Pacific Journal of Mathematics

NOTE ON THE RELATIONS IN THE TAUTOLOGICAL RING OF $\mathcal{M}_g$

SHENGMAO ZHU

Volume 252 No. 2 August 2011

PACIFIC JOURNAL OF MATHEMATICS
Vol. 252, No. 2, 2011

NOTE ON THE RELATIONS IN THE TAUTOLOGICAL RING OF $\mathcal{M}_g$

SHENGMAO ZHU

We give some nontrivial relations in the tautological ring of $\mathcal{M}_g$. These are derived from some new geometric relations obtained by localization on the moduli of stable quotients, which was recently introduced by A. Marian, D. Oprea and R. Pandharipande.

1. Introduction

We denote by $\mathcal{M}_g$ the moduli space of smooth curves of genus $g \geq 2$ over an algebraically closed field. Let $\pi : \mathcal{C}_g \to \mathcal{M}_g$ be its tautological family and ω_π be the dualizing sheaf. We denote by $\mathbb{E} = \pi_* \omega_\pi$ the Hodge bundle. Define $\kappa_i = \pi_*(c_1(\omega_\pi)^{i+1}) \in A^i(\mathcal{M}_g)$, $\lambda_i = c_i(\mathbb{E})$, and in particular, $k_0 = 2g-2$, $k_{-1} = 0$. The tautological ring $R^*(\mathcal{M}_g)$ is defined to be the subring generated by λ-classes and κ-classes. By Mumford's formula [1983], the tautological ring is in fact generated by the κ-classes $\kappa_1, \ldots, \kappa_{g-2}$.

Faber [1999] proposed a series of remarkable conjectures about the structure of $R^*(\mathcal{M}_g)$:

(a) The tautological ring $R^*(\mathcal{M}_g)$ is Gorenstein with socle in degree $g-2$, and when an isomorphism $R^{g-2}(\mathcal{M}_g) = \mathbb{Q}$ is fixed, the natural pairing

$$R^i(\mathcal{M}_g) \times R^{g-2-i}(\mathcal{M}_g) \to R^{g-2}(\mathcal{M}_g) = \mathbb{Q}$$

is perfect.

(b) The $[g/3]$ classes $\kappa_1, \ldots, \kappa_{[g/3]}$ generate the ring $R^*(\mathcal{M}_g)$, with no relations in degrees $\leq [g/3]$.

(c) Let $\sum_{j=1}^n d_j = g-2$ and $d_j \geq 0$. Then

$$\sum_{\sigma \in S_n} \kappa_\sigma = \frac{(2g-3+n)!}{(2g-2)!! \prod_{j=1}^n (2d_j+1)!!} \kappa_{g-2}, \tag{1}$$

MSC2010: primary 14H10; secondary 05A15.
Keywords: tautological relations, moduli space, stable quotients.

where κ_σ is defined as follows: write the permutation $\sigma = \beta_1 \dots \beta_{\nu(\sigma)}$, where we think of the symmetric group S_n as acting on the n-tuple $(d_1, \dots, d_n)$. Denote by $|\beta|$ the sum of the elements of a cycle β. Then $\kappa_\sigma = \kappa_{|\beta_1|}\kappa_{|\beta_2|} \dots \kappa_{|\beta_{\nu(\sigma)}|}$.

By now there are many works related to Faber's conjecture. Looijenga [1995] illustrated that

$$\dim R^k(\mathcal{M}_g) = 0, \quad k > g-2, \quad \text{and} \quad \dim R^{g-2}(\mathcal{M}_g) \le 1.$$

Faber [1999] proved that actually $\dim R^{g-2}(\mathcal{M}_g) = 1$ and thus $R^*(\mathcal{M}_g)$ has the Gorenstein property. But the perfect pairing conjecture is still open.

Part (b) of the conjecture was proved independently by Morita [2003] and Ionel [2005] with different methods.

Part (c) of the conjecture (that is, Faber's intersection number conjecture) is equivalent to a closed formula of the $\lambda_g\lambda_{g-1}$ Hodge integral, itself a consequence of the degree-zero Virasoro conjecture for surfaces [Getzler and Pandharipande 1998]. A short and direct proof of that integral formula can be found in [Liu and Xu 2009]. Recently, Buryak and Shadrin [2009] gave another combinatoric approach to this problem.

Thus, only the perfect pairing conjecture is open in Faber's original conjecture. Liu and Xu [2010] proved some effective recursive relations in the top-degree tautological ring $R^{g-2}(\mathcal{M}_g)$ based on Faber's intersection number conjecture. We know that it is important to find explicit relations in the tautological ring independent of genus. Faber [1999] also proposed a conjecture that all the tautological relations can be generated by the Brill–Noether method. Recently, Faber and Pandharipande have found some counterexamples when $g \ge 24$, and thus Faber's approach may not produce all tautological relations starting from $g = 24$. The Brill–Noether method is an effective way to calculate the tautological relations. Ionel [2005] found some explicit relations in dimension $a = g+b+1-2d$ for each $d \ge 2$, $g \ge 2$ and $b \ge 0$. As an application, Ionel gave a proof for Part (b) of Faber's conjecture.

