–Noether method is an effective way to calculate the tautological relations. Ionel [2005] found some explicit relations in dimension $a = g + b + 1 - 2d$ for each $d \geq 2$, $g \geq 2$ and $b \geq 0$. As an application, Ionel gave a proof for Part (b) of Faber’s conjecture.

Marian, Oprea and Pandharipande [2009] obtained a vanishing theorem via a localization technique on the moduli space of stable quotients. Their result leads to some new geometric relations in the tautological ring. They computed these three special cases of their new geometric relation:

**Case 1.** If $a = 0$, $b = 1$, and $c = 2k$ (for $k \geq 1$), then in $R^{g-2d-1+2k}(\mathcal{M}_g)$,
\[
\rho_*(c_{g-d-1+2k}(\tilde{F}_d)) = 0.
\]

**Case 2.** If $a = 1$, $b = 1$, and $c = 2k$ (for $k \geq 1$), then in $R^{g-2d+2k}(\mathcal{M}_g)$,
\[
\rho_*(2(K_1 + \cdots + K_d) \cdot c_{g-d-1+2k}(\tilde{F}_d) + (2g-2) \cdot c_{g-d+2k}(\tilde{F}_d)) = 0.
\]
Case 3. If \( a = 2, b = 0, \) and \( c = 2k \) (for \( k \geq 1 \)), then in \( R^{g-2d+2k} \mathcal{M}_g \),

\[
\rho_*((-2(K_1 + \cdots + K_d) - 2\Delta \cdot c_{g-d+1+2k}(\bar{F}_d) + 2d \cdot c_{g-d+2k}(\bar{F}_d)) = 0.
\]

Combining (3) and (4), we have

\[
\rho_*((2 \Delta \cdot c_{g-d+1+2k}(\bar{F}_d) + (g + d - 1)c_{g-d+2k}(\bar{F}_d)) = 0 \text{ in } R^{g-2d+2k} \mathcal{M}_g.
\]

The notation in these formulas is explained in Section 2.

In this note, applying the method of [Ionel 2005], we derive new relations for the tautological ring in \( \mathcal{M}_g \) from (2) and (3), which can be considered a partial generalization of the main results in [Ionel 2005]. We also show that (5) is equivalent to (3).

Our main results are given by the following proposition.

**Proposition 1.1.** For each \( g, d \geq 2 \) and \( k \geq 1 \), formula (2) is equivalent to

\[
\exp\left(\frac{1}{t} \pi_* G(tK, w)\right)_{t^{g-2d+1+2k}u^d} = 0,
\]

and (3) is equivalent to

\[
\exp\left(\frac{1}{t} \pi_* G(tK, w)\right) \pi_*((2wG_w(tK, w) + 1)K)_{t^{g-2d+2k}u^d} = 0.
\]

Here \( G(x, w) \) (as in [Ionel 2005, Definition 2.1]) is the unique formal power series in \( x \) and \( w \) that satisfies the recursive formula

\[
xwG_{ww} = w(G_w)^2 + (1 - x)G_{ww} - 1
\]

with

\[
G(x, 0) = -\sum_{a=2}^{\infty} \frac{B_a}{a(a-1)}x^a,
\]

where \( B_a \) denotes the Bernoulli numbers.

**Theorem 1.2.** For each \( g, d \geq 2 \) and \( k \geq 1 \), formulas (6) and (7) give the following relations in \( R^{g-2d+1+2k} \mathcal{M}_g \) and \( R^{g-2d+2k} \mathcal{M}_g \), respectively:

\[
(1 + 4u)^{k-1} \exp\left(-\sum_{a=1}^{\infty} x^a \kappa_a \sum_{j=0}^{a} c_{a,j}u^j\right)_{t^{g-2d+1+2k}u^d} = 0,
\]

