COMPACT SYMMETRIC SPACES, TRIANGULAR
FACTORIZATION, AND CAYLEY COORDINATES

DEREK HABERMAS

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Let U/K represent a connected, compact symmetric space, where θ is an involution of U that fixes K, $\phi : U/K \to U$ is the geodesic Cartan embedding, and G is the complexification of U. We investigate the intersection of $\phi(U/K)$ with the Bruhat decomposition of G corresponding to a θ-stable triangular, or LDU, factorization of the Lie algebra of G. When $g \in \phi(U/K)$ is generic, the corresponding factorization $g = l d(g) u$ is unique, where $l \in N^-$, $d(g) \in H$, and $u \in N^+$. We present an explicit formula for d in Cayley coordinates, compute it in several types of symmetric spaces, and use it to identify representatives of the connected components of the generic part of $\phi(U/K)$. This formula calculates a moment map for a torus action on the highest dimensional symplectic leaves of the Evens–Lu Poisson structure on U/K.

1. Introduction

Let U/K be a connected, irreducible, compact, Riemannian symmetric space on which U acts isometrically. Then K is the fixed point set of an involution θ of U. Let G be the complexification of U and $\mathfrak{g}$ the complexification of the Lie algebra $\mathfrak{u}$ of U. We assume θ can be extended to a holomorphic involution of G and we let θ denote this extension as well as the corresponding involutions of $\mathfrak{u}$ and $\mathfrak{g}$. In this paper we consider the intersection of the image of the Cartan embedding

$$\phi : U/K \to U \subseteq G, \quad uK \mapsto u\theta(u^{-1})$$

with the Bruhat (or triangular, or LDU) decomposition

$$G = \bigsqcup_{w \in W} \Sigma_w^G, \qquad \Sigma_w^G = N^- w H N^+$$

relative to a θ-stable triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$.

For a generic element g in this intersection, $g \in \Sigma_1^G \cap \phi(U/K)$, this yields a unique triangular factorization $g = l d(g) u$. Our main contribution is to produce explicit formulas for the diagonal map d in classical cases when θ is an inner automorphism, using Cayley coordinates. This formula calculates a moment map for a torus action on the highest dimensional symplectic leaves of the Evens–Lu Poisson structure on U/K [Evens and Lu 2001] studied also in [Foth and Otto 2006] and [Caine 2008]. This intersection is also studied in the context of harmonic analysis on symmetric spaces; for more information, see [Pickrell 2006; Borodin and Olshanski 2005].

For each type of symmetric space under consideration, we choose a representation of $\mathfrak{u}$ in $\mathfrak{su}(n)$ and a specific involution θ of $\mathfrak{g}$ such that θ fixes each of the subspaces $\mathfrak{n}^-$, $\mathfrak{h}$, and $\mathfrak{n}^+$ which, in each representation, always consist of strictly lower triangular, diagonal, and strictly upper triangular matrices, respectively. This is made precise in Section 3.

The formulas for d contain determinants such as $\det(1 + X)$, where X is in $i\mathfrak{p}$, the -1-eigenspace of θ acting on the Lie algebra $\mathfrak{u}$. Due to the relatively sparse nature of these matrices, these determinants are often easily calculable, and we illustrate this with examples. The structure of the paper is as follows.

In Section 2 we introduce notation and review relevant background for the intersection $\Sigma_1^G \cap \phi(U/K)$.

In Section 3 we calculate d in Cayley coordinates.

In Section 4 we use this calculation to identify representatives in each connected component of $\Sigma_1^G \cap \phi(U/K)$.

In Section 5 we show explicit calculations for d in low dimensional examples, and we apply the results of Section 4 to five types of compact symmetric spaces.

In the Appendix, some nonstandard representations used in the paper are more fully explained.

2. Background

Here we review the intersection of a compact symmetric space with a compatible Bruhat decomposition; this material is presented in more detail in [Pickrell 2006]. As stated in the introduction, U/K is a connected irreducible compact symmetric space, where U is a connected Lie group acting on the symmetric space isometrically and transitively, G is the complexification of U, and $K \subseteq U$ is the connected component, containing the identity, of the fixed point set of an involution θ of U. In a slight abuse of notation, we also use θ to denote the induced involution on the Lie algebra $\mathfrak{u}$ of U as well as its complex linear extension to the Lie algebra $\mathfrak{g}$ of G. We also assume that θ extends to a holomorphic involution on G which will also be denoted θ. Let $g \mapsto g^{-*}$ denote the Cartan involution of G fixing U,

and let g^θ denote $\theta(g)$; since the inversion map, $*$, and θ commute, this notation should not cause confusion. Let G_0 denote the fixed point set of the involution of G given by $\sigma : g \mapsto g^{-*\theta}$.

We have Cartan embeddings of symmetric spaces as follows:

$$
\begin{array}{ccc}
U/K & \xrightarrow{\ \phi\ } & U \qquad uK \mapsto uu^{-\theta} \\
\big\downarrow & & \big\downarrow \\
G/G_0 & \xrightarrow{\ \phi\ } & G \qquad gG_0 \mapsto gg^{*\theta}
\end{array}
$$

These are totally geodesic embeddings of symmetric spaces. The following result characterizes the images of these embeddings as subsets of G.

Proposition 2.1 [Pickrell 2006, Theorem 1a]. *Let ϕ be the Cartan embedding as stated above. Then we have the following inclusion maps,*

$$
\begin{array}{ccc}
\phi(U/K) = \{g \in G : g^{-1} = g^* = g^\theta\}_0 & \longrightarrow & U = \{g \in G : g^{-1} = g^*\} \\
\big\downarrow & & \big\downarrow \\
\phi(G/G_0) = \{g \in G : g^* = g^\theta\}_0 & \longrightarrow & G
\end{array}
$$

where $\{\,\cdot\,\}_0$ denotes the connected component containing the identity.

Let $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$ be the decomposition of $\mathfrak{u}$ into $+1$ and -1 eigenspaces of θ. By Proposition 2.1 we can use the derivative of the Cartan embedding to identify the tangent space of U/K at $1K$ with

$$
i\mathfrak{p} = \{X \in \mathfrak{g} : -X = X^* = X^\theta\}.
$$

The exponential map of $\mathfrak{g}$ maps $i\mathfrak{p}$ onto $\phi(U/K)$ [Helgason 1978, Chapter VII].

Fix a maximal abelian subalgebra $\mathfrak{t}_0 \subseteq \mathfrak{k}$. We obtain θ-stable Cartan subalgebras

$$
\mathfrak{h}_0 = Z_{\mathfrak{g}_0}(\mathfrak{t}_0) = \mathfrak{t}_0 \oplus \mathfrak{a}_0, \quad \mathfrak{t} = \mathfrak{t}_0 \oplus i\mathfrak{a}_0, \quad \text{and} \quad \mathfrak{h} = \mathfrak{h}_0^{\mathbb{C}}
$$

of $\mathfrak{g}_0$, $\mathfrak{u}$, and $\mathfrak{g}$, respectively, where $Z_{\mathfrak{g}_0}(\mathfrak{t}_0)$ is the centralizer of $\mathfrak{t}_0$ in $\mathfrak{g}_0$ and $\mathfrak{a}_0 \subseteq \mathfrak{p}$; see [Knapp 2002, (6.60)]. Let $T_0 = \exp(\mathfrak{t}_0)$ and $T = \exp(\mathfrak{t})$ correspond to maximal tori in K and U, respectively.

We obtain a θ-stable triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ so that $\sigma(\mathfrak{n}^\pm) = \mathfrak{n}^\mp$; see [Pickrell 2006, p. 709]. Let $N^\pm = \exp(\mathfrak{n}^\pm)$ and $H = \exp(\mathfrak{h})$. We also let $W = W(G, T)$ denote the Weyl group, $W = N_U(T)/T \cong N_G(H)/H$. Corresponding to this triangular decomposition of $\mathfrak{g}$, we have the Bruhat decomposition of the group G:

$$
G = \coprod_{w \in W} \Sigma_w^G, \quad \Sigma_w^G = N^- w H N^+,
$$

where Σ_w^G is diffeomorphic to $(N^- \cap wN^-w^{-1}) \times H \times N^+$. Elements in Σ_1^G are called "generic". Define

$$(2.2) \qquad\qquad d : \Sigma_1^G \to H, \quad g \mapsto d(g) \quad \text{if } g = ld(g)u$$

where $l \in N^-$, $d(g) \in H$ and $u \in N^+$. Since this factorization is unique for generic elements, the map d is well defined.

