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PAUL-ÉMILE PARADAN

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We study the formal geometric quantization of noncompact Hamiltonian manifolds. Our main result is that two quantization processes coincide. Ma and Zhang obtained the same result in a recent preprint by completely different means.

In [Paradan 2009], we studied some functorial properties of the "formal geometric quantization" process $\mathcal{Q}^{-\infty}$, which is defined on *proper Hamiltonian manifolds*, that is, *noncompact* Hamiltonian manifolds with *proper* moment map.

There is another way, denoted $\mathcal{Q}^{\Phi}$, of quantizing proper Hamiltonian manifolds by localizing the index of the Dolbeault Dirac operator on the critical points of the square of the moment map [Paradan 2001; 2003; Ma and Zhang 2008].

The main purpose of this paper is to provide a geometric proof that the quantization processes $\mathcal{Q}^{-\infty}$ and $\mathcal{Q}^{\Phi}$ coincide. Ma and Zhang [2008] proved this by completely different means (see also their note [Ma and Zhang 2009]).

1. Introduction and statement of results

First, we recall the definition of the geometric quantization of a smooth and compact Hamiltonian manifold. Then we show two ways of extending the notion of geometric quantization to the case of a *noncompact* Hamiltonian manifold.

Let K be a compact connected Lie group, with Lie algebra $\mathfrak{k}$. In the Kostant–Souriau framework, a Hamiltonian K-manifold (M, Ω, Φ) is prequantized if there

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is an equivariant Hermitian line bundle L with an invariant Hermitian connection ∇ such that

$$\mathcal{L}(X) - \nabla_{X_M} = i\langle\Phi, X\rangle \quad \text{and} \quad \nabla^2 = -i\Omega \tag{1}$$

for every $X \in \mathfrak{k}$. Here X_M is the vector field on M defined by $X_M(m) = \frac{d}{dt}e^{-tX}m|_0$.

The data (L, ∇) is also called a Kostant–Souriau line bundle, and $\Phi : M \to \mathfrak{k}^*$ is the moment map. Via the equivariant Bianchi formula, the conditions of (1) imply the relations

$$\iota(X_M)\Omega = -d\langle\Phi, X\rangle, \qquad X \in \mathfrak{k}. \tag{2}$$

Recall the notion of geometric quantization when M is *compact*. Choose a K-invariant almost complex structure J on M that is compatible with Ω in the sense that the symmetric bilinear form $\Omega(\cdot, J\cdot)$ is a Riemannian metric. Let $\bar\partial_L$ be the Dolbeault operator with coefficients in L, and let $\bar\partial_L^*$ be its (formal) adjoint. The *Dolbeault–Dirac operator* on M with coefficients in L is $D_L = \sqrt{2}(\bar\partial_L + \bar\partial_L^*)$, considered as an elliptic operator from $\mathcal{A}^{0,\mathrm{even}}(M, L)$ to $\mathcal{A}^{0,\mathrm{odd}}(M, L)$. Let $R(K)$ be the representation ring of K.

Definition 1.1. The geometric quantization of a *compact* Hamiltonian K-manifold (M, Ω, Φ) is the element $\mathcal{Q}_K(M) \in R(K)$ defined as the equivariant index of the Dolbeault–Dirac operator D_L.

Consider the case of a *proper* Hamiltonian K-manifold M: the manifold is (perhaps) *noncompact* but the moment map $\Phi : M \to \mathfrak{k}^*$ is supposed to be proper. Under this properness assumption, one defines the *formal geometric quantization* of M as an element $\mathcal{Q}_K^{-\infty}(M)$ that belongs to $R^{-\infty}(K)$ [Weitsman 2001; Paradan 2009]. Recall the definition:

Let T be a maximal torus of K. Let $\mathfrak{t}^*$ be the dual of the Lie algebra $\mathfrak{t}$ of T containing the weight lattice $\wedge^*$, that is, $\alpha \in \wedge^*$ if $i\alpha : \mathfrak{t} \to i\mathbb{R}$ is the differential of a character of T. Let $\mathfrak{t}^*_+ \subset \mathfrak{t}^*$ be a Weyl chamber, and let $\widehat{K} := \wedge^* \cap \mathfrak{t}^*_+$ be the set of dominant weights. The ring of characters $R(K)$ has a $\mathbb{Z}$-basis V_μ^K, $\mu \in \widehat{K}$:V_μ^K is the irreducible representation of K with highest weight μ.

A representation E of K is *admissible* if it has finite K-multiplicities, that is, $\dim(\hom_K(V_\mu^K, E)) < \infty$ for every $\mu \in \widehat{K}$. Let

$$R^{-\infty}(K) \tag{3}$$

be the Grothendieck group associated to the K-admissible representations. We have an inclusion map $R(K) \hookrightarrow R^{-\infty}(K)$ and $R^{-\infty}(K)$ is canonically identified with $\hom_{\mathbb{Z}}(R(K), \mathbb{Z})$. The tensor product induces an $R(K)$-module structure on $R^{-\infty}(K)$ since $E \otimes V$ is an admissible representation when V and E are, respectively, a finite-dimensional and an admissible representation of K.

For any $\mu \in \widehat{K}$ that is a regular value of the moment map Φ, the reduced space (or symplectic quotient) $M_\mu := \Phi^{-1}(K \cdot \mu)/K$ is a *compact* orbifold equipped with a symplectic structure Ω_μ. Moreover $L_\mu := (L|_{\Phi^{-1}(\mu)} \otimes \mathbb{C}_{-\mu})/K_\mu$ is a Kostant–Souriau line orbibundle over (M_μ, Ω_μ). The definition of the index of the Dolbeault–Dirac operator carries over to the orbifold case, hence $\mathcal{Q}(M_\mu) \in \mathbb{Z}$ is defined. In Section 2C, we explain how this notion of geometric quantization extends further to the case of singular symplectic quotients. So the integer $\mathcal{Q}(M_\mu) \in \mathbb{Z}$ is well defined for every $\mu \in \widehat{K}$: in particular $\mathcal{Q}(M_\mu) = 0$ if $\mu \notin \Phi(M)$.

Definition 1.2. Let (M, Ω, Φ) be a *proper* Hamiltonian K-manifold prequantized by a Kostant–Souriau line bundle L. The formal quantization of (M, Ω, Φ) is the element of $R^{-\infty}(K)$ defined by

$$\mathcal{Q}_K^{-\infty}(M) = \sum_{\mu \in \widehat{K}} \mathcal{Q}(M_\mu)\, V_\mu^K.$$

When M is compact, the fact that

$$\mathcal{Q}_K(M) = \mathcal{Q}_K^{-\infty}(M) \tag{4}$$

is known as the "quantization commutes with reduction" theorem. This was conjectured in [Guillemin and Sternberg 1982] and was first proved in [Meinrenken 1998; Meinrenken and Sjamaar 1999]. Other proofs of (4) were given in [Tian and Zhang 1998; Paradan 2001]. For complete references on the subject, consult [Sjamaar 1996; Vergne 2002].