\[
(1 + 4u)^{k-1} \exp\left(-\sum_{a=1}^{\infty} x^a \kappa_a \sum_{j=0}^{a} c_{a,j}u^j\right) \times \left(g - 1 - \sum_{a=0}^{\infty} x^{a+1} \kappa_{a+1} \sum_{j=0}^{a} q_{a,j}u^{j+1}\right)_{t^{g-2d+2k}u^d} = 0.
\]
Here the positive integers $q_{k,j}$ (as in [Ionel 2005, Definition 1.3]) are defined recursively for $k \geq j \geq 0$ by the relation

\[(12) \quad q_{k,j} = (2k + 4j - 2)q_{k-1,j-1} + (j + 1)q_{k-1,j} + \sum_{m=0}^{k-1} \sum_{l=0}^{j-1} q_{m,l}q_{k-1-m,j-1-l},\]

with initial condition $q_{0,0} = 1$; and the coefficients $c_{k,j}$, for $k \geq 1$ and $k \geq j \geq 0$, by the relation

\[(13) \quad q_{k,j} = (2k + 4j)c_{k,j} + (j + 1)c_{k,j+1},\]

for all $k \geq 1$ and $k \geq j \geq 0$.

When $k = 1$, formulas (10) and (11) are just [Ionel 2005, (1.10) and (1.9)] with $b = 0$ and $b = 1$ respectively.

**Theorem 1.3.** Formula (5) is equivalent to formula (3).

### 2. Preliminaries

In this section, we introduce the notations and results that we use in this paper. We denote by $\mathcal{E}_g^d$ the $d$-fold of not necessarily distinct fiber products of $\mathcal{E}_g$ over $\mathcal{M}_g$, parametrizing smooth curves of genus $g$ with $n$-tuples of necessary distinct points, that is, $\mathcal{E}_g^d = \{(C, x_1, \ldots, x_d) | x_i \in C \}$. Let $\rho : \mathcal{E}_g^d \to \mathcal{M}_g$ be the map forgetting all the points. Then $\rho$ is the composition of morphisms $\pi_i : \mathcal{E}_g^i \to \mathcal{E}_g^{i-1}$ forgetting the $i$-th point, $\rho = \pi_1 \pi_2 \ldots \pi_d$; here $\pi_1 = \pi$.

There are some natural classes in $A^1(\mathcal{E}_g^d)$: $K_i = p_i^*(c_1(\omega_{\pi}))$ where $p_i$ is the $i$-th projection from $\mathcal{E}_g^d$ to $\mathcal{E}_g$. $K_1$ is written as $K$ in the following. $D_{ij}$ is the diagonal class of $\mathcal{E}_g^d$ where the points $x_i = x_j$.

Faber [1999] collected the following $\rho$-rules, due to Harris and Mumford [1982]:

**Formularium.** (a) Every monomial in the classes

\[K_i(1 \leq i \leq d) \text{ and } D_{ij}(1 \leq i < j \leq d)\]

on $\mathcal{E}_g^d$ can be rewritten as monomial $M$ pulled back from $\mathcal{E}_g^{d-1}$ times either a single diagonal $D_{id}$ or a power $K^l_i$ of $K_i$ by a repeated application of the following substitution rules:

\[D_{id}D_{jd} \to D_{ij}D_{id} \quad (i < j < d),\]
\[D_{id}^2 \to -K_iD_{id} \quad (i < d),\]
\[K_dD_{id} \to K_iD_{id} \quad (i < d).\]

(b) For $M$ a monomial pulled back from $\mathcal{E}_g^{d-1}$,

\[(\pi_d)_*(M \cdot D_{id}) = M,\]
\[(\pi_d)_*(M \cdot K^l_i) = M \cdot \rho^*(k_{l-1}).\]
For convenience, Ionel [2005] introduced the more general classes in $A^*(\mathcal{C}_{g}^d)$. If $1 \leq i_1 < \cdots < i_k \leq d$ is a sequence of integers, let $D_{i_1, \ldots, i_k}$ be the class of the stratum of $\mathcal{C}_{g}^d$, where all the points $x_{i_l}$ are equal for $l = 1, \ldots, k$. Given an unordered partition $\{J_1, \ldots, J_k\}$ of $\{x_1, \ldots, x_d\}$, we denote by $\Delta_{J_1, \ldots, J_k} = \prod_{i=1}^{d} D_{i}$ the codimension $d-k$ multidiagonal in $\mathcal{C}_{g}^d$, where all points in each $J_i$ are equal. Given such a stratum $\Delta_{J_1, \ldots, J_k}$, we denote by $x_{J_i}$ any one of the points of $J_i$, and by $K_{J_i}$ its corresponding $K$-class; also $|J_i| > 0$ denotes the number of points in $J_i$.