Intersecting the Bruhat decomposition of G with $\phi(U/K)$ we obtain a decomposition, indexed by W, of the symmetric space. Let $\Sigma_w^{\phi(U/K)}$ denote $\Sigma_w^G \cap \phi(U/K)$. Theorems 2 and 3 from [Pickrell 2006] examine the intersections of the symmetric spaces and varieties mentioned in Proposition 2.1 with Σ_w^G for arbitrary $w \in W$. The following proposition summarizes some facts from these theorems about Σ_1^G and its intersection with $\phi(U/K)$. For any group Γ, let $\Gamma^{(2)}$ denote $\{g \in \Gamma : g^2 = 1\}$.

Proposition 2.3 [Pickrell 2006, Theorems 2(e) and 3]. (1) *Each connected component of $\Sigma_1^{\phi(U/K)}$ contains an element $\mathbf{w} \in T_0^{(2)}$, which is unique up to multiplication by elements in $\exp(i\mathfrak{a}_0)^{(2)}$.*

(2) *If $\mathbf{w} \in T_0^{(2)}$, then $\mathbf{w} \in \phi(U/K)$ if and only if there exists $\mathbf{w}_1 \in N_U(T_0)$ such that $\phi(\mathbf{w}_1 K) = \mathbf{w}$.*

(3) *If $T \subseteq K$, then $i\mathfrak{a}_0 = 0$, and so $\pi_0(\Sigma_1^{\phi(U/K)})$ is in one-to-one correspondence with*

$$\{\mathbf{w} \in T_0^{(2)} : \exists \mathbf{w}_1 \in N_U(T_0) \text{ such that } \phi(\mathbf{w}_1 K) = \mathbf{w}\}.$$

3. The diagonal map in Cayley coordinates

In this section we compute, in Cayley coordinates, the diagonal map $d : \Sigma_1^G \to H$ and its restriction to $\Sigma_1^{\phi(U/K)}$ for compact symmetric spaces U/K of types *AIII*, *DIII*, *CI*, *CII*, and *BDI*. For these, the complexification G of U is either $\mathrm{SL}(n, \mathbb{C})$, $\mathrm{SO}(n, \mathbb{C})$, or $\mathrm{Sp}(n/2, \mathbb{C})$. Let N_{SL}^+, N_{SL}^-, and H_{SL} be the subgroups of $\mathrm{SL}(n, \mathbb{C})$ consisting of upper triangular unipotent, lower triangular unipotent, and diagonal matrices, respectively. Let $\mathcal{T}$ denote the antiholomorphic involution of $\mathrm{SL}(n, \mathbb{C})$ given by $\mathcal{T}(g) = (g^{-1})^*$ where $*$ denotes conjugate transpose. Let τ denote antitranspose (reflection across the antidiagonal), the holomorphic anti-involution of $\mathrm{GL}(n, \mathbb{C})$ given by $g^\tau = J_n g^t J_n^{-1}$ where $J_n \in \mathrm{GL}(n, \mathbb{C})$ has entries equal to 1 on the antidiagonal and 0 elsewhere. Also let τ denote its restriction to any subgroup of $\mathrm{GL}(n, \mathbb{C})$ as well as the derivatives acting on the corresponding Lie algebras.

We embed $\mathrm{SO}(n, \mathbb{C})$ (respectively, $\mathrm{Sp}(n/2, \mathbb{C})$) into $\mathrm{SL}(n, \mathbb{C})$ as the fixed point set of a holomorphic involution Θ of $\mathrm{SL}(n, \mathbb{C})$ such that Θ preserves N_{SL}^+, N_{SL}^-, and H_{SL}, and such that $\mathcal{T}\Theta = \Theta\mathcal{T}$. For $G = \mathrm{SO}(n, \mathbb{C})$, define $\Theta(g) = (g^{-1})^\tau$. For $G = \mathrm{Sp}(n/2, \mathbb{C})$, define $\Theta(g) = I_{\frac{n}{2}, \frac{n}{2}}(g^{-1})^\tau I_{\frac{n}{2}, \frac{n}{2}}^{-1}$, where $I_{\frac{n}{2}, \frac{n}{2}}$ is the $n \times n$ identity matrix with the first $n/2$ diagonal entries negated. Each Θ has the specified

properties; for more information see the Appendix. For each G, let

$$N^+ = G \cap N^+_{\mathrm{SL}}, \quad N^- = G \cap N^-_{\mathrm{SL}} \quad \text{and} \quad H = G \cap H_{\mathrm{SL}},$$

and let $\mathfrak{n}^+$, $\mathfrak{n}^-$ and $\mathfrak{h}$ be their Lie algebras. Then $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ is a triangular decomposition of $\mathfrak{g}$.

For each type of symmetric space U/K we choose a holomorphic involution θ of $\mathrm{SL}(n, \mathbb{C})$ such that it commutes with both $\mathcal{T}$ and Θ, the triangular decomposition is θ-stable, and $G_0 = G^{\mathcal{T}\theta}$. Note that $\mathcal{T}\theta = \sigma$ as defined in Section 2. The restriction of θ to G, the restriction to U, and the corresponding involutions of the Lie algebras $\mathfrak{g}$ and $\mathfrak{u}$ will still be denoted θ. The choices for θ are these:

(3.1)

Type	U/K	θ
AIII	$\mathrm{SU}(m+n)/S(\mathrm{U}(m) \times \mathrm{U}(n))$	$\mathrm{Ad}(I_{m,n})$
DIII	$\mathrm{SO}(2n)/U(n)$	$\mathrm{Ad}(I_{n,n})$
CI	$\mathrm{Sp}(n)/\mathrm{U}(n)$	$\mathrm{Ad}(I_{n,n})$
CII	$\mathrm{Sp}(p+q)/\mathrm{Sp}(p) \times \mathrm{Sp}(q)$	$\mathrm{Ad}(I_{p,2q,p})$
BDI	$\mathrm{SO}(p+q)/\mathrm{SO}(p) \times \mathrm{SO}(q), \quad p$ even	$\mathrm{Ad}(I_{\frac{p}{2},q,\frac{p}{2}})$

The matrix $I_{a,b}$ (or $I_{a,b,a}$) is a diagonal matrix with the first a diagonal entries -1 and the next b diagonal entries 1 (and the next a diagonal entries -1, respectively). (For type *BDI*, if p and q are both odd, θ is an outer automorphism of $\mathrm{SO}(p+q)$. We address this case in Section 5.) In the discussion that follows, if the symmetric space is not specified, we assume that $\theta = \mathrm{Ad}(\hat{I})$ and that matrices have dimension $n \times n$.

Define the Cayley map by

$$\Phi : \mathfrak{u}(n) \to \{g \in \mathrm{U}(n) : -1 \notin \operatorname{spec} g\}, \quad X \mapsto g = (1-X)(1+X)^{-1}.$$

Note that Φ is invertible by $g \mapsto (1-g)(1+g)^{-1}$.