We summarize the main features of the formal geometric quantization $\mathcal{Q}^{-\infty}$:

Theorem 1.3 [Paradan 2009]. (1) (restriction to subgroup) *Let M be a prequantized Hamiltonian K-manifold that is* proper. *Let $H \subset K$ be a closed connected Lie subgroup such that M is still* proper *as a Hamiltonian H-manifold. Then $\mathcal{Q}_K^{-\infty}(M)$ is H-admissible and $\mathcal{Q}_K^{-\infty}(M)|_H = \mathcal{Q}_H^{-\infty}(M)$ in $R^{-\infty}(H)$.*

(2) (product) *Let M and N be prequantized Hamiltonian K-manifolds*, *where M is* proper *and N is* compact. *Then $M \times N$ is a* proper *prequantized Hamiltonian K-manifold and $\mathcal{Q}_K^{-\infty}(M \times N) = \mathcal{Q}_K^{-\infty}(M) \cdot \mathcal{Q}_K(N)$ in $R^{-\infty}(K)$.*

When M is a proper Hamiltonian K-manifold, we can also define another "formal geometric quantization", denoted

$$\mathcal{Q}_K^{\Phi}(M) \in R^{-\infty}(K), \tag{5}$$

by localizing the index of the Dolbeault–Dirac operator D_L on the set $\mathrm{Cr}(\|\Phi\|^2)$ of critical points of the square of the moment map (see Section 2B for the precise definition). This idea of nonabelian localization goes back to Witten [1992]. We proved in [Paradan 2003; 2009] that

$$\mathcal{Q}^{-\infty}_{K}(M) = \mathcal{Q}^{\Phi}_{K}(M) \tag{6}$$

in some situations:

- M is a coadjoint orbit of a semisimple Lie group S that parametrizes a representation of the discrete series of S.
- M is a Hermitian vector space.

In her ICM 2006 plenary lecture, Vergne [2007] conjectured that (6) holds when $\mathrm{Cr}(\|\Phi\|^2)$ is compact. Recently, Ma and Zhang [2008] proved the following generalization of this conjecture.

Theorem 1.4. *The equality* (6) *holds for* any *proper Hamiltonian K-manifold.*

Corollary 1.5. *The formal quantization map $\mathcal{Q}^{\Phi}$ satisfies the functorial properties listed in Theorem 1.3.*

This article is dedicated to the study of the quantization map $\mathcal{Q}^{\Phi}$. In Section 2B, we give the precise definition of the quantization process $\mathcal{Q}^{\Phi}$. In particular, we refine the constant a_{γ} that appears in [Ma and Zhang 2008, Theorem 0.1]. In Section 2D, we explain how to compute the quantization of a point. In Section 3, we give another proof of Theorem 1.4 by using the technique of symplectic cutting developed in [Paradan 2009]. In Section 4, we consider the case where $K = K_1 \times K_2$ acts on M in such a way that the symplectic reduction $M/\!\!/_0 K_1$ is a *smooth* proper K_2-Hamiltonian manifold. We show then that the K_1-invariant part of $\mathcal{Q}^{\Phi}_{K_1\times K_2}(M)$ is equal to $\mathcal{Q}^{\Phi_2}_{K_2}(M/\!\!/_0 K_1)$. In Section 5, we study the example of the cotangent bundle of a homogeneous space: $M = \mathrm{T}^*(K/H)$ where H is a closed subgroup of K.

We finish this introduction by discussing the two proofs of Theorem 1.4 in [Ma and Zhang 2008] and in this paper. Both proofs use the Witten [1992] deformation argument. The work of Ma and Zhang [2008] is analytic and makes a great use of techniques initiated in [Bismut and Lebeau 1991]. One of Ma and Zhang's main tools is an interpretation of the transversal index as an Atiyah–Patodi–Singer type index. In the present work, we stay on the topological/geometrical side. Our main tools are based on localization formulas (see [Paradan 2001]) and on a symplectic cutting technique (see [Paradan 2009]).

The approach of Ma and Zhang [2008] is different from ours, but the results are equivalent. In [Ma and Zhang 2008, Theorem 0.5], they show that the geometric quantization process $\mathcal{Q}^{\Phi}$ is functorial with respect to the *product* (see the second point of Theorem 1.3), and then deduce the equality $\mathcal{Q}^{\Phi} = \mathcal{Q}^{-\infty}$.

I wish to thank the referees for their useful comments.

2. Quantizations of noncompact manifolds

In this section we define the quantization process $\mathcal{Q}^{\Phi}$, and we give another definition of the quantization process $\mathcal{Q}^{-\infty}$ that uses the notion of symplectic cutting [Paradan 2009].

2A. ***Transversally elliptic symbols.*** Here we give the basic definitions from the theory of transversally elliptic symbols (or operators) defined in [Atiyah 1974]. For an axiomatic treatment of the index morphism, see [Berline and Vergne 1996a; 1996b; Paradan and Vergne 2009]. For a short introduction, see [Paradan 2001].

Let $\mathcal{X}$ be a *compact* K-manifold. Let $p : \mathrm{T}\mathcal{X} \to \mathcal{X}$ be the projection, and let $(-,-)_{\mathcal{X}}$ be a K-invariant Riemannian metric. If E^0, E^1 are K-equivariant complex vector bundles over $\mathcal{X}$, a K-equivariant morphism $\sigma \in \Gamma(\mathrm{T}\mathcal{X}, \hom(p^*E^0, p^*E^1))$ is called a *symbol* on $\mathcal{X}$. The subset of all $(x, v) \in \mathrm{T}\mathcal{X}$ where[1] $\sigma(x, v) : E^0_x \to E^1_x$ is not invertible is called the *characteristic set* of σ, and is denoted by $\mathrm{Char}(\sigma)$.

In the following, the product of a symbol σ by a complex vector bundle $F \to M$, is the symbol

$$\sigma \otimes F$$

defined by $\sigma \otimes F(x, v) = \sigma(x, v) \otimes \mathrm{Id}_{F_x}$ from $E^0_x \otimes F_x$ to $E^1_x \otimes F_x$. Note that $\mathrm{Char}(\sigma \otimes F) = \mathrm{Char}(\sigma)$.

Let $\mathrm{T}_K\mathcal{X}$ be the following subset of $\mathrm{T}\mathcal{X}$:

$$\mathrm{T}_K\mathcal{X} = \left\{(x, v) \in \mathrm{T}\mathcal{X} \mid (v, X_{\mathcal{X}}(x))_{\mathcal{X}} = 0 \text{ for all } X \in \mathfrak{k}\right\}.$$

A symbol σ is *elliptic* if σ is invertible outside a compact subset of $\mathrm{T}\mathcal{X}$ (that is, $\mathrm{Char}(\sigma)$ is compact), and is *K-transversally elliptic* if the restriction of σ to $\mathrm{T}_K\mathcal{X}$ is invertible outside a compact subset of $\mathrm{T}_K\mathcal{X}$ (that is, $\mathrm{Char}(\sigma) \cap \mathrm{T}_K\mathcal{X}$ is compact). An elliptic symbol σ defines an element in the equivariant $\mathbf{K}^0$-theory of $\mathrm{T}\mathcal{X}$ with compact support, which is denoted by $\mathbf{K}^0_K(\mathrm{T}\mathcal{X})$, and the index of σ is a virtual finite-dimensional representation of K, which we denote $\mathrm{Index}^K_{\mathcal{X}}(\sigma) \in R(K)$ [Atiyah and Segal 1968; Atiyah and Singer 1968a; 1968b; 1971].

Consider the $R(K)$-submodule

$$R^{-\infty}_{\mathrm{tc}}(K) \subset R^{-\infty}(K)$$

formed by all the infinite sums $\sum_{\mu \in \widehat{K}} m_\mu V^K_\mu$ where the map $\mu \in \widehat{K} \mapsto m_\mu \in \mathbb{Z}$ has at most a *polynomial* growth. The $R(K)$-module $R^{-\infty}_{\mathrm{tc}}(K)$ is the Grothendieck group associated to the *trace class* virtual K-representations. We can associate to any $V \in R^{-\infty}_{\mathrm{tc}}(K)$ its trace, $k \to \mathrm{Tr}(k, V)$, which is a generalized function on K invariant by conjugation. Then the trace defines a morphism of $R(K)$-modules

$$R^{-\infty}_{\mathrm{tc}}(K) \hookrightarrow \mathcal{C}^{-\infty}(K)^{\mathrm{Ad}}, \tag{7}$$

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[1] The map $\sigma(x, v)$ will be also denote $\sigma|_x(v)$

where $\mathscr{C}^{-\infty}(K)^{\mathrm{Ad}}$ is the vector space of generalized function on K invariant by conjugation.