Marian, Oprea and Pandharipande [2009] obtained a vanishing theorem via a localization technique on the moduli space of stable quotients. Their result leads to some new geometric relations in the tautological ring. They computed these three special cases of their new geometric relation:

Case 1. If $a = 0$, $b = 1$, and $c = 2k$ (for $k \ge 1$), then in $R^{g-2d-1+2k}(\mathcal{M}_g)$,

$$\rho_*(c_{g-d-1+2k}(\tilde{\mathbb{F}}_d)) = 0. \tag{2}$$

Case 2. If $a = 1$, $b = 1$, and $c = 2k$ (for $k \ge 1$), then in $R^{g-2d+2k}(\mathcal{M}_g)$,

$$\rho_*\big(2(K_1 + \cdots + K_d) \cdot c_{g-d-1+2k}(\tilde{\mathbb{F}}_d) + (2g-2) \cdot c_{g-d+2k}(\tilde{\mathbb{F}}_d)\big) = 0. \tag{3}$$

Case 3. If $a=2$, $b=0$, and $c=2k$ (for $k\geq 1$), then in $R^{g-2d+2k}(\mathcal{M}_g)$,

$$\rho_*\big(-2(K_1+\cdots+K_d-2\Delta\cdot c_{g-d-1+2k}(\tilde{\mathbb{F}}_d)+2d\cdot c_{g-d+2k}(\tilde{\mathbb{F}}_d)\big)=0. \tag{4}$$

Combining (3) and (4), we have

$$\rho_*\big(2\Delta\cdot c_{g-d-1+2k}(\tilde{\mathbb{F}}_d)+(g+d-1)c_{g-d+2k}(\tilde{\mathbb{F}}_d)\big)=0 \text{ in } R^{g-2d+2k}(\mathcal{M}_g). \tag{5}$$

The notation in these formulas is explained in Section 2.

In this note, applying the method of [Ionel 2005], we derive new relations for the tautological ring in $\mathcal{M}_g$ from (2) and (3), which can be considered a partial generalization of the main results in [Ionel 2005]. We also show that (5) is equivalent to (3).

Our main results are given by the following proposition.

Proposition 1.1. *For each $g, d\geq 2$ and $k\geq 1$, formula (2) is equivalent to*

$$\Big[\exp\Big(\frac{1}{t}\pi_*G(tK,w)\Big)\Big]_{t^{g-2d-1+2k}w^d}=0, \tag{6}$$

and (3) is equivalent to

$$\Big[\exp\Big(\frac{1}{t}\pi_*G(tK,w)\Big)\pi_*\big((2wG_w(tK,w)+1)K\big)\Big]_{t^{g-2d+2k}w^d}=0. \tag{7}$$

Here $G(x,w)$ (as in [Ionel 2005, Definition 2.1]) *is the unique formal power series in x and w that satisfies the recursive formula*

$$xwG_{ww}=w(G_w)^2+(1-x)G_{ww}-1 \tag{8}$$

with

$$G(x,0)=-\sum_{a=2}^{\infty}\frac{B_a}{a(a-1)}x^a, \tag{9}$$

where B_a denotes the Bernoulli numbers.

Theorem 1.2. *For each $g, d\geq 2$ and $k\geq 1$, formulas (6) and (7) give the following relations in $R^{g-2d-1+2k}(\mathcal{M}_g)$ and $R^{g-2d+2k}(\mathcal{M}_g)$, respectively*:

$$\Big[(1+4u)^{k-1}\exp\Big(-\sum_{a=1}^{\infty}x^a\kappa_a\sum_{j=0}^{a}c_{a,j}u^j\Big)\Big]_{x^{g-2d-1+2k}u^d}=0, \tag{10}$$

$$\Big[(1+4u)^{k-1}\exp\Big(-\sum_{a=1}^{\infty}x^a\kappa_a\sum_{j=0}^{a}c_{a,j}u^j\Big) \times\Big(g-1-\sum_{a=0}^{\infty}x^{a+1}\kappa_{a+1}\sum_{j=0}^{a}q_{a,j}u^{j+1}\Big)\Big]_{x^{g-2d+2k}u^d}=0. \tag{11}$$

Here the positive integers $q_{k,j}$ (as in [Ionel 2005, Definition 1.3]) *are defined recursively for* $k \geq j \geq 0$ *by the relation*

$$q_{k,j} = (2k+4j-2)q_{k-1,j-1} + (j+1)q_{k-1,j} + \sum_{m=0}^{k-1}\sum_{l=0}^{j-1} q_{m,l}q_{k-1-m,j-1-l}, \tag{12}$$

with initial condition $q_{0,0} = 1$; *and the coefficients* $c_{k,j}$, *for* $k \geq 1$ *and* $k \geq j \geq 0$, *by the relation*

$$q_{k,j} = (2k+4j)c_{k,j} + (j+1)c_{k,j+1}, \tag{13}$$

for all $k \geq 1$ *and* $k \geq j \geq 0$.

