Let $X \subset \mathbb{P}^N$ be an irreducible, nondegenerate projective variety and let $X^* \subset \mathbb{P}^{N^*}$ be its projective dual. Let $L \subset \mathbb{P}^N$ be a linear space such that $\langle L, T_{X,x} \rangle \neq \mathbb{P}^N$ for all $x \in X_{\text{smooth}}$ and such that the lines in $X$ meeting $L$ do not cover $X$. If $x \in X$ is general, we prove that the multiplicity of $X^*$ at a general point of $\langle L, T_{X,x} \rangle$ is strictly greater than the multiplicity of $X^*$ at a general point of $L^\perp$. This is a strong refinement of Bertini’s theorem.

1. Introduction

1.1. Multiplicities of the projective dual. Let $X \subset \mathbb{P}^N$ be an irreducible projective variety over the field of complex numbers. Let $X^* \subset \mathbb{P}^{N^*}$ be its projective dual, let $L \subset \mathbb{P}^N$ be a linear space and $H$ be a general hyperplane containing $L$. Bertini’s classical theorem asserts that the tangency locus of $H$ with $X$ is included in $X \cap L$.

Very little is known about the hyperplanes whose tangency locus with $X$ lies outside $L \cap X$. It is tempting to think that the multiplicity in $X^*$ of such a hyperplane is strictly larger than the multiplicity of a general hyperplane containing $L$. The following example shows that this is not true for every $L$.

Example 1.1.1. Let $X \subset \mathbb{P}^4$ be a smooth hyperplane section of $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$. The variety $X$ is a ruled surface of degree 3. Its dual is a hypersurface of degree 3 in $\mathbb{P}^{4^*}$ which does not contain any points of multiplicity higher than 2. Let $L$ be the exceptional section of $X$, then $H \cap X = L \cup D_1 \cup D_2$, where $D_1$ and $D_2$ are two distinct lines on $X$ such that $D_1.D_2 = 0$ and $L.D_i = 1$ for $i = 1, 2$. As a consequence, a general point of $L^\perp$ is of multiplicity 2 in $X^*$. Now, let $D \subset X$ be a line such that $D.L = 1$ and let $x \in D$ such that $x \notin L$. The hyperplane containing $L$ and $T_{X,x}$ is a point of multiplicity exactly 2 in $X^*$, that is, the multiplicity of a general point of $L^\perp$.

This example shows that, even for general $x \in X$, the multiplicity in $X^*$ of a hyperplane containing $L$ and tangent to $X$ at $x$ may well be equal to the multiplicity of a general hyperplane containing $L$. Thus, without extra hypotheses on $L$, it
seems hopeless to say something about the multiplicity in $X^*$ of special points of $L^\perp$. For this purpose, we introduce a definition:

**Definition 1.1.2.** Let $X \subset \mathbb{P}^N$ be an irreducible projective variety and let $L \subset \mathbb{P}^N$ be a linear space. Consider the conormal diagram

$$
\begin{array}{ccc}
X^* \subset \mathbb{P}^N^* & \xrightarrow{q} & I(X/\mathbb{P}^N) := \{(H, x) \in \mathbb{P}^N^* \times X_{\text{smooth}} : T_{X,x} \subset H\} \\
\downarrow & & \downarrow p \\
& X \subset \mathbb{P}^N & 
\end{array}
$$

Let $F_1, \ldots, F_m$ be all the irreducible components of $q^{-1}(L^\perp)$ such that the restrictions

$$
q|_{F_i} : F_i \to L^\perp
$$

are surjective. The **contact locus** of $L$ with $X$, which we denote by $\text{Tan}(L, X)$, is the union of the $p(F_i)$, for $1 \leq i \leq m$.

In the case where $L$ is a hyperplane, the contact locus $\text{Tan}(L, X)$ is called the **tangency locus** of $L$ with $X$. A **tangent hyperplane** to $X$ is a hyperplane $H \subset \mathbb{P}^N$ such that $\text{Tan}(H, X) \neq \emptyset$.

The contact locus $\text{Tan}(L, X)$ can be thought as the variety covered by the tangency loci of general hyperplanes containing $L$. In case $L^\perp \not\subset X^*$, this locus is empty. We always have the inclusion

$$
\{x \in X_{\text{smooth}} : T_{X,x} \subset L\} \subset \text{Tan}(L, X),
$$

but if $\dim(L) < N - 1$ or if $X$ is not smooth, the former locus can be strictly smaller than the latter. Note also that Bertini’s theorem says that $\text{Tan}(L, X) \subset L \cap X$.

Finally, the contact locus is well behaved. If for a general hyperplane $H'$ containing $L$, we have $\dim(\text{Tan}(H', X)) > 0$, then

$$
\text{Tan}(H \cap L, H \cap X) = H \cap \text{Tan}(L, X),
$$

for any general hyperplane $H \subset \mathbb{P}^N$.

**Example 1.1.3.** If $X \subset \mathbb{P}^N$ is such that $X^*$ is a hypersurface and $L = T_{X,x}$, where $x \in X$ is a general point, then $\text{Tan}(L, X) = x$.

- If $X = G(1, 7) \subset \mathbb{P}^{27}$ and $L = \langle T_{X,y_1}, T_{X,y_2} \rangle$, where $y_1, y_2 \in G(1, 7)$ are two general points, then $\text{Tan}(L, X) = \{x \in X : T_{X,x} \subset L\}$ is a 4-dimensional quadric, the entry locus of a general point $z \in \langle y_1, y_2 \rangle$.

- If $X = G(1, 4) \subset \mathbb{P}^9$ and $L = T_{X,y}$, for any $y \in X$, then $\dim(\text{Tan}(L, X)) > 0$, whereas $\{x \in X : T_{X,x} \subset L\} = \{y\}$. 


**Definition 1.1.4.** Let \( X \subset \mathbb{P}^N \) be an irreducible projective variety, and let \( L \subset \mathbb{P}^N \) be a linear subspace. The **shadow** of \( L \) on \( X \), which we denote by \( \text{Sh}_X(L) \), is the closed variety covered by the linear spaces \( M \subset X \) such that \( \dim(M) = \text{def}(X) + 1 \) and \( \dim(M \cap \text{Tan}(L, X)) = \text{def}(X) \).

Here \( \text{def}(X) = \text{codim}(X^*) - 1 \). The shadow is also well behaved. Namely, assume that \( \text{def}(X) > 0 \); then

\[
\text{Sh}_L(X) = X \iff \text{Sh}_{H \cap L}(H \cap X) = H \cap X,
\]

for any general hyperplane \( H \subset \mathbb{P}^N \). Note also that if \( x \in X \) is a general point and \( L = T_{X,x} \), then \( \text{Sh}_L(X) \neq X \), unless \( X \) is a linear space. Indeed, if \( X^* \) is a hypersurface, this is obvious since \( \text{Tan}(T_{X,x}, X) = \{ x \} \) for general \( x \in X \). If \( X^* \) is not a hypersurface, take enough general hyperplane sections of \( X \) passing through \( x \), so that the corresponding dual is a hypersurface.

