FORMAL GEOMETRIC QUANTIZATION II

PAUL-ÉMILE PARADAN

We study the formal geometric quantization of noncompact Hamiltonian manifolds. Our main result is that two quantization processes coincide. Ma and Zhang obtained the same result in a recent preprint by completely different means.

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Introduction and statement of results</td>
<td>169</td>
</tr>
<tr>
<td>2</td>
<td>Quantizations of noncompact manifolds</td>
<td>173</td>
</tr>
<tr>
<td>3</td>
<td>Proof of Theorem 1.4</td>
<td>193</td>
</tr>
<tr>
<td>4</td>
<td>Other properties of $\mathcal{H}^\Phi$</td>
<td>202</td>
</tr>
<tr>
<td>5</td>
<td>Example: The cotangent bundle of an orbit</td>
<td>205</td>
</tr>
<tr>
<td></td>
<td>References</td>
<td>210</td>
</tr>
</tbody>
</table>

In [Paradan 2009], we studied some functorial properties of the “formal geometric quantization” process $\mathfrak{H}^{-\infty}$, which is defined on proper Hamiltonian manifolds, that is, noncompact Hamiltonian manifolds with proper moment map.

There is another way, denoted $\mathfrak{H}^\Phi$, of quantizing proper Hamiltonian manifolds by localizing the index of the Dolbeault Dirac operator on the critical points of the square of the moment map [Paradan 2001; 2003; Ma and Zhang 2008].

The main purpose of this paper is to provide a geometric proof that the quantization processes $\mathfrak{H}^{-\infty}$ and $\mathfrak{H}^\Phi$ coincide. Ma and Zhang [2008] proved this by completely different means (see also their note [Ma and Zhang 2009]).

1. Introduction and statement of results

First, we recall the definition of the geometric quantization of a smooth and compact Hamiltonian manifold. Then we show two ways of extending the notion of geometric quantization to the case of a noncompact Hamiltonian manifold.

Let $K$ be a compact connected Lie group, with Lie algebra $\mathfrak{k}$. In the Kostant– Souriau framework, a Hamiltonian $K$-manifold $(M, \Omega, \Phi)$ is prequantized if there

MSC2000: 58F06, 57S15, 19L47, 58J20, 81S10.

Keywords: moment map, reduction, geometric quantization, transversally elliptic symbol.
is an equivariant Hermitian line bundle $L$ with an invariant Hermitian connection $\nabla$ such that
\[
\mathcal{L}(X) - \nabla_{X_M} = i \langle \Phi, X \rangle \quad \text{and} \quad \nabla^2 = -i \Omega
\]
for every $X \in \mathfrak{k}$. Here $X_M$ is the vector field on $M$ defined by $X_M(m) = \frac{d}{dt} e^{-tX} m |_{t=0}$.

The data $(L, \nabla)$ is also called a Kostant–Souriau line bundle, and $\Phi : M \to \mathfrak{k}^*$ is the moment map. Via the equivariant Bianchi formula, the conditions of (1) imply the relations
\[
i(X_M) \Omega = -d \langle \Phi, X \rangle, \quad X \in \mathfrak{k}.
\]

Recall the notion of geometric quantization when $M$ is compact. Choose a $K$-invariant almost complex structure $J$ on $M$ that is compatible with $\Omega$ in the sense that the symmetric bilinear form $\Omega(\cdot, J\cdot)$ is a Riemannian metric. Let $\tilde{\partial}_L$ be the Dolbeault operator with coefficients in $L$, and let $\tilde{\partial}^*_L$ be its (formal) adjoint. The Dolbeault–Dirac operator on $M$ with coefficients in $L$ is $D_L = \sqrt{2}(\tilde{\partial}_L + \tilde{\partial}^*_L)$, considered as an elliptic operator from $\mathcal{A}^{0, \text{even}}(M, L)$ to $\mathcal{A}^{0, \text{odd}}(M, L)$. Let $R(K)$ be the representation ring of $K$.

**Definition 1.1.** The geometric quantization of a compact Hamiltonian $K$-manifold $(M, \Omega, \Phi)$ is the element $\mathcal{Q}_K(M) \in R(K)$ defined as the equivariant index of the Dolbeault–Dirac operator $D_L$.

Consider the case of a proper Hamiltonian $K$-manifold $M$: the manifold is (perhaps) noncompact but the moment map $\Phi : M \to \mathfrak{k}^*$ is supposed to be proper. Under this properness assumption, one defines the formal geometric quantization of $M$ as an element $\mathcal{Q}^{-\infty}_K(M)$ that belongs to $R^{-\infty}(K)$ [Weitsman 2001; Paradan 2009]. Recall the definition:

Let $T$ be a maximal torus of $K$. Let $\mathfrak{t}^*$ be the dual of the Lie algebra $\mathfrak{t}$ of $T$ containing the weight lattice $\wedge^*$, that is, $\alpha \in \wedge^*$ if $i\alpha : t \to i\mathbb{R}$ is the differential of a character of $T$. Let $\mathfrak{t}^*_+ \subset \mathfrak{t}^*$ be a Weyl chamber, and let $\widehat{\mathcal{K}} := \wedge^* \cap \mathfrak{t}^*_+$ be the set of dominant weights. The ring of characters $R(K)$ has a $\mathbb{Z}$-basis $V^K_\mu, \mu \in \widehat{\mathcal{K}}; V^K_\mu$ is the irreducible representation of $K$ with highest weight $\mu$.

A representation $E$ of $K$ is admissible if it has finite $K$-multiplicities, that is, $\dim(\text{hom}_K(V^K_\mu, E)) < \infty$ for every $\mu \in \widehat{\mathcal{K}}$. Let
\[
R^{-\infty}(K)
\]
be the Grothendieck group associated to the $K$-admissible representations. We have an inclusion map $R(K) \hookrightarrow R^{-\infty}(K)$ and $R^{-\infty}(K)$ is canonically identified with $\text{hom}_{\mathbb{Z}}(R(K), \mathbb{Z})$. The tensor product induces an $R(K)$-module structure on $R^{-\infty}(K)$ since $E \otimes V$ is an admissible representation when $V$ and $E$ are, respectively, a finite-dimensional and an admissible representation of $K$. 
For any $\mu \in \hat{K}$ that is a regular value of the moment map $\Phi$, the reduced space (or symplectic quotient) $M_{\mu} := \Phi^{-1}(K \cdot \mu)/K$ is a compact orbifold equipped with a symplectic structure $\Omega_{\mu}$. Moreover $L_{\mu} := (L|_{\Phi^{-1}(\mu)} \otimes \mathbb{C}_{-\mu})/K_{\mu}$ is a Kostant–Souriau line orbibundle over $(M_{\mu}, \Omega_{\mu})$. The definition of the index of the Dolbeault–Dirac operator carries over to the orbifold case, hence $\mathcal{D}(M_{\mu}) \in \mathbb{Z}$ is defined. In Section 2C, we explain how this notion of geometric quantization extends further to the case of singular symplectic quotients. So the integer $\mathcal{D}(M_{\mu}) \in \mathbb{Z}$ is well defined for every $\mu \in \hat{K}$: in particular $\mathcal{D}(M_{\mu}) = 0$ if $\mu \notin \Phi(M)$.

**Definition 1.2.** Let $(M, \Omega, \Phi)$ be a proper Hamiltonian $K$-manifold prequantized by a Kostant–Souriau line bundle $L$. The formal quantization of $(M, \Omega, \Phi)$ is the element of $R^{-\infty}(K)$ defined by

$$\mathcal{D}^{-\infty}_{K}(M) = \sum_{\mu \in \hat{K}} \mathcal{D}(M_{\mu}) V_{\mu}^{K}.$$ 

When $M$ is compact, the fact that

$$\mathcal{D}_{K}(M) = \mathcal{D}^{-\infty}_{K}(M)$$

is known as the “quantization commutes with reduction” theorem. This was conjectured in [Guillemin and Sternberg 1982] and was first proved in [Meinrenken 1998; Meinrenken and Sjamaar 1999]. Other proofs of (4) were given in [Tian and Zhang 1998; Paradan 2001]. For complete references on the subject, consult [Sjamaar 1996; Vergne 2002].

We summarize the main features of the formal geometric quantization $\mathcal{D}^{-\infty}$:

**Theorem 1.3** [Paradan 2009]. (1) (restriction to subgroup) Let $M$ be a prequantized Hamiltonian $K$-manifold that is proper. Let $H \subset K$ be a closed connected Lie subgroup such that $M$ is still proper as a Hamiltonian $H$-manifold. Then $\mathcal{D}^{-\infty}_{K}(M)$ is $H$-admissible and $\mathcal{D}^{-\infty}_{K}(M)|_{H} = \mathcal{D}^{-\infty}_{H}(M) \text{ in } R^{-\infty}(H)$.

(2) (product) Let $M$ and $N$ be prequantized Hamiltonian $K$-manifolds, where $M$ is proper and $N$ is compact. Then $M \times N$ is a proper prequantized Hamiltonian $K$-manifold and $\mathcal{D}^{-\infty}_{K}(M \times N) = \mathcal{D}^{-\infty}_{K}(M) \cdot \mathcal{D}_{K}(N) \text{ in } R^{-\infty}(K)$.

When $M$ is a proper Hamiltonian $K$-manifold, we can also define another “formal geometric quantization”, denoted

$$\mathcal{D}_{K}^{\Phi}(M) \in R^{-\infty}(K),$$

by localizing the index of the Dolbeault–Dirac operator $D_{L}$ on the set $\text{Cr}(\|\Phi\|^{2})$ of critical points of the square of the moment map (see Section 2B for the precise definition). This idea of nonabelian localization goes back to Witten [1992]. We
proved in [Paradan 2003; 2009] that

\[ \mathcal{Q}_K^{-\infty}(M) = \mathcal{Q}_K^\Phi(M) \]

in some situations:

- \( M \) is a coadjoint orbit of a semisimple Lie group \( S \) that parametrizes a representation of the discrete series of \( S \).
- \( M \) is a Hermitian vector space.

In her ICM 2006 plenary lecture, Vergne [2007] conjectured that (6) holds when \( \text{Cr}(\|\Phi\|^2) \) is compact. Recently, Ma and Zhang [2008] proved the following generalization of this conjecture.

**Theorem 1.4.** The equality (6) holds for any proper Hamiltonian \( K \)-manifold.

**Corollary 1.5.** The formal quantization map \( \mathcal{Q}^\Phi \) satisfies the functorial properties listed in Theorem 1.3.

This article is dedicated to the study of the quantization map \( \mathcal{Q}^\Phi \). In Section 2B, we give the precise definition of the quantization process \( \mathcal{Q}^\Phi \). In particular, we refine the constant \( a_\gamma \) that appears in [Ma and Zhang 2008, Theorem 0.1]. In Section 2D, we explain how to compute the quantization of a point. In Section 3, we give another proof of Theorem 1.4 by using the technique of symplectic cutting developed in [Paradan 2009]. In Section 4, we consider the case where \( K = K_1 \times K_2 \) acts on \( M \) in such a way that the symplectic reduction \( M \sslash_0 K_1 \) is a smooth proper \( K_2 \)-Hamiltonian manifold. We show then that the \( K_1 \)-invariant part of \( \mathcal{Q}_K^\Phi(M) \) is equal to \( \mathcal{Q}_K^{\Phi_2}(M \sslash_0 K_1) \). In Section 5, we study the example of the cotangent bundle of a homogeneous space: \( M = T^*(K/H) \) where \( H \) is a closed subgroup of \( K \).

We finish this introduction by discussing the two proofs of Theorem 1.4 in [Ma and Zhang 2008] and in this paper. Both proofs use the Witten [1992] deformation argument. The work of Ma and Zhang [2008] is analytic and makes a great use of techniques initiated in [Bismut and Lebeau 1991]. One of Ma and Zhang’s main tools is an interpretation of the transversal index as an Atiyah–Patodi–Singer type index. In the present work, we stay on the topological/geometrical side. Our main tools are based on localization formulas (see [Paradan 2001]) and on a symplectic cutting technique (see [Paradan 2009]).

The approach of Ma and Zhang [2008] is different from ours, but the results are equivalent. In [Ma and Zhang 2008, Theorem 0.5], they show that the geometric quantization process \( \mathcal{Q}^\Phi \) is functorial with respect to the product (see the second point of Theorem 1.3), and then deduce the equality \( \mathcal{Q}^\Phi = \mathcal{Q}^{-\infty} \).

I wish to thank the referees for their useful comments.
2. Quantizations of noncompact manifolds

In this section we define the quantization process $\mathcal{Q}^\Phi$, and we give another definition of the quantization process $\mathcal{Q}^{-\infty}$ that uses the notion of symplectic cutting [Paradan 2009].

2A. Transversally elliptic symbols. Here we give the basic definitions from the theory of transversally elliptic symbols (or operators) defined in [Atiyah 1974]. For an axiomatic treatment of the index morphism, see [Berline and Vergne 1996a; 1996b; Paradan and Vergne 2009]. For a short introduction, see [Paradan 2001].

Let $\mathcal{X}$ be a compact $K$-manifold. Let $p : T\mathcal{X} \to \mathcal{X}$ be the projection, and let $(-, -)_\mathcal{X}$ be a $K$-invariant Riemannian metric. If $E^0, E^1$ are $K$-equivariant complex vector bundles over $\mathcal{X}$, a $K$-equivariant morphism $\sigma \in \Gamma(T\mathcal{X}, \text{hom}(p^*E^0, p^*E^1))$ is called a symbol on $\mathcal{X}$. The subset of all $(x, v) \in T\mathcal{X}$ where $\sigma(x, v) : E^0_x \to E^1_x$ is not invertible is called the characteristic set of $\sigma$, and is denoted by $\text{Char}(\sigma)$.

In the following, the product of a symbol $\sigma$ by a complex vector bundle $F \to M$, is the symbol $\sigma \otimes F$

defined by $\sigma \otimes F(x, v) = \sigma(x, v) \otimes \text{Id}_{F_x}$ from $E^0 \otimes F_x$ to $E^1 \otimes F_x$. Note that $\text{Char}(\sigma \otimes F) = \text{Char}(\sigma)$.

Let $T_K\mathcal{X}$ be the following subset of $T\mathcal{X}$:

$$T_K\mathcal{X} = \{(x, v) \in T\mathcal{X} \mid (v, X_\mathcal{X}(x))_x = 0 \text{ for all } X \in \mathfrak{k}\}.$$ 

A symbol $\sigma$ is elliptic if $\sigma$ is invertible outside a compact subset of $T\mathcal{X}$ (that is, $\text{Char}(\sigma)$ is compact), and is $K$-transversally elliptic if the restriction of $\sigma$ to $T_K\mathcal{X}$ is invertible outside a compact subset of $T_K\mathcal{X}$ (that is, $\text{Char}(\sigma) \cap T_K\mathcal{X}$ is compact). An elliptic symbol $\sigma$ defines an element in the equivariant $K^0$-theory of $T\mathcal{X}$ with compact support, which is denoted by $K^0_K(T\mathcal{X})$, and the index of $\sigma$ is a virtual finite-dimensional representation of $K$, which we denote $\text{Index}^K_K(\sigma) \in R(K)$ [Atiyah and Segal 1968; Atiyah and Singer 1968a; 1968b; 1971].

Consider the $R(K)$-submodule

$$R^{-\infty} (K) \subset R^{-\infty} (K)$$

formed by all the infinite sums $\sum_{\mu \in \mathcal{K}} m_\mu V^K_\mu$ where the map $\mu \in \hat{K} \mapsto m_\mu \in \mathbb{Z}$ has at most a polynomial growth. The $R(K)$-module $R^{-\infty} (K)$ is the Grothendieck group associated to the trace class virtual $K$-representations. We can associate to any $V \in R^{-\infty} (K)$ its trace, $k \mapsto \text{Tr}(k, V)$, which is a generalized function on $K$ invariant by conjugation. Then the trace defines a morphism of $R(K)$-modules

$$R^{-\infty} (K) \hookrightarrow \mathcal{E}^{-\infty} (K)^{\text{Ad}},$$

The map $\sigma(x, v)$ will be also denote $\sigma|_x(v)$.
where $(\ell \in \mathcal{C}(K))^{\text{Ad}}$ is the vector space of generalized function on $K$ invariant by conjugation.

A $K$-transversally elliptic symbol $\sigma$ defines an element of $\mathcal{K}^0_K(T_K\mathcal{X})$, and the index of $\sigma$ is defined as a trace class virtual representation of $K$, which we still denote $\text{Index}^K_K(\sigma) \in R_{tc}^{-\infty}(K)$ [Atiyah 1974].

Any elliptic symbol of $T\mathcal{X}$ is $K$-transversally elliptic, hence we have a restriction map $\mathcal{K}^0_K(T\mathcal{X}) \to \mathcal{K}^0_K(T_K\mathcal{X})$ and a commutative diagram

\[
\begin{array}{ccc}
\mathcal{K}^0_K(T\mathcal{X}) & \longrightarrow & \mathcal{K}^0_K(T_K\mathcal{X}) \\
\text{Index}_K & \downarrow & \text{Index}_K \\
R(K) & \longrightarrow & R_{tc}^{-\infty}(K).
\end{array}
\]

(8)

Using the excision property, one can easily show that the index map $\text{Index}^K_K : \mathcal{K}^0_K(T_K\mathcal{U}) \to R_{tc}^{-\infty}(K)$ is still defined when $\mathcal{U}$ is a $K$-invariant relatively compact open subset of a $K$-manifold (see [Paradan 2001, Section 3.1]).

Suppose now that the group $K$ is equal to the product $K_1 \times K_2$. When a symbol $\sigma$ is $(K_1 \times K_2)$-transversally elliptic we will be interested in the $K_1$-invariant part of its index, which we denote by

$\left[\text{Index}^K_{K_1 \times K_2}(\sigma)\right]^{K_1} \in R_{tc}^{-\infty}(K_2)$.

An intermediate notion between the “ellipticity” and “$(K_1 \times K_2)$-transversal ellipticity” is “$K_1$-transversal ellipticity”. When a $(K_1 \times K_2)$-equivariant symbol $\sigma$ is $K_1$-transversally elliptic, its index $\text{Index}^K_{K_1 \times K_2}(\sigma) \in R_{tc}^{-\infty}(K_1 \times K_2)$, viewed as a generalized function on $K_1 \times K_2$, is smooth relative to the variable in $K_2$ [Atiyah 1974; Berline and Vergne 1996b; Paradan and Vergne 2009]. It implies that $\text{Index}^K_{K_1 \times K_2}(\sigma) = \sum_\lambda \theta(\lambda) \otimes V^K_\lambda$ with

$\theta(\lambda) \in R(K_2)$ for all $\lambda \in \hat{K}_1$.

In particular, $\left[\text{Index}^K_{K_1 \times K_2}(\sigma)\right]^{K_1} = \theta(0)$ belongs to $R(K_2)$.

Recall the multiplicative property of the index map for the product of manifolds that was proved in [Atiyah 1974]. Consider a compact Lie group $K_2$ acting on two manifolds $\mathcal{X}_1$ and $\mathcal{X}_2$, and assume that another compact Lie group $K_1$ acts on $\mathcal{X}_1$ commuting with the action of $K_2$.