A K*-transversally elliptic* symbol σ defines an element of $\mathbf{K}^0_K(\mathrm{T}_K\mathscr{X})$, and the index of σ is defined as a trace class virtual representation of K, which we still denote $\mathrm{Index}^K_{\mathscr{X}}(\sigma) \in R^{-\infty}_{\mathrm{tc}}(K)$ [Atiyah 1974].

Any elliptic symbol of $\mathrm{T}\mathscr{X}$ is K-transversally elliptic, hence we have a restriction map $\mathbf{K}^0_K(\mathrm{T}\mathscr{X}) \to \mathbf{K}^0_K(\mathrm{T}_K\mathscr{X})$ and a commutative diagram

$$\begin{tikzcd}
\mathbf{K}^0_K(\mathrm{T}\mathscr{X}) \arrow[r] \arrow[d, "\mathrm{Index}^K_{\mathscr{X}}"'] & \mathbf{K}^0_K(\mathrm{T}_K\mathscr{X}) \arrow[d, "\mathrm{Index}^K_{\mathscr{X}}"] \\
R(K) \arrow[r] & R^{-\infty}_{\mathrm{tc}}(K).
\end{tikzcd} \tag{8}$$

Using the *excision property*, one can easily show that the index map

$$\mathrm{Index}^K_{\mathscr{U}} : \mathbf{K}^0_K(\mathrm{T}_K\mathscr{U}) \to R^{-\infty}_{\mathrm{tc}}(K)$$

is still defined when $\mathscr{U}$ is a K-invariant relatively compact open subset of a K-manifold (see [Paradan 2001, Section 3.1]).

Suppose now that the group K is equal to the product $K_1\times K_2$. When a symbol σ is $(K_1\times K_2)$-transversally elliptic we will be interested in the K_1-invariant part of its index, which we denote by

$$[\mathrm{Index}^{K_1\times K_2}_{\mathscr{X}}(\sigma)]^{K_1} \in R^{-\infty}_{\mathrm{tc}}(K_2).$$

An intermediate notion between the "ellipticity" and "$(K_1\times K_2)$-transversal ellipticity" is "K_1-transversal ellipticity". When a $(K_1\times K_2)$-equivariant symbol σ is K_1*-transversally elliptic*, its index $\mathrm{Index}^{K_1\times K_2}_{\mathscr{X}}(\sigma) \in R^{-\infty}_{\mathrm{tc}}(K_1 \times K_2)$, viewed as a generalized function on $K_1 \times K_2$, is *smooth* relative to the variable in K_2 [Atiyah 1974; Berline and Vergne 1996b; Paradan and Vergne 2009]. It implies that $\mathrm{Index}^{K_1\times K_2}_{\mathscr{X}}(\sigma) = \sum_\lambda \theta(\lambda)\otimes V^{K_1}_\lambda$ with

$$\theta(\lambda) \in R(K_2) \quad \text{for all } \lambda \in \widehat{K_1}.$$

In particular, $[\mathrm{Index}^{K_1\times K_2}_{\mathscr{X}}(\sigma)]^{K_1} = \theta(0)$ belongs to $R(K_2)$.

Recall the multiplicative property of the index map for the product of manifolds that was proved in [Atiyah 1974]. Consider a compact Lie group K_2 acting on two manifolds $\mathscr{X}_1$ and $\mathscr{X}_2$, and assume that another compact Lie group K_1 acts on $\mathscr{X}_1$ commuting with the action of K_2.

The external product of complexes on $\mathrm{T}\mathscr{X}_1$ and $\mathrm{T}\mathscr{X}_2$ induces a multiplication (see [Atiyah 1974]):

$$\odot : \mathbf{K}^0_{K_1\times K_2}(\mathrm{T}_{K_1}\mathscr{X}_1) \times \mathbf{K}^0_{K_2}(\mathrm{T}_{K_2}\mathscr{X}_2) \to \mathbf{K}^0_{K_1\times K_2}(\mathrm{T}_{K_1\times K_2}(\mathscr{X}_1 \times \mathscr{X}_2)).$$

Recall the definition of the external product: For $k = 1, 2$, consider equivariant morphisms[2] $\sigma_k : \mathscr{E}_k^+ \to \mathscr{E}_k^-$ on $\mathrm{T}\mathscr{X}_k$. Consider the equivariant morphism on $\mathrm{T}(\mathscr{X}_1 \times \mathscr{X}_2)$

$$\sigma_1 \odot \sigma_2 : \mathscr{E}_1^+ \otimes \mathscr{E}_2^+ \oplus \mathscr{E}_1^- \otimes \mathscr{E}_2^- \to \mathscr{E}_1^- \otimes \mathscr{E}_2^+ \oplus \mathscr{E}_1^+ \otimes \mathscr{E}_2^-$$

defined by

$$\sigma_1 \odot \sigma_2 = \begin{pmatrix} \sigma_1 \otimes \mathrm{Id} & -\mathrm{Id} \otimes \sigma_2^* \\ \mathrm{Id} \otimes \sigma_2 & \sigma_1^* \otimes \mathrm{Id} \end{pmatrix}. \tag{9}$$

We see that the set $\mathrm{Char}(\sigma_1 \odot \sigma_2) \subset \mathrm{T}\mathscr{X}_1 \times \mathrm{T}\mathscr{X}_2$ is equal to $\mathrm{Char}(\sigma_1) \times \mathrm{Char}(\sigma_2)$. Suppose now that the morphisms σ_k are respectively K_k-transversally elliptic. Since $\mathrm{T}_{K_1 \times K_2}(\mathscr{X}_1 \times \mathscr{X}_2) \neq \mathrm{T}_{K_1}\mathscr{X}_1 \times \mathrm{T}_{K_2}\mathscr{X}_2$, the morphism $\sigma_1 \odot \sigma_2$ is not necessarily $(K_1 \times K_2)$-transversally elliptic. Nevertheless, if σ_2 is *almost homogeneous*, then the morphism $\sigma_1 \odot \sigma_2$ is $(K_1 \times K_2)$-transversally elliptic (see [Paradan and Vergne 2009]). So the exterior product $a_1 \odot a_2$ is the $\mathbf{K}^0$-theory class defined by $\sigma_1 \odot \sigma_2$, where $a_k = [\sigma_k]$ and σ_2 is almost homogeneous.

The following property will be used frequently; see [Atiyah 1974, Lecture 3; Paradan and Vergne 2009].

Theorem 2.1 (multiplicative property). *For any* $[\sigma_1] \in \mathbf{K}^0_{K_1 \times K_2}(\mathrm{T}_{K_1}\mathscr{X}_1)$ *and any* $[\sigma_2] \in \mathbf{K}^0_{K_2}(\mathrm{T}_{K_2}\mathscr{X}_2)$ *we have*

$$\mathrm{Index}^{K_1 \times K_2}_{\mathscr{X}_1 \times \mathscr{X}_2}([\sigma_1] \odot [\sigma_2]) = \mathrm{Index}^{K_1 \times K_2}_{\mathscr{X}_1}([\sigma_1]) \otimes \mathrm{Index}^{K_2}_{\mathscr{X}_2}([\sigma_2]).$$

The product of $\mathrm{Index}^{K_1 \times K_2}_{\mathscr{X}_1}([\sigma_1]) \in \mathscr{C}^{-\infty}(K_1 \times K_2)^{\mathrm{Ad}}$ with $\mathrm{Index}^{K_2}_{\mathscr{X}_2}([\sigma_2]) \in \mathscr{C}^{-\infty}(K_2)^{\mathrm{Ad}}$ is well defined since the generalized function

$$(k_1, k_2) \mapsto \mathrm{Index}^{K_1 \times K_2}_{\mathscr{X}_1}([\sigma_1])(k_1, k_2)$$

is smooth relative to the variable $k_2 \in K_2$.