**Main Theorem 1.1.5.** Let \( X \subset \mathbb{P}^N \) be an irreducible, nondegenerate projective variety. Let \( L \subset \mathbb{P}^N \) be a linear space such that \( \text{Sh}_X(L) \neq X \). Then, for all \( x \in X \) smooth such that \( x \notin \text{Sh}_X(L) \) and such that \( \langle L, T_{X,x} \rangle \neq \mathbb{P}^N \), the multiplicity in \( X^* \) of a general hyperplane containing \( \langle L, T_{X,x} \rangle \) is strictly larger than the multiplicity in \( X^* \) of a general hyperplane containing \( L \).

If \( X \) is the ruled cubic surface considered in Example 1.1.1 and \( L \) is the directrix of \( X \), one notices easily that \( \text{Sh}_X(L) = X \). This shows that the hypothesis \( \text{Sh}_X(L) \neq X \) can not be withdrawn. Here is an obvious consequence of Main Theorem 1.1.5:

**Corollary 1.1.6.** Let \( X \subset \mathbb{P}^N \) be an irreducible, nondegenerate projective variety. Let \( L \subset \mathbb{P}^N \) be a linear space such that \( \langle L, T_{X,x} \rangle \neq \mathbb{P}^N \) for general \( x \in X \), and such that the lines in \( X \) meeting \( L \) do not cover \( X \). Then, for general \( x \in X \), the multiplicity in \( X^* \) of a general hyperplane containing \( \langle L, T_{X,x} \rangle \) is strictly larger than the multiplicity in \( X^* \) of a general hyperplane containing \( L \).

1.2. **Variety of multisecant spaces and duals.**

**Definition 1.2.1.** Let \( X \subset \mathbb{P}^N \) be an irreducible projective variety. Let

\[
S^k_X \{ (x_0, \ldots, x_k, u) \in X \times \cdots \times X \times \mathbb{P}^N : \dim\langle x_0, \ldots, x_k \rangle = k, \ u \in \langle x_0, \ldots, x_k \rangle \},
\]

and let \( S^k_X \) be its Zariski closure in \( X \times \cdots \times X \times \mathbb{P}^N \). Denote by \( \phi \) the projection onto \( \mathbb{P}^N \). The variety \( S^k_X(X) = \phi(S^k_X) \) is the \( k \)-th secant variety to \( X \).

**Theorem 1.2.2** (Terracini’s lemma [Zak 1993]). Let \( X \subset \mathbb{P}^N \) be an irreducible projective variety, and let \( (x_0, \ldots, x_k) \in X \times \cdots \times X \), be general points. If \( u \) is general in \( (x_0, \ldots, x_k) \), we have the equality

\[
\langle T_{X,x_0}, \ldots, T_{X,x_k} \rangle = T_{S^k_X(X), u}.
\]
\textbf{Definition 1.2.3.} Let $X \subset \mathbb{P}^N$ be an irreducible, nondegenerate projective variety, and let $k$ be an integer such that $S^k(X) \neq \mathbb{P}^N$. We say that $X$ is \textit{dual $k$-defective} if $\text{def}(S^k(X)) > t(S^k(X))$, where $t(S^k(X))$ is the dimension of the general fiber of the Gauss map of $S^k(X)$.

Note that when $X$ is smooth, then dual 0-defectiveness is the classical dual defectiveness. I don’t know if there exist smooth varieties which are dual $k$-defective for some $k \geq 1$, but which are not dual 0-defective. I believe it would be interesting to find some examples of such varieties.

Note also that the notion of dual $k$-defectiveness seems to be related to that of $R_k$ regularity explored in [Chiantini and Ciliberto 2010].

Here is a consequence of the Main Theorem 1.1.5 and Terracini’s lemma:

\textbf{Proposition 1.2.4.} Let $X \subset \mathbb{P}^N$ be an irreducible, nondegenerate, smooth, projective variety. Assume moreover that for all $k$ such that $S^k(X) \neq \mathbb{P}^N$ the variety $X$ is not dual $k-1$-defective. Then, for any such $k$, we have

$$S^k(X)^* \subset X_{k+1}^*,$$

where $X_{k+1}^*$ is the set of points which have multiplicity at least $k + 1$ in $X^*$.

\textbf{Proof.} The case $k = 0$ is the definition of $S^0(X)^* = X^*$. Let $k \geq 1$ be an integer such that $S^k(X) \neq \mathbb{P}^N$, let $z \in S^{k-1}(X)$ be a general point and $H$ be a general hyperplane containing $T_{S^{k-1}(X),z}$. Let’s prove that

$$\text{Tan}(H, X) = \{x \in X : T_{X,x} \subset T_{S^{k-1}(X),z}\}.$$ 

Let $x_0, \ldots, x_{k-1}$ be $k$ general points in $\text{Tan}(H, X)$. Let $z' \in \text{Tan}(H, S^{k-1}(X))$. But $\text{def}(S^{k-1}(X)) = t(S^{k-1}(X))$ by hypothesis, and this implies

$$z' \in \{y \in S^{k-1}(X)_{\text{smooth}} : T_{S^{k-1}(X),y} \subset T_{S^{k-1}(X),z}\},$$

so that $x_0, \ldots, x_{k-1} \in \{x \in X : T_{X,x} \subset T_{S^{k-1}(X),z}\}$.

We now prove that $\text{Sh}_X(T_{S^{k-1}(X),z}) \neq X$. The argument above shows that

$$\text{Tan}(T_{S^{k-1}(X),z}, X) = \{x \in X : T_{X,x} \subset T_{S^{k-1}(X),z}\}.$$ 

Assume $\text{Sh}_X(T_{S^{k-1}(X),z}) = X$. For all $x'' \in X$, there is $x' \in \{x \in X : T_{X,x} \subset T_{S^k(X),z}\}$ such that the line $\langle x'', x' \rangle$ lies in $X$. But since $X$ is smooth, this line $\langle x'', x' \rangle$ lies in $T_{X,x'}$. So we have $X \subset T_{S^{k-1}(X),z}$, which contradicts the nondegeneracy.
As a consequence of Main Theorem 1.1.5, we get that for a general \( x \in X \), the multiplicity in \( X^* \) of a general hyperplane containing \( \langle T_{S^{k-1}(X), x}, T_{X, x} \rangle \) is strictly larger than the multiplicity in \( X^* \) of a general hyperplane containing \( T_{S^{k-1}(X), x} \). We apply Terracini’s lemma to find that \( S^k(X)^* \subset X^*_{k+1} \). This concludes the proof. \( \square \)

A stronger result than Proposition 1.2.4 has been stated for the first time in [Zak 2004], but no proof was given there.