The external product of complexes on $T\mathcal{X}_1$ and $T\mathcal{X}_2$ induces a multiplication (see [Atiyah 1974]):

$\otimes : \mathcal{K}^0_{K_1 \times K_2}(T_{K_1}\mathcal{X}_1) \times \mathcal{K}^0_{K_2}(T_{K_2}\mathcal{X}_2) \to \mathcal{K}^0_{K_1 \times K_2}(T_{K_1 \times K_2}(\mathcal{X}_1 \times \mathcal{X}_2))$. 
Recall the definition of the external product: For $k = 1, 2$, consider equivariant morphisms\(^2\) $\sigma_k : \mathcal{E}_k^+ \to \mathcal{E}_k^-$ on $T\mathcal{X}_k$. Consider the equivariant morphism on $T(\mathcal{X}_1 \times \mathcal{X}_2)$

$$\sigma_1 \circ \sigma_2 : \mathcal{E}_1^+ \otimes \mathcal{E}_2^+ \oplus \mathcal{E}_1^- \otimes \mathcal{E}_2^- \to \mathcal{E}_1^- \otimes \mathcal{E}_2^+ \oplus \mathcal{E}_1^+ \otimes \mathcal{E}_2^-$$

defined by

$$\sigma_1 \circ \sigma_2 = \begin{pmatrix} \sigma_1 \otimes \Id & -\Id \otimes \sigma_2^* \\ \Id \otimes \sigma_2 & \sigma_1^* \otimes \Id \end{pmatrix}. \tag{9}$$

We see that the set $\text{Char}(\sigma_1 \circ \sigma_2) \subset T\mathcal{X}_1 \times T\mathcal{X}_2$ is equal to $\text{Char}(\sigma_1) \times \text{Char}(\sigma_2)$. Suppose now that the morphisms $\sigma_k$ are respectively $K_k$-transversally elliptic. Since $T_{K_1 \times K_2}(\mathcal{X}_1 \times \mathcal{X}_2) \neq T_{K_1}\mathcal{X}_1 \times T_{K_2}\mathcal{X}_2$, the morphism $\sigma_1 \circ \sigma_2$ is not necessarily $(K_1 \times K_2)$-transversally elliptic. Nevertheless, if $\sigma_2$ is almost homogeneous, then the morphism $\sigma_1 \circ \sigma_2$ is $(K_1 \times K_2)$-transversally elliptic (see [Paradan and Vergne 2009]). So the exterior product $a_1 \circ a_2$ is the $K^0$-theory class defined by $\sigma_1 \circ \sigma_2$, where $a_k = [\sigma_k]$ and $\sigma_2$ is almost homogeneous.

The following property will be used frequently; see [Atiyah 1974, Lecture 3; Paradan and Vergne 2009].

**Theorem 2.1** (multiplicative property). For any $[\sigma_1] \in K_{K_1 \times K_2}^0(T_{K_1}\mathcal{X}_1)$ and any $[\sigma_2] \in K_{K_2}^0(T_{K_2}\mathcal{X}_2)$ we have

$$\text{Index}_{\mathcal{X}_1 \times \mathcal{X}_2}^{K_1 \times K_2}([\sigma_1] \circ [\sigma_2]) = \text{Index}_{\mathcal{X}_1}^{K_1}([\sigma_1]) \otimes \text{Index}_{\mathcal{X}_2}^{K_2}([\sigma_2]).$$

The product of $\text{Index}_{\mathcal{X}_1}^{K_1 \times K_2}([\sigma_1]) \in \mathcal{C}^{−\infty}(K_1 \times K_2)^{Ad}$ with $\text{Index}_{\mathcal{X}_2}^{K_2}([\sigma_2]) \in \mathcal{C}^{−\infty}(K_2)^{Ad}$ is well defined since the generalized function

$$(k_1, k_2) \mapsto \text{Index}_{\mathcal{X}_1}^{K_1 \times K_2}([\sigma_1])(k_1, k_2)$$

is smooth relative to the variable $k_2 \in K_2$.

We finish this section by recalling the notion of limit in $R^{-\infty}(K)$.

**Definition 2.2.** The *support* of $\chi := \sum_{\mu \in \mathcal{K}} a_\mu V^K_\mu \in R^{-\infty}(K)$ is the set of $\mu \in \mathcal{K}$ such that $a_\mu \neq 0$.

We will say that $\chi \in R^{-\infty}(K)$ is supported outside $B \subset t^*$ if the support of $\chi$ does not intersect $B$. Denote by $O(r)$ any character of $R^{-\infty}(K)$ that is supported outside the ball $B_r = \{\xi \in t^* \mid \|\xi\| < r\}$.

**Definition 2.3.** A sequence $\chi_n \in R^{-\infty}(K)$ converges to $\chi_\infty$ when $n$ goes to infinity if for any $r > 0$ there exists $N \in \mathbb{N}$ such that

$$\chi_\infty - \chi_n = O(r)$$

for any $n \geq N$.

---

\(^2\)To simplify notation, we do not distinguish between vector bundles on $T\mathcal{X}$ and on $\mathcal{X}$. 
We will be interested in an infinite sum $\sum_{i \in I} \psi_i$ of generalized characters. Here $\sum_{i \in I} \psi_i$ converges in $R^{-\infty}(K)$ if for any $r > 0$ the set 

$$\{i \in I \mid \text{support}(\psi_i) \cap B_r \neq \emptyset\}$$

is finite.

2B. Definition and first properties of $\mathcal{H}^\Phi$. Let $(M, \Omega, \Phi)$ be a proper Hamiltonian $K$-manifold prequantized by an equivariant line bundle $L$. Let $J$ be an invariant almost complex structure compatible with $\Phi$. Let $p : TM \to M$ be the projection.

To begin, we describe the principal symbol of the Dolbeault–Dirac operator $\sqrt{2(\bar{\partial}_L + \bar{\partial}_L^*)}$. The complex vector bundle $(T^*M)_0, 1$ is $K$-equivariantly identified with the tangent bundle $TM$ equipped with the complex structure $J$. Let $h$ be the Hermitian structure on $(TM, J)$ defined by $h(v, w) = (v, Jw) - i (v, w)$ for $v, w \in TM$. The symbol

$$\text{ Thom}(M, J) \in \Gamma(TM, \text{hom}(p^*(_C^\text{even} \otimes TM), p^*(_C^\text{odd} \otimes TM)))$$

at $(m, v) \in TM$ is equal to the Clifford map

$$c_m(v) : _C^\text{even} T_m M \to _C^\text{odd} T_m M,$$

where $c_m(v).w = v \wedge w - i(v)w$ for $w \in _C^\text{odd} T_m M$. Here $i(v) : _C^\text{odd} T_m M \to _C^{\text{even}} T_m M$ denotes the contraction map relative to $h$. Since $c_m(v)^2 = -\|v\|^2 \text{Id}$, the map $c_m(v)$ is invertible for all $v \neq 0$. Hence the characteristic set of Thom$(M, J)$ corresponds to the 0-section of $TM$.

It is a classical fact that the principal symbol of the Dolbeault–Dirac operator $\sqrt{2(\bar{\partial}_L + \bar{\partial}_L^*)}$ is equal to

$$\text{ Thom}(M, J) \otimes L$$

(see [Berline et al. 2004, Proposition 3.67]). Here also, $\text{Char(Thom}(M, J) \otimes L)$ coincides with the 0-section of $TM$.

Remark 2.4. If the manifold $M$ is a product $M_1 \times M_2$, the symbol $\text{ Thom}(M, J) \otimes L$ is equal to the product $\sigma_1 \circ \sigma_2$ where $\sigma_k = \text{ Thom}(M_k, J_k) \otimes L_k$.

When $M$ is compact, the symbol $\text{ Thom}(M, J) \otimes L$ is elliptic and then defines an element of the equivariant $K$-group of $TM$. The topological index of $\text{ Thom}(M, J) \otimes L \in K^0_K(TM)$ is equal to the analytical index of the Dolbeault–Dirac operator $\sqrt{2(\bar{\partial}_L + \bar{\partial}_L^*)}$:

$$\mathcal{D}_K(M) = \text{Index}_M^K(\text{ Thom}(M, J) \otimes L) \text{ in } R(K).$$

---

3Here we use an identification $T^*M \simeq TM$ given by an invariant Riemannian metric.
When $M$ is not compact the topological index of $\text{Thom}(M, J) \otimes L$ is not defined. To extend the notion of geometric quantization to this setting we deform the symbol $\text{Thom}(M, J) \otimes L$ in the “Witten” way [Paradan 2001; 2003]. Consider the identification $\xi \mapsto \tilde{\xi}, \mathfrak{t}^* \to \mathfrak{t}$ defined by a $K$-invariant scalar product on $\mathfrak{t}^*$. Define the Kirwan vector field on $M$ as

$$\kappa_m = (\Phi(m))_M(m), \quad m \in M.$$  

**Definition 2.5.** The symbol $\text{Thom}(M, J) \otimes L$ pushed by the vector field $\kappa$ is the symbol $c^\kappa$ defined by the relation

$$c^\kappa|_m(v) = \text{Thom}(M, J) \otimes L|_m(v - \kappa_m)$$

for any $(m, v) \in T M$. More generally, if $E \to M$ is an equivariant complex vector bundle, one defines the symbol $c^\kappa_E$ with the same relation (with $E$ at the place of $L$).

Note that $c^\kappa|_m(v)$ is invertible except if $v = \kappa_m$. If furthermore $v$ belongs to the subset $T_K M$ of tangent vectors orthogonal to the $K$-orbits, then $v = 0$ and $\kappa_m = 0$. Indeed $\kappa_m$ is tangent to $K \cdot m$ while $v$ is orthogonal.

Since $\kappa$ is the Hamiltonian vector field of the function $\frac{1}{2} \| \Phi \|^2$, the set of zeros of $\kappa$ coincides with the set $\text{Cr}(\| \Phi \|^2)$ of critical points of $\| \Phi \|^2$. Finally we have

$$\text{Char}(c^\kappa) \cap T_K M \simeq \text{Cr}(\| \Phi \|^2).$$

In general $\text{Cr}(\| \Phi \|^2)$ is not compact, so $c^\kappa$ does not define a transversally elliptic symbol on $M$. To define a kind of index of $c^\kappa$, we proceed as follows. For any invariant open relatively compact subset $U \subset M$ the set

$$\text{Char}(c^\kappa|_U) \cap T_K U \simeq \text{Cr}(\| \Phi \|^2) \cap U$$

is compact when

$$\partial U \cap \text{Cr}(\| \Phi \|^2) = \emptyset. \quad (14)$$

When (14) holds, denote by

$$\mathcal{G}_K^\Phi(U) := \text{Index}^K_U(c^\kappa|_U) \in R_{\text{ic}}^{-\infty}(K) \quad (15)$$

the equivariant index of the transversally elliptic symbol $c^\kappa|_U$.

It will be useful to understand the dependence of the generalized character $\mathcal{G}_K^\Phi(U)$ relative to the data $(U, \Omega, L)$. So consider two proper Hamiltonian $K$-manifolds $(M, \Omega, \Phi)$ and $(M', \Omega', \Phi')$ respectively prequantized by the line bundles $L$ and $L'$. Let $V \subset M$ and $V' \subset M'$ two invariant open subsets.

**Proposition 2.6.** The generalized character $\mathcal{G}_K^\Phi(U)$ does not depend of the choice of an invariant almost complex structure on $U$ compatible with $\Omega|_U$. 

Suppose that there exists an equivariant diffeomorphism \( \Psi : V \to V' \) such that

1. \( \Psi^*(\Phi') = \Phi \),
2. \( \Psi^*(L') = L \),
3. there exists a homotopy of symplectic forms taking \( \Psi^*(\Omega'|_{V'}) \) to \( \Omega|_V \).

Let \( U' \subset \overline{U} \subset V' \) be an invariant open relatively compact subset such that \( \partial U' \) satisfies (14). Take \( U = \Psi^{-1}(U') \). Then \( \partial U \) satisfies (14) and

\[
\mathcal{D}_K^\Phi(U') = \mathcal{D}_K^\Phi(U) \in R^{-\infty}(K).
\]

**Proof.** To prove the first point, let \( c^i|_U, i = 0, 1 \) be the transversally elliptic symbols defined with the compatible almost complex structure \( J_i, i = 0, 1 \). Since the space of compatible almost complex structure is contractible, there exists a homotopy \( J_t, t \in [0, 1] \) of almost complex structures linking \( J_0 \) and \( J_1 \). By [Paradan 2001, Lemma 2.2], there exists an invertible bundle map \( A \in \Gamma(U, \text{End}(TU)) \), homotopic to the identity, such that \( A \circ J_0 = J_1 \circ A \). With the help of \( A \) we prove then that the symbols \( c^0|_U \) and \( c^1|_U \) define the same class in \( K^0_K(TKU) \) (see [Paradan 2001, Lemma 2.2]). Hence their equivariant indices coincide.

To prove the second point, observe that the characters \( \mathcal{D}_K^\Phi(U) \) and \( \mathcal{D}_K^\Phi(U') \) are computed as the equivariant index of the symbols \( c^i|_U \) and \( c^i|_{U'} \). Let \( \tilde{c}^i|_U \) the pull back of \( c^i|_{U'} \) by \( \Psi \). Thanks to conditions (1) and (2), the only thing which differs in the definitions of the symbols \( c^i|_U \) and \( \tilde{c}^i|_U \) are the almost complex structures \( J \) and \( \tilde{J} = \Psi^*(J') \): the first one is compatible with \( \Omega|_V \) and the second one with \( \Psi^*(\Omega'|_{V'}) \). Since these two symplectic structures are homotopic, the almost complex structures \( J \) and \( \tilde{J} \) are also homotopic. So we can conclude as in the first point. \( \square \)

We describe the critical points of \( \|\Phi\|^2 \), when the moment map \( \Phi \) is proper. We know that \( m \in \text{Cr}(\|\Phi\|^2) \) if and only if \( \tilde{\beta}_M(m) = 0 \) for \( \beta = \Phi(m) \). Hence the set \( \text{Cr}(\|\Phi\|^2) \) has the decomposition

\[
\text{Cr}(\|\Phi\|^2) = \bigcup_{\beta \in \mathcal{B}} M^{\tilde{\beta}} \cap \Phi^{-1}(\beta) = \bigcup_{\beta \in \mathcal{B}} K \cdot (M^{\tilde{\beta}} \cap \Phi^{-1}(\beta)),
\]

where \( \mathcal{B} \) is a subset of the Weyl chamber \( t^*_+ \). The set of singular values of \( \|\Phi\|^2 \) is then \( \{\|\beta\|^2, \beta \in \mathcal{B}\} \). Each part \( Z_{\beta} \) is compact, hence \( \text{Cr}(\|\Phi\|^2) \) is compact if \( \mathcal{B} \) is finite. Denote by \( B_r \subset t^* \) the open ball \( \{\xi \in t^* | \|\xi\| < r\} \).

**Proposition 2.7.**

- For any \( r > 0 \), the set \( \mathcal{B} \cap B_r \) is finite.
- \( \text{Cr}(\|\Phi\|^2) \) is compact if and only if \( \mathcal{B} \) is finite.
- The set of singular values of \( \|\Phi\|^2 : M \to \mathbb{R} \) forms a sequence \( 0 \leq r_1 < r_2 < \cdots < r_k < \cdots \) that is finite if and only if \( \text{Cr}(\|\Phi\|^2) \) is compact. In the other case, \( \lim_{k \to \infty} r_k = \infty \).
Proof. To prove the first point, let \( r > 0 \) and consider the relatively compact invariant open subset \( \mathcal{V}_r := \Phi^{-1}(\{\xi \in \mathfrak{t}^* \mid \|\xi\| < r\}) \) and the infinitesimal action of the Lie algebra \( \mathfrak{t} \) on \( \mathcal{V}_r \). For any vector subspace \( a \subset \mathfrak{t} \), define the \( T \)-invariant submanifold

\[
\mathcal{V}_r(a) := \{ x \in \mathcal{V}_r \mid \text{Stabilizer}_t(x) = a \}.
\]

Since \( \mathcal{V}_r \) is relatively compact, it has finitely many types of stabilizers \( a_1, \ldots, a_p \). Hence we have a decomposition \( \mathcal{V}_r = \mathcal{V}_r(a_1) \cup \cdots \cup \mathcal{V}_r(a_p) \) where each \( \mathcal{V}_r(a_k) \) has a finite number, say \( n(r, k) \), of connected components. We will show that

\[
(17) \quad \sum_{k=1}^{p} n(r, k) \geq \# \mathcal{B} \cap B_r.
\]

Let \( \mathcal{C}_r \) be the finite collection formed by the connected components of the manifold \( \mathcal{V}_r(a_k), 1 \leq k \leq p \). Let \( \mathcal{C}_r' \subset \mathcal{C}_r \) be the subset formed by the connected components \( F \) for which \( F \mathcal{B} \cap \Phi^{-1}(\beta) \neq \emptyset \) for some \( \beta \in \mathcal{B} \cap B_r \). The inequality \((17)\) follows from the existence of a surjective map \( \theta : \mathcal{C}_r' \to \mathcal{B} \cap B_r \).

Let \( F \in \mathcal{C}_r' \). Suppose that there exist \( \beta, \beta' \) in \( \mathcal{B} \cap B_r \) such that \( F \mathcal{B} \cap \Phi^{-1}(\beta) \) and \( F \mathcal{B} \cap \Phi^{-1}(\beta') \) are nonempty. It implies first that \( \tilde{\beta}, \tilde{\beta}' \in a_k \). The relation \((2)\) shows that the function \( x \in F \mapsto (\Phi(\chi), Y) \) is constant for any \( Y \in a_k \). If we take \( Y = \tilde{\beta} \), the fact that \( F \) intersects both \( \Phi^{-1}(\beta) \) and \( \Phi^{-1}(\beta') \) gives \( \|\beta\|^2 = (\beta, \beta') \).

By taking \( Y = \tilde{\beta}' \), we have also \( \|\beta'\|^2 = (\beta, \beta') \). Finally

\[
\|\beta - \beta'\|^2 = \|\beta\|^2 + \|\beta'\|^2 - 2(\beta, \beta') = 0,
\]

hence \( \beta = \beta' \). Define \( \theta : \mathcal{C}_r' \to \mathcal{B} \cap B_r \) as follows: \( \theta(F) \) is the unique element \( \beta \in \mathcal{B} \cap B_r \) such that \( F \mathcal{B} \cap \Phi^{-1}(\beta) \neq \emptyset \). It is easy to check that \( \theta \) is onto.

The two other points are a direct consequence of the first one. \( \square \)

To each regular value \( R \) of \( \text{Cr}(|\Phi|^2) \) associate the invariant open subset \( M_{<R} := \{ |\Phi|^2 < R \} \) that satisfies \((14)\). The restriction \( \text{c}^\kappa|_{M_{<R}} \) defines then a transversally elliptic symbol on \( M_{<R} \). Let \( \mathcal{D}_K^\Phi(M_{<R}) \) be its equivariant index. We show that \( \mathcal{D}_K^\Phi(M_{<R}) \) has a limit when \( R \to \infty \).

For any \( \beta \in \mathcal{B} \), consider a relatively compact open invariant neighborhood \( \mathcal{U}_\beta \) of \( Z_\beta \) such that \( \text{Cr}(|\Phi|^2) \cap \mathcal{U}_\beta = Z_\beta \). By the excision property, the generalized character \( \mathcal{D}_K^\Phi(\mathcal{U}_\beta) = \text{Index}_{\mathcal{U}_\beta}^K(\text{c}^\kappa|_{\mathcal{U}_\beta}) \) does not depend of the choice of \( \mathcal{U}_\beta \). To simplify notation, use the following:

**Definition 2.8.** Denote by \( \mathcal{D}_K^\Phi(M) \in R_{w,-\infty}(K) \) the equivariant\(^4\) of the transversally elliptic symbol \( \text{c}^\kappa|_{\mathcal{U}_\beta} \).

When \( E \to M \) is an equivariant complex vector bundle, denote by \( RR_{\beta}^K(M, E) \) the equivariant index of the transversally elliptic symbol \( \text{c}^\kappa|_{\mathcal{U}_\beta} \).

\(^4\)The index of \( \text{c}^\kappa|_{\mathcal{U}_\beta} \) was denoted \( RR_{\beta}^K(M, L) \) in [Paradan 2001].
A simple application of the excision property [Paradan 2001, Section 4] gives

\begin{equation}
\mathcal{D}_K^f(M_{<R}) = \sum_{\beta \in \mathfrak{B} \cap B_\pi} \mathcal{D}_K^\beta(M),
\end{equation}

where the sum is finite, thanks to Proposition 2.7.