If $I, J$ are two subsets of $\{1, \ldots, d\}$ with $I \cap J \neq \emptyset$, then

$$D_I \cdot D_J = (-K_I)^{|I \cap J| - 1} D_{I \cup J}.$$  

Let $f(x_1, \ldots, x_d) \in \mathbb{Q}[x_1, \ldots, x_d]$ be an arbitrary polynomial. Then by (14) and the $\rho$-rules, we have

$$\rho_*(\Delta_{J_1, \ldots, J_k} \cdot f(K_1, \ldots, K_d)) = \pi_* f(K, \ldots, K).$$  

Denote by $\mathbb{F}_d = (\pi_{d+1})_* (\mathcal{C}_{g}^{d+1}(D_{1,d+1} + \cdots + D_{d,d+1})/\mathcal{C}_{g}^{d+1})$ the jet bundle at $d$ points, and let $\mathbb{E}^\vee$ be the dual of the Hodge bundle. By direct calculation,

$$c(\mathbb{F}_d) = c(\mathbb{F}_{d-1})(1 - K_d + D_{1,d} + \cdots + D_{d-1,d}).$$

Let $\mathbb{F}_d = \rho^* \mathbb{E}^\vee - \mathbb{F}_d$. The geometric relation formulated in [Ionel 2005] is

$$(2d + 2g - 2) \cdot \rho_*(c_{g+1-d} (\mathbb{F}_d) K_1^b) = (d-1) \kappa_{b-1} \cdot \rho_*(D_{12} \cdot c_{g+1-d} (\mathbb{F}_d)),$$

for $d \geq 2$, $g \geq 2$ and $b \geq 0$.

Via the main formula [Ionel 2005, Proposition 2.3], we have

$$c_t (\mathbb{F}_d) = \frac{\rho^* (c_t (\mathbb{E}^\vee))}{c_t (\mathbb{F}_d)}$$

$$= \rho^* \exp \left( - \sum_{a=1}^{\infty} \frac{B_{a+1}}{a(a+1)} \kappa_a t^a \right) \cdot \sum_{r=0}^{\infty} \sum_{\Delta_{J_1, \ldots, J_r}} \Delta_{J_1, \ldots, J_r} \prod_{i=1}^{r} H_{|J_i|} (t K_{J_i}),$$

where the last sum is over all (unordered) partitions $\{J_1, \ldots, J_r\}$ of $\{x_1, \ldots, x_d\}$, and the formal power series

$$G(x, w) = \sum_{d=0}^{\infty} H_d(x) \frac{w^d}{d!} = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} a_{kj} x^k w^j$$

satisfies (8) and (9). The main result of [Ionel 2005] is that relation (16) gives the following relation in $R^{g+1+b-2d}(M_g)$:

$$\exp \left( - \sum_{a=1}^{\infty} x^a \kappa_a \sum_{j=0}^{a} c_{a,j} u^j \right) \cdot (\kappa_{b-1} - 2 \sum_{a=0}^{\infty} \kappa_{a+b} x^{a+1} \sum_{j=0}^{a} q_{a,j} u^j) \bigg|_{x^{g+2-2d} u^d} = 0,$$
where $g, d \geq 2$ and $b \geq 0$. For $b = 0$, this relation simplifies to

$$\left[ \exp \left( - \sum_{a=1}^{\infty} \sum_{j=0}^{a} x^{a} \kappa_{a} \sum_{j=0}^{a} c_{a, j} u^{j} \right) \right]_{x^{\gamma+1-2d} u^{d}} = 0. \quad (20)$$

As an application of the relation (19), Part (b) of Faber’s conjecture is proved.