In the second part of this paper we present a proof of Main Theorem 1.1.5, while in the third part we discuss some consequences and open questions.

2. Proof of the Main Theorem

When \( Z \subset \mathbb{P}^N \), we denote by \( \mathcal{E}_Z(Z) \subset \mathbb{P}^N \) the embedded tangent cone to \( Z \) at \( z \) and if \( H \subset \mathbb{P}^N \) is a hyperplane, then \( [h] \) is the corresponding point in \( (\mathbb{P}^N)^* \).

The proof of Main Theorem 1.1.5 is obvious if \( L^\perp \not\subset X^* \). Thus, we only deal with the case where \( L^\perp \subset X^* \). Moreover, we can restrict to the case where \( X^* \) is a hypersurface. Indeed, assume that \( X^* \) has codimension \( p \geq 2 \). Let \( z \in L^\perp \) and \( z_x \in \langle L, T_{X, x} \rangle^\perp \) be general points, let \( M \subset \mathbb{P}^N \) be a general \( \mathbb{P}^{N+1-p} \) passing through \( x \), let \( X' = M \cap X \) and \( L' = M \cap L \). We have \( \text{Sh}_{X'}(L') \neq X' \) and \( \langle T_{X', x}, L' \rangle \neq \mathbb{P}^{N+1-p} \). Moreover, we have

\[
(X')^* = \pi_{M^\perp}(X^*),
\]

where \( \pi_{M^\perp} \) is the projection from \( M^\perp \) in \( \mathbb{P}^N^* \). Since \( M \) is general, the map \( \pi_{M^\perp} \) is locally an isomorphism around \( z_x \). Hence

\[
\text{mult}_{z} X^* = \text{mult}_{z_x} X^* \iff \text{mult}_{\pi_{M^\perp}(z)}(X')^* = \text{mult}_{\pi_{M^\perp}(z_x)}(X')^*.
\]

Finally, note that \( \pi_{M^\perp}(z) \) is a general point of \( (L')^\perp \) and that \( \pi_{M^\perp}(z_x) \) is a general point of \( \langle L', T_{X', x} \rangle^\perp \). As a consequence, it is sufficient to prove the theorem for \( X' \), whose dual is a hypersurface.

Let’s start with a plan of the proof. We assume that \( X^* \) has constant multiplicity along a smooth curve \( S \subset L^\perp \) passing through \( \langle L, T_{X, x} \rangle^\perp \) and through a general point of \( L^\perp \) and we find a contradiction. More precisely:

- We prove that the equimultiplicity of \( X^* \) along \( S \) implies that the family of the tangent cones to \( X^* \) at the points of \( S \) is flat.

- Then, we show that the flatness of the family of the tangent cones to \( X^* \) at the points of \( S \) leads to the flatness of the family of the conormal spaces of these tangent cones. As a consequence, we have \( |\mathcal{E}_S(X^*)|^* \subset L \) for all \( s \in S \).

- Finally, we relate the tangent cone to \( X^* \) at \( z \) to the set of tangent hyperplanes to \( X^* \) at \( z \) (when \( z \) is a smooth point of \( X^* \); this is the reflexivity theorem [Kleiman 1986]). Using the fact that \( \text{Sh}_L(X) \neq X \), we deduce that \( |\mathcal{E}_S(X^*)|^* \not\subset L \) for \( s \in \langle L, T_{X, x} \rangle^\perp \) and thus a contradiction.
2.1. Normal flatness and Lagrangian specialization principle. Let $S \subset Z \subset \mathbb{P}^N$ be two varieties. We recall some properties of the tangent cones $\mathcal{C}_s(Z), s \in S$ when $Z$ is equimultiple along $S$.

**Definition 2.1.1.** Let $S \subset Z$ be two varieties. We say that $Z$ is equimultiple along $S$ if the multiplicity of the local ring $\mathcal{O}_{Z,s}$ is constant for $s \in S$.

**Proposition 2.1.2** [Hironaka 1964, Corollary 2, p. 197]. Let $Z \subset \mathbb{P}^N$ be a hypersurface and $S$ a connected smooth subvariety (not necessarily closed) of $Z$ such that $Z$ is equimultiple along $S$.

Then, for all $s \in S$, there exists an open neighborhood $U$ of $s$ in $S$ containing $s$ and a closed subscheme $\mathcal{G}(Z) \subset \mathbb{P}^N \times U$ such that the natural projection $p : \mathcal{G}(Z) \to U$ is a flat and surjective morphism whose fiber $\mathcal{G}(Z)_s'$ over any $s' \in U$ is $\mathcal{C}_s(Z)$.

We assume that our theorem is not true, that is for general $x \in X$, the multiplicity of $X^*$ at a general point of $\langle L, T_{X,x} \rangle$ is equal to the multiplicity at a general point of $L$.

Let $[h]$ be a general point of $\langle L, T_{X,x} \rangle$ and let $S \subset L$ be a smooth (not necessarily closed) connected curve passing through $[h]$ and through a general point of $L$. We apply the proposition to $X^*$ and $S$. Then there exists a scheme $\mathcal{G}(X^*) \subset \mathbb{P}^N \times S$ such that the natural projection $p : \mathcal{G}(X^*) \to S$ is a flat and surjective morphism whose fiber over $s \in S$ is the tangent cone to $X^*$ at $s$. Let $\Gamma(X^*) = |\mathcal{G}(X^*)|$. The induced morphism $\Gamma(X^*) \to S$ is flat and for general $s \in S$ the fiber $\Gamma(X^*)_s$ is exactly $|\mathcal{C}_s(X^*)|$.

Now we study the family of the duals of the reduced tangent cones of $X^*$ at points of $S$. Applying the Lagrangian specialization principle [Lê and Teissier 1988; Kleiman 1984] to $\Gamma(X^*)$ and $S$, we find:

**Theorem 2.1.3.** Let $S \subset X^*$ be a smooth curve such that $X^*$ is equimultiple along $S$. There exists a variety $I_S(\Gamma(X^*)/\mathbb{P}^N \times S)$ with the following properties.

(i) For general $s \in S$, the following equality holds in $\mathbb{P}^N \times \Gamma(X^*)_s$:

$$I_s(\mathcal{C}_s(X^*)/\mathbb{P}^N) = I_S(\Gamma(X^*)/\mathbb{P}^N \times S)_s.$$ 

(ii) The morphism $I_S(\Gamma(X^*)/\mathbb{P}^N \times S) \to S$ is flat and surjective.