We finish this section by recalling the notion of limit in $R^{-\infty}(K)$.

Definition 2.2. The *support* of $\chi := \sum_{\mu \in \widehat{K}} a_\mu V_\mu^K \in R^{-\infty}(K)$ is the set of $\mu \in \widehat{K}$ such that $a_\mu \neq 0$.

We will say that $\chi \in R^{-\infty}(K)$ is supported outside $B \subset \mathfrak{t}^*$ if the support of χ does not intersect B. Denote by $O(r)$ any character of $R^{-\infty}(K)$ that is supported outside the ball $B_r = \{\xi \in \mathfrak{t}^* \mid \|\xi\| < r\}$.

Definition 2.3. A sequence $\chi_n \in R^{-\infty}(K)$ converges to χ_∞ when n goes to infinity if for any $r > 0$ there exists $N \in \mathbb{N}$ such that

$$\chi_\infty - \chi_n = O(r)$$

for any $n \geq N$.

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[2] To simplify notation, we do not distinguish between vector bundles on $\mathrm{T}\mathscr{X}$ and on $\mathscr{X}$.

We will be interested in an infinite sum $\sum_{i\in I}\psi_i$ of generalized characters. Here $\sum_{i\in I}\psi_i$ converges in $R^{-\infty}(K)$ if for any $r>0$ the set

$$\{i\in I \mid \mathrm{support}(\psi_i)\cap B_r\neq\varnothing\}$$

is finite.

2B. ***Definition and first properties of*** $\mathcal{Q}^{\Phi}$**.** Let (M,Ω,Φ) be a proper Hamiltonian K-manifold prequantized by an equivariant line bundle L. Let J be an invariant almost complex structure compatible with Ω. Let $p:\mathrm{T}M\to M$ be the projection.

To begin, we describe the principal symbol of the Dolbeault–Dirac operator $\sqrt{2}(\bar\partial_L+\bar\partial_L^*)$. The complex vector bundle $(\mathrm{T}^*M)^{0,1}$ is K-equivariantly identified with the tangent bundle $\mathrm{T}M$ equipped with the complex structure J. Let h be the Hermitian structure on $(\mathrm{T}M,J)$ defined by $h(v,w)=\Omega(v,Jw)-i\Omega(v,w)$ for $v,w\in\mathrm{T}M$. The symbol

$$\mathrm{Thom}(M,J)\in\Gamma\left(\mathrm{T}M,\hom(p^*(\wedge_{\mathbb{C}}^{\mathrm{even}}\mathrm{T}M),\ p^*(\wedge_{\mathbb{C}}^{\mathrm{odd}}\mathrm{T}M))\right)$$

at $(m,v)\in\mathrm{T}M$ is equal to the Clifford map

$$\mathbf{c}_m(v):\wedge_{\mathbb{C}}^{\mathrm{even}}\mathrm{T}_mM\to\wedge_{\mathbb{C}}^{\mathrm{odd}}\mathrm{T}_mM, \tag{10}$$

where $\mathbf{c}_m(v).w=v\wedge w-\iota(v)w$ for $w\in\wedge_{\mathbb{C}}^{\bullet}\mathrm{T}_mM$. Here $\iota(v):\wedge_{\mathbb{C}}^{\bullet}\mathrm{T}_mM\to\wedge_{\mathbb{C}}^{\bullet-1}\mathrm{T}_mM$ denotes the contraction map relative to h. Since $\mathbf{c}_m(v)^2=-\|v\|^2\mathrm{Id}$, the map $\mathbf{c}_m(v)$ is invertible for all $v\neq0$. Hence the characteristic set of $\mathrm{Thom}(M,J)$ corresponds to the 0-section of $\mathrm{T}M$.

It is a classical fact that the principal symbol of the Dolbeault–Dirac operator $\sqrt{2}(\bar\partial_L+\bar\partial_L^*)$ is equal to[3]

$$\mathrm{Thom}(M,J)\otimes L \tag{11}$$

(see [Berline et al. 2004, Proposition 3.67]). Here also, $\mathrm{Char}(\mathrm{Thom}(M,J)\otimes L)$ coincides with the 0-section of $\mathrm{T}M$.

Remark 2.4. If the manifold M is a product $M_1\times M_2$, the symbol $\mathrm{Thom}(M,J)\otimes L$ is equal to the product $\sigma_1\odot\sigma_2$ where $\sigma_k=\mathrm{Thom}(M_k,J_k)\otimes L_k$.

When M is compact, the symbol $\mathrm{Thom}(M,J)\otimes L$ is elliptic and then defines an element of the equivariant $\mathbf{K}$-group of $\mathrm{T}M$. The topological index of $\mathrm{Thom}(M,J)\otimes L\in\mathbf{K}_K^0(\mathrm{T}M)$ is equal to the analytical index of the Dolbeault–Dirac operator $\sqrt{2}(\bar\partial_L+\bar\partial_L^*)$:

$$\mathcal{Q}_K(M)=\mathrm{Index}_M^K(\mathrm{Thom}(M,J)\otimes L)\quad\text{in } R(K). \tag{12}$$

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[3] Here we use an identification $\mathrm{T}^*M\simeq\mathrm{T}M$ given by an invariant Riemannian metric.

When M is not compact the topological index of $\mathrm{Thom}(M, J)\otimes L$ is not defined. To extend the notion of geometric quantization to this setting we deform the symbol $\mathrm{Thom}(M, J)\otimes L$ in the "Witten" way [Paradan 2001; 2003]. Consider the identification $\xi \mapsto \widetilde{\xi}$, $\mathfrak{k}^* \to \mathfrak{k}$ defined by a K-invariant scalar product on $\mathfrak{k}^*$. Define the *Kirwan vector field* on M as

$$\kappa_m = (\widetilde{\Phi(m)})_M(m), \quad m \in M. \tag{13}$$

Definition 2.5. The symbol $\mathrm{Thom}(M, J)\otimes L$ pushed by the vector field κ is the symbol $\mathbf{c}^\kappa$ defined by the relation

$$\mathbf{c}^\kappa|_m(v) = \mathrm{Thom}(M, J)\otimes L|_m(v-\kappa_m)$$

for any $(m, v) \in \mathrm{T}M$. More generally, if $E \to M$ is an equivariant complex vector bundle, one defines the symbol $\mathbf{c}^\kappa_E$ with the same relation (with E at the place of L).

Note that $\mathbf{c}^\kappa|_m(v)$ is invertible except if $v = \kappa_m$. If furthermore v belongs to the subset $\mathrm{T}_K M$ of tangent vectors orthogonal to the K-orbits, then $v = 0$ and $\kappa_m = 0$. Indeed κ_m is tangent to $K\cdot m$ while v is orthogonal.

Since κ is the Hamiltonian vector field of the function $\frac{-1}{2}\|\Phi\|^2$, the set of zeros of κ coincides with the set $\mathrm{Cr}(\|\Phi\|^2)$ of critical points of $\|\Phi\|^2$. Finally we have

$$\mathrm{Char}(\mathbf{c}^\kappa)\cap \mathrm{T}_K M \simeq \mathrm{Cr}(\|\Phi\|^2).$$

In general $\mathrm{Cr}(\|\Phi\|^2)$ is not compact, so $\mathbf{c}^\kappa$ does not define a transversally elliptic symbol on M. To define a kind of index of $\mathbf{c}^\kappa$, we proceed as follows. For any invariant open relatively compact subset $U \subset M$ the set

$$\mathrm{Char}(\mathbf{c}^\kappa|_U)\cap \mathrm{T}_K U \simeq \mathrm{Cr}(\|\Phi\|^2)\cap U$$

is compact when

$$\partial U \cap \mathrm{Cr}(\|\Phi\|^2) = \varnothing. \tag{14}$$

When (14) holds, denote by

$$\mathcal{Q}^\Phi_K(U) := \mathrm{Index}^K_U(\mathbf{c}^\kappa|_U) \in R^{-\infty}_{\mathrm{tc}}(K) \tag{15}$$

the equivariant index of the transversally elliptic symbol $\mathbf{c}^\kappa|_U$.