Marian, Oprea and Pandharipande [Marian et al. 2009] obtained a vanishing theorem via a localization technique on the moduli space of stable quotients. We describe their main statement for the reader’s convenience.

Given an element $[C, \hat{p}_{1}, \ldots, \hat{p}_{d}] \in \mathcal{C}^{d}_{g}$, there is a canonically associated stable quotient

$$0 \to \mathcal{O}_{C}(- \sum_{j=1}^{d} \hat{p}_{j}) \to \mathcal{O}_{C} \to Q \to 0. \quad (21)$$

Consider the universal curve $\pi : U \to \mathcal{C}^{d}_{g}$ with universal quotient sequence

$$0 \to S_{U} \to \mathcal{O}_{U} \to Q_{U} \to 0$$

obtained from (21). Let $\tilde{\mathcal{F}}_{d} = - R\pi_{*}(S^{*}_{U}) \in K(\mathcal{C}^{d}_{g})$ be the class in $K$-theory. With some computations, we have

$$c(\mathcal{F}_{d}) = c(\tilde{\mathcal{F}}_{d}) = \frac{\rho^{*}(c(\mathbb{E}^{\vee}))}{c(\mathcal{F}_{d})}. \quad (22)$$

Consider the proper morphism

$$\nu : Q_{g}(\mathbb{P}^{1}, d) \to \mathcal{M}_{g}.$$ 

The universal curve

$$\Pi : U \to Q_{g}(\mathbb{P}^{1}, d)$$

carries the basic divisor classes $s = c_{1}(S^{*}_{U})$ and $\omega = c_{1}(\omega_{\Pi})$.

Let $c > 0$ and $a, b \geq 0$. Then by (22), the geometric relation in the tautological ring shows that [Marian et al. 2009, Proposition 5]

$$\rho_{*}\left( \Pi_{*}(s^{a} \omega^{b}) \cdot c_{g-d-1+c}(\tilde{\mathcal{F}}_{d}) \right) + (-1)^{g-d-1} \left[ \Pi_{*}\left( (s-1)^{a} \omega^{b} \right) \cdot c_{-}(\tilde{\mathcal{F}}_{d}) \right]^{g-d-2+a+b+c} = 0 \quad (23)$$

in $R^{*}(\mathcal{M}_{g})$, where $c_{-}(\tilde{\mathcal{F}})$ denotes the total Chern class of $\tilde{\mathcal{F}}_{d}$ evaluated at $-1$. Then they obtained the following three special cases of (23).

**Case 1.** If $a = 0$, $b = 1$, and $c = 2k$ (for $k \geq 1$), then in $R^{g-2d-1+2k}(\mathcal{M}_{g})$,

$$\rho_{*}(c_{g-d-1+2k}(\tilde{\mathcal{F}}_{d})) = 0. \quad (24)$$
Case 2. If \( a = 1, b = 1, \) and \( c = 2k \) (for \( k \geq 1 \)), then in \( R^{g-2d+2k} (\mathcal{M}_g) \),
\[
\rho_*(2(K_1 + \cdots + K_d) \cdot c_{g-d-1+2k}(\tilde{F}_d) + (2g-2) \cdot c_{g-d+2k}(\tilde{F}_d)) = 0.
\]

Case 3. If \( a = 2, b = 0, \) and \( c = 2k \) (for \( k \geq 1 \)), then in \( R^{g-2d+2k} (\mathcal{M}_g) \),
\[
\rho_*(-2(K_1 + \cdots + K_d - 2\Delta \cdot c_{g-d-1+2k}(\tilde{F}_d) + 2d \cdot c_{g-d+2k}(\tilde{F}_d)) = 0,
\]
where \( \Delta = \sum_{1 \leq i < j \leq d} D_{ij} \).

Combining (25) and (26), we have
\[
\rho_* (2(1 + \cdots + K_d) \cdot c_g - d + 2\Delta \cdot c_{g-d+2k}(\tilde{F}_d)) = 0 \text{ in } R^{g-2d+2k} (\mathcal{M}_g).
\]

In the next section, we show how to get similar results with (19) and (20) by the same combinatorial method as in [Ionel 2005].