(iii) For all $s \in S$, the conormal space $I_s(\mathcal{C}_s(X^*)/\mathbb{P}^N \times S)$ is a union of irreducible components of the reduced fiber $|I_S(\Gamma(X^*)/\mathbb{P}^N \times S)_s|$.

As a consequence, the image in $\mathbb{P}^N$ of the fiber $I_S(\Gamma(X^*)/\mathbb{P}^N \times S)_s$, for general $s \in S$, is $|\mathcal{C}_s(X^*)|$. Moreover, for any $s \in S$, the image of the reduced fiber $|I_S(\Gamma(X^*)/\mathbb{P}^N \times S)_s|$ contains $|\mathcal{C}_s(X^*)|$.
2.2. **Polar varieties and duals of tangent cones.** We discuss an extension of the reflexivity theorem proved in [Lê and Teissier 1988]. The main results of this section will be applied to $X^*$ when it is a hypersurface, so we restrict our study to that case.

**Definition 2.2.1.** Let $Z \subset \mathbb{P}^N$ be a reduced and irreducible hypersurface and let $D \subset \mathbb{P}^N$ be a linear space. The *polar variety* of $Z$ associated to $D$, which we denote by $P(Z, D)$, is the closure of the set \{z $\in Z_{\text{smooth}} : D \subset T_{Z, z}$\}.

If $D = \emptyset$ (that is, $D$ has dimension $-1$), then we put $P(Z, D) = Z$.

**Remark 2.2.2.** If $Z$ is normal, if $u = [u_0, \ldots, u_N]$ in an homogeneous system of coordinates on $\mathbb{P}^N$ and $f$ is an equation of $Z$ in this system then $P(Z, u)$ is given by the equations $f = 0$ and $u_0 \frac{\partial f}{\partial x_0} + \cdots + u_N \frac{\partial f}{\partial x_N} = 0$.

If $Z$ is not normal, then all irreducible components of $Z_{\text{sing}}$ which are of dimension $N - 2$ are irreducible components of the scheme defined by $f = 0$ and $u_0 \frac{\partial f}{\partial x_0} + \cdots + u_N \frac{\partial f}{\partial x_N} = 0$, but they are not irreducible components of $P(X, u)$.

**Proposition 2.2.3.** Let $Z \subset \mathbb{P}^N$ be a reduced, irreducible hypersurface and let $D \subset \mathbb{P}^N$ be a general linear space of dimension $k$. Then $P(Z, D)$ is empty or of codimension $k + 1$ in $Z$.

We state a result of Lê and Teissier which relates the duals of the tangent cones at $z$ of some polar varieties of $Z$ with the tangency locus of $z^\perp$ with $Z^*$. See [Lê and Teissier 1988, Proposition 2.2.1]. For any $z \in Z$, recall that $\text{Tan}(z^\perp, Z^*)$ is the tangency locus of $z$ along $Z^*$ (see conormal diagram on page 2).

**Theorem 2.2.4.** Let $Z \subset \mathbb{P}^N$ be a reduced and irreducible hypersurface and let $z \in Z$ be a point.

(i) The dual of $|\mathcal{C}_z(Z)|$ is a union of reduced spaces underlying (possibly embedded) components of $\text{Tan}(z^\perp, Z^*)$.

(ii) Any irreducible component of $|\text{Tan}(z^\perp, Z^*)|$ is dual to an irreducible component of $|\mathcal{C}_z(P(Z, D))|$ for general $D \in \mathcal{G}(k, N)$ and for some integer $k \in \{-1, \ldots, N - 2\}$.

**Remark 2.2.5.** Part (ii) of the theorem has to be explained. Assume that there is an irreducible component (say $T$) of $|\text{Tan}(z^\perp, Z^*)|$ which is not dual to an irreducible component of $|\mathcal{C}_z(Z)|$. Then, there is $k \in \{0, \ldots, N - 2\}$ such that for general $D \in \mathcal{G}(k, N)$, we have $z \in P(Z, D)$. Moreover, as $D$ varies in a dense open subset of $\mathcal{G}(k, D)$, the cones $\mathcal{C}_z(P(D, Z))$ have a fixed irreducible component in common whose reduced locus is $T^*$. Note also that if $z \in Z_{\text{smooth}}$ then for $k \geq 0$ and for $D$ general in $\mathcal{G}(k, N)$, we have $z \notin P(Z, D)$. As a consequence of the (ii) of the above theorem, we find
Tan(z⊥, Z*) = T_{Z,z}^⊥ for z ∈ Z_{smooth}. This is the way the (obvious corollary of the) reflexivity theorem is often stated.

When Tan(z⊥, Z*) is irreducible, one may expect |ₖ(z)(Z)|* = |Tan(z⊥, Z*)|. But this is not true:

**Example 2.2.6.** Let X ⊂ P³ be the smooth ruled surface of degree 3 considered in example 1.1.1 and let X* its dual. The hypersurface X* has also degree 3 and its singular locus is a P², the dual of the exceptional section of X (which we denote by L). Let C ⊂ L⊥ = X_{sing}^* be the conic corresponding to the hyperplanes which are tangent to X along a ruling of X and let z ∈ C.

The tangent cone λ(z(X*)) is a doubled P³ so that |ₖ(z(X*))|* ≠ Tan(z⊥, X). We also note that the scheme-theoretic tangency locus of z⊥ along X is a line with an embedded point. The embedded point is dual to |ₖ(z(X*))| and the line is dual to |ₖ(z(P(X*, u)))|, for general u ∈ P⁴*.

**Notations 2.2.7.** Let f : Y → T be a quasiprojective morphism between quasi-projective schemes, let T′ ⊂ T be a smooth variety and let s ∈ T′ be any point. Let Y₁, . . . , Yₘ be the irreducible components of f⁻¹(T′) such that the restrictions

$$f|_{Y_i} : Y_i \rightarrow T'$$

are surjective. Define the scheme

$$\lim_{\text{flat}}_{(t \rightarrow s \in T')} f^{-1}(t) := f|_{Y_1 \cup \ldots \cup Y_m}^{-1}(s).$$

If dim(T') = 1 and the Yᵢ are all reduced, this is the classical flat limit taken along a smooth curve. If f|⁻¹(T′) : f⁻¹(T′) → T′ is flat, then

$$\lim_{\text{flat}}_{(t \rightarrow s \in T')} f^{-1}(t) = f|_{f^{-1}(T')}^{-1}(s).$$

**Proof of Main Theorem 1.1.5.** We recall the setting for the convenience of the reader. The projective variety X ⊂ Pᴺ is irreducible and nondegenerate. The linear space L ⊂ Pᴺ is such that Shₓ(X) ≠ X and ⟨L, Tₓ,X⟩ ≠ Pᴺ for all x ∈ X_{smooth}. We want to prove that for all x ∈ X_{smooth} such that x ∉ Shₓ(L), the multiplicity in X* of a general hyperplane containing ⟨L, Tₓ,X⟩ is strictly greater than that of a general hyperplane containing L.