For a dominant weight \( \gamma \in \hat{K} \), the positive number

\[ a_\gamma = \|\gamma + \rho\|^2 - \|\rho\|^2 \geq \|\gamma\|^2 \]

corresponds to the eigenvalue of the Casimir operator acting on the irreducible representation \( V_\gamma^K \). Ma and Zhang [2008, Theorem 2.1] prove that the support of the generalized character \( \mathcal{D}_K^\beta(M) \) is contained in \( \{ \gamma \in \hat{K} | a_\gamma \geq \|\beta\|^2 \} \).

We propose another proof which refines Ma and Zhang’s result and uses a different method. They used Atiyah–Patodi–Singer index theory on manifolds with boundary whereas we use localization and induction formulae for our transversally elliptic index.

**Theorem 2.9.** The generalized character \( \mathcal{D}_K^\beta(M) \) is supported outside the open ball \( B_{\|\beta\|} \).

**Proof.** The proof uses computations done in [Paradan 2001].

First consider the case where \( \beta \neq 0 \) is a \( K \)-invariant element of \( \mathfrak{B} \). Let \( i : \mathbb{T}_\beta \hookrightarrow T \) be the compact torus generated by \( \beta \). If \( F \) is \( \mathbb{Z} \)-module denote by \( F \widehat{\otimes} R^{-\infty}(\mathbb{T}_\beta) \) the \( \mathbb{Z} \)-module formed by the infinite formal sums \( \sum_a E_a h^a \) taken over the set of weights of \( \mathbb{T}_\beta \), where \( E_a \in F \) for every \( a \).

Since \( \mathbb{T}_\beta \) lies in the center of \( K \), the morphism \( \pi : (k, t) \in K \times \mathbb{T}_\beta \mapsto kt \in K \) induces a map \( \pi^* : R^{-\infty}(K) \rightarrow R^{-\infty}(K) \widehat{\otimes} R^{-\infty}(\mathbb{T}_\beta) \).

The normal bundle \( \mathcal{N} \) of \( M^{\mathbb{B}} \) in \( M \) inherits a canonical complex structure \( J_\mathcal{N} \) on the fibers. Denote by \( \mathcal{N} \rightarrow M^{\mathbb{B}} \) the complex vector bundle with the opposite complex structure. The torus \( \mathbb{T}_\beta \) is included in the center of \( K \), so the bundle \( \mathcal{N} \) and the virtual bundle \( 0 : \wedge^* \mathcal{N} := \wedge_{\mathbb{C}}^{\text{even}} \mathcal{N} \rightarrow \wedge_{\mathbb{C}}^{\text{odd}} \mathcal{N} \) carry a \( K \times \mathbb{T}_\beta \)-action. Thus they can be considered as elements of \( K_0^K(K, \mathcal{N}) = K_0^K(M^{\mathbb{B}}) \otimes R(\mathbb{T}_\beta) \).

In [Paradan 2001], we defined an inverse of \( \wedge_{\mathbb{C}}^* \mathcal{N} \),

\[ \left[ \wedge_{\mathbb{C}}^* \mathcal{N} \right]^{-1}_\beta = K_0^K(M^{\mathbb{B}}) \widehat{\otimes} R^{-\infty}(\mathbb{T}_\beta), \]

which is polarized by \( \beta \). This means that \( \left[ \wedge_{\mathbb{C}}^* \mathcal{N} \right]^{-1}_\beta = \sum_a N_a h^a \) with \( N_a \neq 0 \) only if \( (a, \beta) \geq 0 \). [Paradan 2001, Theorem 5.8] proved the localization formula

\begin{equation}
\pi^* [\mathcal{D}_K^\beta(M)] = RR_{\rho^\mathbb{T}_\beta}(M^{\mathbb{B}}), L_{M^{\mathbb{B}}} \otimes [\wedge_{\mathbb{C}}^* \mathcal{N}]^{-1}_\beta,}
\end{equation}
Lemma 2.10. The irreducible representation $V^K_{\mu}$ occurs in $RR^K_{\beta} (M, E)$ only if $\text{Hom}_{\beta} (\mathbb{C}, E|Z) \neq 0$ for some $Z \in s_\beta$.

Thus $V^K_{\mu}$ occurs in the character $RR^K_{\beta} (M, E)$ only if $\text{Hom}_{\beta} (\mathbb{C}, E|Z)$ is nonzero only if $(\mu, \beta) \geq \beta$. Hence $V^K_{\mu}$ occurs in the character $\mathcal{G}^\beta_K (M)$ only if $(\mu, \beta) \geq \beta$.

Now consider the case were $\beta \in \mathcal{B}$ is not a $K$-invariant element. Let $\sigma$ be the unique open face of the Weyl chamber $t_+^\sigma$ which contains $\beta$. Let $K_\sigma$ be the corresponding stabilizer subgroup. Following [Guillemin and Sternberg 1984], we introduce a $K_\sigma$-invariant open subset $U_\sigma$ of $t_+^\sigma$ as $U_\sigma := K_\sigma \cdot \{ y \in t_+^\sigma \mid K_y \subset K_\sigma \}$. By construction, $U_\sigma$ is a slice for the coadjoint action at any $\xi \in \sigma$; see [Lerman et al. 1998, Definition 3.1]. This means that the map $K \times U_\sigma \to t^\sigma, (k, \xi) \mapsto k \cdot \xi$ factors through an inclusion $K \times K_\sigma U_\sigma \to t^\sigma$. The symplectic cross-section theorem [Guillemin and Sternberg 1984] asserts that the preimage

$$\mathcal{Y}_\sigma := \Phi^{-1} (U_\sigma)$$

is a $K_\sigma$-invariant symplectic submanifold prequantized by the line bundle $L|_{\mathcal{Y}_\sigma}$. The restriction of $\Phi$ to $\mathcal{Y}_\sigma$ is a moment map $\Phi_\sigma : \mathcal{Y}_\sigma \to t^\sigma$ that is proper as a map from $\mathcal{Y}_\sigma$ into $U_\sigma$. The set

$$K_\sigma \cdot (\mathcal{Y}_\sigma \cap \Phi^{-1}_\sigma (\beta)) = M^\beta \cap \Phi^{-1} (\beta)$$

is a component of $\text{Cr}(\|\Phi_\sigma\|^2)$. Let $\mathcal{G}^\beta_{K_\sigma} (\mathcal{Y}_\sigma) \in R_{\text{ic}}^{-\infty} (K_\sigma)$ be the corresponding character (see Definition 2.8).

In [Paradan 2001, Theorem 7.5], we proved the induction formula

$$\mathcal{G}^\beta_K (M) = \text{Hol}^K_{K_\sigma} (\mathcal{G}^\beta_{K_\sigma} (\mathcal{Y}_\sigma)),$$

where $\text{Hol}^K_{K_\sigma} : R^{-\infty} (K_\sigma) \to R^{-\infty} (K)$ is the holomorphic induction map. See [Paradan 2001, Appendix] for the definition and properties of these induction maps.
We know from the previous case that
\[ \mathcal{D}_{K_\sigma}^\beta (\mathcal{Y}_\sigma) = \sum_{\mu \in \hat{K}_\sigma} m_\mu V_{\mu}^{K_\sigma}, \]
where \( m_\mu \neq 0 \) implies (\( \mu, \beta \)) \( \geq \| \beta \|^2 \). Then, with (20), we get
\[ \mathcal{D}_{K}^\beta (M) = \sum_{(\mu, \beta) \geq \| \beta \|^2} m_\mu \text{Hol}_{K_\sigma}^K (V_{\mu}^{K_\sigma}) = \sum_{(\mu, \beta) \geq \| \beta \|^2} m_\mu \text{Hol}_T^K (t^\mu), \]
where \( \text{Hol}_T^K : R^{-\infty} (T) \to R^{-\infty} (K) \) is the holomorphic induction map. Here we use that \( \text{Hol}_T^K = \text{Hol}_{K_\sigma}^K \circ \text{Hol}_T^{K_\sigma} \) and that \( V_{\mu}^{K_\sigma} = \text{Hol}_T^{K_\sigma} (t^\mu) \) for \( \mu \in \hat{K}_\sigma \subset \wedge^* \) (see [Paradan 2001, Appendix]).

Let \( \rho \) be half the sum of the positive roots. The term \( \text{Hol}_T^K (t^\mu) \) is equal to 0 when \( \mu + \rho \) is not a regular element of \( t^* \). When \( \mu + \rho \) is a regular element of \( t^* \), we have \( \text{Hol}_T^K (t^\mu) = (-1)^{\{\mu\}} V_{\mu_\omega}^{K}, \) where
\[ \mu_\omega = \omega (\mu + \rho) - \rho \]
is dominant for a unique element \( \omega \) of the Weyl group.

Finally, a representation \( V_{\lambda}^{K} \) appears in the character \( \mathcal{D}_{K}^\beta (M) \) only if \( \lambda = \mu_\omega \) for a weight \( \mu \) satisfying (\( \mu, \beta \)) \( \geq \| \beta \|^2 \). Hence, for such \( \lambda \), we have
\[ \| \lambda \| = \| \mu + \rho - \omega^{-1} \rho \| \]
\[ \geq \left( \mu + \rho - \omega^{-1} \rho, \frac{\beta}{\| \beta \|} \right) \]
\[ \geq \| \beta \|. \]
In the last inequality we use that (\( \rho - \omega^{-1} \rho, \beta \)) \( \geq 0 \) since \( \rho - \omega^{-1} \rho \) is a sum of positive roots, and \( \beta \in t^*_+ \). \( \square \)

With the help of Theorem 2.9 and decomposition (18), we see that the multiplicity of \( V_{\gamma}^{K} \) in \( \mathcal{D}_{K}^\Phi (M_{< R}) \) does not depend on the regular value \( R > \| \gamma \|^2 \).

**Definition 2.11.** The generalized character \( \mathcal{D}_{K}^\Phi (M) \) is defined as the limit of characters \( \mathcal{D}_{K}^\Phi (M_{< R}) \) in \( R^{-\infty} (K) \) when \( R \) goes to infinity. In other words
\[ (21) \quad \mathcal{D}_{K}^\Phi (M) = \sum_{\beta \in \Phi} \mathcal{D}_{K}^\beta (M). \]

For any regular value \( R \) of \( \| \Phi \|^2 \) we have the useful relation
\[ (22) \quad \mathcal{D}_{K}^\Phi (M) = \mathcal{D}_{K}^\Phi (M_{< R}) + O(\sqrt{R}). \]
2C. Quantization of a symplectic quotient. We will now explain how to define the geometric quantization of singular compact Hamiltonian manifolds, where “singular” means that the manifold is obtained by symplectic reduction.

Let \((N, \Omega)\) be a smooth symplectic manifold equipped with a Hamiltonian action of \(K_1 \times K_2\). Denote by \((\Phi_1, \Phi_2) : N \to \mathfrak{t}_1^* \times \mathfrak{t}_2^*\) the corresponding moment map. Assume that \(N\) is prequantized by a \((K_1 \times K_2)\)-equivariant line bundle \(L\) and suppose that the map \(\Phi_1\) is proper. One wants to define the geometric quantization of the compact symplectic quotient

\[N//_0 K_1 := \Phi_1^{-1}(0)/K_1\]

which is in general singular.

Let \(\kappa_1\) be the Kirwan vector field attached to the moment map \(\Phi_1\). Denote by \(e^{\kappa_1}\) the symbol \(\text{Thom}(N, J) \otimes L\) pushed by the vector field \(\kappa_1\). For any regular value \(R_1\) of \(\|\Phi_1\|^2\), consider the restriction \(e^{\kappa_1}|_{N < R_1}\) to the invariant, open subset \(N < R_1 := \{\|\Phi_1\|^2 < R_1\}\). The symbol \(e^{\kappa_1}|_{N < R_1}\) is \((K_1 \times K_2)\)-equivariant and \(K_1\)-transversally elliptic, hence we can consider its index

\[\text{Index}_{N < R_1}^{K_1 \times K_2}(e^{\kappa_1}|_{N < R_1}) \in R^{-\infty}(K_1 \times K_2),\]

which is smooth relative to the parameter in \(K_2\). Consider the following extension of Definition 2.11.

Definition 2.12. The generalized character \(\varrho_{K_1 \times K_2}^{\Phi_1}(N)\) is defined as the limit in \(R^{-\infty}(K_1 \times K_2)\) of \(\text{Index}_{N < R_1}^{K_1 \times K_2}(e^{\kappa_1}|_{N < R_1})\) when \(R_1\) goes to infinity.

Here \(\text{Cr}(\|\Phi_1\|^2)\) is equal to the disjoint union of the compact \((K_1 \times K_2)\)-invariant subsets \(Z_{\beta_1} := K_1.(\mathfrak{m}_{\beta_1} \cap \Phi_1^{-1}(\beta_1))\), \(\beta_1 \in \mathfrak{b}_1\). For \(\beta_1 \in \mathfrak{b}_1\), consider an invariant relatively compact open subset \(\mathcal{U}_{\beta_1}\) such that: \(Z_{\beta_1} \subset \mathcal{U}_{\beta_1}\) and \(Z_{\beta_1} = \text{Cr}(\|\Phi_1\|^2) \cap \mathcal{U}_{\beta_1}\). Let \(\varrho_{K_1 \times K_2}^{\beta_1}(N) \in R^{-\infty}(K_1 \times K_2)\) be the equivariant index of the \(K_1\)-transversally elliptic symbol \(e^{\kappa_1}|_{\mathcal{U}_{\beta_1}}\). The \(K_1\)-transversality condition imposes that \(\varrho_{K_1 \times K_2}^{\beta_1}(N) = \sum_{\lambda \in \hat{K}_1} \theta_{\beta_1}(\lambda) \otimes V_{\lambda}^{K_1}\) with

\[\theta_{\beta_1}(\lambda) \in R(K_2)\quad \text{for all } \lambda \in \hat{K}_1.\]

We have the following extension of Theorem 2.9:

Theorem 2.13. We have

\[\varrho_{K_1 \times K_2}^{\beta_1}(N) = \sum_{\lambda \in \hat{K}_1} \theta_{\beta_1}(\lambda) \otimes V_{\lambda}^{K_1},\]

where \(\theta_{\beta_1}(\lambda) \neq 0\) only if \(\|\lambda\| \geq \|\beta_1\|\).

Proof. The proof works exactly like that of Theorem 2.9. \(\square\)
We now explain the “quantization commutes with reduction theorem”, or why we can consider the geometric quantization of

\[ \Phi_1^{-1}(0)/K_1 \]

as the \(K_1\)-invariant part of \(\mathcal{D}_{K_1 \times K_2}(N)\).

First suppose that 0 is a regular value of \(\Phi_1\). Then \(N//0K_1\) is a compact symplectic orbifold equipped with a Hamiltonian action of \(K_2\): the corresponding moment map is induced by the restriction of \(\Phi_2\) to \(\Phi_1^{-1}(0)\). The symplectic quotient \(N//0K_1\) is prequantized by the line orbibundle \(L_0 := (L|_{\Phi_1^{-1}(0)})/K_1\).

Definition 1.1 extends to the orbifold case. We can still define the geometric quantization of \(N//0K_1\) as the index of an elliptic operator and denote it by \(\mathcal{D}_{K_2}(N//0K_1) \in R(K_2)\).

**Theorem 2.14.** If 0 is a regular value of \(\Phi_1\), the \(K_1\)-invariant part of \(\mathcal{D}_{K_1 \times K_2}(N)\) is equal to \(\mathcal{D}_{K_2}(N//0K_1) \in R(K_2)\).

Suppose now that 0 is not a regular value of \(\Phi_1\). Let \(T_1\) be a maximal torus of \(K_1\), and let \(t_1^* \subset t_1^*\) be a Weyl chamber. Since \(\Phi_1\) is proper, the convexity theorem says that the image of \(\Phi_1\) intersects \(t_1^*\) in a closed locally polyhedral convex set, which we denote by \(\Delta_{K_1}(N)\) [Lerman et al. 1998].

Consider an element \(a \in \Delta_{K_1}(N)\) which is generic and sufficiently close to \(0 \in \Delta_{K_1}(N)\). Denote by \((K_1)_a\) the subgroup of \(K_1\) which stabilizes \(a\). When \(a \in \Delta_{K_1}(N)\) is generic, one can show (see [Meinrenken and Sjamaar 1999]) that

\[ N//aK_1 := \Phi_{K_1}^{-1}(a)/(K_1)_a \]

is a compact Hamiltonian \(K_2\)-orbifold, and that

\[ L_a := (L|_{\Phi_{K_1}^{-1}(a)})/(K_1)_a \]

is a \(K_2\)-equivariant line orbibundle over \(N//aK_1\). We can then define, like in Definition 1.1, the element \(\mathcal{D}_{K_2}(N//aK_1) \in R(K_2)\) as the equivariant index of the Dolbeault–Dirac operator on \(N//aK_1\) (with coefficients in \(L_a\)).

**Theorem 2.15.** The \(K_1\)-invariant part of \(\mathcal{D}_{K_1 \times K_2}(M)\) is equal to \(\mathcal{D}_{K_2}(N//aK_1) \in R(K_2)\). In particular, the elements \(\mathcal{D}_{K_2}(N//aK_1)\) do not depend on the choice of the generic element \(a \in \Delta_{K_1}(N)\), when \(a\) is sufficiently close to 0.

**Proofs of Theorems 2.14 and 2.15.** When \(N\) is compact and \(K_2 = \{e\}\), the proofs can be found in [Meinrenken and Sjamaar 1999; Paradan 2001]. We explain briefly...
how the $K^0$-theoretic proof of [Paradan 2001] extends naturally to our case. Like in Definition 2.11, we have the decomposition

$$\mathcal{D}_{K_1 \times K_2}(N) = \sum_{\beta \in \mathcal{B}} \mathcal{D}_{K_1 \times K_2}(N),$$

and Theorem 2.13 implies $[\mathcal{D}_{K_1 \times K_2}(N)]_{K_1} = 0$ if $\beta_1 \neq 0$. This proves the first step:

$$[\mathcal{D}_{K_1 \times K_2}(N)]_{K_1} = [\mathcal{D}_{K_1 \times K_2}(N)]_{K_1}^0.$$

The analysis of the term $[\mathcal{D}_{K_1 \times K_2}(N)]_{K_1}$ is undertaken in [Paradan 2001] when $K_2 = \{e\}$: this term is equal either to $\mathcal{D}(N \|_0 K_1)$ when 0 is a regular value (see [Paradan 2001, Section 6.2]), or to $\mathcal{D}(N \|_a K_1)$ with $a$ generic (see [Paradan 2001, Section 7.4]). It works similarly with an action of a compact Lie group $K_2$.

**Definition 2.16.** The geometric quantization of $N \|_0 K_1 := \Phi_1^{-1}(0)/K_1$ is taken as the $K_1$-invariant part of $\mathcal{D}_{K_1 \times K_2}(N)$. Denote it by $\mathcal{D}_{K_2}(N \|_0 K_1) \in R(K_2)$.

**2D. Quantization of points.** Let $(M, \Omega, \Phi)$ be a proper Hamiltonian $K$-manifold prequantized by a Kostant–Souriau line bundle $L$. Let $\mu \in \widehat{K}$ be a dominant weight such that $\Phi^{-1}(K \cdot \mu)$ is a $K$-orbit in $M$. Let $m^\sigma \in \Phi^{-1}(\mu)$ so that

$$\Phi^{-1}(K \cdot \mu) = K \cdot m^\sigma.$$

Then the reduced space $M_\mu := \Phi^{-1}(K \cdot \mu)/K$ is a point. The aim of this section is to compute the quantization of $M_\mu$: $\mathcal{D}(M_\mu) \in \mathbb{Z}$.