3. Proof of the main results

With a minor modification of [Ionel 2005, Lemma 2.5], we have:

**Lemma 3.1.** In terms of the generating function \( G(x, w) \) defined by (8) and (9), we have
\[
\sum_{d=0}^{\infty} \frac{w^d t^{-d}}{d!} \rho_* c_t(\tilde{F}_d) = \exp \left( \frac{1}{t} \rho_* G(t K, w) \right)
\]
and
\[
\sum_{d=1}^{\infty} \frac{w^{d-1} t^{-d}}{d!} \rho_* (c_t(\tilde{F}_d) K_j) = \exp \left( \frac{1}{t} \rho_* G(t K, w) \right) \cdot \frac{1}{t} \rho_* (G_w(t K, w) K_j),
\]
for \( j = 1, \ldots, d \).

**Proof.** The \( c_t(\tilde{F}_d) \), after being pushed forward by \( \rho \), depends only on the lengths \( l_i \) of sets \( J_i \). By (17), and some combinatoric enumeration, it is easy to get (28) and (29); see [Ionel 2005] for details.

Therefore, by Lemma 3.1, identities (2) and (3) give rise to (6) and (7) in Proposition 1.1, respectively.

In order to get Theorem 1.2, we need to better understand the structure of the function \( G(x, w) \) defined by (8) and (9). Ionel [2005, Lemmas 3.1 and 3.2] obtained the following expansions for \( G_w(x, w) \) and \( G(x, w) \):
\[
G_w(x, w) = \frac{-1 + \sqrt{1 + 4w}}{2w} + \frac{x}{1 + 4w} + \sum_{k=1}^{\infty} \sum_{j=0}^{k} x^{k+1} q_{k-j} (-w)^j (1 + 4w)^{-j-k/2-1},
\]
where the coefficients \( q_{k,j} \) are defined by (12) and

\[
G(x, w) = G(0, w) + \frac{x}{4} \ln(1 + 4w) - \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} x^{k+1} c_{k,j} (-w)^j (1 + 4w)^{-j-k/2},
\]

where the coefficients \( c_{k,j} \) are related to the coefficients \( q_{k,j} \) by (13). Also, we need the variable transformation formula used in [Ionel 2005].

**Lemma 3.2 [Ionel 2005, Lemma 3.3].** Let \( P(x, w) \) be a formal power series in \( x \) and \( w \). Denote by \( \hat{P}(y, u) \) the formal power series in \( y \) and \( u \) obtained from \( P(x, w) \) after the change of variables \( u = -w/(1 + 4w) \) and \( y = x/\sqrt{1 + 4w} \).

\[
[P(x, w)]_{x^aw^d} = (-1)^d [(1 + 4u)^{(a+2d-2)/2} \hat{P}(y, u)]_{y^au^d}.
\]

By the expansion (31),

\[
\frac{1}{t} \pi_* G(tK, w) = \frac{\kappa_0}{4} \ln(1 + 4w) - \sum_{a=1}^{\infty} t^a \kappa_a \sum_{j=0}^{\infty} c_{a,j} (-w)^j (1 + 4w)^{-j-a/2}.
\]

Using the change of variables,

\[
t \to (1 + 4w)^{\frac{1}{2}} y, \quad w \to \frac{-u}{1 + 4u}, \quad (1 + 4w) \to \frac{1}{1 + 4u},
\]

we have

\[
\exp\left(\frac{1}{t} \pi_* (G(tK, w))\right) = (1 + 4u)^{-\kappa_0/4} \exp\left(-\sum_{a=1}^{\infty} y^a \kappa_a \sum_{j=0}^{\infty} c_{a,j} u^j\right).
\]