It will be useful to understand the dependence of the generalized character $\mathcal{Q}^\Phi_K(U)$ relative to the data (U, Ω, L). So consider two proper Hamiltonian K-manifolds (M, Ω, Φ) and (M', Ω', Φ') respectively prequantized by the line bundles L and L'. Let $V \subset M$ and $V' \subset M'$ two invariant open subsets.

Proposition 2.6. • *The generalized character $\mathcal{Q}^\Phi_K(U)$ does not depend of the choice of an invariant almost complex structure on U compatible with $\Omega|_U$.*

- *Suppose that there exists an equivariant diffeomorphism* $\Psi : V \to V'$ *such that*
  (1) $\Psi^*(\Phi') = \Phi$,
  (2) $\Psi^*(L') = L$,
  (3) *there exists a homotopy of symplectic forms taking* $\Psi^*(\Omega'|_{V'})$ *to* $\Omega|_V$.

  Let $U' \subset \overline{U'} \subset V'$ *be an invariant open relatively compact subset such that* $\partial U'$ *satisfies* (14). *Take* $U = \Psi^{-1}(U')$. *Then* ∂U *satisfies* (14) *and*

$$\mathcal{Q}_K^{\Phi'}(U') = \mathcal{Q}_K^{\Phi}(U) \in R^{-\infty}(K).$$

Proof. To prove the first point, let $\mathbf{c}_i^\kappa|_U$, $i = 0, 1$ be the transversally elliptic symbols defined with the compatible almost complex structure J_i, $i = 0, 1$. Since the space of compatible almost complex structure is contractible, there exists a homotopy J_t, $t \in [0, 1]$ of almost complex structures linking J_0 and J_1. By [Paradan 2001, Lemma 2.2], there exists an invertible bundle map $A \in \Gamma(U, \mathrm{End}(\mathrm{T}U))$, homotopic to the identity, such that $A \circ J_0 = J_1 \circ A$. With the help of A we prove then that the symbols $\mathbf{c}_0^\kappa|_U$ and $\mathbf{c}_1^\kappa|_U$ define the same class in $\mathbf{K}_K^0(\mathrm{T}_K U)$ (see [Paradan 2001, Lemma 2.2]). Hence their equivariant indices coincide.

To prove the second point, observe that the characters $\mathcal{Q}_K^{\Phi}(U)$ and $\mathcal{Q}_K^{\Phi'}(U')$ are computed as the equivariant index of the symbols $\mathbf{c}^\kappa|_U$ and $\mathbf{c}^{\kappa'}|_{U'}$. Let $\tilde{\mathbf{c}}^\kappa|_U$ the pull back of $\mathbf{c}^{\kappa'}|_{U'}$ by Ψ. Thanks to conditions (1) and (2), the only thing which differs in the definitions of the symbols $\mathbf{c}^\kappa|_U$ and $\tilde{\mathbf{c}}^\kappa|_U$ are the almost complex structures J and $\widetilde{J} = \Psi^*(J')$: the first one is compatible with $\Omega|_V$ and the second one with $\Psi^*(\Omega'|_{V'})$. Since these two symplectic structures are homotopic, the almost complex structures J and $\widetilde{J}$ are also homotopic. So we can conclude as in the first point. ☐

We describe the critical points of $\|\Phi\|^2$, when the moment map Φ is proper. We know that $m \in \mathrm{Cr}(\|\Phi\|^2)$ if and only if $\widetilde{\beta}_M(m) = 0$ for $\beta = \Phi(m)$. Hence the set $\mathrm{Cr}(\|\Phi\|^2)$ has the decomposition

$$\mathrm{Cr}(\|\Phi\|^2) = \bigcup_{\beta \in \mathfrak{t}^*} M^{\widetilde{\beta}} \cap \Phi^{-1}(\beta) = \bigcup_{\beta \in \mathcal{B}} \underbrace{K \cdot (M^{\widetilde{\beta}} \cap \Phi^{-1}(\beta))}_{Z_\beta}, \tag{16}$$

where $\mathcal{B}$ is a subset of the Weyl chamber $\mathfrak{t}^*_+$. The set of singular values of $\|\Phi\|^2$ is then $\{\|\beta\|^2, \beta \in \mathcal{B}\}$. Each part Z_β is compact, hence $\mathrm{Cr}(\|\Phi\|^2)$ is compact if $\mathcal{B}$ is finite. Denote by $B_r \subset \mathfrak{t}^*$ the open ball $\{\xi \in \mathfrak{t}^* \mid \|\xi\| < r\}$.

Proposition 2.7.
- *For any* $r > 0$, *the set* $\mathcal{B} \cap B_r$ *is finite.*
- $\mathrm{Cr}(\|\Phi\|^2)$ *is compact if and only if* $\mathcal{B}$ *is finite.*
- *The set of singular values of* $\|\Phi\|^2 : M \to \mathbb{R}$ *forms a sequence* $0 \le r_1 < r_2 < \cdots < r_k < \cdots$ *that is finite if and only if* $\mathrm{Cr}(\|\Phi\|^2)$ *is compact. In the other case*, $\lim_{k\to\infty} r_k = \infty$.

Proof. To prove the first point, let $r > 0$ and consider the relatively compact invariant open subset $\mathcal{V}_r := \Phi^{-1}(\{\xi \in \mathfrak{k}^* \mid \|\xi\| < r\})$ and the infinitesimal action of the Lie algebra $\mathfrak{t}$ on $\mathcal{V}_r$. For any vector subspace $\mathfrak{a} \subset \mathfrak{t}$, define the T-invariant submanifold

$$\mathcal{V}_r(\mathfrak{a}) := \{x \in \mathcal{V}_r \mid \mathrm{Stabilizer}_{\mathfrak{t}}(x) = \mathfrak{a}\}.$$

Since $\mathcal{V}_r$ is relatively compact, it has finitely many types of stabilizers $\mathfrak{a}_1, \dots, \mathfrak{a}_p$. Hence we have a decomposition $\mathcal{V}_r = \mathcal{V}_r(\mathfrak{a}_1) \cup \cdots \cup \mathcal{V}_r(\mathfrak{a}_p)$ where each $\mathcal{V}_r(\mathfrak{a}_k)$ has a finite number, say $n(r, k)$, of connected components. We will show that

$$\sum_{k=1}^{p} n(r, k) \geq \# \mathcal{B} \cap B_r. \tag{17}$$

Let $\mathcal{C}_r$ be the finite collection formed by the connected components of the manifold $\mathcal{V}_r(\mathfrak{a}_k)$, $1 \leq k \leq p$. Let $\mathcal{C}'_r \subset \mathcal{C}_r$ be the subset formed by the connected components F for which $F^{\widetilde{\beta}} \cap \Phi^{-1}(\beta) \neq \emptyset$ for some $\beta \in \mathcal{B} \cap B_r$. The inequality (17) follows from the existence of a surjective map $\theta : \mathcal{C}'_r \to \mathcal{B} \cap B_r$