The stabilizer subgroup $H$ of $m^\sigma$ is contained in the subgroup $K_\mu \subset K$ that fixes $\mu \in \mathfrak{t}^*$. We have a linear action of $H$ on the 1-dimensional vector space $L_{m^\sigma} \subset L$.

Let $\mathfrak{k}_\mu$ be the Lie algebra of $K_\mu$. We recall why the Lie algebra morphism $i_\mu : \mathfrak{t}_\mu \to i\mathbb{R}$ integrates in a character $\chi_\mu$ of $K_\mu$. The group $K_\mu$, which is connected, decomposes as $K_\mu = [K_\mu, K_\mu]Z_\mu$, where $Z_\mu$ is the connected component of the center of $K_\mu$. For the maximal torus $T$, we have $T = T_\mu Z_\mu$ with $T_\mu = T \cap [K_\mu, K_\mu] = \exp(t \cap [\mathfrak{k}_\mu, \mathfrak{k}_\mu])$. The morphism $i_\mu : t \to i\mathbb{R}$ integrates in a character $\chi_\mu^T$ of $T$ which is trivial on $T_\mu$ since $\langle \mu, [t, \mathfrak{k}_\mu] \rangle = 0$. Hence we can define the character $\chi_\mu$ as being trivial on $[K_\mu, K_\mu]$, and equal to $\chi_\mu^T$ on $Z_\mu$.

Let $G_{-\mu}$ be the 1-dimensional representation of $K_\mu$ associated to the character $\chi_\mu^{-1}$. Denote by $\chi$ the character of $H$ defined by the 1-dimensional representation $\mathbb{C}_\chi := L_{m^\sigma} \otimes \mathbb{C}_{-\mu}$. We know from the Kostant formula (1) that $\chi = 1$ on the identity component $H^0 \subset H$.

**Theorem 2.17.** We have

$$\mathcal{D}(M_\mu) = \begin{cases} 1 & \text{if } \chi = 1 \text{ on } H, \\ 0 & \text{otherwise.} \end{cases}$$

(23)
This theorem tells us in particular that \( \mathcal{D}(M_\mu) = 1 \) when the stabilizer subgroup \( H \subset K \) of a point \( m^o \in \Phi^{-1}(\mu) \) is connected.

**Proof.** Let \( N = M \times K \cdot \mu \) be the proper Hamiltonian \( K \)-manifold which is pre-quantized by the line bundle \( L_N := L \otimes [\mathbb{C}_-\mu] \). Denote by \( \Phi_N \) the moment map on \( N \). Since \( \Phi^{-1}(K \cdot \mu) \) is a \( K \)-orbit in \( M \), we see that \( \Phi_N^{-1}(0) \) is the \( K \)-orbit through \( n^o := (m^o, \mu) \) where \( m^o \in \Phi^{-1}(\mu) \). Note that \( H \) is the stabilizer subgroup of \( n^o \).

Let \( \mathcal{D}^{\Phi_N}(N) \in \mathbb{R}^{-\infty}(K) \) be the formal quantization of \( N \) through the proper map \( \Phi_N \). We know by Theorem 2.15 and Definition 2.16 that

\[
\mathcal{D}(M_\mu) = [\mathcal{D}^{\Phi_N}(N)]^K = [\mathcal{D}^0(N)]^K,
\]

where \( \mathcal{D}^0(N) \) depends only on a neighborhood of \( \Phi_N^{-1}(0) \).

The orbit \( K \cdot n^o \to N \) is an isotropic embedding since it is the 0-level of the moment map \( \Phi_N \). To describe a \( K \)-invariant neighborhood of \( K \cdot n^o \) in \( N \) we can use the normal-form recipe of Marle, Guillemin and Sternberg.

First consider, following [Weinstein 1979], the symplectic normal bundle

\[
\mathcal{V} := T(K \cdot n^o)^\perp/T(K \cdot n^o),
\]

where the orthogonal \( ^\perp \) is taken relative to the symplectic 2-form. We have

\[
\mathcal{V} = K \times_H V,
\]

where the vector space \( V := T_{n^o}(K \cdot n^o)^\perp/T_{n^o}(K \cdot n^o) \) inherits a canonical symplectic structure \( \Omega_V \) and a Hamiltonian action of the group \( H \). Let \( \Phi_V : V \to \mathfrak{h}^* \) be the corresponding moment map.

Consider now the symplectic manifold

\[
\tilde{N} := \mathcal{V} \oplus T^*(K/H) = K \times_H (\mathfrak{k}/\mathfrak{h})^* \oplus V.
\]

The action of \( K \) on \( \tilde{N} \) is Hamiltonian with moment map \( \Phi_{\tilde{N}} : \tilde{N} \to \mathfrak{k}^* \) given by

\[
\Phi_{\tilde{N}}([k; \xi, v]) = k \cdot (\xi + \Phi_V(v)) \quad \text{for } k \in K, \; \xi \in (\mathfrak{k}/\mathfrak{h})^*, \; v \in V.
\]

The Hamiltonian \( K \)-manifold \( \tilde{N} \) is prequantized by the line bundle \( L_{\tilde{N}} := K \times_H \mathbb{C}_\chi \).

The **local normal form** theorem (see [Guillemin and Sternberg 1984; Sjamaar and Lerman 1991, Proposition 2.5]) tells us that there exists a \( K \)-Hamiltonian isomorphism \( \Upsilon : \mathfrak{u}_1 \to \mathfrak{u}_2 \) between a \( K \)-invariant neighborhood \( \mathfrak{u}_1 \) of \( K \cdot n^o \) in \( N \), and a \( K \)-invariant neighborhood \( \mathfrak{u}_2 \) of \( K/H \) in \( \tilde{N} \). This isomorphism \( \Upsilon \), when restricted to \( K \cdot n^o \), corresponds to the natural isomorphism \( K \cdot n^o \to K/H \).

We check that

\[
\{\Phi_{\tilde{N}} = 0\} = K \times_H \{\Phi_V = 0\} \quad \text{and} \quad \text{Cr}(\|\Phi_{\tilde{N}}\|^2) = K \times_H \text{Cr}(\|\Phi_V\|^2).
\]
See (30). Let $\kappa_V$ be the Kirwan vector field associated to the Hamiltonian action of $H$ on $(V, \Omega_V)$. A simple computation gives

$$\Omega(\kappa_V(v), v) = -2\|\Phi_V(v)\|^2,$$

which implies that $\text{Cr}(\|\Phi_V\|^2) = \{\Phi_V = 0\}$ and then $\text{Cr}(\|\Phi_N\|^2) = \{\Phi_N = 0\}$. Note that $\{\Phi_V = 0\}$ is a cone in $V$ since the map $\Phi_V$ is quadratic. The map $\Upsilon$ sends $\{\Phi_N = 0\} \cap \mathcal{U}_1$ onto $\{\Phi_N = 0\} \cap \mathcal{U}_2$. Our hypothesis imposes that $\{\Phi_N = 0\}$ is reduced to a $K$-orbit, therefore the cone $\{\Phi_V = 0\}$ is reduced to $\{0\}$; this last point is equivalent to the fact that $\Phi_V$ (and then $\Phi_N$) is proper map (see [Paradan 2009, Lemma 5.2]).

We get the equalities

$$\mathcal{O}_K^0(N) = \mathcal{O}_K^0(\tilde{N}) = \mathcal{O}_K^\Phi(\tilde{N}).$$

The first equality follows from Proposition 2.6 (applied to the isomorphism $\Upsilon$), and the second one is due to the fact that $\text{Cr}(\|\Phi_N\|^2) = \Phi_N^{-1}(0)$.

Let $\text{Ind}_H^K: R^{-\infty}(H) \to R^{-\infty}(K)$ be the induction map that is defined by the relation $\langle \text{Ind}_H^K(\varphi), E \rangle = \langle \varphi, E|_H \rangle$ for any $\varphi \in R^{-\infty}(H)$ and $E \in R(K)$. Note that

$$[\text{Ind}_H^K(\varphi)]^K = \langle \text{Ind}_H^K(\varphi), C \rangle = \langle \varphi, C \rangle = [\varphi]^H.$$ 

Since $\Phi_V: V \to \mathfrak{h}^*$ is proper, one can consider the quantization of the vector space $V$ through the moment map $\Phi_V: \mathcal{O}_V^\Phi(V) \in R^{-\infty}(H)$. In the next proposition we consider an $H$-invariant complex structure $J_V$ on $V$ which is compatible with the symplectic structure $\Omega_V$, and $V^*$ denotes the complex $H$-module $\text{hom}_\mathbb{C}(V, \mathbb{C})$.

**Proposition 2.18.**

- We have

$$\mathcal{O}_K^\Phi(\tilde{N}) = \text{Ind}_H^K(\mathcal{O}_H^\Phi(V) \otimes C_\chi).$$

- The formal quantization $\mathcal{O}_H^\Phi(V)$ coincides, as a generalized $H$-module, to the $H$-module $S(V^*)$ of complex polynomial function on $V$.

- The set $[S(V^*)]^H$ of polynomials invariant by the connected component $H^0$ is reduced to the scalars.

With Proposition 2.18, we can finish proving Theorem 2.17 with a calculation:

$$\mathcal{O}(M_\mu) = [\mathcal{O}_K^\Phi(N)]^K = [\mathcal{O}_K^\Phi(\tilde{N})]^K = [\mathcal{O}_H^\Phi(V) \otimes C_\chi]^H = [S(V^*) \otimes C_\chi]^H = [C_\chi]^H.$$

**Proof of Proposition 2.18.** The first point is implied by the induction property defined by Atiyah (see [Paradan 2001, Section 3.4]) by the following argument: We
work with the $H$-manifold $\mathcal{Y} = \mathfrak{k} / \mathfrak{h} \oplus V$ and the $H$-equivariant map $j : \mathcal{Y} \hookrightarrow \widetilde{N} := K \times_H \mathcal{Y}$, $y \mapsto [e, y]$.

Notice that $T\widetilde{N} \simeq K \times_H (\mathfrak{k} / \mathfrak{h} \oplus T\mathcal{Y})$ and that $T_K \widetilde{N} \simeq K \times_H (T_H \mathcal{Y})$. Hence the map $j$ induces an isomorphism $j_* : K^0_H(T_H \mathcal{Y}) \rightarrow K^0_K(T_K \widetilde{N})$. By [Atiyah 1974, Theorem 4.1], the diagram

$$
\begin{array}{c}
\text{Index}_H^R \downarrow \\
K^0_H(T_H \mathcal{Y}) \xrightarrow{j_*} K^0_K(T_K \widetilde{N}) \\
\text{Ind}_H^K \downarrow \\
R^{-\infty}(H) \xrightarrow{\text{Ind}_H^K} R^{-\infty}(K)
\end{array}
$$

is commutative. The tangent bundle $T\widetilde{N}$ is equivariantly diffeomorphic to $K \times_H [\mathfrak{k} / \mathfrak{h} \oplus T(\mathfrak{k} / \mathfrak{h}) \oplus TV] \simeq K \times_H [\mathcal{Y} \times ((\mathfrak{k} / \mathfrak{h})_C \oplus V)]$, where $(\mathfrak{k} / \mathfrak{h})_C$ is the complexification of the real vector space $\mathfrak{k} / \mathfrak{h}$. Consider on $\widetilde{N}$ the almost complex structure $J\widetilde{N} = (i, J_V)$ for $i$ the complex structure on $(\mathfrak{k} / \mathfrak{h})_C$. Note that $J\widetilde{N}$ is compatible with the symplectic structure on a neighborhood $U$ of the 0-section of the bundle $\widetilde{N} \rightarrow K / H$.

We compute the Kirwan vector field $\kappa_\widetilde{N}$ on $\widetilde{N}$. If we take $Y = k \cdot X$ and $\widetilde{n} = [k; \xi \oplus v] \in \widetilde{N}$ we have the following relations in $T_{\widetilde{n}} \widetilde{N} \simeq (\mathfrak{k} / \mathfrak{h})_C \oplus V$:

- $Y_{\widetilde{N}}(\widetilde{n}) = -X$ when $X \in \mathfrak{k} / \mathfrak{h}$,
- $Y_{\widetilde{N}}(\widetilde{n}) = i[\xi, X] \oplus -X \cdot v$ when $X \in \mathfrak{h}$.

By taking $Y = \Phi_\widetilde{N}([k; \xi, v]) = k \cdot (\xi + \Phi_V(v))$ we get

$$
(30) \quad \kappa_\widetilde{N}([k; \xi, v]) = -\xi + i[\xi, \Phi_V(v)] \oplus \kappa_V(v) \in (\mathfrak{k} / \mathfrak{h})_C \oplus V.
$$

Since $\kappa_V$ vanishes only on $\{0\} \subset V$, the vector field $\kappa_\widetilde{N}$ vanishes exactly on the 0-section of the bundle $\widetilde{N} \rightarrow K / H$.

Let $\mathbf{e}^{\kappa_\widetilde{N}}$ be the symbol $\text{Thom}(\widetilde{N}, J_{\widetilde{N}}) \otimes L_{\widetilde{N}}$ pushed by the vector field $\kappa_\widetilde{N}$. The generalized character $\mathcal{O}_K^\Phi(\widetilde{N})$ is either computed as the equivariant index of the symbols $\mathbf{e}^{\kappa_\widetilde{N}}$ or $\mathbf{e}^{\kappa_\widetilde{N}}|_U$.

**Remark 2.19.** The fact that $J_\widetilde{N}$ is not compatible on the entire manifold $\widetilde{N}$ is not problematic, since $J_\widetilde{N}$ is compatible in a neighborhood $U$ of the set where $\kappa_\widetilde{N}$ vanishes. See the first point of Proposition 2.6.

---

5We have an $H$-equivariant identification $(\mathfrak{k} / \mathfrak{h})^* \simeq \mathfrak{k} / \mathfrak{h}$.

6 These identities come from the following $(K \times H)$-equivariant isomorphism of vector bundles over $K \times \mathcal{Y}$: $T_H(K \times \mathcal{Y}) \rightarrow K \times (\mathfrak{k} / \mathfrak{h} \times T\mathcal{Y})$, $(k, m; \frac{d}{dt})_{t=0}([ke^tX] \oplus v_m) \mapsto (k, m; \text{pr}_{\mathfrak{k} / \mathfrak{h}}(X) + v_m)$, where $\text{pr}_{\mathfrak{k} / \mathfrak{h}} : \mathfrak{k} \rightarrow \mathfrak{k} / \mathfrak{h}$ is the orthogonal projection.
For \((X + i\eta, w) \in T_{[\xi, v]}N \simeq (\mathfrak{g}/\mathfrak{h})_C \times V\), the map
\[
\mathbf{c}^{\kappa_N}(X + i\eta, w) = \mathbf{c}(X + \xi + i\eta - i[\xi, \Phi_V(v)]) \odot \mathbf{c}(w - \kappa_V(v))
\]
acts on the vector space \(\wedge \mathbb{C}(\mathfrak{g}/\mathfrak{h})_C \otimes \wedge \mathbb{C} V \otimes \mathbb{C}_\chi\). We see that
\[
\mathbf{c}^{\kappa_N} = j_*(\mathbf{c}^\mathcal{Y}),
\]
where \(\mathbf{c}^\mathcal{Y}\) is the symbol on \(\mathcal{Y}\) defined as follows. For \((\xi, v) \in \mathcal{Y} = \mathfrak{g}/\mathfrak{h} \times V\), the map \(\mathbf{c}\mathcal{Y}|_{(\xi, v)}(\eta, w)\) acts on \(\wedge \mathbb{C}(\mathfrak{g}/\mathfrak{h})_C \otimes \wedge \mathbb{C} V \otimes \mathbb{C}_\chi\) as the product
\[
\mathbf{c}(\xi + i\eta - i[\xi, \Phi_V(v)]) \odot \mathbf{c}(w - \kappa_V(v)).
\]

Let \(\text{Bott}(\mathfrak{g}/\mathfrak{h})\) be the Bott symbol on the vector space \(\mathfrak{g}/\mathfrak{h}\). It is an elliptic morphism defined by
\[
\text{Bott}(\mathfrak{g}/\mathfrak{h})|_\xi(\eta) = \mathbf{c}(\xi + i\eta) \text{ acting on } \wedge \mathbb{C} (\mathfrak{g}/\mathfrak{h})_C,
\]
for \(\eta \in T_\xi(\mathfrak{g}/\mathfrak{h})\). Let \(\mathbf{c}^{\mathbf{K}_V}\) be the symbol \(\text{Thom}(V, J_V)\) pushed by the vector field \(\kappa_V\).

**Lemma 2.20.** We have
\[
\mathbf{c}^{\kappa_N} = j_*(\text{Bott}(\mathfrak{g}/\mathfrak{h}) \odot \mathbf{c}^{\mathbf{K}_V} \otimes \mathbb{C}_\chi).
\]

**Proof.** We work with the family of symbols \(\sigma^T, T \in [0, 1]\), on \(\mathcal{Y} = \mathfrak{g}/\mathfrak{h} \times V\) defined for \((\eta, w) \in T_{(\xi, v)}\mathcal{Y}\) as the map
\[
\sigma^T|_{(\xi, v)}(\eta, w) = \mathbf{c}(\xi + i\eta - iT[\xi, \Phi_V(v)]) \odot \mathbf{c}(w - \kappa_V(v))
\]
acting on the vector space \(\wedge \mathbb{C}(\mathfrak{g}/\mathfrak{h})_C \otimes \wedge \mathbb{C} V \otimes \mathbb{C}_\chi\). Note \(\sigma^0 = \text{Bott}(\mathfrak{g}/\mathfrak{h}) \odot \mathbf{c}^{\mathbf{K}_V} \otimes \mathbb{C}_\chi\), and \(\sigma^1 = \mathbf{c}^\mathcal{Y}\). It is now easy to check that
\[
\text{Char}(\sigma^T) = \{(0, 0) \in T(\mathfrak{g}/\mathfrak{h})\} \times \{(v, \kappa_V(v)), v \in V\} \subset T\mathcal{Y}
\]
and that \(\text{Char}(\sigma^T) \cap T_H\mathcal{Y} = \{(0, 0) \in T(\mathfrak{g}/\mathfrak{h})\} \times \{(0, 0) \in TV\} \) for any \(T \in [0, 1]\). Hence \(\sigma^T, T \in [0, 1]\), is a homotopy of \(H\)-transversally elliptic symbols on \(\mathfrak{g}/\mathfrak{h} \times V\). It gives finally that \(\mathbf{c}^{\kappa_N} = j_*(\mathbf{c}^\mathcal{Y}) = j_*(\sigma^0)\).