When $k = 1$, formulas (10) and (11) are just [Ionel 2005, (1.10) and (1.9)] with $b = 0$ and $b = 1$ respectively.

Theorem 1.3. *Formula (5) is equivalent to formula (3).*

2. Preliminaries

In this section, we introduce the notations and results that we use in this paper. We denote by $\mathscr{C}_g^d$ the d-fold of not necessarily distinct fiber products of $\mathscr{C}_g$ over $\mathscr{M}_g$, parametrizing smooth curves of genus g with n-tuples of necessary distinct points, that is, $\mathscr{C}_g^d = \{(C, x_1, \dots, x_d) | x_i \in C\}$. Let $\rho : \mathscr{C}_g^d \to \mathscr{M}_g$ be the map forgetting all the points. Then ρ is the composition of morphisms $\pi_i \colon \mathscr{C}_g^i \to \mathscr{C}_g^{i-1}$ forgetting the i-th point, $\rho = \pi_1\pi_2 \dots \pi_d$; here $\pi_1 = \pi$.

There are some natural classes in $A^1(\mathscr{C}_g^d)$: $K_i = p_i^*(c_1(\omega_\pi))$ where p_i is the i-th projection from $\mathscr{C}_g^d$ to $\mathscr{C}_g$. K_1 is written as K in the following. D_{ij} is the diagonal class of $\mathscr{C}_g^d$ where the points $x_i = x_j$.

Faber [1999] collected the following ρ-rules, due to Harris and Mumford [1982]:

Formularium. (a) *Every monomial in the classes*

$$K_i (1 \leq i \leq d) \ \ \text{and} \ \ D_{ij} (1 \leq i < j \leq d)$$

on $\mathscr{C}_g^d$ *can be rewritten as monomial* M *pulled back from* $\mathscr{C}_g^{d-1}$ *times either a single diagonal* D_{id} *or a power* K_d^l *of* K_d *by a repeated application of the following substitution rules*:

$$\begin{aligned} D_{id}D_{jd} &\to D_{ij}D_{id} \quad (i < j < d), \\ D_{id}^2 &\to -K_iD_{id} \quad (i < d), \\ K_dD_{id} &\to K_iD_{id} \quad (i < d). \end{aligned}$$

(b) *For* M *a monomial pulled back from* $\mathscr{C}_g^{d-1}$,

$$\begin{aligned} (\pi_d)_*(M \cdot D_{id}) &= M, \\ (\pi_d)_*(M \cdot K_d^l) &= M \cdot \rho^*(k_{l-1}). \end{aligned}$$

For convenience, Ionel [2005] introduced the more general classes in $A^*(\mathscr{C}_g^d)$. If $1 \le i_1 < \cdots < i_k \le d$ is a sequence of integers, let $D_{i_1,\dots,i_k}$ be the class of the stratum of $\mathscr{C}_g^d$, where all the points x_{i_l} are equal for $l = 1, \dots, k$. Given an unordered partition $\{J_1, \dots, J_k\}$ of $\{x_1, \dots, x_d\}$, we denote by $\Delta_{J_1,\dots,J_k} = \prod_{i=1}^d D_{J_i}$ the codimension $d-k$ multidiagonal in $\mathscr{C}_d^d$, where all points in each J_i are equal. Given such a stratum $\Delta_{J_1,\dots,J_k}$, we denote by x_{J_i} any one of the points of J_i, and by K_{J_i} its corresponding K-class; also $|J_i| > 0$ denotes the number of points in J_i. If I, J are two subsets of $\{1, \dots, d\}$ with $I \cap J \neq \varnothing$, then

$$D_I \cdot D_J = (-K_I)^{|I\cap J|-1} D_{I\cup J}. \tag{14}$$

Let $f(x_1, \dots, x_d) \in \mathbb{Q}[x_1, \dots, x_d]$ be an arbitrary polynomial. Then by (14) and the ρ-rules, we have

$$\rho_*(\Delta_{J_1,\dots,J_k} \cdot f(K_1, \dots, K_d)) = \pi_* f(K, \dots, K). \tag{15}$$