Let $F \in \mathcal{C}'_r$. Suppose that there exist β, β' in $\mathcal{B} \cap B_r$ such that $F^{\widetilde{\beta}} \cap \Phi^{-1}(\beta)$ and $F^{\widetilde{\beta}'} \cap \Phi^{-1}(\beta')$ are nonempty. It implies first that $\widetilde{\beta}, \widetilde{\beta}' \in \mathfrak{a}_k$. The relation (2) shows that the function $x \in F \mapsto \langle \Phi(x), Y \rangle$ is constant for any $Y \in \mathfrak{a}_k$. If we take $Y = \widetilde{\beta}$, the fact that F intersects both $\Phi^{-1}(\beta)$ and $\Phi^{-1}(\beta')$ gives $\|\beta\|^2 = (\beta, \beta')$. By taking $Y = \widetilde{\beta}'$, we have also $\|\beta'\|^2 = (\beta, \beta')$. Finally

$$\|\beta - \beta'\|^2 = \|\beta\|^2 + \|\beta'\|^2 - 2(\beta, \beta') = 0,$$

hence $\beta = \beta'$. Define $\theta : \mathcal{C}'_r \to \mathcal{B} \cap B_r$ as follows: $\theta(F)$ is the unique element $\beta \in \mathcal{B} \cap B_r$ such that $F^{\widetilde{\beta}} \cap \Phi^{-1}(\beta) \neq \emptyset$. It is easy to check that θ is onto.

The two other points are a direct consequence of the first one. □

To each regular value R of $\mathrm{Cr}(\|\Phi\|^2)$ associate the invariant open subset $M_{<R} := \{\|\Phi\|^2 < R\}$ that satisfies (14). The restriction $\mathbf{c}^\kappa|_{M_{<R}}$ defines then a transversally elliptic symbol on $M_{<R}$. Let $\mathcal{Q}_K^\Phi(M_{<R})$ be its equivariant index. We show that $\mathcal{Q}_K^\Phi(M_{<R})$ has a limit when $R \to \infty$.

For any $\beta \in \mathcal{B}$, consider a relatively compact open invariant neighborhood $\mathcal{U}_\beta$ of Z_β such that $\mathrm{Cr}(\|\Phi\|^2) \cap \overline{\mathcal{U}_\beta} = Z_\beta$. By the excision property, the generalized character $\mathcal{Q}_K^\Phi(\mathcal{U}_\beta) = \mathrm{Index}^K_{\mathcal{U}_\beta}(\mathbf{c}^\kappa|_{\mathcal{U}_\beta})$ does not depend of the choice of $\mathcal{U}_\beta$. To simplify notation, use the following:

Definition 2.8. Denote by $\mathcal{Q}_K^\beta(M) \in R^{-\infty}_{\mathrm{tc}}(K)$ the equivariant[4] of the transversally elliptic symbol $\mathbf{c}^\kappa|_{\mathcal{U}_\beta}$.

When $E \to M$ is an equivariant complex vector bundle, denote by $RR^K_\beta(M, E)$ the equivariant index of the transversally elliptic symbol $\mathbf{c}^\kappa_E|_{\mathcal{U}_\beta}$.

---

[4] The index of $\mathbf{c}^\kappa|_{\mathcal{U}_\beta}$ was denoted $RR^K_\beta(M, L)$ in [Paradan 2001].

A simple application of the excision property [Paradan 2001, Section 4] gives

$$\mathcal{Q}_K^{\Phi}(M_{<R}) = \sum_{\beta\in\mathcal{B}\cap B_{\sqrt{R}}} \mathcal{Q}_K^{\beta}(M), \tag{18}$$

where the sum is finite, thanks to Proposition 2.7.

For a dominant weight $\gamma \in \widehat{K}$, the positive number

$$a_\gamma = \|\gamma+\rho\|^2 - \|\rho\|^2 \geq \|\gamma\|^2$$

corresponds to the eigenvalue of the Casimir operator acting on the irreducible representation V_γ^K. Ma and Zhang [2008, Theorem 2.1] prove that the support of the generalized character $\mathcal{Q}_K^{\beta}(M)$ is contained in $\{\gamma\in\widehat{K} \mid a_\gamma \geq \|\beta\|^2\}$.

We propose another proof which refines Ma and Zhang's result and uses a different method. They used Atiyah–Patodi–Singer index theory on manifolds with boundary whereas we use localization and induction formulae for our transversally elliptic index.

Theorem 2.9. *The generalized character $\mathcal{Q}_K^{\beta}(M)$ is supported outside the open ball $B_{\|\beta\|}$.*

Proof. The proof uses computations done in [Paradan 2001].

First consider the case where $\beta\neq 0$ is a K-invariant element of $\mathcal{B}$. Let $i:\mathbb{T}_\beta\hookrightarrow T$ be the compact torus generated by β. If F is $\mathbb{Z}$-module denote by $F\widehat{\otimes}R^{-\infty}(\mathbb{T}_\beta)$ the $\mathbb{Z}$-module formed by the infinite formal sums $\sum_a E_a\, h^a$ taken over the set of weights of $\mathbb{T}_\beta$, where $E_a\in F$ for every a.

Since $\mathbb{T}_\beta$ lies in the center of K, the morphism $\pi:(k,t)\in K\times\mathbb{T}_\beta\mapsto kt\in K$ induces a map $\pi^*:R^{-\infty}(K)\to R^{-\infty}(K)\,\widehat{\otimes}\,R^{-\infty}(\mathbb{T}_\beta)$.

The normal bundle $\mathcal{N}$ of $M^{\tilde\beta}$ in M inherits a canonical complex structure $J_{\mathcal{N}}$ on the fibers. Denote by $\overline{\mathcal{N}}\to M^{\tilde\beta}$ the complex vector bundle with the opposite complex structure. The torus $\mathbb{T}_\beta$ is included in the center of K, so the bundle $\overline{\mathcal{N}}$ and the virtual bundle $0:\wedge_{\mathbb{C}}^{\bullet}\overline{\mathcal{N}} := \wedge_{\mathbb{C}}^{\mathrm{even}}\overline{\mathcal{N}}\to\wedge_{\mathbb{C}}^{\mathrm{odd}}\overline{\mathcal{N}}$ carry a $K\times\mathbb{T}_\beta$-action. Thus they can be considered as elements of $\mathbf{K}^0_{K\times\mathbb{T}_\beta}(M^{\tilde\beta}) = \mathbf{K}^0_K(M^{\tilde\beta})\otimes R(\mathbb{T}_\beta)$.

In [Paradan 2001], we defined an inverse of $\wedge_{\mathbb{C}}^{\bullet}\overline{\mathcal{N}}$,

$$[\wedge_{\mathbb{C}}^{\bullet}\overline{\mathcal{N}}]_\beta^{-1}\in\mathbf{K}^0_K(M^{\tilde\beta})\,\widehat{\otimes}\,R^{-\infty}(\mathbb{T}_\beta),$$

which is polarized by β. This means that $[\wedge_{\mathbb{C}}^{\bullet}\overline{\mathcal{N}}]_\beta^{-1} = \sum_a N_a\, h^a$ with $N_a\neq 0$ only if $(a,\beta)\geq 0$. [Paradan 2001, Theorem 5.8] proved the localization formula

$$\pi^*[\mathcal{Q}_K^{\beta}(M)] = RR_\beta^{K\times\mathbb{T}_\beta}\big(M^{\tilde\beta}, L|_{M^{\tilde\beta}}\otimes[\wedge_{\mathbb{C}}^{\bullet}\overline{\mathcal{N}}]_\beta^{-1}\big), \tag{19}$$

as an equality in $R^{-\infty}(K)\,\widehat{\otimes}\,R^{-\infty}(\mathbb{T}_\beta)$. Let $\mathcal{A}$ be the set of connected components of $M^{\widetilde{\beta}}$ that intersect $\Phi^{-1}(\beta)$. For any equivariant vector bundle E on $M^{\widetilde{\beta}}$, we have

$$RR_\beta^{K\times\mathbb{T}_\beta}(M^{\widetilde{\beta}}, E) = \sum_{Z\in\mathcal{A}} RR_\beta^{K\times\mathbb{T}_\beta}(Z, E|_Z).$$

For any weight μ, denote by $\mathbb{C}_\mu$ the 1-dimensional representation of the maximal torus T (which contains $\mathbb{T}_\beta$). We use now the crucial lemma which is a direct consequence of [Paradan 2001, Lemma 9.4].