The commutative diagram (29) and the last lemma give
\[
\mathcal{L}^{\Phi_V}(N) = \text{Ind}^K_H(\mathbf{c}^{\kappa_N})
\]
\[
= \text{Ind}^K_H\left(\text{Ind}^H_{\mathfrak{g}/\mathfrak{h} \times V}(\text{Bott}(\mathfrak{g}/\mathfrak{h}) \odot \mathbf{c}^{\mathbf{K}_V}) \otimes \mathbb{C}_\chi\right)
\]
\[
= \text{Ind}^K_H\left(\text{Ind}^H_{\mathfrak{g}/\mathfrak{h}}(\text{Bott}(\mathfrak{g}/\mathfrak{h})) \otimes \text{Ind}^H_V(\mathbf{c}^{\mathbf{K}_V}) \otimes \mathbb{C}_\chi\right)
\]
\[
= \text{Ind}^K_H(\mathcal{L}^{\Phi_V}(V) \otimes \mathbb{C}_\chi).
\]

We have used here that the \(H\)-equivariant index of \(\text{Bott}(\mathfrak{g}/\mathfrak{h})\) is equal to 1, that is, the trivial representation of \(H\); see [Paradan and Vergne 2009, Section 2.4.1].
We now prove the second point of Proposition 2.18. Since the Kirwan vector field \( \kappa_V \) satisfies the relations
\[
(\kappa_V(v), J_V v) = -\Omega(\kappa_V(v), v) = 2\|\Phi_V(v)\|^2,
\]
we have
\[
(\kappa_V(v), J_V v) > 0
\]
for \( v \neq 0 \). Consider on \( V \) the family of symbols \( \sigma^s \):
\[
\sigma^s|_v(w) = c(w - s\kappa_V(v) - (1 - s)J_V v)
\]
viewed as a map from \( \wedge^\text{even}_C V \) to \( \wedge^\text{odd}_C V \). By (32), one sees that \( \sigma^s \) is a family of \( H \)-transversally elliptic symbols on \( V \). Hence \( \sigma^1 = c\kappa_V \) and \( \sigma^0 = c(w - J_V v) \) defines the same class in the group \( K^0_H(T_H V) \). The symbol \( \sigma^0 \) was first studied by Atiyah [1974] when \( \dim_C V = 1 \). [Paradan 2001, Proposition 5.4] considered the general case. We have
\[
\text{Index}^H_{V}(\sigma^0) = S(V^*) \quad \text{in} \quad R^{-\infty}(H).
\]
The last point of Proposition 2.18 is a consequence of the properness of the moment map \( \Phi_V \); see [Paradan 2009, Section 5]. \( \square \)

This completes the proof of Theorem 2.17. \( \square \)

**Example 2.21** [Paradan 2009]. Consider the action of the unitary group \( U_n \) on \( C^n \). The symplectic form on \( C^n \) is defined by \( \Omega(v, w) = \frac{i}{2} \sum_k v_k \overline{w_k} - \overline{v_k} w_k \). Identify the Lie algebra \( u_n \) with its dual through the trace map. The moment map \( \Phi : C^n \to u_n \) is defined by \( \Phi(v) = (1/2i)v \otimes v^* \) where \( v \otimes v^* : C^n \to C^n \) is the linear map \( w \mapsto (\sum_k \overline{v_k} w_k) v \). One checks easily that the pullback by \( \Phi \) of a \( U_n \)-orbit in \( u_n \) is either empty or a \( U_n \)-orbit in \( C^n \). We know also that the stabilizer subgroup of a nonzero vector of \( C^n \) is connected since it is diffeomorphic to \( U_{n-1} \). Finally,
\[
(33) \quad \mathcal{Q}((C^n)^*) = \begin{cases} 
1 & \text{if } \mu \in \widehat{U_n} \text{ belongs to the image of } \Phi, \\
0 & \text{if } \mu \notin \widehat{U_n} \text{ does not belong to the image of } \Phi.
\end{cases}
\]
Then one can check that \( \mathcal{Q}^{-\infty}(C^n) \) coincides in \( R^{-\infty}(U_n) \) with the algebra \( S((C^n)^*) \) of polynomial function on \( C^n \).

**Example 2.22** [Paradan 2003]. Consider the Lie group \( SL_2(R) \) and its compact torus of dimension 1 denoted by \( T \). The Lie algebra \( sl_2(R) \) is identified with its dual through the trace map, and the Lie algebra \( t \) is naturally identified with \( sl_2(R)^T \). For \( l \in Z \setminus \{0\} \), consider the character \( \chi_l \) of \( T \) defined by
\[
\chi_l \left( \begin{array}{cc} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{array} \right) = e^{il\theta}.
\]
Its differential \( \frac{1}{i} d \chi_l \in t^* \) corresponds (through the trace map) to the matrix
\[
X_l = \begin{pmatrix}
0 & l/2 \\
-l/2 & 0
\end{pmatrix}.
\]

Let \( \mathcal{O}_l \) be the coadjoint orbit of the group \( SL_2(\mathbb{R}) \) through the matrix \( X_l \). It is a Hamiltonian \( SL_2(\mathbb{R}) \)-manifold prequantized by the \( SL_2(\mathbb{R}) \)-equivariant line bundle \( L_l \cong SL_2(\mathbb{R}) \times_T \mathcal{C}_l \), where \( \mathcal{C}_l \) is the \( T \)-module associated to the character \( \chi_l \). We look at the Hamiltonian action of \( T \) on \( \mathcal{O}_l \). Let \( \Phi_T : \mathcal{O}_l \to t^* \) be the corresponding moment map. This moment map \( \Phi_T \) is proper and its image is equal to the half-line \( \{ aX_l, a \geq 1 \} \subset t^* \).

We check that for each \( \xi \in \{ aX_l, a \geq 1 \} \) the fiber \( \Phi_T^{-1}(\xi) \) is equal to a \( T \)-orbit in \( \mathcal{O}_l \). For \( k \in \mathbb{Z} \), denote by \( (\mathcal{O}_l)_k \) the symplectic reduction of \( \mathcal{O}_l \) at the level \( X_k \). We know that \( (\mathcal{O}_l)_k = \emptyset \) if \( k \notin \{ al, a \geq 1 \} \), and that \( (\mathcal{O}_l)_k \) is a point if \( k \in \{ al, a \geq 1 \} \).

To compute \( \mathcal{Q}(\mathcal{O}_l)_k \), we look at the stabilizer subgroup \( T_m := \{ t \in T \mid t \cdot m = m \} \) for each point \( m \in \mathcal{O}_l \). One sees that \( T_m = T \) if \( m = X_l \) and \( T_m \) is equal to the center \( \{ \pm \text{Id} \} \) of \( SL_2(\mathbb{R}) \), when \( m \neq X_l \).

Theorem 2.17 gives in this setting that, for \( k \in \{ al, a \geq 1 \} \),
\[
\mathcal{Q}(\mathcal{O}_l)_k = \begin{cases}
1 & \text{if } l - k \text{ is even}, \\
0 & \text{if } l - k \text{ is odd}.
\end{cases}
\]

Hence the formal geometric quantization of the proper \( T \)-manifold \( \mathcal{O}_l \) is
\[
\mathcal{Q}_T^{-\infty}(\mathcal{O}_l) = \begin{cases}
\mathcal{C}_l \cdot \sum_{p \geq 0} \mathcal{C}_{2p} & \text{if } l > 0, \\
\mathcal{C}_l \cdot \sum_{p \geq 0} \mathcal{C}_{-2p} & \text{if } l < 0.
\end{cases}
\]

Here the quantization \( \mathcal{Q}_T^{-\infty}(\mathcal{O}_l) \) coincides with the restriction of the holomorphic (respectively antiholomorphic) discrete series representation \( \Theta_l \) to the group \( T \) when \( l > 0 \) (respectively \( l < 0 \)).

2E. Wonderful compactifications and symplectic cuts. Another equivalent definition of the quantization \( \mathcal{Q}_T^{-\infty} \) uses a generalization of the technique of symplectic cutting (originally due to [Lerman 1995]) that was introduced in [Paradan 2009] and was motivated by the wonderful compactifications of [De Concini and Procesi 1983; 1985]; see also [Brion 1998].

Recall that \( T \) is a maximal torus of the compact connected Lie group \( K \), and \( W \) is the corresponding Weyl group. Define a \( K \)-adapted polytope in \( t^* \) to be a \( W \)-invariant Delzant polytope \( P \) in \( t^* \) whose vertices are regular elements of the weight lattice \( \wedge^* \). If \( \{ \lambda_1, \ldots, \lambda_N \} \) are the dominant weights lying in the union of all the closed one-dimensional faces of \( P \), then there is a \((G \times G)\)-equivariant
embedding of $G = K_C$ into

$$\mathbb{P}\left( \bigoplus_{i=1}^{N} (V_{\lambda_i}^K)^* \otimes V_{\lambda_i}^K \right)$$

associating to $g \in G$ its representation on $\bigoplus_{i=1}^{N} V_{\lambda_i}^K$. The closure $\mathcal{X}_P$ of the image of $G$ in this projective space is smooth and is equipped with a $(K \times K)$-action

$$(k_1, k_2) \cdot x = k_2 \cdot x \cdot k_1^{-1}.$$ 

The restriction of the canonical Kähler structure on $\mathbb{P}$ defines a symplectic 2-form $\omega_\mathcal{X}_P$. Recall briefly the different properties of $(\mathcal{X}_P, \omega_\mathcal{X}_P)$ — all the details can be found in [Paradan 2009].

1. $\mathcal{X}_P$ is equipped with an Hamiltonian action of $K \times K$. Let $\Phi := (\Phi_l, \Phi_r) : \mathcal{X}_P \to T^* \times T^*$ be the corresponding moment map.

2. The image of $\Phi$ is equal to $\{(k \cdot \xi, -k' \cdot \xi) \mid \xi \in P$ and $k, k' \in K\}$. 

3. The Hamiltonian $(K \times K)$-manifold $(\mathcal{X}_P, \omega_\mathcal{X}_P)$ has no multiplicities: the pullback by $\Phi$ of a $(K \times K)$-orbit in the image is a $(K \times K)$-orbit in $\mathcal{X}_P$.

Let $\mathcal{U}_P := K \cdot P^\circ$, where $P^\circ$ is the interior of $P$. Define

$$\mathcal{X}_P^\circ := \Phi_l^{-1}(\mathcal{U}_P),$$

which is an invariant, open and dense subset of $\mathcal{X}_P$. We have the following important properties concerning $\mathcal{X}_P^\circ$.

4. There exists an equivariant diffeomorphism $\Upsilon : K \times \mathcal{U}_P \to \mathcal{X}_P^\circ$ such that $\Upsilon^*(\Phi_l)(k, \xi) = k \cdot \xi$ and $\Upsilon^*(\Phi_r)(k, \xi) = -\xi$.

5. This diffeomorphism $\Upsilon$ is a quasisymplectomorphism in the sense that there is a homotopy of symplectic forms taking the symplectic form on the open subset $K \times \mathcal{U}_P$ of the cotangent bundle $T^* K$ to the pullback of the symplectic form $\Omega_{\mathcal{X}_P}$ on $\mathcal{X}_P^\circ$.

6. The symplectic manifold $(\mathcal{X}_P, \omega_\mathcal{X}_P)$ is prequantized by the restriction of the hyperplane line bundle $\mathcal{O}(1) \to \mathbb{P}(\bigoplus_{i=1}^{N} (V_{\lambda_i}^K)^* \otimes V_{\lambda_i}^K)$ to $\mathcal{X}_P$: denote by $L_P$ the corresponding $(K \times K)$-equivariant line bundle.

7. The pullback of the line bundle $L_P$ by the map $\Upsilon : K \times \mathcal{U}_P \hookrightarrow \mathcal{X}_P$ is trivial.

Let $(M, \Omega_M, \Phi_M)$ be a proper Hamiltonian $K$-manifold and $\mathcal{X}_P$ be the Hamiltonian $(K \times K)$-manifold associated to a $K$-adapted polytope $P$. Consider now the product $M \times \mathcal{X}_P$ with the following $K \times K$ action:
the action \( k \cdot_1 (m, x) = (k \cdot m, x \cdot k^{-1}) \), with corresponding moment map 
\( \Phi_1(m, x) = \Phi_M(m) + \Phi_f(x) \),

- the action \( k \cdot_2 (m, x) = (m, k \cdot x) \), with corresponding moment map 
\( \Phi_2(m, x) = \Phi_f(x) \).

**Definition 2.23.** Denote by \( M_P \) the symplectic reduction at 0 of \( M \times \mathbb{P} \) for the action \( \cdot_1: M_P := (\Phi_1)^{-1}(0)/(K, \cdot_1) \).

Then \( M_P \) inherits a Hamiltonian \( K \)-action with moment map 
\( \Phi_M: M_P \to \mathfrak{k}^* \) whose image is \( \Phi_M(M) \cap K \cdot P \).

In [Paradan 2009], we proved that \( M_P \) contains an open and dense subset of smooth points which is quasisymplectomorphic to the open subset \( (\Phi_M)^{-1}(\mathbb{P}) \).

If the polytope \( P \) is fixed, we can work with the dilated polytopes \( nP \) for \( n \geq 1 \). We have then the family of compact, perhaps singular, \( K \)-Hamiltonian manifolds \( M_nP, n \geq 1 \). In Section 2C, we explained how their geometric quantization was defined:

\[
\mathcal{Q}_K(M_M) := \mathcal{Q}_{K \times K}(M \times \mathbb{P}_n) \in R(K).
\]

We have the following convenient property of \( \mathcal{Q}_{-\infty} \).

**Proposition 2.24** [Paradan 2009]. The following equality in \( R^{-\infty}(K) \) holds:

\[
\mathcal{Q}_K^{-\infty}(M) = \lim_{n \to \infty} \mathcal{Q}_K(M_nP).
\]

Here the limit is taken using the convention of Definition 2.3.

### 3. Proof of Theorem 1.4

The main result of this section is:

**Theorem 3.1.** Let \( r_P := \inf_{\xi \in \partial P} \|\xi\| > 0 \). The generalized character 
\[
\mathcal{Q}_K^{\Phi}(M) - \mathcal{Q}_K(M_P) \in R^{-\infty}(K)
\]

is supported outside the ball \( B_{r_P} \).

Then, for the dilated polytope \( nP, n \geq 1 \), the character \( \mathcal{Q}_K^{\Phi}(M) - \mathcal{Q}_K(M_nP) \) is supported outside the ball \( B_{nr_P} \). Taking the limit as \( n \) goes to infinity gives

\[
\mathcal{Q}_K^{\Phi}(M) = \lim_{n \to \infty} \mathcal{Q}_K(M_nP).
\]

Finally, identity (6) of Theorem 1.4,

\[
\mathcal{Q}_K^{\Phi}(M) = \mathcal{Q}_K^{-\infty}(M),
\]

is a direct consequence of (36) and (37).

Recall that \( O(r) \in R^{-\infty}(K) \) denotes any generalized character supported outside the ball \( B_r \).
Theorem 3.1 follows from the comparison of three different geometrical situations. All of them concern Hamiltonian actions of $K_1 \times K_2$, where $K_1$ and $K_2$ are two copies of $K$.

First setting. We work with the Hamiltonian $(K_1 \times K_2)$-manifold $M \times \mathcal{X}_p$, where $K_1$ acts both on $M$ and on $\mathcal{X}_p$. Since the moment map $\Phi_1$ (relative to the $K_1$-action) is proper, we may “quantize” $M \times \mathcal{X}_p$ via the map $\|\Phi_1\|^2$. Denote the corresponding generalized character by

$$\mathfrak{Q}_{K_1 \times K_2}(M \times \mathcal{X}_p) \in R^{-\infty}(K_1 \times K_2).$$

Recall that $\mathfrak{Q}_{K_2}(M_p)$ is equal to $[\mathfrak{Q}_{K_1 \times K_2}(M \times \mathcal{X}_p)]^{K_1}$.

Second setting. Consider as before the Hamiltonian action of $K_1 \times K_2$ on $M \times \mathcal{X}_p$, but “quantize” $M \times \mathcal{X}_p$ through the global moment map $\Phi = (\Phi_1, \Phi_2)$. Here we have some liberty in the choice of the scalar product on $\mathfrak{t}_1^* \times \mathfrak{t}_2^*$. If $\|\xi\|^2$ is an invariant Euclidean norm on $\mathfrak{t}^*$, we take on $\mathfrak{t}_1^* \times \mathfrak{t}_2^*$ the Euclidean norm

$$\| (\xi_1, \xi_2) \|^2 = \|\xi_1\|^2 + \rho \|\xi_2\|^2 \tag{38}$$

depending on a parameter $\rho > 0$. Consider the quantization of $M \times \mathcal{X}_p$ via the map $\|\Phi\|^2_{\rho}$:

$$\mathfrak{Q}_{K_1 \times K_2}(M \times \mathcal{X}_p) \in R^{-\infty}(K_1 \times K_2).$$

Third setting. Consider the cotangent bundle $T^*K$ with the Hamiltonian action of $K_1 \times K_2$, where $K_1$ acts by right translations and $K_2$ by left translations. Consider the Hamiltonian action of $K_1 \times K_2$ on $M \times T^*K$, where $K_1$ acts both on $M$ and on $T^*K$. Let $\Phi = (\Phi_1, \Phi_2)$ be the global moment map on $M \times T^*K$. Since the moment map $\Phi$ is proper we can “quantize” $M \times T^*K$ via the map $\|\Phi\|^2_{\rho}$. Let

$$\mathfrak{Q}_{K_1 \times K_2}(M \times T^*K) \in R^{-\infty}(K_1 \times K_2)$$

be the corresponding generalized character.

Theorem 3.1 is a consequence of the following propositions.

First we compare $\mathfrak{Q}_{K_2}(M)$ with the $K_1$-invariant part of $\mathfrak{Q}_{K_1 \times K_2}(M \times T^*K)$.

**Proposition 3.2.** For any $\rho \in [0, 1]$, we have

$$[\mathfrak{Q}_{K_1 \times K_2}(M \times T^*K)]^{K_1} = \mathfrak{Q}_{K_2}(M) \quad \text{in } R^{-\infty}(K_2). \tag{39}$$

Next we compare the $K_1$-invariant parts of the generalized characters

$$\mathfrak{Q}_{K_1 \times K_2}(M \times T^*K) \quad \text{and} \quad \mathfrak{Q}_{K_1 \times K_2}(M \times \mathcal{X}_p).$$

**Proposition 3.3.** For any $\rho \in [0, 1]$, we have the following relation in $R^{-\infty}(K_2)$

$$[\mathfrak{Q}_{K_1 \times K_2}(M \times \mathcal{X}_p)]^{K_1} - [\mathfrak{Q}_{K_1 \times K_2}(M \times T^*K)]^{K_1} = O(r_\rho), \tag{40}$$

where $r_\rho := \inf_{\xi \in \mathcal{X}_p} \|\xi\| > 0$. 

Finally we compare the $K_1$-invariant parts of the generalized characters
\[
\mathfrak{g}^{\Phi,\rho}_{K_1 \times K_2} (M \times \mathcal{X}_\rho) \quad \text{and} \quad \mathfrak{g}^{\Phi}_{K_1 \times K_2} (M \times \mathcal{X}_\rho).
\]

**Proposition 3.4.** There exists $\epsilon > 0$ such that
\[
\mathfrak{g}_{K_2}(M_\rho) - \left[ \mathfrak{g}^{\Phi,\rho}_{K_1 \times K_2} (M \times \mathcal{X}_\rho) \right]^{K_1} = O((\epsilon/\rho)^{1/2}) \quad \text{in } R^{-\infty}(K_2),
\]
if $\rho > 0$ is small enough.

If we sum the relations (39), (40) and (41) we get
\[
\mathfrak{g}_{K_2}(M) = \mathfrak{g}_{K_2}(M_\rho) + O(r_\rho) + O((\epsilon/\rho)^{1/2})
\]
if $\rho$ is small enough. So Theorem 3.1 follows by taking $(\epsilon/\rho)^{1/2} \geq r_\rho$.

**3A. Proof of Proposition 3.2.** The cotangent bundle $T^*K$ is identified with $K \times \mathfrak{k}^*$. The data is then (see Section 5A):

- the Liouville 1-form $\lambda = \sum_j \omega_j \otimes E_j$, where $(E_j)$ is a basis of $\mathfrak{k}$ with dual basis $(E_j^*)$ and $\omega_j$ is the left invariant 1-form on $K$ defined by $\omega_j(\frac{d}{dt} a e^{t X}|_0) = \langle E_j^*, X \rangle$.
- the symplectic form $\Omega := -d\lambda$.
- the action of $K_1 \times K_2$ on $K \times \mathfrak{k}^*$ given by $(k_1, k_2) \cdot (a, \xi) = (k_2 ak_1^{-1}, k_1 \cdot \xi)$,
- the moment map relative to the $K_1$-action $\Phi_1(a, \xi) = -\xi$,
- the moment map relative to the $K_2$-action $\Phi_2(a, \xi) = a \cdot \xi$.