The result is obvious if L⊥ ∉ X* and we have already seen that we can restrict to the case where X* is a hypersurface. So we only consider the case where L⊥ ⊂ X* and X* is a hypersurface and we assume that our result is not true. Let x ∈ X_{smooth} with x ∉ Shₓ(L) and let [h] be a general point in ⟨L, Tₓ,X⟩⊥. By the results of the previous section, there exists a smooth (not necessarily closed) curve S ⊂ L⊥ with [h] ∈ S and a flat morphism

$$I_S(\Gamma(X^*)/\mathbb{P}^N^* \times S) \rightarrow S,$$
whose fiber \( I_{S}(\Gamma(X^*)/\mathbb{P}^N^* \times S), s \) is the conormal space of \(|\ell_{s}(X^*)|\), for general \( s \in S \). Further, the conormal space of \(|\ell_{s}(X^*)|\) is included in \(|I_{S}(\Gamma(X^*)/\mathbb{P}^N^* \times S), s \)| for all \( s \in S \).

Theorem 2.2.4(i) implies that
\[
|\ell_{s}(X^*)|^* \subset p(|q^{-1}(s)|),
\]
for all \( s \in S \), where \( p \) and \( q \) are as in the conormal diagram of page 2. The flatness of \( I_{S}(\Gamma(X^*)/\mathbb{P}^N^* \times S) \to S \) gives the inclusion
\[
|\ell_{[h]}(X^*)|^* \subset p(\limflat_{[s \to [h], s \in S]} |q^{-1}(s)|).
\]
By Definition 1.1.2, the right-hand side is contained in \( \text{Tan}(L, X) \subset L \).

Let \( \mathcal{F} \) be an irreducible component of \( \text{Tan}(H, X) \) passing through \( x \). By Theorem 2.2.4, there is an integer \( k \in \{-1, \ldots, N-2\} \) such that \( |\mathcal{F}| \) is dual to an irreducible component of \(|\ell_{[h]}(P(X^*, D))|\), for general \( D \in \mathbb{G}(k, N) \). Since \(|\ell_{[h]}(X^*)|^* \subset L \), we have \( k \geq 0 \).

Let \( x_0 \in \mathcal{F} \) be a general point. Duality implies \( T_{[\ell_{[h]}(P(X^*, D))], z} \subset x_0^\perp \) for some general \( z \) in the irreducible component of \(|\ell_{[h]}(P(X^*, D))|\) whose reduced locus is \(|\mathcal{F}|^* \). Note that \(|\ell_{[h]}(P(X^*, D))| \subset \ell_{[h]}(X^*)\). Let \( T_{[\ell_{[h]}(X^*)], z} \) be a limit of tangent spaces to \(|\ell_{[h]}(X^*)|\) at \( z \). The point \( z \) is general in \(|\ell_{[h]}(P(X^*, D))|\), so \( T_{[\ell_{[h]}(P(X^*, D))], z} \subset T_{[\ell_{[h]}(X^*)], z} \).

As a consequence of this, we have \( T_{[\ell_{[h]}(P(X^*, D))], z} \subset x_0^\perp \cap T_{[\ell_{[h]}(X^*)], z} \). That is,
\[
(x_0, T_{[\ell_{[h]}(X^*)], z}) \subset \mathcal{F} \subset X.
\]
But \(|\ell_{[h]}(X^*)|^* \subset \text{Tan}(L, X) \), so \( T_{[\ell_{[h]}(X^*)], z} \in \text{Tan}(L, X) \), and the inclusion above says that \( x_0 \in \text{Sh}_X(L) \). This is a contradiction. \( \square \)

### 3. Corollaries and open questions

We present here some corollaries of the Main Theorem and related open questions.

#### 3.1. Zak’s conjecture on varieties with minimal codegree

Let \( X \subset \mathbb{P}^N \) be an irreducible, nondegenerate projective variety. We recall, following Zak, that the order of \( X \) is \( \text{ord } X = \min\{k, S^{k-1}(X) = \mathbb{P}^N\} \) and the \( k \)-th secant-defect is \( \delta_k = \dim X + \dim S^{k-1}(X) + 1 - \dim S^k(X) \), for all \( k \leq \text{ord } X - 1 \).

Zak [1993] proved an important result related to secant defects.

**Theorem 3.1.1** (Zak’s superadditivity theorem). *Let \( X \subset \mathbb{P}^N \) be an irreducible, nondegenerate projective variety such that \( \delta_1 > 0 \). For all \( k \leq \text{ord } X - 1 \), we have the inequality*
\[
\delta_k \geq \delta_{k-1} + \delta_1.
\]

The varieties on the boundary are called Scorza varieties. More precisely:
Definition 3.1.2. An irreducible, smooth, nondegenerate projective variety $X \subset \mathbb{P}^N$ is a Scorza variety if the following conditions hold:

(i) $\delta_1 > 0$ and $N > 2n + 1 - \delta_1$,
(ii) $\delta_k = \delta_{k-1} + \delta_1$ for all $k \leq \text{ord } X - 1$,
(iii) $\text{ord } X - 1 = [\dim X / \delta_1]$, where $[ \cdot ]$ denotes the integral part.

Theorem 3.1.3 (Classification of Scorza varieties [Zak 1993]). Any Scorza variety $X$ is of one of the following types:

(i) $X = v_2(\mathbb{P}^n) \subset \mathbb{P}^{n(n+3)/2}$ (2nd Veronese) and $\deg X^* = n + 1$;
(ii) $X = \mathbb{P}^n \times \mathbb{P}^n \subset \mathbb{P}^{n(n+2)}$ and $\deg X^* = n + 1$;
(iii) $X = G(1, 2n+1) \subset \mathbb{P}(\Lambda^2 \mathbb{C}^{2n+2})$ and $\deg (X^*) = n + 1$;
(iv) $X \subset \mathbb{P}^{26}$ is the 16-dimensional variety corresponding to the orbit of highest weight vector in the lowest nontrivial representation of the group of type $E_6$ and $\deg X^* = 3$.

In [Zak 2004] an important consequence of the assertion $S_k(X^*) \subset X^*_{k+1}$ (where $X^*_k$ is the set of points of multiplicity at least $k$ in $X^*$) was discovered. We state that result in the setting where we are able to prove it.