Lemma 3.2. *Suppose that $\psi : \mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$ is an automorphism or an anti-automorphism and that ψ can be extended to a linear operator $\bar{\psi}$ on $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$. Let $X \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$ be in the tangent space to $\mathrm{GL}(n, \mathbb{C})$ at 1. If $-X = X^* = \bar{\psi}(X)$, then X is in the domain of the Cayley map Φ, and*

$$\Phi(X) \in \{g \in \mathrm{GL}(n, \mathbb{C}) : g^{-1} = g^* = \psi(g)\}.$$

Proof. Let ψ be an automorphism (or an anti-automorphism) of $\mathrm{GL}(n, \mathbb{C})$. Then since $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$ is complete, and $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^n) \backslash \mathrm{GL}(n, \mathbb{C})$ has measure zero, we have $\bar{\psi}(XY) = \bar{\psi}(X)\bar{\psi}(Y)$ (or $\bar{\psi}(XY) = \bar{\psi}(Y)\bar{\psi}(X)$) for all $X, Y \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$. Now suppose $-X = X^* = \bar{\psi}(X)$ and let $g = \Phi(X)$. Then X is skew Hermitian and g is

unitary. So,

$$(3.3) \qquad \psi(g) = \bar{\psi}\left((1-X)(1+X)^{-1}\right)$$
$$= \bar{\psi}(1-X)\left(\bar{\psi}(1+X)\right)^{-1} = (1+X)(1-X)^{-1} = g^{-1}.$$

Note that if ψ is an anti-automorphism, (3.3) follows from the fact that $1 - X$ and $(1 + X)^{-1}$ commute. $\qquad \square$

Proposition 3.4. *Let U/K be one of the symmetric spaces in table* (3.1) *with corresponding involution θ. Then $\Phi(i\mathfrak{p}) \subseteq \phi(U/K)$.*

Proof. By Proposition 2.1, we must show that, for each U/K,

$$\Phi(i\mathfrak{p}) \subseteq \{g \in U : g^{-1} = g^{\theta}\}_0.$$

Each involution θ meets the criteria of Lemma 3.2. Therefore, since $i\mathfrak{p}$ is connected, by continuity of Φ we have

$$\Phi(i\mathfrak{p}) \subseteq \{g \in \mathrm{U}(n) : g^{-1} = g^{\theta}\}_0.$$

Furthermore, since the determinant is fixed under conjugation, we have $\det(g) = \det(g^{\theta}) = \det(g^{-1}) = (\det(g))^{-1}$ which implies that $\det(g) = \pm 1$. By continuity of Φ, and since $0 \in i\mathfrak{p}$, we have $\det(g) = 1$. So,

$$\Phi(i\mathfrak{p}) \subseteq \{g \in \mathrm{SU}(n) : g^{-1} = g^{\theta}\}_0.$$

All that remains to be shown is that $\Phi(i\mathfrak{p}) \subseteq U$. In the case where $U = \mathrm{SU}(n)$, we are done. For $U = \mathrm{SO}(n)$, note that τ meets the criteria of Lemma 3.2, since our representation of $\mathfrak{so}(n)$ lies in the -1 eigenspace of τ. Therefore,

$$\Phi(i\mathfrak{p}) \subseteq \{g \in \mathrm{SU}(n) : g^{-1} = g^{\tau}\} = U.$$

The case where $U = \mathrm{Sp}(n/2)$ follows similarly, since our representation of $\mathfrak{sp}(n/2)$ lies in the -1 eigenspace of $\mathrm{Ad}(I_{\frac{n}{2},\frac{n}{2}}) \circ \tau$. $\qquad \square$

Multiplication of a matrix A by $\hat{I} = I_{a,b}$ (or $I_{a,b,a}$) on the left has the effect of negating the first a rows of A and fixing the next b rows (and negating the following a rows, for $I_{a,b,a}$), and multiplication by $\hat{I}$ on the right has the corresponding effect on columns of A. Thus, conjugation of A by $\hat{I}$ fixes the top left $a \times a$ block of A, negates the $a \times b$ block to its right, and so on. We will refer to these alternately as the blocks fixed by θ and the blocks negated by θ. For example, in type *AIII*, the $m \times n$ and $n \times m$ off-diagonal blocks of $g \in U = \mathrm{SU}(m+n)$ are negated by θ, and the $m \times m$ and $n \times n$ diagonal blocks are fixed by θ, corresponding to $K \cong S(\mathrm{U}(m) \times \mathrm{U}(n))$.

To simplify the notation, let I_k denote $I_{k,n-k}$ when it is understood from context to be an $n \times n$ matrix; in particular, let $I_0 = 1$ and $I_n = -1$. Also, let $A[k]$ denote the principal $k \times k$ block of the matrix A.

Lemma 3.5. *Let* $X \in \mathfrak{su}(n)$ *and* $g = \Phi(X)$. *Then, for* $1 \le k \le n$,

$$\det(g[k]) = \frac{\det(1 + I_k X)}{\det(1 + X)}.$$

Proof. Write $X = \left[\begin{array}{c|c} X_1 & X_2 \\ \hline X_3 & X_4 \end{array}\right]$ and $(1+X)^{-1} = \left[\begin{array}{c|c} Y_1 & Y_2 \\ \hline Y_3 & Y_4 \end{array}\right]$, where $X_1, Y_1 \in M_{k \times k}(\mathbb{C})$, and so on. Then

$$(1 + I_k X)(1 + X)^{-1} = \left[\begin{array}{c|c} (1 - X_1)Y_1 - X_2 Y_2 & * \\ \hline X_3 Y_1 + (1 + X_4)Y_3 & X_3 Y_2 + (1 + X_4)Y_4 \end{array}\right]$$

$$= \left[\begin{array}{c|c} g[k] & * \\ \hline 0_{(n-k) \times k} & 1_{(n-k) \times (n-k)} \end{array}\right].$$

Taking determinants, the claim follows. $\qquad\qquad\square$

Consider the diagonal map $d : \Sigma_1^{\mathrm{SL}(n,\mathbb{C})} \to H_{\mathrm{SL}}$ defined by

$$d(g) = \gamma \in H_{\mathrm{SL}} \quad \Longleftrightarrow \quad g = l\gamma u \quad \text{for some } l \in N_{\mathrm{SL}}^-, u \in N_{\mathrm{SL}}^+.$$

Since $G \subseteq \mathrm{SL}(n, \mathbb{C})$ for each symmetric space under consideration, we shall let d also denote the restriction of this map to Σ_1^G to correspond with (2.2).

Lemma 3.6. *Let* $g \in \Sigma_1^{\mathrm{SL}(n,\mathbb{C})}$ *such that* $g = \Phi(X)$ *for some* $X \in \mathfrak{su}(n)$. *Then* $\det(1 + I_k X) \ne 0$ *for* $1 \le k \le n$, *and*

$$d(g) = \mathrm{diag}\left(\frac{\det(1 + I_1 X)}{\det(1 + X)}, \frac{\det(1 + I_2 X)}{\det(1 + I_1 X)}, \frac{\det(1 + I_3 X)}{\det(1 + I_2 X)}, \right.$$
$$\left. \dots, \frac{\det(1 + I_n X)}{\det(1 + I_{n-1}X)} = \frac{\det(1 - X)}{\det(1 + I_{n-1}X)} \right).$$

Proof. Let $X \in \mathfrak{su}(n)$ such that $g = \Phi(X)$ is generic, and let $g = ld(g)u$ as described above. Then, by Gaussian elimination, $\det(g[k]) \ne 0$ for $1 \le k \le n$, and

$$(3.7) \qquad d(g) = \mathrm{diag}\left(\det(g[1]), \frac{\det(g[2])}{\det(g[1])}, \frac{\det(g[3])}{\det(g[2])}, \dots, \frac{\det(g)}{\det(g[n-1])} \right).$$

Then the proposition follows immediately from Lemma 3.5. $\qquad\qquad\square$

4. Identifying $T_0^{(2)} \cap \phi(U/K)$ with Cayley coordinates

In this section, motivated by Proposition 2.3, we use Cayley coordinates to explicitly identify $T_0^{(2)} \cap \phi(U/K)$ for each type of symmetric space in (3.1) using Lemma 3.6. This is not completely straightforward, as no $\boldsymbol{w} \in T_0^{(2)} \cap \phi(U/K)$ but the identity is in the image of the Cayley map Φ.