Similarly, by the expansion (30),

\[
\pi_* \left((2wG_w(tK, w) + 1)K\right)
\]

\[
= \pi_* \left((1 + 4w)^{1/2} K + \frac{2w}{1 + 4w} t K^2 \right.
\]

\[
+ \sum_{a=1}^{\infty} \sum_{j=0}^{a} t^{a+1} K^{a+2} q_{a,j} (-w)^j 2w (1 + 4w)^{-j-a/2-1}\right)
\]

\[
= (1 + 4w)^{\frac{1}{2}} \left(\kappa_0 - 2 \sum_{a=0}^{\infty} t^{a+1} \kappa_{a+1} \sum_{j=0}^{a} q_{a,j} (-w)^j (1 + 4w)^{-j-(a+1)/2-1}\right).
\]

By the change of variables

\[
t \to (1 + 4w)^{\frac{1}{2}} y, \quad w \to \frac{-u}{1 + 4u}, \quad (1 + 4w) \to \frac{1}{1 + 4u}.
\]
we get

\[ (37) \quad \pi_*(2w G_w(tK, w) + 1)K) = (1 + 4u)^{-1/2} \left( \kappa_0 - 2 \sum_{a=0}^{\infty} y^{a+1} \kappa_{a+1} \sum_{j=0}^{a} q_{a,j} u^{j+1} \right). \]

By Lemma 3.2, (35) and (37), we have

\[
\left[ \exp \left( \frac{1}{t} \pi_* G(tK, w) \right) \right]_{t^g-2d-1+2k w^d} = (-1)^d \left[ (1 + 4u)^{k-1} \exp \left( - \sum_{a=1}^{\infty} x^a \kappa_a \sum_{j=0}^{a} c_{a,j} u^j \right) \right]_{t^g-2d-1+2k w^d},
\]

\[
\left[ \exp \left( \frac{1}{t} \pi_* G(tK, w) \right) \pi_* \left( (2w G_w(tK, w) + 1)K \right) \right]_{t^g-2d-1+2k w^d} = \left[ (1 + 4u)^{k-1} \exp \left( - \sum_{a=1}^{\infty} x^a \kappa_a \sum_{j=0}^{a} c_{a,j} u^j \right) \left( g - 1 - \sum_{a=0}^{\infty} y^{a+1} \kappa_{a+1} \sum_{j=0}^{a} q_{a,j} u^{j+1} \right) \right]_{x^g-2d+2k w^d}.
\]

By formulas (6) and (7), Theorem 1.2 is proved.

**Theorem 3.3.** Formula (3) is equivalent to formula (5).

**Proof.** By the definition of \( \pi^*_d \),

\[ \pi^*_d(c_i(\mathbb{F}_{d-1})) = c_i(\mathbb{F}_d)(1 - tK_d + t D_{1,d} + \cdots + t D_{d-1,d}). \]

After being pushed forward by \( \rho \),

\[ 0 = \rho_* \pi^*_d(c_i(\mathbb{F}_{d-1})) = \rho_* (c_i(\mathbb{F}_d)) - t \rho_* (K_d \cdot c_i(\mathbb{F}_d)) + (d - 1) t \rho_* (D_{1,d} \cdot c_i(\mathbb{F}_d)). \]

In particular,

\[ (d - 1) \left\{ \rho_* (D_{1,d} \cdot c_i(\mathbb{F}_d)) \right\}_{t^g-d-1+2k} = \left\{ \rho_* (K_d \cdot c_i(\mathbb{F}_d)) \right\}_{t^g-d-1+2k} - \left\{ \rho_* (c_i(\mathbb{F}_d)) \right\}_{t^g-d-1+2k}. \]

In fact

\[ \rho_* (D_{i,j} \cdot c_i(\mathbb{F}_d)) = \rho_* (D_{1,d} \cdot c_i(\mathbb{F}_d)), \]

and the equivalence of formulas (3) and (5) is deduced from the identity

\[ 2 \left\{ \rho_* (\Delta \cdot c_i(\mathbb{F}_d)) \right\}_{t^g-d-1+2k} = d (d - 1) \left\{ \rho_* (D_{1,d} \cdot c_i(\mathbb{F}_d)) \right\}_{t^g-d-1+2k} = d \left\{ \rho_* (K_d \cdot c_i(\mathbb{F}_d)) \right\}_{t^g-d-1+2k} - d \left\{ \rho_* (c_i(\mathbb{F}_d)) \right\}_{t^g-d-1+2k} = - (g - 1) \left\{ \rho_* (c_i(\mathbb{F}_d)) \right\}_{t^g-d+2k} - d \left\{ \rho_* (c_i(\mathbb{F}_d)) \right\}_{t^g-d+2k} = - (g + d - 1) \left\{ \rho_* (c_i(\mathbb{F}_d)) \right\}_{t^g-d+2k}. \]