Denote by $\mathbb{F}_d = (\pi_{d+1})_*\big(\mathbb{O}_{\mathscr{C}_g^{d+1}}(D_{1,d+1} + \cdots + D_{d,d+1})/\mathbb{O}_{\mathscr{C}_g^{d+1}}\big)$ the jet bundle at d points, and let $\mathbb{E}^\vee$ be the dual of the Hodge bundle. By direct calculation,

$$c(\mathbb{F}_d) = c(\mathbb{F}_{d-1})(1 - K_d + D_{1,d} + \cdots + D_{d-1,d}).$$

Let $\tilde{\mathbb{F}}_d = \rho^*\mathbb{E}^\vee - \mathbb{F}_d$. The geometric relation formulated in [Ionel 2005] is

$$(2d+2g-2) \cdot \rho_*(c_{g+1-d}(\tilde{\mathbb{F}}_d)K_1^b) = (d-1)\kappa_{b-1} \cdot \rho_*(D_{12} \cdot c_{g+1-d}(\tilde{\mathbb{F}}_d)), \tag{16}$$

for $d \ge 2$, $g \ge 2$ and $b \ge 0$.

Via the main formula [Ionel 2005, Proposition 2.3], we have

$$\begin{aligned} c_t(\tilde{\mathbb{F}}_d) &= \frac{\rho^*(c_t(\mathbb{E}^\vee))}{c_t(\mathbb{F}_d)} \\ &= \rho^* \exp\Big(-\sum_{a=1}^{\infty} \frac{B_{a+1}}{a(a+1)}\kappa_a t^a\Big) \cdot \sum_{r=0}^{\infty} \sum_{\{J_1,\dots,J_r\}} t^{d-r} \Delta_{J_1,\dots,J_r} \prod_{i=1}^{r} H_{|J_i|}(tK_{J_i}), \end{aligned} \tag{17}$$

where the last sum is over all (unordered) partitions $\{J_1, \dots, J_r\}$ of $\{x_1, \dots, x_d\}$, and the formal power series

$$G(x, w) = \sum_{d=0}^{\infty} H_d(x)\frac{w^d}{d!} = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} a_{kj} x^k w^j \tag{18}$$

satisfies (8) and (9). The main result of [Ionel 2005] is that relation (16) gives the following relation in $R^{g+1+b-2d}(\mathcal{M}_g)$:

(19)

$$\Big[\exp\Big(-\sum_{a=1}^{\infty} x^a \kappa_a \sum_{j=0}^{a} c_{a,j} u^j\Big) \cdot \Big(\kappa_{b-1} - 2\sum_{a=0}^{\infty} \kappa_{a+b} x^{a+1} \sum_{j=0}^{a} q_{a,j} u^{j+1}\Big)\Big]_{x^{g+2-2d}u^d} = 0,$$

where $g, d \geq 2$ and $b \geq 0$. For $b = 0$, this relation simplifies to

$$\Big[\exp\Big(-\sum_{a=1}^{\infty} x^a \kappa_a \sum_{j=0}^{a} c_{a,j} u^j\Big)\Big]_{x^{g+1-2d}u^d} = 0. \tag{20}$$

As an application of the relation (19), Part (b) of Faber's conjecture is proved.

Marian, Oprea and Pandharipande [Marian et al. 2009] obtained a vanishing theorem via a localization technique on the moduli space of stable quotients. We describe their main statement for the reader's convenience.

Given an element $[C, \hat{p}_1, \ldots, \hat{p}_d] \in \mathscr{C}_g^d$, there is a canonically associated stable quotient

$$0 \to \mathbb{O}_C(-\sum_{j=1}^{d} \hat{p}_j) \to \mathbb{O}_C \to Q \to 0. \tag{21}$$

Consider the universal curve $\pi : U \to \mathscr{C}_g^d$ with universal quotient sequence

$$0 \to S_U \to \mathbb{O}_U \to Q_U \to 0$$

obtained from (21). Let $\overline{\mathbb{F}}_d = -R\pi_*(S_U^*) \in K(\mathscr{C}_g^d)$ be the class in K-theory. With some computations, we have

$$c(\overline{\mathbb{F}}_d) = c(\tilde{\mathbb{F}}_d) = \frac{\rho^*(c(\mathbb{E}^\vee))}{c(\mathbb{F}_d)}. \tag{22}$$

Consider the proper morphism

$$\nu : Q_g(\mathbb{P}^1, d) \to \mathcal{M}_g.$$

The universal curve

$$\Pi : U \to Q_g(\mathbb{P}^1, d)$$

carries the basic divisor classes $s = c_1(S_U^*)$ and $\omega = c_1(\omega_\pi)$.