Example 4.1. Let $U/K = SU(2)/U(1) \cong \mathbb{C}P^1 \cong S^2$. Let

$$X = \begin{bmatrix} 0 & z \\ -\bar{z} & 0 \end{bmatrix} \in i\mathfrak{p} \subseteq \mathfrak{su}(n).$$

and let $g = \Phi(X)$. If g is generic, then

$$d(g) = \begin{bmatrix} \frac{1-|z|^2}{1+|z|^2} & 0 \\ 0 & \frac{1+|z|^2}{1-|z|^2} \end{bmatrix}.$$

There are two connected components of $\Sigma_1^{\phi(SU(2)/U(1))}$. By Proposition 2.3, these are indexed by

$$T_0^{(2)} = \left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Obviously, $\Phi(0) = +1$, but $-1 \notin \Phi(i\mathfrak{p})$. However, letting $|z|$ tend to infinity, we see that

$$\lim_{|z| \to \infty} \Phi\left(\begin{bmatrix} 0 & z \\ -\bar{z} & 0 \end{bmatrix} \right) = \lim_{|z| \to \infty} \begin{bmatrix} \frac{1-|z|^2}{1+|z|^2} & 0 \\ 0 & \frac{1+|z|^2}{1-|z|^2} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Since $\phi(SU(2)/U(1))$ is connected and complete, this calculation verifies that $-1 \in \phi(SU(2)/U(1))$. Note that $\det(X)$ appears (up to a sign) as a summand in $\det(1 + I_k X)$. The next theorem generalizes this technique.

First we need some notation to let us talk precisely about submatrices. Let A be an $n \times n$ matrix, and let $0 \leq l \leq n$. Let $Q_{l,n}$ denote the set of all subsets of $\{1, \ldots, n\}$ with cardinality l. Let $\alpha = \{i_1, \ldots, i_l\} \in Q_{l,n}$, and let $A[\alpha]$ denote the $l \times l$ submatrix consisting of the intersection of rows $i_1, \ldots, i_l$ and columns $i_1, \ldots, i_l$ of A.

Now, viewing A as an operator on $\mathbb{C}^n$, from Fredholm theory we have

$$\det(1 + A) = \sum_{l=0}^{n} \mathrm{Tr} \, \textstyle\bigwedge^l(A) = \sum_{l=0}^{n} \sum_{1 \leq i_1 < \cdots < i_l \leq n} \left\langle \textstyle\bigwedge^l(A) e_{i_1} \wedge \cdots \wedge e_{i_l}, \, e_{i_1} \wedge \cdots \wedge e_{i_l} \right\rangle$$

$$= \sum_{l=0}^{n} \sum_{\alpha \in Q_{l,n}} \det A[\alpha].$$

(For convenience, we define $\det A[\varnothing] = 1$.) Applying this calculation to the result of Lemma 3.6, we find the kth entry of $d(g)$ can be written

$$(4.2) \qquad [d(g)]_{kk} = \frac{\det(1 + I_k X)}{\det(1 + I_{k-1} X)} = \frac{\displaystyle\sum_{l=0}^{n} \sum_{\alpha \in Q_{l,n}} \det(I_k X)[\alpha]}{\displaystyle\sum_{l=0}^{n} \sum_{\alpha \in Q_{l,n}} \det(I_{k-1} X)[\alpha]}.$$

This shows that the nonzero entries of $d(g)$ are ratios of sums of determinants of submatrices of $I_k X$, for $0 \leq k \leq n$. Furthermore, since multiplication of X by I_k on the left negates the first k rows of X, $\det(1 + I_k X)$ has the same summands for each $0 \leq k \leq n$, up to a sign.

Next we recall some facts from linear algebra and Lie theory, and establish notation. Recall that, since T_0 is contained in H_{SL}, it consists of diagonal matrices, and so $T_0^{(2)} = \{w \in T_0 : w^2 = 1\}$ consists of diagonal matrices whose diagonal entries are ± 1. Let $w \in T_0^{(2)}$. If w has exactly l entries equal to -1, then define $\alpha_w \in Q_{l,n}$ by $i \in \alpha_w$ if and only if $[w]_{ii} = -1$. Then multiplication of a diagonal matrix A by w (on the left or the right) negates the ith diagonal entry of A if and only if $i \in \alpha_w$. Also, if $\theta = \mathrm{Ad}(\hat{I})$, define α_θ by $i \in \alpha_\theta$ if and only if $[\hat{I}]_{ii} = -1$. It follows that $[A]_{i,j}$ is in a block negated by θ if and only if either $i \in \alpha_\theta$ and $j \notin \alpha_\theta$, or $i \notin \alpha_\theta$ and $j \in \alpha_\theta$. Finally, the Weyl group of U acts on T by conjugation; that is, for $w_1 \in W$, let $\boldsymbol{w}_1 \in w_1$, then conjugation of a (diagonal) element $A \in T_0$ by $\boldsymbol{w}_1$ performs a permutation σ_1 on the diagonal entries of A. In particular, $\boldsymbol{w}_1$ can be obtained by performing σ_1 on the rows of an element of T_0, possibly with sign changes.

Theorem 4.3. *Let U/K be one of the symmetric spaces in (3.1) with $\theta = \mathrm{Ad}(\hat{I})$ where $\hat{I}$ has the form $I_{a,b}$ or $I_{a,b,a}$, and let $w \in T_0^{(2)}$ with the block structure induced by θ. Then $w \in \phi(U/K)$ if and only if the number of -1 entries in the $b \times b$ diagonal block of w is the same as the number of -1 entries outside the $b \times b$ diagonal block. More precisely, $w \in \phi(U/K)$ if and only if exactly half of α_w is contained in α_θ; that is, $|\alpha_w \cap \alpha_\theta| = |\alpha_w \backslash \alpha_\theta|$.*

Proof. Suppose $w \in \phi(U/K)$. By Proposition 2.3 there exists $\boldsymbol{w}_1 \in N_U(T_0)$ such that $w = \boldsymbol{w}_1 \boldsymbol{w}_1^{-\theta} = \boldsymbol{w}_1 \hat{I} \boldsymbol{w}_1^{-1} \hat{I}^{-1}$, and so $w\hat{I} = \boldsymbol{w}_1 \hat{I} \boldsymbol{w}_1^{-1}$. Since conjugation of $\hat{I}$ by $\boldsymbol{w}_1$ permutes the diagonal entries of $\hat{I}$, thus fixing the number of negative entries of $\hat{I}$, and since multiplication of $\hat{I}$ by w changes the sign of $[\hat{I}]_{i,i}$ if and only if $i \in \alpha_w$, we have $|\alpha_w \cap \alpha_\theta| = |\alpha_w \backslash \alpha_\theta|$.

Conversely, let $w \in T_0^{(2)}$ such that $|\alpha_w \cap \alpha_\theta| = |\alpha_w \backslash \alpha_\theta|$. We shall construct $X \in i\mathfrak{p}$ such that $\lim_{t \to \infty} d(\Phi(tX)) = w$. This will suffice, as $\Phi(i\mathfrak{p}) \subseteq \phi(U/K)$, and $\phi(U/K)$ is complete.

Let $s = |\alpha_w \cap \alpha_\theta| = |\alpha_w \backslash \alpha_\theta|$. If $s = 0$, then $w = 1 \in \phi(U/K)$, so assume $s \geq 1$. Let $\alpha_w \cap \alpha_\theta = \{i_1, \ldots, i_s\}$ and $\alpha_w \backslash \alpha_\theta = \{j_1, \ldots, j_s\}$ such that they are each enumerated in ascending order. Whether $\hat{I}$ has the form $I_{a,b}$ or $I_{a,b,a}$, $[w]_{i_r,i_r} = -1$ is in an $a \times a$ diagonal block, and $[w]_{j_r,j_r} = -1$ is in the $b \times b$ diagonal block, for all $1 \leq r \leq s$.

Case 1: $\mathfrak{u} = \mathfrak{su}(n) = \{X \in \mathfrak{sl}(n, \mathbb{C}) : -X = X^*\}$. Then U/K is of type *AIII* and $\theta = \mathrm{Ad}(I_{m,n})$. Choose $X \in \mathfrak{sl}(n, \mathbb{C})$ by $[X]_{i_r,j_r} = 1$ and $[X]_{j_r,i_r} = -1$ for all $1 \leq r \leq s$, with all other entries zero. Then X is skew-Hermitian by construction,

and all nonzero entries of X are in blocks negated by θ, since $i_r \in \alpha_\theta$ and $j_r \notin \alpha_\theta$ for all $1 \leq r \leq s$. Therefore, $-X = X^* = X^\theta$; that is, $X \in i\mathfrak{p}$.