Proposition 3.1.4. Let $X \subset \mathbb{P}^N$ be an irreducible, nondegenerate, smooth, projective variety. Assume that $X$ is not $k$ dual defective for $k < \text{ord } X - 1$, then

$$\deg X^* \geq \text{ord } X.$$ 

Proof. With the assumptions above, Proposition 1.2.4 implies that there is a point of multiplicity $\text{ord } X - 1$ in $X^*$. Since $X$ is nondegenerate, its dual is not a cone and so $\deg X^* \geq \text{ord } X$. \qed

If $X$ is a Scorza variety then $\deg X^* = \text{ord } X$. The converse statement in conjectured in [Zak 2004]. We formulate the conjecture in the setting where we can prove the inequality: $\deg X^* \geq \text{ord } X$.

Conjecture 3.1.5 [Zak 2004]. Let $X \subset \mathbb{P}^N$ be an irreducible, smooth, nondegenerate, projective variety. Assume that $X$ is not $k$ dual defective for all $k < \text{ord } X$ and that $\deg X^* = \text{ord } X + 1$, then $X$ is a hyperquadric or a Scorza variety.

It is proved in [Zak 1993], without any hypothesis on the dual defectiveness of $X$, that smooth varieties with $\deg (X^*) = 3$ and $\text{ord } X = 3$ are Severi varieties. In particular, they are Scorza varieties. Note, however, that the smoothness assumption seems to be necessary in his proof. I believe it would be very interesting to have a classification of all varieties whose duals have degree 3.

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3.2. Varieties with unexpected equisingular linear spaces. We come back to our usual setting. Let \( L \subset \mathbb{P}^N \) be a linear space such that for all \( x \in X_{\text{smooth}} \), we have \( \langle L, T_{X,x} \rangle \neq \mathbb{P}^N \). We have seen in example 1.1.1 that a hyperplane containing the join \( \langle L, T_{X,x} \rangle \) may have the same multiplicity in \( X^* \) as the general hyperplane containing \( L \), even if \( x \) is a general point of \( X \). The following definition is convenient to describe this situation.

**Definition 3.2.1.** Let \( X \subset \mathbb{P}^N \) be an irreducible, nondegenerate projective variety such that \( X^* \) is a hypersurface. Let \( L \subset \mathbb{P}^N \) be a linear space such that for all \( x \in X_{\text{smooth}} \), we have \( \langle L, T_{X,x} \rangle \neq \mathbb{P}^N \). We say that \( L \perp \) is an unexpected equisingular linear space in \( X^* \) if for all \( x \in X_{\text{smooth}} \), the general hyperplane containing \( \langle L, T_{X,x} \rangle \) has the same multiplicity in \( X^* \) as the general hyperplane containing \( L \).

The variety in Example 1.1.1 is rather special since it is a scroll surface (see [Zak 2004] for interesting discussions about this variety). It is not a coincidence that the directrix of this variety is an unexpected equisingular linear space in its dual. Indeed, we have:

**Theorem 3.2.2.** Let \( X \subset \mathbb{P}^N \) be an irreducible, smooth, nondegenerate projective variety such that \( X^* \) is a hypersurface. Let \( L \subset X \) be a linear space with \( \dim(L) = \dim(X) - 1 \). Assume that \( L \perp \) is an unexpected equisingular linear space in \( X^* \) such that \( \text{mult}_{L \perp} X^* = 2 \). Then \( X \) is the cubic scroll surface in \( \mathbb{P}^4 \).

Here \( \text{mult}_{L \perp} X^* \) denotes the multiplicity in \( X^* \) of a general point of \( L \perp \). Before diving into the proof of Theorem 3.2.2, we describe the tangency locus of any point \([h] \subset X^* \), such that \( \text{mult}_{[h]} X^* = 2 \).

**Proposition 3.2.3.** Let \( X \subset \mathbb{P}^N \) be a smooth, irreducible, nondegenerate projective variety such that \( X^* \) is a hypersurface. Let \([h] \in X^* \) be such that \( \text{mult}_{[h]} X^* = 2 \). The scheme theoretic tangency locus of \( H \) with \( X \) is either

(i) an irreducible hyperquadric and in this case \( |\mathcal{E}_{[h]}(X^*)|^{*} = \text{Tan}(H, X) \),

(ii) the union of two (not necessarily distinct) linear spaces,

(iii) a linear space with at least one embedded component.

We postpone the proof of this result to the Appendix.

**Proof of Theorem 3.2.2.** Let \( H \) be a general hyperplane containing \( L \). We have \( H \cap X = L \cup D_H \), where \( D_H \) is a divisor such that

\[ D_H \cap L = \text{Tan}(H, X). \]

Let \( x \in X \) be a general point and let \( H_x \) be a general hyperplane containing \( \langle L, T_{X,x} \rangle \). Then \( \text{Tan}(H_x, X) \) contains \( x \) and

\[ \xi := p(\limflat_{[h] \to [h_x], [h] \in L \perp} q^{-1}([h])). \]
By hypothesis, we have
\[ \text{mult}_{[h,1]} X^* = \text{mult}_{[h]} X^* = 2, \]
for all \([h] \in L^\perp\). Proposition 3.2.3 hence implies that the irreducible component of \(\text{Tan}(H_x, X)\) containing \(x\), which we denote by \(R_{H_x}\), also contains \(\xi\). Moreover, \(\xi \subset L\), so \(\dim R_{H_x} > \dim \xi\), for general \([h] \in L^\perp\). As a consequence, \(\dim R_{H_x} = n - 1\).

On the other hand, since
\[ \text{mult}_{[h,1]} X^* = \text{mult}_{[h]} X^* = 2, \]
for all \([h] \in L^\perp\), we have \(|\mathcal{E}_{[h,1]}(X^*)|^* \neq |R_{H_x}|\). We apply again Proposition 3.2.3 and we find that \(|R_{H_x}|\) is necessarily a linear space of dimension \(n - 1\). Thus,
\[ \dim(L, T_{X,x}) = n + 1. \]

Note that Bertini’s theorem implies that
\[ R_{H_x} \subset \langle L, T_{X,x} \rangle \cap X, \]
for general \(H_x\) containing \(\langle L, T_{X,x} \rangle\). As a consequence \(R_{H_x}\) is an irreducible component of \(\langle L, T_{X,x} \rangle \cap X\), for general \(H_x\). Thus \(R_{H_x}\) does not depend on \(H_x\), for general \(H_x\) containing \(\langle L, T_{X,x} \rangle\). We deduce that \(\langle L, T_{X,x} \rangle\) is tangent to \(X\) along a linear space of dimension \(n - 1\). By the theorem on tangencies, we have \(n - 1 \leq 1\), that is \(n = 2\) (obviously, \(X\) is not a curve). So \(X \subset \mathbb{P}^N\) is a nondegenerate surface containing a distinguished line \(L\), such that for general \(x \in X\), there is a \(\mathbb{P}^3\) tangent to \(X\) along a line passing through \(x\) and meeting \(L\). This means that \(X\) is the projection of a scroll of type \(S_1, d - 1\). By hypothesis, we have \(\text{mult}_{L^\perp} X^* = 2\), hence of [Ciliberto et al. 2008, Proposition 1.6] implies that \(X = S_{1,2} \subset \mathbb{P}^4\). □

**Appendix: Tangency loci of points of multiplicity 2 in the dual**

The goal of this appendix is to prove the following proposition.