Let $c > 0$ and $a, b \geq 0$. Then by (22), the geometric relation in the tautological ring shows that [Marian et al. 2009, Proposition 5]

$$\rho_*\Big(\Pi_*(s^a\omega^b)\cdot c_{g-d-1+c}(\tilde{\mathbb{F}}_d) + (-1)^{g-d-1}\big[\Pi_*((s-1)^a\omega^b)\cdot c_-(\tilde{\mathbb{F}}_d)\big]^{g-d-2+a+b+c}\Big) = 0 \tag{23}$$

in $R^*(\mathcal{M}_g)$, where $c_-(\tilde{\mathbb{F}})$ denotes the total Chern class of $\tilde{\mathbb{F}}_d$ evaluated at -1. Then they obtained the following three special cases of (23).

Case 1. If $a = 0$, $b = 1$, and $c = 2k$ (for $k \geq 1$), then in $R^{g-2d-1+2k}(\mathcal{M}_g)$,

$$\rho_*(c_{g-d-1+2k}(\tilde{\mathbb{F}}_d)) = 0. \tag{24}$$

Case 2. If $a=1$, $b=1$, and $c=2k$ (for $k\geq 1$), then in $R^{g-2d+2k}(\mathcal{M}_g)$,

$$\rho_*\big(2(K_1+\cdots+K_d)\cdot c_{g-d-1+2k}(\tilde{\mathbb{F}}_d)+(2g-2)\cdot c_{g-d+2k}(\tilde{\mathbb{F}}_d)\big)=0. \tag{25}$$

Case 3. If $a=2$, $b=0$, and $c=2k$ (for $k\geq 1$), then in $R^{g-2d+2k}(\mathcal{M}_g)$,

$$\rho_*\big(-2(K_1+\cdots+K_d-2\Delta\cdot c_{g-d-1+2k}(\tilde{\mathbb{F}}_d)+2d\cdot c_{g-d+2k}(\tilde{\mathbb{F}}_d)\big)=0, \tag{26}$$

where $\Delta=\sum_{1\leq i<j\leq d}D_{ij}$.

Combining (25) and (26), we have

$$\rho_*\big(2\Delta\cdot c_{g-d-1+2k}(\tilde{\mathbb{F}}_d)+(g+d-1)c_{g-d+2k}(\tilde{\mathbb{F}}_d)\big)=0 \text{ in } R^{g-2d+2k}(\mathcal{M}_g). \tag{27}$$

In the next section, we show how to get similar results with (19) and (20) by the same combinatorial method as in [Ionel 2005].

3. Proof of the main results

With a minor modification of [Ionel 2005, Lemma 2.5], we have:

Lemma 3.1. *In terms of the generating function $G(x,w)$ defined by (8) and (9), we have*

$$\sum_{d=0}^{\infty}\frac{w^d t^{-d}}{d!}\rho_* c_t(\tilde{\mathbb{F}}_d)=\exp\Big(\frac{1}{t}\rho_* G(tK,w)\Big) \tag{28}$$

and

$$\sum_{d=1}^{\infty}\frac{w^{d-1}t^{-d}}{d!}\rho_*(c_t(\tilde{\mathbb{F}}_d)K_j)=\exp\Big(\frac{1}{t}\rho_* G(tK,w)\Big)\cdot\frac{1}{t}\rho_*\big(G_w(tK,w)K_j\big), \tag{29}$$

for $j=1,\dots,d$.

Proof. The $c_t(\tilde{\mathbb{F}}_d)$, after being pushed forward by ρ, depends only on the lengths l_i of sets J_i. By (17), and some combinatoric enumeration, it is easy to get (28) and (29); see [Ionel 2005] for details. □

Therefore, by Lemma 3.1, identities (2) and (3) give rise to (6) and (7) in Proposition 1.1, respectively.

In order to get Theorem 1.2, we need to better understand the structure of the function $G(x,w)$ defined by (8) and (9). Ionel [2005, Lemmas 3.1 and 3.2] obtained the following expansions for $G_w(x,w)$ and $G(x,w)$:

$$G_w(x,w) =\frac{-1+\sqrt{1+4w}}{2w}+\frac{x}{1+4w}+\sum_{k=1}^{\infty}\sum_{j=0}^{k}x^{k+1}q_{k,j}(-w)^j(1+4w)^{-j-k/2-1}, \tag{30}$$

where the coefficients $q_{k,j}$ are defined by (12) and

$$G(x,w)=G(0,w)+\frac{x}{4}\ln(1+4w)-\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}x^{k+1}c_{k,j}(-w)^j(1+4w)^{-j-k/2}, \tag{31}$$

where the coefficients $c_{k,j}$ are related to the coefficients $q_{k,j}$ by (13). Also, we need the variable transformation formula used in [Ionel 2005].