Let $t \in \mathbb{R}$. Then $[d(\Phi(tX))]_{k,k} = \det(1 + t I_k X)/\det(1 + t I_{k-1} X)$ for $1 \leq k \leq n$ by Lemma 3.6. By (4.2), this is a ratio of polynomials in t whose terms are identical up to a sign. By construction, $X[\alpha_w]$ is the largest submatrix of X with nonzero determinant. Thus, the leading term of $\det(1 + t I_k X)$ is $\det(t I_k X)[\alpha_w] = \pm t^{2s}$. Furthermore, the leading terms of the numerator and denominator of $[d(\Phi(tX))]_{k,k}$, $\det(t I_k X)[\alpha_w]$ and $\det(t I_{k-1} X)[\alpha_w]$, respectively, differ by a sign exactly when $k \in \alpha_w$. Hence,

$$\lim_{t \to \infty} [d(\Phi(tX))]_{k,k} = \frac{\lim_{t \to \infty} \det(t I_k X)[\alpha_w]}{\lim_{t \to \infty} \det(t I_{k-1} X)[\alpha_w]} = \begin{cases} -1 & \text{if } k \in \alpha_w \\ 1 & \text{if } k \notin \alpha_w \end{cases}$$
$$= [w]_{k,k}.$$

As d and Φ are continuous, $w = \lim_{t \to \infty} d(\Phi(tX)) \in \phi(U/K)$.

<u>Case 2</u>: $\mathfrak{u} = \mathfrak{sp}(n/2) = \{X \in \mathfrak{sl}(n, \mathbb{C}) : -X = X^* = \mathrm{Ad}(I_{\frac{n}{2}, \frac{n}{2}})X^\tau\}$ (see Appendix). Suppose U/K is of type CI; then $\theta = \mathrm{Ad}(I_{\frac{n}{2}, \frac{n}{2}})$. Choose $X \in \mathfrak{sl}(n, \mathbb{C})$ by $[X]_{i_r, j_r} = 1$ and $[X]_{j_r, i_r} = -1$ with all other entries zero. As above, we have $-X = X^* = X^\theta$, so we must show that $-X = \mathrm{Ad}(I_{\frac{n}{2}, \frac{n}{2}})X^\tau$. Note that since $w \in \mathrm{Sp}(n/2)$, $w^* = I_{\frac{n}{2}, \frac{n}{2}} w^\tau I_{\frac{n}{2}, \frac{n}{2}}^{-1} = w^{\tau\theta}$. Also, as diagonal blocks are fixed by θ and w is diagonal and real, $w = w^\tau$; that is, w is symmetric across the antidiagonal. Since $\theta = \mathrm{Ad}(I_{\frac{n}{2}, \frac{n}{2}})$, it follows immediately that $\alpha_{w^\tau} \cap \alpha_\theta = \alpha_w \setminus \alpha_\theta$ and $\alpha_{w^\tau} \setminus \alpha_\theta = \alpha_w \cap \alpha_\theta$. Since $\alpha_w \cap \alpha_\theta$ and $\alpha_w \setminus \alpha_\theta$ are enumerated in ascending order, $X = X^\tau$, and so $-X = X^{\tau\theta} = \mathrm{Ad}(I_{\frac{n}{2}, \frac{n}{2}})X^\tau$. Therefore, $X \in i\mathfrak{p}$.

Suppose U/K is of type CII, then $\theta = \mathrm{Ad}(I_{p, 2q, p})$. (So $n = 2p + 2q$.) Choose $X \in \mathfrak{sl}(n, \mathbb{C})$ by $[X]_{i_r, j_r} = 1$ and $[X]_{j_r, i_r} = -1$ with all other entries zero. Again, we have $-X = X^* = X^\theta$, and we must show that $-X = \mathrm{Ad}(I_{\frac{n}{2}, \frac{n}{2}})X^\tau$. First note that, as in type CI, $w = w^\tau$, and so the $2q \times 2q$ middle block of w is symmetric across the antidiagonal. Thus, s is even, and it follows that $i_1, \ldots, i_{\frac{s}{2}}, j_1, \ldots, j_{\frac{s}{2}} \leq n/2$, and $i_{\frac{s}{2}+1}, \ldots, i_s, j_{\frac{s}{2}+1}, \ldots, j_s > n/2$. Hence, the nonzero entries of X are in blocks fixed by $\mathrm{Ad}(I_{\frac{n}{2}, \frac{n}{2}})$. Furthermore, since $\theta = \mathrm{Ad}(I_{p, 2q, p})$, in contrast to the type CI case we have $\alpha_{w^\tau} \cap \alpha_\theta = \alpha_w \cap \alpha_\theta$ and $\alpha_{w^\tau} \setminus \alpha_\theta = \alpha_w \setminus \alpha_\theta$. Thus, by construction, $-X = X^\tau = \mathrm{Ad}(I_{\frac{n}{2}, \frac{n}{2}})X^\tau$. Therefore, $X \in i\mathfrak{p}$.

By the same argument as in case 1, $w = \lim_{t \to \infty} d(\Phi(tX)) \in \phi(U/K)$.

<u>Case 3</u>: $\mathfrak{u} = \mathfrak{so}(n) = \{X \in \mathfrak{sl}(n, \mathbb{C}) : -X = X^* = X^\tau\}$ (see Appendix). Suppose U/K is of type BDI, where $n = p + q$ and p is even. Then $\theta = \mathrm{Ad}(I_{\frac{p}{2}, q, \frac{p}{2}})$. Choose $X \in \mathfrak{sl}(n, \mathbb{C})$ by $[X]_{i_r, j_r} = 1$ and $[X]_{j_r, i_r} = -1$ for all $1 \leq r \leq s$ with all other entries zero. Then $-X = X^* = X^\theta$. The argument here is similar to the one for type CII; s is even, and since $\alpha_{w^\tau} \cap \alpha_\theta = \alpha_w \cap \alpha_\theta$ and $\alpha_{w^\tau} \setminus \alpha_\theta = \alpha_w \setminus \alpha_\theta$, we have $-X = X^\tau$. So $X \in i\mathfrak{p}$.

Suppose U/K is of type *DIII*; then $\theta = \mathrm{Ad}(I_{\frac{n}{2}, \frac{n}{2}})$. We have $\boldsymbol{w} = \boldsymbol{w}_1 \boldsymbol{w}_1^\theta$ where $\boldsymbol{w}_1 \in w_1 \in W$. It is well known that the Weyl group of $U = \mathrm{SO}(2n)$ acts by even permutations, in this case, on the diagonal entries of elements in T_0, via conjugation. Since w_1 has order two, it can be written as a product of disjoint transpositions. As each transposition in w_1 corresponds to two negative entries of $\boldsymbol{w}$, it follows that s is even. Therefore, choose $X \in \mathfrak{sl}(n, \mathbb{C})$ by

$$[X]_{i_r, j_r} = \begin{cases} 1 & \text{if } i_r \leq \frac{s}{2}, \\ -1 & \text{if } i_r > \frac{s}{2}, \end{cases} \qquad [X]_{j_r, i_r} = \begin{cases} -1 & \text{if } i_r \leq \frac{s}{2}, \\ 1 & \text{if } i_r > \frac{s}{2}, \end{cases}$$

with all other entries of X zero. By construction, $-X = X^* = X^\theta$. Since $\theta = \mathrm{Ad}(I_{\frac{n}{2}, \frac{n}{2}})$, we have $\alpha_{\boldsymbol{w}^\tau} \cap \alpha_\theta = \alpha_{\boldsymbol{w}} \setminus \alpha_\theta$ and $\alpha_{\boldsymbol{w}^\tau} \setminus \alpha_\theta = \alpha_{\boldsymbol{w}} \cap \alpha_\theta$. Thus, $-X = X^\tau$, and so $X \in i\mathfrak{p}$. As in the first two cases, $\boldsymbol{w} = \lim_{t \to \infty} d(\Phi(tX)) \in \phi(U/K)$. $\qquad\square$

5. Explicit calculations of d and $T_0^{(2)} \cap \phi(U/K)$

We now apply the results of Sections 3 and 4 for each type of symmetric space in (3.1). Throughout this section, $X \in i\mathfrak{p}$ and $g = \Phi(X)$. As noted in Section 4, in each representation, each $\boldsymbol{w} \in T_0^{(2)}$ is a diagonal matrix with diagonal entries ± 1.