We work now with the Hamiltonian action of $K_1 \times K_2$ on $M \times T^*K$ given by
\[
(k_1, k_2) \cdot (m, a, \xi) = (k_1 \cdot m, k_2 ak_1^{-1}, k_1 \cdot \xi).
\]
The corresponding moment map is $\Phi = (\Phi_1, \Phi_2)$: $\Phi_1(m, a, \xi) = \Phi_M(m) = \xi$ and $\Phi_2(m, a, \xi) = a \cdot \xi$.

Let $c_1$ be a symbol $\text{Thom}(M, J_1) \otimes L$ attached to the prequantized Hamiltonian $K_1$-manifold $(M, \Omega)$. The cotangent bundle $T^*K$ is prequantized by the trivial line bundle. Let $c_2$ be the symbol $\text{Thom}(T^*K, J_2)$ attached to the prequantized Hamiltonian $(K_1 \times K_2)$-manifold $T^*K$. The product $c = c_1 \otimes c_2$ corresponds to the symbol $\text{Thom}(N, J) \otimes L$ on $N = M \times T^*K$.

For the rest of this section we fix $\rho > 0$. Let $\kappa_\rho$ be the Kirwan vector field associated to the map $\|\Phi\|^2_{\rho} : M \times T^*K \to \mathbb{R}$. We check that $\|\Phi\|^2_{\rho}(m, k, \xi) = \|\Phi_M(m) - \xi\|^2 + \rho \|\xi\|^2$, and
\[
\kappa_\rho(m, k, \xi) = \left( \Phi_M(m) - \xi; \Phi_M(m) - (1 + \rho)\tilde{\xi}; -\Phi_M(m), \tilde{\xi} \right).
\]
Here $T_{(m,k,\xi)}(M \times T^* K) \simeq T_m M \times \mathfrak{k} \times \mathfrak{k}$. We have

$$\text{Cr}(\|\Phi\|_{\rho}^2) = \{\kappa_{\rho} = 0\}$$

$$= \bigcup_{\beta \in \mathfrak{b}} K_1 \times K_2 \cdot \left[ M_{\tilde{\beta}} \cap \Phi_{-1}^{-1}(\beta) \times \{1\} \times \{\tilde{\beta} \rho + 1\} \right].$$

where $\mathfrak{b} \subset \mathfrak{k}^*$ parametrizes $\text{Cr}(\|\Phi_M\|^2)$. One can check that

$$\|\Phi\|_{\rho}^2(Z_{\beta}) = \left(\frac{\rho}{\rho + 1}\right)\|\beta\|^2$$

and $\|\Phi_M\|_{\rho}^2(Z_{\beta}) = \|\beta\|^2$ for $\beta \in \mathfrak{b}$.

Let $c^{\kappa_{\rho}}$ be the symbol $c$ pushed by the vector field $\kappa_{\rho}$. We have

$$c^{\kappa_{\rho}}(v; X; Y) = c_1(v - \kappa_I) \odot c_2(X - \kappa_{II, \rho}; Y - \kappa_{III})$$

for $(v; X; Y) \in T_{(m,k,\xi)}(M \times T^* K) \simeq T_m M \times \mathfrak{k} \times \mathfrak{k}$.

For a real $R > 0$, define the open invariant subsets of $M \times T^* K$

$$U_R := \{\|\Phi\|_{\rho}^2 < R\},$$

$$V_R := \{\|\Phi_M\|^2 < R\} \times T^* K.$$ 

We see that $Z_{\beta} \subset U_R$ if and only if $(\rho/(\rho + 1))\|\beta\|^2 < R$ and $Z_{\beta} \subset V_R$ if and only if $\|\beta\|^2 < R$. By Definition 2.11, the generalized index $\mathcal{D}_{K_1 \times K_2}^{\Phi,\rho}(M \times T^* K)$ is defined as the limit of the equivariant index

$$\mathcal{D}_{K_1 \times K_2}^{\Phi,\rho}(U_R) := \text{Index}_{U_R}^{K_1 \times K_2}(c^{\kappa_{\rho}}|_{U_R}) = \sum_{(\rho/(\rho + 1))\|\beta\|^2 < R} \mathcal{D}_{K_1 \times K_2}^{\beta,\rho}(M \times T^* K)$$

when $R$ goes to infinity (and stays outside the critical values of $\|\Phi\|_{\rho}^2$).

On the other hand, when $R'$ is a regular value of $\|\Phi_M\|_{\rho}^2$, the symbol $c^{\kappa_{\rho}}|_{V_{R'}}$ is $(K_1 \times K_2)$-transversally elliptic since

$$\text{Cr}(\|\Phi\|_{\rho}^2) \cap V_{R'} = \bigcup_{\|\beta\|^2 < R'} Z_{\beta}$$

is compact. The index map is well-defined on $V_{R'} = \{\|\Phi_M\|_{\rho}^2 < R'\} \times T^* K$ since $T^* K$ can be seen as an invariant open subset of a compact $(K_1 \times K_2)$-manifold.

**Lemma 3.5.** The character $\mathcal{D}_{K_1 \times K_2}^{\Phi,\rho}(M \times T^* K)$ is equal to the limit of

$$\text{Index}_{V_{R'}}^{K_1 \times K_2}(c^{\kappa_{\rho}}|_{V_{R'}})$$

when $R'$ goes to infinity (and stays outside the critical values of $\|\Phi_M\|_{\rho}^2$).
Proof. Thanks to (42) and to the excision property we have

\[
\text{Index}_{V_{K'}}^{K_1 \times K_2} (c^{\kappa_\rho}|_{V_{K'}}) = \sum_{\|\beta\|^2 < R'} \partial_{K_1 \times K_2}^{\beta, \rho}(M \times T^* K),
\]

and then

\[
\partial_{K_1 \times K_2}^{\Phi, \rho}(M \times T^* K) = \text{Index}_{V_{K'}}^{K_1 \times K_2} (c^{\kappa_\rho}|_{V_{K'}}) = \sum_{\|\beta\|^2 \geq R'} \partial_{K_1 \times K_2}^{\beta, \rho}(M \times T^* K).
\]

By Definition 2.11, the support of \(\partial_{K_1 \times K_2}^{\beta, \rho}(M \times T^* K)\) is contained in

\[
\left\{ (\gamma_1, \gamma_2) \in \hat{K} \times \hat{K} \left| \|\gamma_1\|^2 + \rho \|\gamma_2\|^2 \geq \frac{\rho}{\rho + 1} \|\beta\|^2 \right. \right\}
\]

\[
\subset \left\{ (\gamma_1, \gamma_2) \in \hat{K} \times \hat{K} \left| \|\gamma_1\|^2 + \|\gamma_2\|^2 \geq \frac{\rho}{(\rho + 1)^2} \|\beta\|^2 \right. \right\}.
\]

Finally we have proved that

\[
\partial_{K_1 \times K_2}^{\Phi, \rho}(M \times T^* K) - \text{Index}_{V_{K'}}^{K_1 \times K_2} (c^{\kappa_\rho}|_{V_{K'}}) = \sum_{(\gamma_1, \gamma_2)} m_{(\gamma_1, \gamma_2)} R' \, V_{\gamma_1} K_1 \otimes V_{\gamma_2} K_2
\]

with \(m_{(\gamma_1, \gamma_2)} = 0\) if \(\|\gamma_1\|^2 + \|\gamma_2\|^2 \leq (\rho/(\rho + 1))^2 R'.\) Hence the right hand side of the last equation tends to 0 in \(R^{-\infty}(K_1 \times K_2)\) when \(R' \to \infty.\)

Look now to the deformation \(\kappa_\rho(s) = (\kappa_1^s; \kappa_2^s; \kappa_3^s),\ s \in [0, 1],\) where

\[
\kappa_1^s(m, \xi) = (\Phi_M(m) - s\xi) \cdot m \quad \text{and} \quad \kappa_2^s(m, \xi) = s\Phi_M(m) - (1 + s\rho)\tilde{\xi}.
\]

Let \(c^{\kappa_\rho(s)}\) be the symbol \(c\) pushed by the vector field \(\kappa_\rho(s).\)

**Lemma 3.6.** Let \(R'\) be a regular value of \(\|\Phi_M\|^2\).

- The family \(c^{\kappa_\rho(s)}|_{V_{R'}}\), \(s \in [0, 1],\) defines a homotopy of \((K_1 \times K_2)\)-transversally elliptic symbols on \(V_{R'}\).
- The \(K_1\)-invariant part of \(\text{Index}_{V_{R'}}^{K_1 \times K_2} (c^{\kappa_\rho(0)}|_{V_{R'}})\) is equal to \(\partial_{K_2}^{\Phi}(M \subset R')\).

**Proof.** The first point follows from the fact that \(\text{Char}(c^{\kappa_\rho(s)}|_{V_{R'}}) \cap T_{K_1 \times K_2}(V_{R'})\), which is equal to

\[
\left\{ (m, k, \frac{s}{1 + s\rho}\Phi_M(m)) \left| k \in K \text{ and } m \in \text{Cr}(\|\Phi_M\|^2) \cap \|\Phi_M\|^2 < R' \right. \right\},
\]

stays in a compact set when \(s \in [0, 1].\)

The symbol \(c^{\kappa_\rho(0)}|_{V_{R'}}\) is equal to the product of the symbol \(c_1^s|_{M < R'}\), which is \(K_1\)-transversally elliptic, with the symbol

\[
c_2^s(X; Y) = c_2(X + \xi; Y),
\]

where

\[
f(X, Y) = X + \xi.
\]
which is a \(K_2\)-transversally elliptic on \(T^*K\). A basic computation in Section 5A1 gives that

\[
\text{Index}_{T^*K}^{K_1 \times K_2}(c_2^k) = \text{L}^2(K) = \sum_{\mu \in \hat{K}} (V_{\mu}^{K_1})^* \otimes V_{\mu}^{K_2}
\]

in \(R^{-\infty}(K_1 \times K_2)\). Finally the multiplicative property (Theorem 2.1) gives

\[
\text{Index}_{V_{K'}^{K_1 \times K_2}(c_2^k) \mid V_{K'}}^{K_1 \times K_2} = \text{Index}_{M < R'}^{K_1 \times K_2}(c_2^k) = \sum_{\mu \in \hat{K}} \text{L}^2(K_{< R'}) = \sum_{\mu \in \hat{K}} (V_{\mu}^{K_1})^* \otimes V_{\mu}^{K_2}\]

Taking the \(K_1\)-invariant part completes the proof of the second point. \(\square\)

Finally we have proved that the generalized character \([\text{Index}_{V_{K'}^{K_1 \times K_2}(c_2^k) \mid V_{K'}}^{K_1 \times K_2}]\] is equal to \(\text{L}^2(K_{< R'})\). Taking the limit \(R' \to \infty\) gives

\[
\left[ \text{L}^2(K_{< R'}) \right]^{K_1} = \lim_{R' \to \infty} \left[ \text{Index}_{V_{K'}^{K_1 \times K_2}(c_2^k) \mid V_{K'}}^{K_1 \times K_2} \right]^{K_1} = \lim_{R' \to \infty} \text{L}^2(K_{< R'}) = \text{L}^2(K).
\]

**3B. Proof of Proposition 3.3.** We work here with the Hamiltonian action of the product \(K_1 \times K_2\) on \(M \times \mathcal{B}_p\). The action is \((k_1, k_2) \cdot (m, x) = (k_1 \cdot m, k_2 \cdot x \cdot k_1^{-1})\) and the corresponding moment map is \(\Phi = (\Phi_1, \Phi_2)\) with \(\Phi_1(m, x) = \Phi_M(m) + \Phi_r(x)\) and \(\Phi_2(m, x) = \Phi_t(x)\). Let \(\|\xi_1, \xi_2\|^2 = \|\xi_1\|^2 + \|\xi_2\|^2\) be the Euclidean norm \(t_1^* \times t_2^*\) attached to \(\rho > 0\).

Consider the quantization of \(M \times \mathcal{B}_p\) via the map \(\Phi\):

\[
\text{L}^2(K_1 \times K_2) \subset \mathcal{B}_p \subset R^{-\infty}(K_1 \times K_2).
\]

The critical set \(\text{Cr}(\|\Phi\|^2_{\rho})\) admits the decomposition

\[
\text{Cr}(\|\Phi\|^2_{\rho}) = \bigcup_{\gamma \in \mathcal{B}_p} K_1 \times K_2 \cdot e_{\gamma},
\]

where \((m, x) \in e_{\gamma}\) if and only if \(\gamma = (\gamma_1, \gamma_2)\) with

\[
\begin{cases}
\Phi_M(m) + \Phi_r(x) = \gamma_1, \\
\Phi_t(x) = \gamma_2, \\
\gamma_1 \cdot m = 0, \\
\gamma_1 \cdot r + \rho \gamma_2 \cdot l = 0.
\end{cases}
\]

Here \(\mathcal{B}_p \subset t_1^* \times t_2^*\) is defined as the set of elements \(\gamma = (\gamma_1, \gamma_2) \in t_1^* \times t_2^*\) where the equations (44) have solutions in \(M \times \mathcal{B}_p\).
We have

\[(45) \quad \mathcal{Q}^{\Phi,\rho}_{K_1 \times K_2}(M \times \mathcal{X}_P) = \sum_{\gamma \in \mathcal{B}_\rho} \mathcal{Q}^{\gamma,\rho}_{K_1 \times K_2}(M \times \mathcal{X}_P),\]

where the generalized character \(\mathcal{Q}^{\gamma,\rho}_{K_1 \times K_2}(M \times \mathcal{X}_P)\) is computed as an index of a transversally elliptic symbol in a neighborhood of

\[K_1 \times K_2 \cdot \mathcal{C}_\gamma \subset M \times \Phi_i^{-1}(K_2 \cdot \gamma_2)\]

By Theorem 2.9, the support of the generalized character \(\mathcal{Q}^{\gamma,\rho}_{K_1 \times K_2}(M \times \mathcal{X}_P)\) is contained in \(\{(a, b) \in \overline{K}_1 \times \overline{K}_2 \mid a \|^2 + \rho \|^2 \geq \| \gamma \|^2\}\). Hence

\[\text{support}(\mathcal{Q}^{\gamma,\rho}_{K_1 \times K_2}(M \times \mathcal{X}_P)|_{K_1}) \subset \{b \in \overline{K}_2 \mid \rho \|^2 \geq \| \gamma \|^2\}\].

Let \(r_P = \inf_{\xi \in \partial P} \| \xi \|\). We know then that

\[\mathcal{Q}^{\Phi,\rho}_{K_1 \times K_2}(M \times \mathcal{X}_P)|_{K_1} = \sum_{\gamma \in \mathcal{B}_\rho} \mathcal{Q}^{\gamma,\rho}_{K_1 \times K_2}(M \times \mathcal{X}_P)|_{K_1} + O(r_P).\]

Let \(R_P < \rho r_P^2\) be a regular value of \(\| \Phi \|^2 : M \times \mathcal{X}_P \to \mathbb{R}\) such that for all \(\gamma \in \mathcal{B}_\rho\), we have \(\| \gamma \|^2 < \rho r_P^2\) if and only if \(\| \gamma \|^2 < R_P\). Then

\[(46) \quad \mathcal{Q}^{\Phi,\rho}_{K_1 \times K_2}(M \times \mathcal{X}_P)|_{K_1} = \mathcal{Q}^{\Phi,\rho}_{K_1 \times K_2}((M \times \mathcal{X}_P)_{<R_P})|_{K_1} + O(r_P).\]

For the generalized index \(\mathcal{Q}^{\Phi,\rho}_{K_1 \times K_2}(M \times T^*K)\) we have also a decomposition

\[\mathcal{Q}^{\Phi,\rho}_{K_1 \times K_2}(M \times T^*K) = \sum_{\gamma \in \mathcal{B}_\rho'} \mathcal{Q}^{\gamma,\rho}_{K_1 \times K_2}(M \times T^*K),\]

where \(\mathcal{B}_\rho'\) parametrizes the critical set of \(\| \Phi \|^2 : M \times T^*K \to \mathbb{R}\). As before,

\[(47) \quad \mathcal{Q}^{\Phi,\rho}_{K_1 \times K_2}(M \times T^*K)|_{K_1} = \mathcal{Q}^{\Phi,\rho}_{K_1 \times K_2}((M \times T^*K)_{<R'_P})|_{K_1} + O(r_P).\]

Here \(R'_P < \rho r_P^2\) is a regular value of \(\| \Phi \|^2 : M \times T^*K \to \mathbb{R}\) such that for all \(\gamma \in \mathcal{B}_\rho'\), we have \(\| \gamma \|^2 < \rho r_P^2\) if and only if \(\| \gamma \|^2 < R'_P\).

**Lemma 3.7.** We have

\[(48) \quad \mathcal{Q}^{\Phi,\rho}_{K_1 \times K_2}((M \times \mathcal{X}_P)_{<R_P}) = \mathcal{Q}^{\Phi,\rho}_{K_1 \times K_2}((M \times T^*K)_{<R'_P}).\]

**Proof.** The lemma follows from Proposition 2.6. We take here \(V' = M \times \mathcal{X}_P\), \(V = M \times K \times \mathfrak{u}_P \subset M \times T^*K\) and the equivariant diffeomorphism \(\Psi : V \to V'\) equal to \(\text{Id} \times \Upsilon\) where \(\Upsilon\) was introduced in Section 2E. The map \(\Psi\) satisfies points (1)–(3) of Proposition 2.6.

The inequality \(\| \Phi(m, x) \|^2 < \rho r_P^2\) implies that \(\| \Phi_i(x) \| < r_P\) and then \(x \in \mathcal{X}_P\). Hence the open subset \(U' := (M \times \mathcal{X}_P)_{<R_P}\) is contained in \(V' = M \times \mathcal{X}_P\). In
the same way the open subset $U := (M \times T^* K)_{< R_p}$ is contained in $V$. We have $\Psi(U) = U'$ if $R_p = R_p'$.

We have proved that (48) is a consequence of Proposition 2.6. $\square$

Finally, taking the difference between (46) and (47) gives

$$\left[ \mathcal{D}_{K_1 \times K_2}(M \times \mathcal{X}_p) \right]^{K_1} \mid K_1 - \left[ \mathcal{D}_{K_1 \times K_2}(M \times T^* K) \right]^{K_1} = O(r_p),$$

which is the relation of Proposition 3.3.

**3C. Proof of Proposition 3.4.** Here we want to compare the $K_1$-invariant part of the characters $\mathcal{D}_{K_1 \times K_2}(M \times \mathcal{X}_p)$ and $\mathcal{D}_{K_1 \times K_2}(M \times \mathcal{X}_p)$.

By Theorem 2.15,

$$\mathcal{D}_{K_2}(M_p) = \left[ \mathcal{D}_{K_1 \times K_2}(M \times \mathcal{X}_p) \right]^{K_1} = \left[ \mathcal{D}_{K_1 \times K_2}(U_\epsilon) \right]^{K_1}$$

when $\epsilon > 0$ is any regular value of $\|\Phi_1\|^2$, and $U_\epsilon := \{\|\Phi_1\|^2 < \epsilon\} \subset M \times \mathcal{X}_p$.

In this section we fix once and for all $\epsilon > 0$ small enough so that

$$(49) \quad \text{Cr}(\|\Phi_1\|^2) \cap \{\|\Phi_1\|^2 \leq \epsilon\} = \{\Phi_1 = 0\}.$$

Let $c_1$ be the symbol $\text{Thom}(M, J_1) \otimes L$ attached to the prequantized Hamiltonian $K_1$-manifold $(M, \Omega)$. Let $c_3$ be the symbol $\text{Thom}(\mathcal{X}_p, J_3) \otimes L_p$ attached to the prequantized Hamiltonian $(K_1 \times K_2)$-manifold $\mathcal{X}_p$. The product $c = c_1 \otimes c_3$ corresponds to the symbol $\text{Thom}(N, J) \otimes L$ on $N = M \times \mathcal{X}_p$.