Lemma 3.2 [Ionel 2005, Lemma 3.3]. *Let $P(x,w)$ be a formal power series in x and w. Denote by $\hat{P}(y,u)$ the formal power series in y and u obtained from $P(x,w)$ after the change of variables $u=-w/(1+4w)$ and $y=x/\sqrt{1+4w}$.*

$$[P(x,w)]_{x^a w^d}=(-1)^d[(1+4u)^{(a+2d-2)/2}\hat{P}(y,u)]_{y^a u^d}. \tag{32}$$

By the expansion (31),

$$\frac{1}{t}\pi_*G(tK,w)=\frac{\kappa_0}{4}\ln(1+4w)-\sum_{a=1}^{\infty}t^a\kappa_a\sum_{j=0}^{\infty}c_{a,j}(-w)^j(1+4w)^{-j-a/2}. \tag{33}$$

Using the change of variables,

$$t\to(1+4w)^{\frac{1}{2}}y,\quad w\to\frac{-u}{1+4u},\quad (1+4w)\to\frac{1}{1+4u}, \tag{34}$$

we have

$$\exp\Big(\frac{1}{t}\pi_*(G(tK,w))\Big)=(1+4u)^{-\kappa_0/4}\exp\Big(-\sum_{a=1}^{\infty}y^a\kappa_a\sum_{j=0}^{\infty}c_{a,j}u^j\Big). \tag{35}$$

Similarly, by the expansion (30),

$$\begin{aligned}
\pi_*\big((2wG_w(tK,w)+1)K\big)
&=\pi_*\Big((1+4w)^{1/2}K+\frac{2w}{1+4w}tK^2\\
&\qquad+\sum_{a=1}^{\infty}\sum_{j=0}^{a}t^{a+1}K^{a+2}q_{a,j}(-w)^j2w(1+4w)^{-j-a/2-1}\Big)\\
&=(1+4w)^{\frac{1}{2}}\Big(\kappa_0-2\sum_{a=0}^{\infty}t^{a+1}\kappa_{a+1}\sum_{j=0}^{a}q_{a,j}(-w)^{j+1}(1+4w)^{-j-(a+1)/2-1}\Big).
\end{aligned}$$

By the change of variables

$$t\to(1+4w)^{\frac{1}{2}}y,\quad w\to\frac{-u}{1+4u},\quad (1+4w)\to\frac{1}{1+4u}, \tag{36}$$

we get

$$\text{(37)}\quad \pi_*\big((2wG_w(tK,w)+1)K\big) = (1+4u)^{-1/2}\Big(\kappa_0 - 2\sum_{a=0}^{\infty} y^{a+1}\kappa_{a+1}\sum_{j=0}^{a} q_{a,j}u^{j+1}\Big).$$

By Lemma 3.2, (35) and (37), we have

$$\Big[\exp\Big(\frac{1}{t}\pi_* G(tK,w)\Big)\Big]_{t^{g-2d-1+2k}w^d} = (-1)^d\Big[(1+4u)^{k-1}\exp\Big(-\sum_{a=1}^{\infty} y^a\kappa_a\sum_{j=0}^{\infty} c_{a,j}u^j\Big)\Big]_{y^{g-2d-1+2k}u^d},$$

$$\Big[\exp\Big(\frac{1}{t}\pi_* G(tK,w)\Big)\pi_*\big((2wG_w(tK,w)+1)K\big)\Big]_{t^{g-2d+2k}w^d} = \Big[(1+4u)^{k-1}\exp\Big(-\sum_{a=1}^{\infty} x^a\kappa_a\sum_{j=0}^{a} c_{a,j}u^j\Big) \Big(g-1-\sum_{a=0}^{\infty} x^{a+1}\kappa_{a+1}\sum_{j=0}^{a} q_{a,j}u^{j+1}\Big)\Big]_{x^{g-2d+2k}u^d}.$$

By formulas (6) and (7), Theorem 1.2 is proved.

Theorem 3.3. *Formula* (3) *is equivalent to formula* (5).

Proof. By the definition of $\tilde{\mathbb{F}}_d$,

$$\pi_d^*(c_t(\tilde{\mathbb{F}}_{d-1})) = c_t(\tilde{\mathbb{F}}_d)(1 - tK_d + tD_{1,d} + \cdots + tD_{d-1,d}).$$

After being pushed forward by ρ,

$$0 = \rho_*\pi_d^*(c_t(\tilde{\mathbb{F}}_{d-1})) = \rho_*(c_t(\tilde{\mathbb{F}}_d)) - t\rho_*(K_d\cdot c_t(\tilde{\mathbb{F}}_d)) + (d-1)t\rho_*(D_{1,d}\cdot c_t(\tilde{\mathbb{F}}_d)).$$