Type AIII. Symmetric space: $\mathrm{SU}(m+n)/S(\mathrm{U}(m) \times \mathrm{U}(n)) \cong \mathrm{Gr}(m, \mathbb{C}^{m+n})$.
Involution: $\theta : X \mapsto \mathrm{Ad}(I_{m,n})(X)$.
Block structure:

$$(5.1) \qquad \mathfrak{k} = \left\{ \begin{bmatrix} A & 0 \\ \hline 0 & B \end{bmatrix} : \text{trace} = 0 \right\}, \qquad i\mathfrak{p} = \left\{ \begin{bmatrix} 0 & Z \\ \hline -Z^* & 0 \end{bmatrix} \right\},$$

where $A \in \mathfrak{u}(m)$, $B \in \mathfrak{u}(n)$, and $Z \in M_{m \times n}(\mathbb{C})$. Note that Z in (5.1) is the graph coordinate for $\mathrm{Gr}(n, \mathbb{C}^{m+n})$.

For example, if $m = 1$ then $U/K \cong \mathbb{C}P^n$, so

$$X = \begin{bmatrix} 0 & Z \\ \hline -Z^* & 0 \end{bmatrix} \quad \text{where } Z = \begin{bmatrix} z_1 & \cdots & z_n \end{bmatrix}.$$

By (4.2), we have $[d(g)]_{1,1} = \dfrac{1 - \sum |z_i|^2}{1 + \sum |z_i|^2}$, and

$$[d(g)]_{k,k} = \frac{1 + \sum_{i=1}^{k-1} |z_i|^2 - \sum_{j=k}^{n} |z_j|^2}{1 + \sum_{i=1}^{k-2} |z_i|^2 - \sum_{j=k-1}^{n} |z_j|^2}, \qquad 2 \leq k \leq n+1.$$

For $\mathbb{C}P^1 \cong S^2$, we have $i\mathfrak{p} \cong \mathbb{C}$. In Cayley coordinates, the formula above yields $[d(g)]_{1,1} = (1 - |z|^2)/(1 + |z|^2)$, which is the height function in stereographic coordinates (under projection from the south pole) or in the z coordinate on the Riemann sphere.

By Theorem 4.3, $\mathbf{w} \in T_0^{(2)} \cap \phi(U/K)$ if and only if the number of -1 entries in the $m \times m$ upper diagonal block is equal to the number in the $n \times n$ lower diagonal block; there are $\binom{m+n}{m}$ such $\mathbf{w}$.

Type CI. Symmetric space: $\mathrm{Sp}(n)/\mathrm{U}(n)$.
Involution: $\theta : X \mapsto \mathrm{Ad}(I_{n,n})(X)$.
Block structure:

$$\mathfrak{k} = \left\{ \left[\begin{array}{c|c} A & 0 \\ \hline 0 & -A^\tau \end{array} \right] \right\}, \qquad i\mathfrak{p} = \left\{ \left[\begin{array}{c|c} 0 & Z \\ \hline -Z^* & 0 \end{array} \right] \right\},$$

where $A \in \mathfrak{u}(n)$, and $Z \in M_{n \times n}(\mathbb{C})$ such that $Z = Z^\tau$.

This is a subspace of $\mathrm{Gr}(n, \mathbb{C}^{2n})$; the condition $-X = \mathrm{Ad}(I_{n,n})X^\tau$ restricts $\Phi(X)$ to $\mathrm{Sp}(n)$. For example, if $n = 2$, then

$$X = \left[\begin{array}{c|c} 0 & Z \\ \hline -Z^* & 0 \end{array} \right], \quad \text{where } Z = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{11} \end{bmatrix},$$

and by (4.2), we have

$$d(g) = \mathrm{diag}\left(\frac{1 - |z_{12}|^2 + |z_{21}|^2 - \det ZZ^*}{1 + 2|z_{11}|^2 + |z_{12}|^2 + |z_{21}|^2 + \det ZZ^*}, \frac{1 - 2|z_{11}|^2 - |z_{12}|^2 - |z_{21}|^2 + \det ZZ^*}{1 - |z_{12}|^2 + |z_{21}|^2 - \det ZZ^*}, \right.$$
$$\left. \frac{1 - |z_{12}|^2 + |z_{21}|^2 - \det ZZ^*}{1 - 2|z_{11}|^2 - |z_{12}|^2 - |z_{21}|^2 + \det ZZ^*}, \frac{1 + 2|z_{11}|^2 + |z_{12}|^2 + |z_{21}|^2 + \det ZZ^*}{1 - |z_{12}|^2 + |z_{21}|^2 - \det ZZ^*} \right).$$

As noted in the proof of Theorem 4.3, if $\mathbf{w} \in T_0^{(2)}$ then $\mathbf{w}^\tau = \mathbf{w}$. Therefore, all $\mathbf{w}$ satisfying the condition of Theorem 4.3 that are symmetric across the antidiagonal are in $T_0^{(2)} \cap \phi(\mathrm{Sp}(n)/\mathrm{U}(n))$; there are 2^n such $\mathbf{w}$.

Type CII. Symmetric space: $\mathrm{Sp}(p + q)/\mathrm{Sp}(p) \times \mathrm{Sp}(q) \cong \mathrm{Gr}(p, \mathbb{H}^{p+q})$.
Involution: $\theta : X \mapsto \mathrm{Ad}(I_{p,2q,p})X$.
Block structure:

$$\mathfrak{k} = \left\{ \begin{bmatrix} A & 0 & 0 & B \\ 0 & C & D & 0 \\ 0 & -D^* & -C^\tau & 0 \\ -B^* & 0 & 0 & -A^\tau \end{bmatrix} \right\}, \quad i\mathfrak{p} = \left\{ \begin{bmatrix} 0 & Z_1 & Z_2 & 0 \\ -Z_1^* & 0 & 0 & Z_2^\tau \\ -Z_2^* & 0 & 0 & -Z_1^\tau \\ 0 & -Z_2^{*\tau} & Z_1^{*\tau} & 0 \end{bmatrix} \right\},$$

where $A = -A^*$, $B = B^\tau$, $C = -C^*$, $D = D^\tau$, $Z_1, Z_2 \in M_{p \times q}(\mathbb{C})$. That is, $\begin{bmatrix} A & B \\ -B^* & -A^\tau \end{bmatrix} \in \mathfrak{sp}(p)$, and $\begin{bmatrix} C & D \\ -D^* & -C^\tau \end{bmatrix} \in \mathfrak{sp}(q)$. So $\mathfrak{k} \cong \mathfrak{sp}(p) \oplus \mathfrak{sp}(q)$.