Lemma 2.10. *The irreducible representation V_μ^K occurs in $RR_\beta^{K\times\mathbb{T}_\beta}(Z, E|_Z)$ only if the vector bundle* $\mathrm{Hom}_{\mathbb{T}_\beta}(\mathbb{C}_\mu, E|_Z)$ *is nonzero.*

Thus V_μ^K occurs in the character $RR_\beta^{K\times\mathbb{T}_\beta}(M^{\widetilde{\beta}}, E)$ only if $\mathrm{Hom}_{\mathbb{T}_\beta}(\mathbb{C}_\mu, E|_Z)\neq 0$ for some $Z\in\mathcal{A}$.

For $E = L|_{M^{\widetilde{\beta}}}\otimes[\wedge_{\mathbb{C}}^{\bullet}\overline{\mathcal{N}}]_\beta^{-1}$ and any $Z\in\mathcal{A}$, we check that the vector bundle $\mathrm{Hom}_{\mathbb{T}_\beta}(\mathbb{C}_\mu, E|_Z)$ is nonzero only if $(\mu,\beta)\geq\|\beta\|^2$. Hence V_μ^K occurs in the character $\mathcal{Q}_K^\beta(M)$ only if $(\mu,\beta)\geq\|\beta\|^2$.

Now consider the case were $\beta\in\mathcal{B}$ is not a K-invariant element. Let σ be the unique open face of the Weyl chamber $\mathfrak{t}_+^*$ which contains β. Let K_σ be the corresponding stabilizer subgroup. Following [Guillemin and Sternberg 1984], we introduce a K_σ-invariant open subset U_σ of $\mathfrak{k}_\sigma^*$ as $U_\sigma := K_\sigma\cdot\{y\in\mathfrak{t}_+^* \mid K_y\subset K_\sigma\}$. By construction, U_σ is a slice for the coadjoint action at any $\xi\in\sigma$; see [Lerman et al. 1998, Definition 3.1]. This means that the map $K\times U_\sigma\to\mathfrak{k}^*$, $(k,\xi)\mapsto k\cdot\xi$ factors through an inclusion $K\times_{K_\sigma}U_\sigma\hookrightarrow\mathfrak{k}^*$. The symplectic cross-section theorem [Guillemin and Sternberg 1984] asserts that the preimage

$$\mathcal{Y}_\sigma := \Phi^{-1}(U_\sigma)$$

is a K_σ-invariant symplectic submanifold prequantized by the line bundle $L|_{\mathcal{Y}_\sigma}$. The restriction of Φ to $\mathcal{Y}_\sigma$ is a moment map $\Phi_\sigma:\mathcal{Y}_\sigma\to\mathfrak{k}_\sigma^*$ that is proper as a map from $\mathcal{Y}_\sigma$ into U_σ. The set

$$K_\sigma\cdot(\mathcal{Y}_\sigma^{\widetilde{\beta}}\cap\Phi_\sigma^{-1}(\beta)) = M^{\widetilde{\beta}}\cap\Phi^{-1}(\beta)$$

is a component of $\mathrm{Cr}(\|\Phi_\sigma\|^2)$. Let $\mathcal{Q}_{K_\sigma}^\beta(\mathcal{Y}_\sigma)\in R_{\mathrm{tc}}^{-\infty}(K_\sigma)$ be the corresponding character (see Definition 2.8).

In [Paradan 2001, Theorem 7.5], we proved the induction formula

$$\mathcal{Q}_K^\beta(M) = \mathrm{Hol}_{K_\sigma}^K\big(\mathcal{Q}_{K_\sigma}^\beta(\mathcal{Y}_\sigma)\big), \tag{20}$$

where $\mathrm{Hol}_{K_\sigma}^K : R^{-\infty}(K_\sigma)\to R^{-\infty}(K)$ is the holomorphic induction map. See [Paradan 2001, Appendix] for the definition and properties of these induction maps.

We know from the previous case that

$$\mathcal{Q}^{\beta}_{K_\sigma}(\mathcal{Y}_\sigma) = \sum_{\mu \in \widehat{K_\sigma}} m_\mu V^{K_\sigma}_\mu,$$

where $m_\mu \neq 0$ implies $(\mu, \beta) \geq \|\beta\|^2$. Then, with (20), we get

$$\begin{aligned}\mathcal{Q}^{\beta}_{K}(M) &= \sum_{(\mu,\beta)\geq \|\beta\|^2} m_\mu \operatorname{Hol}^K_{K_\sigma}(V^{K_\sigma}_\mu)\\ &= \sum_{(\mu,\beta)\geq \|\beta\|^2} m_\mu \operatorname{Hol}^K_T(t^\mu),\end{aligned}$$

where $\operatorname{Hol}^K_T : R^{-\infty}(T) \to R^{-\infty}(K)$ is the holomorphic induction map. Here we use that $\operatorname{Hol}^K_T = \operatorname{Hol}^K_{K_\sigma} \circ \operatorname{Hol}^{K_\sigma}_T$ and that $V^{K_\sigma}_\mu = \operatorname{Hol}^{K_\sigma}_T(t^\mu)$ for $\mu \in \widehat{K_\sigma} \subset \wedge^*$ (see [Paradan 2001, Appendix]).

Let ρ be half the sum of the positive roots. The term $\operatorname{Hol}^K_T(t^\mu)$ is equal to 0 when $\mu + \rho$ is not a regular element of $\mathfrak{t}^*$. When $\mu + \rho$ is a regular element of $\mathfrak{t}^*$, we have $\operatorname{Hol}^K_T(t^\mu) = (-1)^{|\omega|} V^K_{\mu_\omega}$, where

$$\mu_\omega = \omega(\mu + \rho) - \rho$$

is dominant for a unique element ω of the Weyl group.

Finally, a representation V^K_λ appears in the character $\mathcal{Q}^{\beta}_{K}(M)$ only if $\lambda = \mu_\omega$ for a weight μ satisfying $(\mu, \beta) \geq \|\beta\|^2$. Hence, for such λ, we have

$$\begin{aligned}\|\lambda\| &= \|\mu + \rho - \omega^{-1}\rho\|\\ &\geq \left(\mu + \rho - \omega^{-1}\rho, \frac{\beta}{\|\beta\|}\right)\\ &\geq \|\beta\|.\end{aligned}$$

In the last inequality we use that $(\rho - \omega^{-1}\rho, \beta) \geq 0$ since $\rho - \omega^{-1}\rho$ is a sum of positive roots, and $\beta \in \mathfrak{t}^*_+$. ☐

With the help of Theorem 2.9 and decomposition (18), we see that the multiplicity of V^K_γ in $\mathcal{Q}^{\Phi}_K(M_{<R})$ does not depend on the regular value $R > \|\gamma\|^2$.