**Proposition 3.2.3.** Let \(X \subset \mathbb{P}^N\) be a smooth, irreducible, nondegenerate projective variety such that \(X^*\) is a hypersurface. Let \([h] \in X^*\) be such that \(\text{mult}_{[h]} X^* = 2\). The scheme theoretic tangency locus of \(H\) with \(X\) is either

(i) an irreducible hyperquadric and in this case \(|\mathcal{E}_{[h]}(X^*)|^* = \text{Tan}(H, X)\),

(ii) the union of two (not necessarily distinct) linear spaces, or

(iii) a linear space with at least one embedded component.

**Example A.1.** All three cases are encountered in nature:

(i) If \(X = v_2(\mathbb{P}^2) \subset \mathbb{P}^5\), then for all \([h] \in v_2(\mathbb{P}^2^*) \subset X^*\), we have \(\text{mult}_{[h]} X^* = 2\) and \(\text{Tan}(H, X)\) is a smooth conic.
(ii) If $X$ is a complete intersection of large multidegree and large codimension, then there are points $[h_1], [h_2] \in X^*$ such that $\text{mult}_{[h_i]} X^* = 2$ and $\text{Tan}(H_1, X)$ is exactly two distinct points, whereas $\text{Tan}(H_2, X)$ is a single double point.

(iii) If $X$ is the cubic scroll of Example 1.1.1, then there is a conic $C \subset X^*$, such that for all $[h] \in C$, we have $\text{mult}_{[h]} X^* = 2$ and $\text{Tan}(H, X)$ is a line with an embedded point.

A doubled linear space will be considered as the union of two (not distinct) linear spaces. By Theorem 2.2.4, we know that the irreducible components of $\text{Tan}(H, X)$ are dual to irreducible components of the reduced spaces underlying some $\mathcal{C}_{[h]}(P(X^*, D_k))$ for general $D_k \in \mathbb{G}(k, N)$. When $\text{mult}_{[h]} X^* = 2$, the cones $\mathcal{C}_{[h]}(P(X^*, D_k))$ are rather easy to describe. Let’s start with some notation.

**Notations A.2.** Let $Z \subset \mathbb{P}^N$ be a reduced and irreducible hypersurface. Let $D \in \mathbb{G}(k, N)$ and let $f_Z$ be an equation for $Z$ in some coordinate system of $\mathbb{P}^N$. We denote by $P(f_Z, D)$ the subscheme of $\mathbb{P}^N$ whose ideal is generated by the equations

$$u_0 \frac{\partial f_Z}{\partial t_0} + \cdots + u_N \frac{\partial f_Z}{\partial t_N},$$

for $u = [u_0, \ldots, u_N]$ varying in $D$.

Let $D \in \mathbb{G}(k, N)$ be a general $k$-plane. Note that if $\dim(Z_{\text{sing}}) < \dim P(Z, D)$ (that is $\dim Z_{\text{sing}} \leq N - k - 3$), then $P(Z, D) = P(f_Z, D) \cap Z$. In the other case, the irreducible components of maximal dimension of $Z_{\text{sing}}$ are irreducible components of $P(f_Z, D) \cap Z$.

**Lemma A.3.** Let $Z \subset \mathbb{P}^N$ be an irreducible and reduced hypersurface. Let $z \in Z$ and let $k \in \{-1, \ldots, N - 2\}$. Then, for general $D \in \mathbb{G}(k, N)$, we have

1. $z \notin P(Z, D)$, or
2. $\text{mult}_z P(Z, D) = \text{mult}_z P(f_Z, D)$, if $\dim(Z_{\text{sing}}^{(z)}) < \dim P(Z, D)$, where $Z_{\text{sing}}^{(z)}$ is an irreducible component of $Z_{\text{sing}}$ of maximal dimension passing through $z$, or
3. $\text{mult}_z P(Z, D) < \text{mult}_z P(f_Z, D)$, if $\dim(Z_{\text{sing}}^{(z)}) \geq \dim P(Z, D)$, where $Z_{\text{sing}}^{(z)}$ is an irreducible component of $Z_{\text{sing}}$ of maximal dimension passing through $z$.

**Proof.** If $z \in P(Z, D)$ for general $D \in \mathbb{G}(k, N)$, we will prove the lemma only in the case $P(f_Z, D)$ is smooth at $z$, for two reasons. The general case is obtained by the same methods, this is only more technical, and we will use the result only in the case $P(f_Z, D)$ is smooth at $z$.

Moreover if $z \in P(Z, D)$ for general $D$, we will only concentrate on the case $\dim(Z_{\text{sing}}^{(z)}) < \dim P(Z, D)$. In this case, we have locally around $z$ the equality
\[ P(Z, D) = P(f_Z, D) \cap Z \] for general \( D \in \mathbb{G}(k, N) \). The situation where an irreducible component \( Z_{\text{sing}} \) containing \( z \) is an irreducible component of \( P(f_Z, D) \cap Z \) — this is case (3) of the lemma — is dealt with exactly in the same way.

Now, we work locally around \( z \), so that \( P(f_Z, D) \cap Z = P(Z, D) \subset \mathbb{A}^N \), for general \( D \in \mathbb{G}(k, N) \). Let \( (Z_i)_{i \in I} \) be a stratification of \( Z \) such that \( Z_i \) is smooth and \( Z \) is normally flat along \( Z_i \), for all \( i \in I \). Such a stratification exists, due to the open nature of normal flatness (see [Hironaka 1964, Chapter II]). Consider the Gauss map \( G : Z \to \mathbb{P}^{N*} \). It restricts to a map \( G_i : Z_i \to \mathbb{P}^{N*} \). We have

\[ P(f_Z, D) \cap Z = P(Z, D) = G^{-1}(D^\perp), \]

so that \( P(f_Z, D) \cap Z_i = G_i^{-1}(D^\perp) \), for all \( i \).