For example, if $p = q = 1$, then $U/K \cong \mathbb{H}P^1$, so

$$X = \begin{bmatrix} 0 & Z & 0 \\ -Z^* & 0 & -Z^\tau \\ 0 & Z^{*\tau} & 0 \end{bmatrix}, \quad \text{where } Z = \begin{bmatrix} z_1 & z_2 \end{bmatrix},$$

and, using (4.2), $d(g)$ simplifies to

$$d(g) = \text{diag}\left(\frac{1 - |z_1|^2 - |z_2|^2}{1 + |z_1|^2 + |z_2|^2}, \frac{1 + |z_1|^2 - |z_2|^2 + (|z_1|^2 + |z_2|^2)^2}{1 - (|z_1|^2 + |z_2|^2)^2}, \right.$$
$$\left. \frac{1 - (|z_1|^2 + |z_2|^2)^2}{1 + |z_1|^2 - |z_2|^2 + (|z_1|^2 + |z_2|^2)^2}, \frac{1 + |z_1|^2 + |z_2|^2}{1 - |z_1|^2 - |z_2|^2} \right).$$

By Theorem 4.3 we have $\boldsymbol{w} \in T_0^{(2)} \cap \phi(\text{Sp}(p+q)/\text{Sp}(p) \times \text{Sp}(q))$ if and only if $\boldsymbol{w}$ is symmetric across the antidiagonal (as in type CI), and has an equal (even) number of -1 entries in the $2q \times 2q$ center block, the "Sp(q) part", as in the $p \times p$ outer blocks combined, the "Sp(p) part". There are $\binom{p+q}{p}$ such $\boldsymbol{w}$.

Type DIII. Symmetric space: $\text{SO}(2n)/\text{U}(n)$.
Involution: $\theta : X \mapsto \text{Ad}(I_{n,n})(X)$.
Block structure:

$$\mathfrak{k} = \left\{ \begin{bmatrix} A & 0 \\ 0 & -A^\tau \end{bmatrix} \right\}, \qquad i\mathfrak{p} = \left\{ \begin{bmatrix} 0 & Z \\ -Z^* & 0 \end{bmatrix} \right\},$$

where $A \in \mathfrak{u}(n)$, and $Z \in M_{n \times n}(\mathbb{C})$ such that $Z = -Z^\tau$.

This is also a subspace of $\text{Gr}(n, \mathbb{C}^{2n})$; the condition $-X = X^\tau$ restricts $\Phi(X)$ to $\text{SO}(2n)$. In particular, the antidiagonal entries of X must be zero for all $X \in i\mathfrak{p}$. For example, if $n = 3$, we have

$$X = \begin{bmatrix} 0 & Z \\ -Z^* & 0 \end{bmatrix}, \quad \text{where } Z = \begin{bmatrix} z_{11} & z_{12} & 0 \\ z_{21} & 0 & -z_{12} \\ 0 & -z_{21} & -z_{11} \end{bmatrix},$$

and, using (4.2), $d(g)$ simplifies to

$$d(g) = \text{diag}\left(\frac{1 - |z_{11}|^2 + |z_{21}|^2 - |z_{12}|^2}{1 + |z_{11}|^2 + |z_{21}|^2 + |z_{12}|^2}, \frac{(1 + |z_{11}|^2 + |z_{21}|^2 - |z_{12}|^2)(1 - |z_{11}|^2 - |z_{21}|^2 - |z_{12}|^2)}{(1 - |z_{11}|^2 + |z_{21}|^2 - |z_{12}|^2)(1 + |z_{11}|^2 + |z_{21}|^2 + |z_{12}|^2)}, \right.$$
$$\frac{1 - |z_{11}|^2 - |z_{21}|^2 - |z_{12}|^2}{1 + |z_{11}|^2 + |z_{21}|^2 - |z_{12}|^2}, \frac{1 + |z_{11}|^2 + |z_{21}|^2 - |z_{12}|^2}{1 - |z_{11}|^2 - |z_{21}|^2 - |z_{12}|^2},$$
$$\left. \frac{(1 - |z_{11}|^2 + |z_{21}|^2 - |z_{12}|^2)(1 + |z_{11}|^2 + |z_{21}|^2 + |z_{12}|^2)}{(1 + |z_{11}|^2 + |z_{21}|^2 - |z_{12}|^2)(1 - |z_{11}|^2 - |z_{21}|^2 - |z_{12}|^2)}, \frac{1 + |z_{11}|^2 + |z_{21}|^2 + |z_{12}|^2}{1 - |z_{11}|^2 + |z_{21}|^2 - |z_{12}|^2} \right).$$

By Theorem 4.3, for $\boldsymbol{w} \in T_0^{(2)} \cap \phi(\text{SO}(2n)/\text{U}(n))$, there are as many -1 entries in the first diagonal block as in the second. Also, as noted in the proof of Theorem 4.3, $|\alpha_{\boldsymbol{w}} \cap \alpha_\theta|$ $|\alpha_{\boldsymbol{w}} \setminus \alpha_\theta|$ are even. Therefore, $\boldsymbol{w} \in T_0^{(2)} \cap \phi(\text{SO}(2n)/\text{U}(n))$ if and only

if w is symmetric across the antidiagonal, and has an even number of -1 entries in each $n \times n$ diagonal block; there are 2^{n-1} such w.

Type BDI. Symmetric space: $SO(p+q)/SO(p) \times SO(q) \cong Gr(p, \mathbb{R}^{p+q})$.
Involution: $\theta : X \mapsto Ad(\hat{I})X$ (inner if and only if pq is even).

Case 1: p and q are not both odd. Without loss of generality, assume p is even.
Then $\hat{I} = I_{\frac{p}{2}, q, \frac{p}{2}}$.
Block structure:

$$\mathfrak{k} = \left\{ \begin{bmatrix} A & 0 & B \\ 0 & C & 0 \\ -B^* & 0 & -A^\tau \end{bmatrix} \right\}, \quad i\mathfrak{p} = \left\{ \begin{bmatrix} 0 & Z & 0 \\ -Z^* & 0 & -Z^\tau \\ 0 & Z^{*\tau} & 0 \end{bmatrix} \right\},$$

where $A = -A^*$, $B = -B^\tau$, $C = -C^* = -C^\tau$, $Z \in M_{\frac{p}{2} \times q}(\mathbb{C})$. That is, $C \in \mathfrak{so}(q)$,
and $\begin{bmatrix} A & B \\ -B^* & -A^\tau \end{bmatrix} \in \mathfrak{so}(p)$. So $\mathfrak{k} \cong \mathfrak{so}(p) \oplus \mathfrak{so}(q)$.

For example, if $p = 6$ and $q = 1$, then $U/K \cong \mathbb{R}P^6$, so

$$X = \begin{bmatrix} 0 & Z & 0 \\ \hline -Z^* & 0 & -Z^\tau \\ \hline 0 & Z^{*\tau} & 0 \end{bmatrix}, \quad \text{where } Z = \begin{bmatrix} z_3 \\ z_2 \\ z_1 \end{bmatrix}.$$

By (4.2), we have

$$d(g) = \text{diag}\left(\frac{1 + 2|z_1|^2 + 2|z_2|^2}{1 + 2|z_1|^2 + 2|z_2|^2 + 2|z_3|^2}, \frac{1 + 2|z_1|^2}{1 + 2|z_1|^2 + 2|z_2|^2}, \frac{1}{1 + 2|z_1|^2}, 1, \right.$$
$$\left. \frac{1 + 2|z_1|^2}{1}, \frac{1 + 2|z_1|^2 + 2|z_2|^2}{1 + 2|z_1|^2}, \frac{1 + 2|z_1|^2 + 2|z_2|^2 + 2|z_3|^2}{1 + 2|z_1|^2 + 2|z_2|^2} \right).$$

The form of $i\mathfrak{p}$ here is similar to that of the quaternionic Grassmannian, type
CII; we have

$$w \in T_0^{(2)} \cap \phi(SO(p+q)/SO(p) \times SO(q))$$

if and only if w is symmetric across the antidiagonal and has an equal (even)
number of -1 entries in the middle $q \times q$ block, the "$SO(q)$ part", as in the outer
$p/2 \times p/2$ outer blocks combined, the "$SO(p)$ part". (Notice that if q is odd, the
middle diagonal entry must be $+1$.) There are $\binom{p/2 + \lfloor q/2 \rfloor}{p/2}$ such w.

If we restrict our attention to even-dimensional real projective space,

$$SO(2n+1)/SO(2n) \times SO(1) \cong \mathbb{R}P^{2n},$$

then by the reasoning above, there can be no -1 entries in any $w \in T_0^{(2)}$ in $\phi(\mathbb{R}P^{2n})$.
Thus, the only w present for $\mathbb{R}P^{2n}$ is the identity matrix, verifying that $\Sigma_1^{\phi(\mathbb{R}P^{2n})}$
is connected.