In particular,

$$(d-1)\big[\rho_*(D_{1,d}\cdot c_t(\tilde{\mathbb{F}}_d))\big]_{t^{g-d-1+2k}} = \big[\rho_*(K_d\cdot c_t(\tilde{\mathbb{F}}_d))\big]_{t^{g+d-1+2k}} - \big[\rho_*(c_t(\tilde{\mathbb{F}}_d))\big]_{t^{g-d+2k}}.$$

In fact

$$\rho_*(D_{i,j}\cdot c_t(\tilde{\mathbb{F}}_d)) = \rho_*(D_{1,d}\cdot c_t(\tilde{\mathbb{F}}_d)),$$

and the equivalence of formulas (3) and (5) is deduced from the identity

$$\begin{aligned} 2\big[\rho_*(\Delta\cdot c_t(\tilde{\mathbb{F}}_d))\big]_{t^{g-d-1+2k}} &= d(d-1)\big[\rho_*(D_{1,d}\cdot c_t(\tilde{\mathbb{F}}_d))\big]_{t^{g-d-1+2k}} \\ &= d\big[\rho_*(K_d\cdot c_t(\tilde{\mathbb{F}}_d))\big]_{t^{g-d-1+2k}} - d\big[\rho_*(c_t(\tilde{\mathbb{F}}_d))\big]_{t^{g-d+2k}} \\ &= -(g-1)\big[\rho_*(c_t(\tilde{\mathbb{F}}_d))\big]_{t^{g-d+2k}} - d\big[\rho_*(c_t(\tilde{\mathbb{F}}_d))\big]_{t^{g-d+2k}} \\ &= -(g+d-1)\big[\rho_*(c_t(\tilde{\mathbb{F}}_d))\big]_{t^{g-d+2k}}. \end{aligned}$$

□

Example 3.4. We give some low genus examples for Theorem 1.2. By the recursion relations (12) and (13) of constants $q_{k,j}$, $c_{k,j}$, we get

$$\begin{aligned}
&q_{0,0}=1,\\
&q_{1,0}=1,\quad q_{1,1}=5,\\
&q_{2,0}=1,\quad q_{2,1}=18,\quad q_{2,2}=60,\\
&q_{3,0}=1,\quad q_{3,1}=47,\quad q_{3,2}=442,\quad q_{3,3}=1105,\\
&\dots
\end{aligned}$$

and

$$\begin{aligned}
&c_{1,0}=\tfrac{1}{12},\qquad c_{1,1}=\tfrac{5}{6},\\
&c_{2,0}=0,\qquad c_{2,1}=1,\qquad c_{2,2}=5,\\
&c_{3,0}=-\tfrac{1}{360},\quad c_{3,1}=\tfrac{61}{60},\quad c_{3,2}=\tfrac{221}{12},\quad c_{3,3}=\tfrac{1105}{18},\\
&\dots
\end{aligned}$$

Taking $g=5$, $d=3$, $k=2$, formula (10) gives a relation in $R^2(\mathcal{M}_5)$:

$$\tfrac{25}{18}\kappa_1^2-20\kappa_2=0. \tag{38}$$

Taking $g=6$, $d=3$, $k=2$, formula (10) gives a relation in $R^3(\mathcal{M}_6)$:

$$-\tfrac{275}{1296}\kappa_1^3+\tfrac{55}{6}\kappa_1\kappa_2-\tfrac{2431}{18}\kappa_3=0. \tag{39}$$

It is easy to check that the relations (38) and (39) match the results in [Faber 1999].

We have written a Maple program to calculate more relations through Theorem 1.2. Unfortunately, it is difficult to determine if they contain the new tautological relations in high genus beyond those obtained by Faber.

4. Conclusion

The new relations in the tautological ring obtained in this note, that is, formulas (10) and (11), can be regarded as a partial generalization of formula (19) (which is [Ionel 2005, (1.9)]) in the special cases $b=0,1$.

When $b=0$, formula (19) is just (20):

$$\Big[\exp\Big(-\sum_{a=1}^{\infty}x^a\kappa_a\sum_{j=0}^{a}c_{a,j}u^j\Big)\Big]_{x^{g+1-2d}u^d}=0,$$

which is the special case of formula (10) with $k=1$.

For $b=1$, (19) becomes

$$\Big[\exp\Big(-\sum_{a=1}^{\infty}x^a\kappa_a\sum_{j=0}^{a}c_{a,j}u^j\Big)\cdot\Big(g-1-\sum_{a=0}^{\infty}\kappa_{a+1}x^{a+1}\sum_{j=0}^{a}q_{a,j}u^{j+1}\Big)\Big]_{x^{g+2-2d}u^d}=0,$$

which is the special case of (11) with $k = 1$.