Let $\kappa_0$ and $\kappa_\rho$ be the Kirwan vector fields associated to the functions $\|\Phi_1\|^2$ and $\|\Phi\|^2_\rho$ on $M \times \mathcal{X}_p$:

$$\kappa_0(m, x) = \left( \widetilde{\Phi}_1(m, x) \cdot \kappa_1 \left( \Phi_1(m, x) \cdot \kappa_3 \right) \right),$$

$$\kappa_\rho(m, x) = \kappa_0(m, x) + \rho \left( \widetilde{\Phi}_1(m, x) \cdot \kappa_3 \right).$$

Let $c^{\kappa_\rho}$ be the symbol $c$ pushed by the vector field $\kappa_\rho$. Then

$$c^{\kappa_\rho}(v; \eta) = c_1(v - \kappa_1) \odot c_3(\eta - \kappa_3 - \rho \kappa_3)$$

for $(v; \eta) \in T_{(m, x)}(M \times \mathcal{X}_p)$.

The character $\mathcal{D}_{K_1 \times K_2}(U_\epsilon)$ is given by the index of the $K_1$-transversally elliptic symbol $c^{\kappa_0}|_{U_\epsilon}$. The character $\mathcal{D}_{K_1 \times K_2}(M \times \mathcal{X}_p)$ is given by the index of the $(K_1 \times K_2)$-transversally elliptic symbol $c^{\kappa_\rho}$.

**Lemma 3.8.** There exists $\rho(\epsilon) > 0$ such that

$$\text{Cr}(\|\Phi\|^2_\rho) \cap \{\|\Phi_1\|^2 \leq \epsilon\} \subset \{\|\Phi_1\|^2 \leq \frac{\epsilon}{2}\}$$

for any $0 \leq \rho \leq \rho(\epsilon)$.  

Proof. With the help of Riemannian metrics on $M$ and $\mathcal{X}_p$, define

$$a(\epsilon) := \inf_{\epsilon/2 \leq \|\Phi_1(m, x)\| \leq \epsilon} \|\kappa_0(m, x)\|,$$

$$b := \sup_{x \in \mathcal{X}_p} \|\Phi_1(x) \cdot x\|.$$

We have $a(\epsilon) > 0$ thanks to (49), and $b < \infty$ since $\mathcal{X}_p$ is compact. It is now easy to check that $\{\kappa_0 = 0\} \cap \{\epsilon/2 \leq \|\Phi_1\| \leq \epsilon\} = \emptyset$ if $0 \leq \rho < a(\epsilon)/b$.

The symbols $\mathcal{E}_\rho|_{U_\epsilon}$, $\rho \in [0, \rho(\epsilon)]$, are $(K_1 \times K_2)$-transversally elliptic, and they define the same class in $K_{K_1 \times K_2}(T_{K_1 \times K_2}U_\epsilon)$. Hence $\mathcal{G}_{K_2}(M \times \mathcal{X}_p)$ can be computed as the $K_1$-invariant part of

$$\mathcal{G}_{K_1 \times K_2}(U_\epsilon) := \text{Index}_{U_\epsilon}^{K_1 \times K_2}(\mathcal{E}_\rho|_{U_\epsilon}) \in R^{-\infty}(K_1 \times K_2)$$

for any $\rho \in [0, \rho(\epsilon)]$.

Let $\rho \in [0, \rho(\epsilon)]$. A component $K_1 \times K_2 \cdot \mathcal{E}_\rho$ of $\text{Cr}(\|\Phi\|_\rho)$ is contained in $U_\epsilon$ if and only if $\|\gamma_1\|^2 < \epsilon$, so the decomposition (45) for the character $\mathcal{G}_{K_1 \times K_2}(M \times \mathcal{X}_p)$ gives

$$\mathcal{G}_{K_1 \times K_2}(M \times \mathcal{X}_p) = \mathcal{G}_{K_1 \times K_2}(U_\epsilon) + \sum_{\gamma \in \mathcal{B}_\rho \|\gamma\|^2 \geq \epsilon} \mathcal{G}_{K_1 \times K_2}(M \times \mathcal{X}_p),$$

where

$$\mathcal{G}_{K_1 \times K_2}(U_\epsilon) = \sum_{\gamma \in \mathcal{B}_\rho \|\gamma\|^2 < \epsilon} \mathcal{G}_{K_1 \times K_2}(M \times \mathcal{X}_p).$$

Taking the $K_1$-invariant gives

$$[\mathcal{G}_{K_1 \times K_2}(M \times \mathcal{X}_p)]^{K_1} = \mathcal{G}_{K_2}(M \times \mathcal{X}_p) + \sum_{\gamma \in \mathcal{B}_\rho \|\gamma\|^2 \geq \epsilon} [\mathcal{G}_{K_1 \times K_2}(M \times \mathcal{X}_p)]^{K_1}.$$  

By Theorem 2.9 the support of the generalized character $[\mathcal{G}_{K_1 \times K_2}(M \times \mathcal{X}_p)]^{K_1}$ is included in $\{b \in \mathcal{K}_2 \mid \rho \|b\|^2 \geq \|\gamma_1, \gamma_2\|_{\rho}^2\}$. When $\|\gamma_1\|^2 \geq \epsilon$ we have then that the support of $[\mathcal{G}_{K_1 \times K_2}(M \times \mathcal{X}_p)]^{K_1}$ is contained outside the ball $B_{(\epsilon/\rho)^{1/2}}$.

Finally (50) imposes that

$$[\mathcal{G}_{K_1 \times K_2}(M \times \mathcal{X}_p)]^{K_1} = \mathcal{G}_{K_2}(M \times \mathcal{X}_p) + O((\epsilon/\rho)^{1/2})$$

when $0 < \rho \leq \rho(\epsilon)$, which is the precise content of Proposition 3.4.
4. Other properties of $\mathcal{Q}_\Phi$

Let $(M, \omega, \Phi)$ be a proper Hamiltonian $K$-manifold that is prequantized by a line bundle $L$. The character $\mathcal{Q}_K(M)$ is computed by means of a scalar product on $\mathfrak{t}^\ast$. The fact that $\mathcal{Q}_K(M) = \mathcal{Q}_K^{-\infty}(M)$ gives the following:

**Proposition 4.1.** The character $\mathcal{Q}_K(M)$ does not depend of the choice of an invariant scalar product on $\mathfrak{t}^\ast$.

In this section we work in the setting where $K = K_1 \times K_2$. Let $\Phi_1$ be the moment map relative to the $K_1$-action.

**4A. $\Phi_1$ is proper.** In this subsection, suppose that the moment map $\Phi_1$ relative to the $K_1$-action is proper. Fix an invariant Euclidean norm $\| \cdot \|^2$ on $\mathfrak{k}$ in such a way that $\mathfrak{k}_1 = \mathfrak{k}_2^\perp$.

To “quantize” $(M, \Omega)$ via the invariant proper function $\| \Phi_1 \|^2$, let

$$\mathcal{Q}_{\Phi_1,\rho}^0(M) = R^{-\infty}(K_1 \times K_2)$$

be the corresponding generalized character.

**Theorem 4.2.** We have

$$\mathcal{Q}_{K_1 \times K_2}(M) = \mathcal{Q}_{\Phi_1,\rho}^0(M) \quad \text{in } R^{-\infty}(K_1 \times K_2).$$

**Proof.** On $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ we may consider the family of invariant Euclidean norms: $\| X_1 \oplus X_2 \|^2 = \| X_1 \|^2 + \rho \| X_2 \|^2$ for $X_j \in \mathfrak{k}_j$. Let

$$\mathcal{Q}_{\Phi,\rho}^0(M) = R^{-\infty}(K_1 \times K_2)$$

be the quantization of $M$ computed via the map $\| \Phi \|^2 = \| \Phi_1 \|^2 + \rho \| \Phi_2 \|^2$. By definition, $\mathcal{Q}_{K_1 \times K_2}(M)$ is equal to $\mathcal{Q}_{\Phi_1,\rho}^0(M)$, and Proposition 4.1 implies that $\mathcal{Q}_{K_1 \times K_2}(M)$ coincides with the generalized character $\mathcal{Q}_{\Phi,\rho}^0(M) \in R^{-\infty}(K)$ for any $\rho > 0$.

Denote by $O(r) \in R^{-\infty}(K_1 \times K_2)$ any generalized character supported outside the ball $\{ \xi \in \mathfrak{k}_1 \times \mathfrak{k}_2 \mid \| \xi_1 \|^2 + \| \xi_2 \|^2 < r^2 \}$. Also, denote by $O_1(r) \in R^{-\infty}(K_1 \times K_2)$ any generalized character supported outside the $\{ \xi \in \mathfrak{k}_1 \times \mathfrak{k}_2 \mid \| \xi_1 \| < r \}$.

Let $R_1 > 0$ be a regular value of $\| \Phi_1 \|^2$, and let $M_{\leq R_1}$ be the open subset $\{ \| \Phi_1 \|^2 < R_1 \}$, which is relatively compact. Theorem 2.13 tells us that

$$\mathcal{Q}_{K_1 \times K_2}(M) = \mathcal{Q}_{K_1 \times K_2}(M_{\leq R_1}) + O_1(\sqrt{R_1}).$$

As in Lemma 3.8, there exists $\rho(R_1) \in ]0, 1[$ small enough such that

$$\text{Cr}(\| \Phi \|^2) \cap \{ \| \Phi_1 \|^2 = R_1 \} = \emptyset \quad \text{for } \rho \in [0, \rho(R_1)].$$

\[7\text{In this section the Lie groups } K_1 \text{ and } K_2 \text{ are not identical.} \]
Let $\rho \in [0, \rho(R_1)]$. The identity (52) first implies that
\[
\mathcal{D}^{\Phi,\rho}_{K_1 \times K_2}(M) = \sum_{\gamma \in \mathbb{H}_\rho} \mathcal{D}^{\gamma,\rho}_{K_1 \times K_2}(M) + \sum_{\gamma \in \mathbb{H}_\rho, \|\gamma\|^2 > R_1} \mathcal{D}^{\gamma,\rho}_{K_1 \times K_2}(M) = \mathcal{D}^{\Phi,\rho}_{K_1 \times K_2}(M_{< R_1}) + O(\sqrt{R_1}),
\]
where the second equality uses that $\mathcal{D}^{\gamma,\rho}_{K_1 \times K_2}(M) = O(\sqrt{R_1})$ when $\|\gamma\|^2 > R_1$, since the ball $\{(\xi_1, \xi_2) \in t_1^* \times t_2^* \mid \|\xi_1\|^2 + \|\xi_2\|^2 < R_1\}$ is contained in
\[
\{(\xi_1, \xi_2) \in t_1^* \times t_2^* \mid (\xi_1, \xi_2)\|_\rho < (\gamma_1, \gamma_2)\|_\rho^2\}.\]
The identity (52) shows also that the symbols $c^{\kappa,\rho}|_{M_{< R_1}}$ are homotopic for $\rho \in [0, \rho(R_1)]$. Hence
\[
\mathcal{D}^{\Phi,\rho}_{K_1 \times K_2}(M_{< R_1}) = \mathcal{D}^{\Phi_1}_{K_1 \times K_2}(M_{< R_1})
\]
if $\rho > 0$ is small enough. Finally,
\[
\mathcal{D}^{\Phi,\rho}_{K_1 \times K_2}(M) = \mathcal{D}^{\Phi_1}_{K_1 \times K_2}(M) = O(\sqrt{R_1}) + O(\sqrt{R_1})
\]
for any regular value $R_1$ of $\|\Phi_1\|^2$, when $\rho \in [0, \rho(R_1)]$. Since the generalized character $\mathcal{D}^{\Phi,\rho}_{K_1 \times K_2}(M)$ does not depend of $\rho > 0$ (see Proposition 4.1),
\[
\mathcal{D}^{\Phi}_{K_1 \times K_2}(M) = \mathcal{D}^{\Phi,\rho}_{K_1 \times K_2}(M) = \mathcal{D}^{\Phi_1}_{K_1 \times K_2}(M). \quad \square
\]

We explain how Theorem 4.2 contains the identity that we called “quantization commutes with reduction in the singular setting” in [Paradan 2009]. By definition the $K_1$-invariant part of the right hand side of (51) is equal to the geometric quantization of the (possibly singular) compact Hamiltonian $K_2$-manifold
\[
M_0 := \Phi^{-1}(0)/K_1.
\]
Using now the fact that the left hand side of (51) is equal to $\mathcal{D}^{-\infty}_{K_1 \times K_2}(M)$, we see that the multiplicity of $\nu_{\mu}^{K_2}$ in $\mathcal{D}_{K_2}(M_0/K_1)$ is equal to the geometric quantization of the (possibly singular) compact manifold
\[
M = \frac{K_2 \times (0,0)K_1 \times K_2}{\mu}. \quad \square
\]

4B. The symplectic reduction $M_0/K_1$ is smooth. Let $(M_0, \Omega)$ be an Hamiltonian $(K_1 \times K_2)$-manifold with proper moment map $\Phi = (\Phi_1, \Phi_2)$. In this section, suppose that $0$ is a regular value of $\Phi_1$ and that $K_1$ acts freely on $\Phi^{-1}(0)$. We work then with the smooth Hamiltonian $K_2$-manifold
\[
N := \Phi_1^{-1}(0)/K_1.
\]
Continue to denote by $\Phi_2 : N \to \mathfrak{k}_2^*$ the moment map relative to the $K_2$-action; note that this map is proper. Hence we can quantize the $K_2$-action on $N$ via the map $\Phi_2$. Let $\mathfrak{g}_{K_2}^e(N) \in R^{-\infty}(K_2)$ be the corresponding character.

**Proposition 4.3.** We have

\begin{equation}
[\mathfrak{g}_{K_1 \times K_2}^e(M)]^{K_1} = \mathfrak{g}_{K_2}^e(N) \quad \text{in} \quad R^{-\infty}(K_2).
\end{equation}

**Proof.** When $\Phi_1$ is proper, the manifold $N$ is compact. Then the right hand side of (53) is equal to $\mathfrak{g}_{\mathfrak{g}_1}^e(N)$, and we know from Theorem 4.2 that the left hand side of (53) is equal to $[\mathfrak{g}_{K_1 \times K_2}^e(M)]^{K_1}$. In this case (53) becomes $[\mathfrak{g}_{K_1 \times K_2}^e(M)]^{K_1} = \mathfrak{g}_{K_2}^e(M//_{0}K_1)$ which is the content of Theorem 2.14.

Consider the general case where $\Phi_1$ is not proper. By Theorem 1.4, the multiplicities of $V_{\mu}^{K_2}$ in $\mathfrak{g}_{K_1 \times K_2}^e(M)$ and in $\mathfrak{g}_{K_2}^e(N)$ are respectively equal to the quantization of the (possibly singular) symplectic reductions

$$M_{\mu} := M \times _{K_2} \cdot \mu//_{(0,0)}K_1 \times K_2,$$

$$M_{\mu}' := N \times _{K_2} \cdot \mu//_{0}K_2 \quad \text{with} \quad N = M//_{0}K_1.$$

Note that $M_{\mu}$ and $M_{\mu}'$ coincide as symplectic reduced space. Their geometric quantizations are identical also. The proof will be done for $\mu = 0$: the other cases follow from the shifting trick.

Let $c$ be the $(K_1 \times K_2)$-equivariant symbol Thom$(M, J) \otimes L_M$. Let $\kappa$ be the Kirwan vector field attached to the moment map $\Phi = (\Phi_1, \Phi_2)$. Let $c^{K_1 \times K_2}$ be the symbol $c$ pushed by $\kappa$. Denote by $M_{< \epsilon}$ the open subset $\{||\Phi||^2 < \epsilon\}$. For $\epsilon > 0$ small enough, the symbol $c^{K_1 \times K_2}_{M_{< \epsilon}}$ is $(K_1 \times K_2)$-transversally elliptic, and $\mathfrak{g}_{\kappa}(M_{0})$ is the $(K_1 \times K_2)$-invariant part of $\text{Index}_{M_{< \epsilon}}(c^{K_1 \times K_2}_{M_{< \epsilon}})$.

Let $c_2$ be the $K_2$-equivariant symbol Thom$(N, J) \otimes L_N$. Let $\kappa_2$ be the Kirwan vector field attached to the moment map $\Phi_2$. Let $c^{K_2}_{N_{< \epsilon}}$ be the symbol $c_2$ pushed by $\kappa_2$. Denote by $N_{< \epsilon}$ the open subset $\{||\Phi_2||^2 < \epsilon\}$. For $\epsilon > 0$ small enough, the symbol $c^{K_2}_{N_{< \epsilon}}_{M_{< \epsilon}}$ is $K_2$-transversally elliptic, and $\mathfrak{g}_{\kappa_2}(M_{0}')$ is the $K_2$-invariant part of $\text{Index}_{N_{< \epsilon}}(c^{K_2}_{N_{< \epsilon}})$.

Our proof follows from the comparison of the classes

$$[c^{K_1 \times K_2}_{M_{< \epsilon}}] \in \mathbb{K}^0_{K_1 \times K_2}(T_{K_1 \times K_2}M_{< \epsilon}),$$

$$[c^{K_2}_{N_{< \epsilon}}] \in \mathbb{K}^0_{K_2}(T_{K_2}N_{< \epsilon}).$$

A neighborhood of the smooth submanifold $Z := \Phi_1^{-1}(0)$ in $M$ is diffeomorphic to a neighborhood of the 0-section of the bundle $Z \times \mathfrak{k}_1^* \to Z$. Let $Z_{< \epsilon} = Z \cap M_{< \epsilon}$ so that $N_{< \epsilon} = Z_{< \epsilon} / K_1$. Hence $[c^{K_1 \times K_2}_{M_{< \epsilon}}]$ can be seen naturally a class in the $K$-group $\mathbb{K}^0_{K_1 \times K_2}(T_{K_1 \times K_2}(Z_{< \epsilon} \times \mathfrak{k}_1^*)).$
Following [Atiyah 1974, Theorem 4.3], the inclusion map \( j : Z_{<\varepsilon} \hookrightarrow Z_{<\varepsilon} \times \mathfrak{k}_1^* \) induces the Thom isomorphism

\[ j! : K^0_{K_1 \times K_2} (T_{K_1 \times K_2} Z_{<\varepsilon}) \to K^0_{K_1 \times K_2} (T_{K_1 \times K_2} (Z_{<\varepsilon} \times \mathfrak{k}_1^*)) , \]

with the commutative diagram

\[
\begin{array}{ccc}
K^0_{K_1 \times K_2} (T_{K_1 \times K_2} Z_{<\varepsilon}) & \xrightarrow{j!} & K^0_{K_1 \times K_2} (T_{K_1 \times K_2} (Z_{<\varepsilon} \times \mathfrak{k}_1^*)) \\
\text{Index}_{Z_{<\varepsilon}}^{K_1 \times K_2} & \downarrow & \text{Index}_{Z_{<\varepsilon} \times \mathfrak{k}_1^*}^{K_1 \times K_2} \\
& \to & R^{-\infty} (K_1 \times K_2) .
\end{array}
\]

\[ (54) \]

Let \( \pi_1 : Z_{<\varepsilon} \to N_{<\varepsilon} \) be the quotient relative to the free action of \( K_1 \). The corresponding isomorphism

\[ \pi_1^* : K^0_{K_2} (T_{K_2} N_{<\varepsilon}) \to K^0_{K_1 \times K_2} (T_{K_1 \times K_2} Z_{<\varepsilon}) \]

satisfies the rule

\[ \text{Index}_{Z_{<\varepsilon}}^{K_1 \times K_2} (\pi_1^* \theta)^{K_1} = \text{Index}_{N_{<\varepsilon}}^{K_2} (\theta) \]

for any \( \theta \in K^0_{K_2} (T_{K_2} N_{<\varepsilon}) \).