<u>Case 2</u>: p and q are both odd. Now $\theta = \mathrm{Ad}(\hat{I})$, where $\hat{I} =$

$$
\begin{bmatrix}
1_{\frac{p-1}{2} \times \frac{p-1}{2}} & 0 & 0 & 0 & 0 & 0 \\
0 & -1_{\frac{q-1}{2} \times \frac{q-1}{2}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1_{\frac{q-1}{2} \times \frac{q-1}{2}} & 0 \\
0 & 0 & 0 & 0 & 0 & 1_{\frac{p-1}{2} \times \frac{p-1}{2}}
\end{bmatrix}
$$

The automorphism θ is an outer automorphism, since $\lambda \hat{I} \notin \mathrm{SO}(p+q)$ for all $\lambda \in \mathbb{C}$. Still, θ meets the criteria of Lemma 3.2, and so the proof of Proposition 3.4 may be applied to this case.

Block structure:

$$
\mathfrak{k} = \left\{
\begin{bmatrix}
A & 0 & u & u & 0 & B \\
0 & C & -v & v & D & 0 \\
-u^* & v^* & 0 & 0 & -v^\tau & -u^\tau \\
-u^* & -v^* & 0 & 0 & v^\tau & -u^\tau \\
0 & -D^* & v^{*\tau} & -v^{*\tau} & -C^\tau & 0 \\
-B^* & 0 & u^{*\tau} & u^{*\tau} & 0 & -A^\tau
\end{bmatrix}
\right\},
$$

where $A = -A^*$, $C = -C^*$, $B = -B^\tau$, $D = -D^\tau$. That is,

$$
\begin{bmatrix}
A & u & B \\
-u^* & 0 & -u^\tau \\
-B^* & u^{*\tau} & -A^\tau
\end{bmatrix} \in \mathfrak{so}(p) \quad \text{and} \quad
\begin{bmatrix}
C & v & D \\
-v^* & 0 & -v^\tau \\
-D^* & v^{*\tau} & -C^\tau
\end{bmatrix} \in \mathfrak{so}(q).
$$

$$
i\mathfrak{p} = \left\{
\begin{bmatrix}
0 & Z_1 & w_1 & -w_1 & Z_2 & 0 \\
-Z_1^* & 0 & w_2 & w_2 & 0 & -Z_2^\tau \\
-w_1^* & -w_2^* & is & 0 & -w_2^\tau & w_1^\tau \\
w_1^* & -w_2^* & 0 & -is & -w_2^\tau & -w_1^\tau \\
-Z_2^* & 0 & w_2^{*\tau} & w_2^{*\tau} & 0 & -Z_1^\tau \\
0 & Z_2^{*\tau} & -w_1^{*\tau} & w_1^{*\tau} & Z_1^{*\tau} & 0
\end{bmatrix}
\right\},
$$

where $Z_1, Z_2 \in M_{\frac{p-1}{2} \times \frac{q-1}{2}}(\mathbb{C})$, $w_1 \in \mathbb{C}^{\frac{p-1}{2}}$, $w_2 \in \mathbb{C}^{\frac{q-1}{2}}$.

The form we have chosen for $i\mathfrak{p}$ reveals the presence of $i\mathfrak{a}_0$ in the center two diagonal entries. For example, if $p = 5$ and $q = 1$ then $U/K \cong \mathbb{R}P^5$, so by (4.2)

we have $d(g) =$

$$\text{diag}\left(\frac{1+s^2+4|z_1|^2}{1+s^2+4|z_1|^2+4|z_2|^2}, \frac{1+s^2}{1+s^2+4|z_1|^2}, \frac{1+is}{1-is}, \right.$$

$$\left. \frac{1-is}{1+is}, \frac{1+s^2+4|z_1|^2}{1+s^2}, \frac{1+s^2+4|z_1|^2+4|z_2|^2}{1+s^2+4|z_1|^2} \right).$$

By Proposition 2.3, the connected components of $\Sigma_1^{\phi(U/K)}$ are indexed by

$$T_0^{(2)} / \exp(i \mathfrak{a}_0)^{(2)}.$$

Therefore, to identify a unique representative for each connected component, we set $s = 0$; that is, we require that the middle two diagonal entries of $\boldsymbol{w}$ are positive. These representatives of elements in $T_0^{(2)} / \exp(i \mathfrak{a}_0)^{(2)}$ are those $\boldsymbol{w}$ that are symmetric across the antidiagonal and have the same number of -1 entries in the inner $\lfloor q/2 \rfloor \times \lfloor q/2 \rfloor$ blocks, the "$O(q)$ part", as in the outer $\lfloor p/2 \rfloor \times \lfloor p/2 \rfloor$ blocks, the "$O(p)$ part". There are $\binom{\lfloor p/2 \rfloor + \lfloor q/2 \rfloor}{\lfloor p/2 \rfloor}$ such $\boldsymbol{w}$.

It follows that for odd-dimensional real projective space,

$$\mathbb{R}P^{2n+1} \cong SO(2n+2)/(SO(1) \times SO(2n+1)),$$

the space $\Sigma_1^{\phi(\mathbb{R}P^{2n+1})}$ is connected.

Appendix: θ-stable representations

We are motivated to use the following representations of $\mathfrak{so}(n)$ and $\mathfrak{sp}(n)$ in $\mathfrak{su}(n)$ because they are the fixed point sets of involutions that preserve the triangular decomposition of $\mathfrak{sl}(n, \mathbb{C}) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ where $\mathfrak{h}$ consists of diagonal matrices, and $\mathfrak{n}^+$ ($\mathfrak{n}^-$) consists of upper (lower) triangular matrices.

Let $\tau : \mathfrak{sl}(n, \mathbb{C}) \to \mathfrak{sl}(n, \mathbb{C})$ be the antitranspose map given by reflection across the antidiagonal; that is, $X^\tau = J X^t J^{-1}$ where J is the $n \times n$ matrix whose entries are ones on the antidiagonal and zeros elsewhere. Then $X \mapsto -X^\tau$ is an involution of $\mathfrak{sl}(n, \mathbb{C})$ that stabilizes the above triangular decomposition. The restriction to $\mathfrak{su}(n)$ is also such an involution.

Proposition 5.2. $\mathfrak{so}(n) \cong \{X \in \mathfrak{su}(n) : -X^\tau = X\}$,

$$\mathfrak{sp}(n) \cong \{X \in \mathfrak{su}(2n) : -X^\tau = \text{Ad}(I_n)X\}.$$

Proof. Define involutions on $\mathfrak{sl}(n, \mathbb{C})$ by

$$\Theta_0(X) = -X^t, \quad \Theta_1(X) = -X^\tau = -JX^tJ, \quad \text{for} \quad X \in \mathfrak{sl}(n, \mathbb{C})$$

Then $\Theta_1 = \text{Ad}_J \circ \Theta_0$. Let $P = \frac{1}{\sqrt{2}}(J + i\mathbb{1})$. A straightforward calculation shows that

$$\Theta_1 = \text{Ad}_P \circ \Theta_0 \circ \text{Ad}_{P^{-1}}.$$

Thus the Lie algebra isomorphism $\mathrm{Ad}_P : \mathfrak{sl}(n, \mathbb{C}) \to \mathfrak{sl}(n, \mathbb{C})$ maps the fixed point subalgebra of Θ_0 to that of Θ_1. This completes the proof for $\mathfrak{so}(n)$; the proof for $\mathfrak{sp}(n)$ follows similarly. $\qquad\square$

Note: This representation of $\mathfrak{so}(n)$ is the space of infinitesimal isometries of the n-dimensional subspace of the real vector space $\mathbb{C}^n$ consisting of elements

$$\begin{bmatrix} x_n - i x_1 \\ \vdots \\ x_1 - i x_n \end{bmatrix},$$

where $x_1, \ldots, x_n \in \mathbb{R}$.

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References

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