Now, we apply Kleiman’s transversality theorem to find that for all \( i \) and for general \( D \in \mathbb{G}(k, N) \), the inverse images \( G_i^{-1}(D^\perp) \) are either empty or smooth of the expected dimension.

Let \( i \) such that \( z \) is in \( Z_i \). If \( z \notin G_i^{-1}(D^\perp) \) for general \( D \in \mathbb{G}(k, N) \), then \( z \notin P(Z, D) \) and we are in the case 1 of the lemma. Otherwise, \( z \) is a smooth point of \( G_i^{-1}(D^\perp) \), so \( T_{P(f_Z, D), z} \) and \( T_{Z_i, z} \) are transverse.

Assume that \( \text{mult}_z P(Z, D) > \text{mult}_z Z, \text{mult}_z P(f_Z, D). \) Since \( P(f_Z, D) \) is smooth at \( z \), this implies that \( T_{P(f_Z, D), z} \) and \( \mathcal{E}_z(Z) \) are not transverse. In particular, the linear spaces \( T_{P(f_Z, D), z} \) and \( \text{Vert}(\mathcal{E}_z(Z)) \) are not transverse (here \( \text{Vert}(\mathcal{E}_z(Z)) \) is the vertex of the cone \( \mathcal{E}_z(Z) \)). But \( Z \) is normally flat along \( Z_i \), so we have \( T_{Z_i, z} \subset \text{Vert}(\mathcal{E}_z(Z)) \) (see [Hironaka 1964, Theorem 2, p. 195]). This is a contradiction.

**Corollary A.4.** Let \( Z \subset \mathbb{P}^N \) be a reduced, irreducible hypersurface. Let \( z \in Z \) such that \( \text{mult}_z Z = 2 \) and let \( k \in \{-1, \ldots, N-2\} \). Then, for general \( D \in \mathbb{G}(k, N) \), we have

\[ \text{mult}_z P(Z, D) \leq 2. \]

**Proof:** The result is obvious for \( k = -1 \), since in this case \( P(Z, D) = Z \). Assume that \( k \geq 0 \) and let \( D \in \mathbb{G}(k, N) \) be a general \( k \)-plane. Let \( u \in D \) be a general point ans let \( \pi_u \) be the projection from \( u \). Then, the projections

\[ \pi_u|_{P(Z, u)} : P(Z, u) \to \pi_u(P(Z, u)) \]

and

\[ \pi_u|_{P(Z, D)} : P(Z, D) \to \pi_u(P(Z, D)) \]

are locally isomorphisms around \( z \). Moreover, we have the following equality (see [Teissier 1982]):

\[ \pi_u(P(Z, D)) = P(\pi_u(P(Z, u)), \pi_u(D)). \]
As a consequence, it is sufficient to prove the result for \( k = 0 \). But in this case, this is an obvious application of the lemma above. Indeed, for general \( u \in \mathbb{P}^N \),

\[
\mult_z P(f_Z, u) = \mult_z Z - 1 = 1.
\]

\( \square \)

We also need the following result.

**Proposition A.5.** Let \( X \subset \mathbb{P}^N \) be an irreducible projective variety such that \( X^* \) is a hypersurface. Let \([h] \in X^*\) be such that \( \Tan(H, X) \) has \( m \) components (some of which may be embedded components), then there exists \( k \in \{-1, \ldots, N - 2\}, \) such that for general \( D \in G(k, N) \), we have

\[
\mult_{[h]} P(X^*, D) \geq m.
\]

*Proof.* We only prove the result when \( \Tan(H, X) \) is reduced and pure dimensional. The general case is done using the same ideas; it’s just more technical.

Assume that \( \Tan(H, X) = Y_1 \cup \cdots \cup Y_m \), where the \( Y_i \) have the same codimension, say \( c \). Let \( D \subset \mathbb{P}^{N^*} \) be a general \( \mathbb{P}^{N-1-c} \).

Then

\[
\pi_D(P(X^*, D)) = (D^\perp \cap X)^*,
\]

where \( \pi_D \) is the projection from \( D \). Moreover, we have \([h] \in P(X^*, D)\) and

\[
\Tan(D^\perp \cap H, D^\perp \cap X) = D^\perp \cap \Tan(H, X).
\]

As a consequence, \( \Tan(D^\perp \cap H, D^\perp \cap X) \) is a 0-dimensional scheme of degree at least \( m \). In this case, it is clear that

\[
\mult_{\pi_D([h])} \pi_D(P(X^*, D)) \geq m.
\]

On the other hand, since \( D \) is general, the morphism

\[
\pi_D : P(X^*, D) \to \pi_D(P(X^*, D))
\]

is locally an isomorphism around \([h]\), so that

\[
\mult_{[h]} P(X^*, D) \geq m.
\]

\( \square \)

**Proof of Proposition 3.2.3.** Let \( T_1 \cup \cdots \cup T_m \) be the decomposition of \( \Tan(H, X) \) into irreducible components. If \( m \geq 3 \), then Proposition A.5 implies that \( \mult_{[h]}(X^*) \geq 3 \), this is impossible, so that \( m \leq 2 \).

Assume that \( m = 2 \). The proof of Proposition A.5 shows that these two irreducible components are scheme-theoretically linear spaces.

Assume that \( m = 1 \) and let \( k \in \{-1, \ldots, N - 2\} \) such that \( T_1 \) is dual to some irreducible component of the reduced space underlying \( ^c \ell_{[h]} P(X^*, D) \), for general \( D \in G(k, N) \). By Corollary A.4, the cone \( ^c \ell_{[h]} P(X^*, D) \) is either a hyperquadric
or a linear space. Assume that it is an irreducible hyperquadric. If \( k \geq 0 \), we know by Theorem 2.2.4 that \(|\ell_{[k]}(X^*)|^*\) is the reduced space underlying some embedded component of \( \text{Tan}(H, X) \). Taking \( q = \dim \text{Tan}(H, X) \) general hyperplane sections of \( \text{Tan}(H, X) \) passing through \(|\ell_{[k]}(X^*)|^*\), we see as in the proof of Proposition A.5 that for general \( D' \in \mathbb{G}(q - 1, N) \), we have

\[
\text{mult}_{[k]} P(X^*, D') \geq 3.
\]

This is impossible by Corollary A.4. Thus, if \( \ell_{[k]} P(X^*, D) \) is an irreducible hyperquadric, then \( k = -1 \), and we are in the case 1 of the proposition.

Finally, if \( \ell_{[k]} P(X^*, D) \) is a the union of two linear spaces or a unique linear space, then we are in case 2 or 3 of the proposition. This concludes the proof of Proposition 3.2.3.

\[ \square \]

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