**Lemma 4.4** [Paradan 2001]. We have

\[ j! \circ \pi_1^* ([c^K_{2 \mid N_{<\varepsilon}}]) = [c^K_{M_{<\varepsilon}}] \]

in \( K^0_{K_1 \times K_2} (T_{K_1 \times K_2} (Z_{<\varepsilon} \times \mathfrak{k}_1^*)) \).

**Proof.** This lemma is proven in [Paradan 2001, Section 6.2] when the group \( K_2 \) is trivial. It is easy to check that the proof extends naturally to our setting. \( \square \)

Using Lemma 4.4 together with (54) and (55), we get that

\[ \mathcal{Q}(\mathcal{M}_0) = \left[ \text{Index}_{Z_{<\varepsilon} \times \mathfrak{k}_1^*}^{K_1 \times K_2} (c^K_{M_{<\varepsilon}}) \right]^{K_1 \times K_2} = \left[ \text{Index}_{N_{<\varepsilon}}^{K_2} (c^K_{2 \mid N_{<\varepsilon}}) \right]^{K_2} = \mathcal{Q}(\mathcal{M}_0'). \]

\[ \square \]

5. **Example: The cotangent bundle of an orbit**

**5A. The formal quantization of \( T^* K \).** Let \( K \) be a compact connected Lie group equipped with the action of two copies of \( K \) given by \( (k_1, k_2) \cdot a = k_2 a k_1^{-1} \). Then we have a Hamiltonian action of \( K_1 \times K_2 \) on the cotangent bundle \( T^* K \). In this section, we check that each formal geometric quantization of \( T^* K \), \( \mathcal{Q}^{-\infty}_{K_1 \times K_2} (T^* K) \) and \( \mathcal{Q}_{K_1 \times K_2} (T^* K) \) are both equal to the \( (K_1 \times K_2) \)-module \( L^2 (K) \).

The tangent bundle \( TK \) is identified with \( K \times \mathfrak{k} \) through the right translations: to \( (a, X) \in K \times \mathfrak{k} \), associate \( \frac{d}{dt} a e^{tX} \mid_0 \). The action of \( K_1 \times K_2 \) on the cotangent bundle \( T^* K \simeq K \times \mathfrak{k}^* \) is then

\[ (k_1, k_2) \cdot (a, \xi) = (k_2 a k_1^{-1}, k_1 \cdot \xi) . \]
The symplectic form on $T^*K$ is $\Omega := -d\lambda$, where $\lambda$ is the Liouville 1-form. We compute these two forms in coordinates. The tangent bundle of $T^*K \cong T \times \mathfrak{t}^*$ is identified with $T^*K \times \mathfrak{t} \times \mathfrak{t}^*$: for each $(a, \xi) \in T^*K$, we have a two-form $\Omega_{(a, \xi)}$ on $\mathfrak{t} \times \mathfrak{t}^*$. A direct computation gives

$$\Omega_{(a, \xi)}(X, X') = \langle \xi, [X, X'] \rangle, \quad \Omega_{(a, \xi)}(\eta, \eta') = 0, \quad \Omega_{(a, \xi)}(X, \eta) = \langle \eta, X \rangle$$

for $X, X' \in \mathfrak{t}$ and $\eta, \eta' \in \mathfrak{t}^*$. So $\Omega_{(a, \xi)} = \Omega_0 + \pi_\xi$ where $\Omega_0$ is the canonical (constant) symplectic form on $\mathfrak{t} \times \mathfrak{t}^*$ and $\pi_\xi$ is the closed two-form on $\mathfrak{t}$ defined by $\pi_\xi(X, Y) = \langle \xi, [X, Y] \rangle$.

If we identify $\mathfrak{t} \cong \mathfrak{t}^*$ through an invariant Euclidean norm, the symplectic structure on $T_{(a, \xi)}(T^*K) \cong \mathfrak{t} \times \mathfrak{t}^*$ is given by a skew-symmetric matrix

$$A_\xi := \begin{pmatrix} \text{ad}(\xi) & -I_n \\ I_n & 0 \end{pmatrix},$$

so that

$$\Omega_{(a, \xi)}((X, \eta), (X', \eta')) = (A_\xi(X, \eta), (X', \eta')) = \langle \xi, [X, X'] \rangle + (X, \eta') - (X', \eta).$$

We will work with the following compatible almost complex structure on the tangent bundle of $T^*K$: $J_\xi = A_\xi(-A_\xi^2)^{-1/2}$. When $\xi = 0$, the complex structure $J_0$ on $\mathfrak{t} \times \mathfrak{t}^*$ is defined by the matrix

$$J_0 := \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

Hence the complex $K$-module $(\mathfrak{t} \times \mathfrak{t}^*, J_0)$ is naturally identified with the complexification $\mathfrak{t}_C$ of $\mathfrak{t}$.

It is easy to check that the moment map relative to the $(K_1 \times K_2)$-action is the proper map $\Phi : T^*K \to \mathfrak{t}^*_1 \times \mathfrak{t}^*_2$ defined by $\Phi(a, \xi) = (-\xi, a \cdot \xi)$.

Here the symplectic manifold $T^*K$ is prequantized by the trivial line bundle.

**5A1. Computation of $\mathfrak{g}^{-\infty}_{K_1 \times K_2}(T^*K)$.** Let $\mathcal{O}_1 \times \mathcal{O}_2$ be a coadjoint orbit of $K_1 \times K_2$ in $\mathfrak{t}^*_1 \times \mathfrak{t}^*_2$. One checks that

$$\Phi^{-1}(\mathcal{O}_1 \times \mathcal{O}_2) = \begin{cases} \emptyset & \text{if } \mathcal{O}_1 \neq -\mathcal{O}_2, \\ \text{a } (K_1 \times K_2)\text{-orbit} & \text{if } \mathcal{O}_1 = -\mathcal{O}_2. \end{cases}$$

We know that the stabilizer subgroup $K_\xi$ of an element $\xi \in \mathfrak{t}^*$ is connected. Then the stabilizer subgroup $(K_1 \times K_2)_{(a, \xi)} = \{(k_1, ak_1a^{-1}), \ k_1 \in K_\xi \}$ is also connected.

Let $(T^*K)_{(\mu, \lambda)}$ be the symplectic reduction of $T^*K$ at the level $(\mu, \lambda) \in \widehat{K}^2$. For any $\mu \in \widehat{K}$, define $\mu^* \in \widehat{K}$ by the relation $-K \cdot \mu = K \cdot \mu^*$; note that $V^K_{\mu^*} \cong (V^K_{\mu})^*$. Using Theorem 2.17 gives

$$\mathfrak{g}((T^*K)_{(\mu, \lambda)}) = \begin{cases} 0 & \text{if } \lambda \neq \mu^*, \\ 1 & \text{if } \lambda = \mu^*. \end{cases}$$
Finally
\[ \partial_{K_1 \times K_2}^{-\infty} (T^* K) = \sum_{(\mu, \lambda) \in \tilde{K} \times \tilde{K}} \partial (T^* K)_{(\mu, \lambda)} V_{\mu}^{K_1} \otimes V_{\lambda}^{K_2} \]
\[ = \sum_{\mu \in \tilde{K}} V_{\mu}^{K_1} \otimes (V_{\mu}^{K_2})^* = L^2 (K). \]

**5A1. Computation of** \( \partial_{K_1 \times K_2}^{\Phi} (T^* K) \). The Kirwan vector field on \( T^* K \) is
\[ \kappa(a, \xi) = -2\xi \in \mathfrak{k}. \]

Let \( c^\kappa \) be the symbol \( \text{Thom}(T^* K, J) \) pushed by the vector field \( \frac{1}{2} \kappa \). At each \((a, \xi)\) in \( T^* K \), the map \( c_{(a, \xi)}^\kappa (X \oplus \eta) \) from \( \wedge^\text{even}_J (\mathfrak{k} \times \mathfrak{k}^*) \) to \( \wedge^\text{odd}_J (\mathfrak{k} \times \mathfrak{k}^*) \) is equal to the Clifford map \( c(X + \xi \oplus \eta) \). Note that \( c^\kappa \) is a \( K_2 \)-transversally elliptic symbol on \( T^* K \): we have \( \text{Char}(c^\kappa) \cap T_{K_2} (T^* K) = \{(1, 0)\} \). We will now compute the equivariant index of \( c^\kappa \).

First consider the homotopy \( t \in [0, 1] \to J_{t\xi} \) of symplectic structure on \( T^* K \). Let \( \tilde{c}^\kappa \) be the symbol acting on \( \wedge^*_{J_0} (\mathfrak{k} \times \mathfrak{k}^*) = \wedge^*_{\xi} \mathfrak{k} \). Proposition 2.6 shows that the symbols \( c^\kappa \) and \( \tilde{c}^\kappa \) define the same class in \( K^0_{K_1 \times K_2} (T_{K_2} (T^* K)) \).

The projection \( \pi : T^* K \to \mathfrak{k}^* \) corresponds to the quotient map relative to the free action of \( K_2 \). At the level of \( K^0 \)-groups we get an isomorphism
\[ \pi_* : K^0_{K_1 \times K_2} (T_{K_2} (T^* K)) \to K^0_{K_1} (T \mathfrak{k}^*). \]

The free action property (see [Atiyah 1974, Theorem 3.1]) gives that
\[ \text{Index}^K_{T^* K} (\sigma) = \sum_{\mu \in \tilde{K}} \text{Index}^K_{\mathfrak{k}^*} (\pi_* (\sigma \otimes V_{\mu}^{K_2})) \otimes (V_{\mu}^{K_2})^* \]
for any class \( \sigma \in K^0_{K_1 \times K_2} (T_{K_2} (T^* K)) \). In our case the symbol \( \pi_* (\tilde{c}^\kappa) \) is equal to the Bott symbol \( \text{Bott}(\mathfrak{k}^*) \), and for any \( K_2 \)-module \( E_2 \) we have
\[ \pi_* (\tilde{c}^\kappa \otimes E_2) = \text{Bott}(\mathfrak{k}^*) \otimes E_1, \]
where \( E_1 \) is the module \( E_2 \) with the action of \( K_1 \). Then
\[ \partial_{K_1 \times K_2}^{\Phi} (T^* K) = \text{Index}^K_{T^* K} (\tilde{c}^\kappa) \]
\[ = \sum_{\mu \in \tilde{K}} \text{Index}^K_{\mathfrak{k}^*} (\text{Bott}(\mathfrak{k}^*) \otimes V_{\mu}^{K_1}) \otimes (V_{\mu}^{K_2})^* \]
\[ = \sum_{\mu \in \tilde{K}} V_{\mu}^{K_1} \otimes (V_{\mu}^{K_2})^* = L^2 (K), \]
since \( \text{Index}^K_{\mathfrak{k}^*} (\text{Bott}(\mathfrak{k}^*)) = 1 \).
5B. The formal quantization of $T^*(K/H)$. Let $H$ be a closed connected subgroup of $K$. We look at $T^*K$ as a Hamiltonian manifold relative to the action of $H \times K \subset K_1 \times K_2$. The moment map $\Phi = (\Phi_H, \Phi_K)$ is defined by: $\Phi_H(a, \xi) = -\text{pr}(\xi)$ and $\Phi_K(a, \xi) = a \cdot \xi$, where $\text{pr} : \mathfrak{t}^* \to \mathfrak{h}^*$ is the projection. Note that $\Phi$ is a proper map.

The cotangent bundle $T^*(K/H)$, viewed as $K$-manifold, is equal to the symplectic reduction of $T^*K$ relative to the $H$-action: if the kernel of the projection $\text{pr}$ is denoted $\mathfrak{h}^\perp$, we have

$$\Phi_H^{-1}(0)/H = K \times_H \mathfrak{h}^\perp = T^*(K/H).$$

This is the setting of Section 4B. The reduction of the $H \times K$ proper Hamiltonian manifold $T^*K$ relative to the $H$-action is smooth. Then its formal quantization is computed as

$$(58) \quad \mathcal{D}_K^\Phi(T^*(K/H)) = [\mathcal{D}_{H \times K}^\Phi(T^*K)]^H_H = [\mathcal{D}_{K_1 \times K_2}^\Phi(T^*K)|_{H \times K}]^H_H = [L^2(K)]^H_H = L^2(K/H).$$

Here the fact that $\mathcal{D}_{H \times K}^\Phi(T^*K)$ is equal to the restriction of $\mathcal{D}_{K_1 \times K_2}^\Phi(T^*K) = L^2(K)$ to $H \times K$ is a consequence of Theorem 1.3.

Denote by $[T^*(K/H)]_\mu$ the symplectic reduction at $\mu \in \hat{K}$ of the $K$-Hamiltonian manifold $T^*(K/H)$. Theorem 1.4 together with (58) gives:

**Corollary 5.1.** For any $\mu \in \hat{K}$, we have

$$\mathcal{D}([T^*(K/H)]_\mu) = \dim[V^K_\mu]^H_H,$$

where $[V^K_\mu]^H_H$ is the subspace of $H$-invariant vector.

5C. The formal quantization of $T^*(K/H)$ relative to the action of $G$. Let $G$ be a closed connected subgroup of $K$. We look at the Hamiltonian action of $G$ on $T^*(K/H)$. Let $\Phi_G : T^*(K/H) \to \mathfrak{g}^*$ be the moment map. Consider also the restriction of the $K$-module $L^2(K/H)$ to the subgroup $G$.

**Proposition 5.2.** The following statements are equivalent:

1. The moment map $\Phi_G : T^*(K/H) \to \mathfrak{g}^*$ is proper.
2. $\Phi_G^{-1}(0)$ is equal to the zero section.
3. $k \cdot \mathfrak{g} + \mathfrak{h} = \mathfrak{k}$ for any $k \in K$.
4. $\mathfrak{g} + \mathfrak{h} = \mathfrak{k}$.
5. $G$ acts transitively on $K/H$.
6. $[L^2(K/H)]^G \simeq \mathbb{C}$.
7. $L^2(K/H)|_G$ is an admissible $G$-representation.
Proof. The implication (1) $\Rightarrow$ (7) is a consequence of Theorem 1.3. To prove (7) $\Rightarrow$ (6), suppose now that

$$L^2(K/H)|_G = \sum_{\mu \in \hat{K}} [V^K_{\mu}]^H \otimes (V^K_{\mu})^*|_G$$

is an admissible $G$-representation. This means that for any $\lambda \in \hat{G}$, the set

$$A_\lambda := \{\mu \in \hat{K} \mid [V^K_{\mu}]^H \neq 0 \text{ and } (V^K_{\lambda})^* \otimes (V^K_{\mu})^*|_G \neq 0\}$$

is finite. Then the vector space $[L^2(K/H)]^G$ is equal to the finite-dimensional vector space $\sum_{\mu \in A_0} [V^K_{\mu}]^H \otimes [(V^K_{\mu})^*]^G$. For any irreducible representation $V^K_{\mu}$ we have, for any $k \geq 1$, a canonical $K$-equivariant inclusion

$$V^K_{\mu} \otimes \cdots \otimes V^K_{\mu} \hookrightarrow V^K_{k\mu}.$$ 

Hence $[V^K_{\mu}]^H \neq 0$ gives $[V^K_{k\mu}]^H \neq 0$ for any $k \geq 1$. Then if $\mu \in A_0$, we have $k\mu \in A_0$ for $k \geq 1$. Finally the fact that $A_0$ is finite implies that $A_0$ is reduced to $\mu = 0$. Hence the only $G$-invariant functions on $K/H$ are the scalars.

The equivalences (6) $\Leftrightarrow$ (5) $\Leftrightarrow$ (4) $\Leftrightarrow$ (3) are a general fact concerning smooth actions of a compact connected Lie group $G$ on a compact connected manifold $M$. The manifold $M$ does not have $G$-invariant functions which are not scalar if and only if the action of $G$ on $M$ is transitive. Also, given a point $m \in M$, the orbit $G \cdot m$ is all of $M$ if and only if tangent spaces $T_m(G \cdot m)$ and $T_mM$ are equal. If we take $m = \overline{k^{-1}}$ in $M = K/H$, the condition $T_m(G \cdot m) = T_mM$ is equivalent to $k \cdot g + h = \mathfrak{t}$.

To check the implication (3) $\Rightarrow$ (2), let $[k, \xi] \in K \times \mathfrak{h} \subseteq T^*(K/H)$. We have $\Phi_G([k, \xi]) = 0$ if and only if $k \cdot \xi \in \mathfrak{g}^\perp$. Therefore the vector $\xi$ belongs to $k^{-1} \cdot \mathfrak{g}^\perp \cap \mathfrak{h}^\perp = (k^{-1} \cdot \mathfrak{g} + \mathfrak{h})^\perp$. Hence condition (3) imposes that $\xi = 0$.

The implication (2) $\Leftrightarrow$ (1) comes from the fact that $\Phi_G$ is a homogeneous map of degree one between the vector bundle $T^*(K/H)$ and the vector space $\mathfrak{g}^*$.

Suppose now that the cotangent bundle $T^*(K/H)$ is a proper Hamiltonian $G$-manifold. Denote by $[T^*(K/H)]_{\mu,G}$ the (compact) symplectic reduction at $\mu \in \hat{G}$ of the $G$-Hamiltonian manifold $T^*(K/H)$.

**Corollary 5.3.** The multiplicity of $V^K_{\mu}$ in $L^2(K/H)$ is equal to the quantization of the reduced space $[T^*(K/H)]_{\mu,G}$.

**Proof.** Using Theorem 1.3, Equation (58) gives

$$\mathcal{D}_G^{-\infty}(T^*(K/H)) = \mathcal{D}_K^{-\infty}(T^*(K/H))|_G = [L^2(K/H)]_G.$$ 

In other words, the multiplicity of $V^K_{\mu}$ in $L^2(K/H)$ is equal to the quantization of the reduced space $[T^*(K/H)]_{\mu,G}$. 

\hspace{1cm} $\square$
References


Received July 2, 2009. Revised October 17, 2011.

PAUL-ÉMILE PARADAN
INSTITUT DE MATHÉMATIQUES ET DE MODÉLISATION DE MONTPELLIER (I3M)
UNIVERSITÉ MONTPELLIER 2
PLACE EUGÈNE BATAILLON
34095 MONTPELLIER CEDEX 5
FRANCE
paradan@math.univ-montp2.fr
http://www.math.univ-montp2.fr/~paradan/
Singularities of the projective dual variety
ROLAND ABUAF

Eigenvalue estimates for hypersurfaces in $\mathbb{H}^m \times \mathbb{R}$ and applications
PIERRE BÉRARD, PHILIPPE CASTILLON and MARCOS CAVALCANTE

Conformal Invariants associated to a measure: Conformally covariant operators
SUN-YUNG A. CHANG, MATTHEW J. GURSKY and PAUL YANG

Compact symmetric spaces, triangular factorization, and Cayley coordinates
DEREK HABERMAS

Automorphisms of the three-torus preserving a genus-three Heegaard splitting
JESSE JOHNSON

The rationality problem for purely monomial group actions
HIDETAKA KITAYAMA

On a Neumann problem with $p$-Laplacian and noncoercive resonant nonlinearity
SALVATORE A. MARANO and NIKOLAOS S. PAPAGEORGIOU

Minimal ramification in nilpotent extensions
NADYA MARKIN and STEPHEN V. ULLOM

Regularity of weakly harmonic maps from a Finsler surface into an $n$-sphere
XIAOHUAN MO and LIANG ZHAO

On the sum of powered distances to certain sets of points on the circle
NIKOLAI NIKOLOV and RAFAEL RAFAILOV

Formal geometric quantization II
PAUL-ÉMILE PARADAN

Embedded constant-curvature curves on convex surfaces
HAROLD ROSENBERG and MATTHIAS SCHNEIDER

A topological construction for all two-row Springer varieties
HEATHER M. RUSSELL