\[ \square \]
Example 3.4. We give some low genus examples for Theorem 1.2. By the recursion relations (12) and (13) of constants \(q_{k,j}, c_{k,j}\), we get

\[
\begin{align*}
q_{0,0} &= 1, \\
q_{1,0} &= 1, \quad q_{1,1} = 5, \\
q_{2,0} &= 1, \quad q_{2,1} = 18, \quad q_{2,2} = 60, \\
q_{3,0} &= 1, \quad q_{3,1} = 47, \quad q_{3,2} = 442, \quad q_{3,3} = 1105,
\end{align*}
\]

... and

\[
\begin{align*}
c_{1,0} &= \frac{1}{12}, \quad c_{1,1} = \frac{5}{6}, \\
c_{2,0} &= 0, \quad c_{2,1} = 1, \quad c_{2,2} = 5, \\
c_{3,0} &= -\frac{1}{360}, \quad c_{3,1} = \frac{61}{60}, \quad c_{3,2} = \frac{221}{12}, \quad c_{3,3} = \frac{1105}{18},
\end{align*}
\]

... Taking \(g = 5, d = 3, k = 2\), formula (10) gives a relation in \(R^2(\mathcal{M}_5)\):

\[(38) \quad \frac{25}{18} \kappa_1^2 - 20 \kappa_2 = 0.\]

Taking \(g = 6, d = 3, k = 2\), formula (10) gives a relation in \(R^3(\mathcal{M}_6)\):

\[(39) \quad -\frac{275}{1296} \kappa_1^3 + \frac{55}{6} \kappa_1 \kappa_2 - \frac{2431}{18} \kappa_3 = 0.\]

It is easy to check that the relations (38) and (39) match the results in [Faber 1999].

We have written a Maple program to calculate more relations through Theorem 1.2. Unfortunately, it is difficult to determine if they contain the new tautological relations in high genus beyond those obtained by Faber.

4. Conclusion

The new relations in the tautological ring obtained in this note, that is, formulas (10) and (11), can be regarded as a partial generalization of formula (19) (which is [Ionel 2005, (1.9)]) in the special cases \(b = 0, 1\).

When \(b = 0\), formula (19) is just (20):

\[
\left[\exp\left(\sum_{a=1}^{\infty} x^a \kappa_a \sum_{j=0}^{a} c_{a,j} u^j\right)\right]_{x^g+1-2d} = 0,
\]

which is the special case of formula (10) with \(k = 1\).

For \(b = 1\), (19) becomes

\[
\left[\exp\left(\sum_{a=1}^{\infty} x^a \kappa_a \sum_{j=0}^{a} c_{a,j} u^j\right) \cdot \left( g - \sum_{a=0}^{\infty} \kappa_{a+1} x^{a+1} \sum_{j=0}^{a} q_{a,j} u^{j+1}\right)\right]_{x^{g+2-2d}} = 0,
\]
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which is the special case of (11) with $k = 1$.

However, our results can not cover formula (19) when $b \geq 2$. In this note, we only consider three special cases of (23) to deduce our main results. We hope that one can obtain a more general formula, like (11), from formula (23), with the same method.\(^1\)

As mentioned in the introduction, it is important to find explicit relations in the tautological ring in studying Faber’s conjecture. From this note, we see that the stable quotient method introduced by Marian, Oprea and Pandharipande [2009] provides a new and effective way to obtain the relations in the tautological ring. With this method, recently, Pandharipande [2009a; 2009b] introduced the $\kappa$ ring of the moduli of curves of compact type and studied its algebraic structure. In our further study, we hope to find more applications of the stable quotient method.

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References


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