However, our results can not cover formula (19) when $b \geq 2$. In this note, we only consider three special cases of (23) to deduce our main results. We hope that one can obtain a more general formula, like (11), from formula (23), with the same method.[1]

As mentioned in the introduction, it is important to find explicit relations in the tautological ring in studying Faber's conjecture. From this note, we see that the stable quotient method introduced by Marian, Oprea and Pandharipande [2009] provides a new and effective way to obtain the relations in the tautological ring. With this method, recently, Pandharipande [2009a; 2009b] introduced the κ ring of the moduli of curves of compact type and studied its algebraic structure. In our further study, we hope to find more applications of the stable quotient method.

Acknowledgements

The author would like to thank Professor Kefeng Liu for invaluable discussions and Dr. Hao Xu for telling him some recent results on the tautological ring of the moduli space of curves.

References

[Buryak and Shadrin 2009] A. Buryak and S. Shadrin, "A new proof of Faber's intesection number conjecture", preprint, 2009. arXiv 0912.5115

[Faber 1999] C. Faber, "A conjectural description of the tautological ring of the moduli space of curves", pp. 109–129 in *Moduli of curves and abelian varieties*, edited by C. Faber and E. Looijenga, Aspects Math. **E33**, Vieweg, Braunschweig, 1999. MR 2000j:14044 Zbl 0978.14029

[Getzler and Pandharipande 1998] E. Getzler and R. Pandharipande, "Virasoro constraints and the Chern classes of the Hodge bundle", *Nuclear Phys. B* **530**:3 (1998), 701–714. MR 2000b:14073 Zbl 0957.14038

[Harris and Mumford 1982] J. Harris and D. Mumford, "On the Kodaira dimension of the moduli space of curves", *Invent. Math.* **67**:1 (1982), 23–88. MR 83i:14018 Zbl 0506.14016

[Ionel 2005] E.-N. Ionel, "Relations in the tautological ring of $\mathcal{M}_g$", *Duke Math. J.* **129**:1 (2005), 157–186. MR 2006c:14040 Zbl 1086.14023

[Liu and Xu 2009] K. Liu and H. Xu, "A proof of the Faber intersection number conjecture", *J. Differential Geom.* **83**:2 (2009), 313–335. MR 2011d:14051 Zbl 1206.14079

[Liu and Xu 2010] K. Liu and H. Xu, "Computing top intersections in the tautological ring of $\mathcal{M}_g$", preprint, 2010. To appear in *Math. Z.* arXiv 1001.4498

[Looijenga 1995] E. Looijenga, "On the tautological ring of $\mathcal{M}_g$", *Invent. Math.* **121**:2 (1995), 411–419. MR 96g:14021 Zbl 0851.14017

[Marian et al. 2009] A. Marian, D. Oprea, and R. Pandharipande, "The moduli space of Stable quotients", preprint, 2009. arXiv 0904.2992

[1] After the submission of this paper, Professor R. Pandharipande wrote to the author that he and A. Pixton had got the general results [Pandharipande and Pixton 2011].

[Morita 2003] S. Morita, "Generators for the tautological algebra of the moduli space of curves", *Topology* **42**:4 (2003), 787–819. MR 2004g:14029 Zbl 1054.32008

[Mumford 1983] D. Mumford, "Towards an enumerative geometry of the moduli space of curves", pp. 271–328 in *Arithmetic and geometry, II*, edited by M. Artin and J. Tate, Progr. Math. **36**, Birkhäuser, Boston, MA, 1983. MR 85j:14046 Zbl 0554.14008

[Pandharipande 2009a] R. Pandharipande, "The κ ring of the moduli of curves of compact type: I", preprint, 2009. arXiv 0906.2657

[Pandharipande 2009b] R. Pandharipande, "The κ ring of the moduli of curves of compact type: II", preprint, 2009. arXiv 0906.2658

[Pandharipande and Pixton 2011] R. Pandharipande and A. Pixton, "Relations in the tautological ring", preprint, 2011. arXiv 1101.2236

Received December 3, 2010. Revised March 2, 2011.

SHENGMAO ZHU
DEPARTMENT OF MATHEMATICS AND CENTER OF MATHEMATICAL SCIENCES
ZHEJIANG UNIVERSITY
HANGZHOU, ZHEJIANG 310027
CHINA

zhushengmao@gmail.com

PACIFIC JOURNAL OF MATHEMATICS

http://www.pjmath.org

Founded in 1951 by
E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

See inside back cover or www.pjmath.org for submission instructions.

The subscription price for 2011 is US $420/year for the electronic version, and $485/year for print and electronic.
Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. Prior back issues are obtainable from Periodicals Service Company, 11 Main Street, Germantown, NY 12526-5635. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and the Science Citation Index.

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

PACIFIC JOURNAL OF MATHEMATICS

Volume 252 No. 2 August 2011