Definition 2.11. The generalized character $\mathcal{Q}^{\Phi}_K(M)$ is defined as the limit of characters $\mathcal{Q}^{\Phi}_K(M_{<R})$ in $R^{-\infty}(K)$ when R goes to infinity. In other words

$$\mathcal{Q}^{\Phi}_K(M) = \sum_{\beta \in \mathcal{B}} \mathcal{Q}^{\beta}_K(M). \tag{21}$$

For any regular value R of $\|\Phi\|^2$ we have the useful relation

$$\mathcal{Q}^{\Phi}_K(M) = \mathcal{Q}^{\Phi}_K(M_{<R}) + O(\sqrt{R}). \tag{22}$$

2C. ***Quantization of a symplectic quotient.*** We will now explain how to define the geometric quantization of *singular* compact Hamiltonian manifolds, where "singular" means that the manifold is obtained by symplectic reduction.

Let (N, Ω) be a smooth symplectic manifold equipped with a Hamiltonian action of $K_1 \times K_2$. Denote by $(\Phi_1, \Phi_2) : N \to \mathfrak{k}_1^* \times \mathfrak{k}_2^*$ the corresponding moment map. Assume that N is prequantized by a $(K_1 \times K_2)$-equivariant line bundle L and suppose that the map Φ_1 is *proper*. One wants to define the geometric quantization of the compact symplectic quotient

$$N/\!\!/_0 K_1 := \Phi_1^{-1}(0)/K_1$$

which is in general singular.

Let κ_1 be the Kirwan vector field attached to the moment map Φ_1. Denote by $\mathbf{c}^{\kappa_1}$ the symbol $\mathrm{Thom}(N, J) \otimes L$ pushed by the vector field κ_1. For any regular value R_1 of $\|\Phi_1\|^2$, consider the restriction $\mathbf{c}^{\kappa_1}|_{N_{<R_1}}$ to the invariant, open subset $N_{<R_1} := \{\|\Phi_1\|^2 < R_1\}$. The symbol $\mathbf{c}^{\kappa_1}|_{N_{<R_1}}$ is $(K_1 \times K_2)$-equivariant and K_1-transversally elliptic, hence we can consider its index

$$\mathrm{Index}^{K_1\times K_2}_{N_{<R_1}}(\mathbf{c}^{\kappa_1}|_{N_{<R_1}}) \in R^{-\infty}(K_1 \times K_2),$$

which is smooth relative to the parameter in K_2. Consider the following extension of Definition 2.11.

Definition 2.12. The generalized character $\mathcal{Q}^{\Phi_1}_{K_1\times K_2}(N)$ is defined as the limit in $R^{-\infty}(K_1 \times K_2)$ of $\mathrm{Index}^{K_1\times K_2}_{N_{<R_1}}(\mathbf{c}^{\kappa_1}|_{N_{<R_1}})$ when R_1 goes to infinity.

Here $\mathrm{Cr}(\|\Phi_1\|^2)$ is equal to the disjoint union of the compact $(K_1 \times K_2)$-invariant subsets $Z_{\beta_1} := K_1\cdot(\widetilde{M^{\beta_1}}\cap\Phi_1^{-1}(\beta_1))$, $\beta_1 \in \mathcal{B}_1$. For $\beta_1 \in \mathcal{B}_1$, consider an invariant relatively compact open subset $\mathcal{U}_{\beta_1}$ such that: $Z_{\beta_1} \subset \mathcal{U}_{\beta_1}$ and $Z_{\beta_1} = \mathrm{Cr}(\|\Phi_1\|^2) \cap \overline{\mathcal{U}_{\beta_1}}$. Let $\mathcal{Q}^{\beta_1}_{K_1\times K_2}(N) \in R^{-\infty}(K_1 \times K_2)$ be the equivariant index of the K_1-transversally elliptic symbol $\mathbf{c}^{\kappa_1}|_{\mathcal{U}_{\beta_1}}$. The K_1-transversality condition imposes that $\mathcal{Q}^{\beta_1}_{K_1\times K_2}(N) = \sum_\lambda \theta^{\beta_1}(\lambda) \otimes V^{K_1}_\lambda$ with

$$\theta^{\beta_1}(\lambda) \in R(K_2) \quad \text{for all } \lambda \in \widehat{K_1}.$$

We have the following extension of Theorem 2.9:

Theorem 2.13. *We have*

$$\mathcal{Q}^{\beta_1}_{K_1\times K_2}(N) = \sum_{\lambda\in\widehat{K_1}} \theta^{\beta_1}(\lambda) \otimes V^{K_1}_\lambda,$$

where $\theta^{\beta_1}(\lambda) \neq 0$ *only if* $\|\lambda\| \geq \|\beta_1\|$.

Proof. The proof works exactly like that of Theorem 2.9. □

We now explain the "quantization commutes with reduction theorem", or why we can consider the geometric quantization of

$$N/\!\!/_0 K_1 := \Phi_1^{-1}(0)/K_1$$

as the K_1-invariant part of $\mathcal{Q}^{\Phi_1}_{K_1\times K_2}(N)$.

First suppose that 0 is a regular value of Φ_1. Then $N/\!\!/_0 K_1$ is a compact symplectic *orbifold* equipped with a Hamiltonian action of K_2: the corresponding moment map is induced by the restriction of Φ_2 to $\Phi_1^{-1}(0)$. The symplectic quotient $N/\!\!/_0 K_1$ is prequantized by the line orbibundle

$$L_0 := (L|_{\Phi_1^{-1}(0)})/K_1.$$

Definition 1.1 extends to the orbifold case. We can still define the geometric quantization of $N/\!\!/_0 K_1$ as the index of an elliptic operator and denote it by $\mathcal{Q}_{K_2}(N/\!\!/_0 K_1) \in R(K_2)$.

Theorem 2.14. *If* 0 *is a regular value of* Φ_1, *the* K_1*-invariant part of* $\mathcal{Q}^{\Phi_1}_{K_1\times K_2}(N)$ *is equal to* $\mathcal{Q}_{K_2}(N/\!\!/_0 K_1) \in R(K_2)$.

Suppose now that 0 is not a regular value of Φ_1. Let T_1 be a maximal torus of K_1, and let $\mathfrak{t}^*_{1,+} \subset \mathfrak{t}^*_1$ be a Weyl chamber. Since Φ_1 is proper, the convexity theorem says that the image of Φ_1 intersects $\mathfrak{t}^*_{1,+}$ in a closed locally polyhedral convex set, which we denote by $\Delta_{K_1}(N)$ [Lerman et al. 1998].

Consider an element $a \in \Delta_{K_1}(N)$ which is generic and sufficiently close to $0 \in \Delta_{K_1}(N)$. Denote by $(K_1)_a$ the subgroup of K_1 which stabilizes a. When $a \in \Delta_{K_1}(N)$ is generic, one can show (see [Meinrenken and Sjamaar 1999]) that

$$N/\!\!/_a K_1 := \Phi_{K_1}^{-1}(a)/(K_1)_a$$

is a compact Hamiltonian K_2-orbifold, and that

$$L_a := (L|_{\Phi_{K_1}^{-1}(a)})/(K_1)_a.$$

is a K_2-equivariant line orbibundle over $N/\!\!/_a K_1$. We can then define, like in Definition 1.1, the element $\mathcal{Q}_{K_2}(N/\!\!/_a K_1) \in R(K_2)$ as the equivariant index of the Dolbeault–Dirac operator on $N/\!\!/_a K_1$ (with coefficients in L_a).

Theorem 2.15. *The* K_1*-invariant part of* $\mathcal{Q}^{\Phi_1}_{K_1\times K_2}(M)$ *is equal to* $\mathcal{Q}_{K_2}(N/\!\!/_a K_1) \in R(K_2)$. *In particular, the elements* $\mathcal{Q}_{K_2}(N/\!\!/_a K_1)$ *do not depend on the choice of the generic element* $a \in \Delta_{K_1}(N)$, *when* a *is sufficiently close to* 0.

Proofs of Theorems 2.14 and 2.15. When N is compact and $K_2 = \{e\}$, the proofs can be found in [Meinrenken and Sjamaar 1999; Paradan 2001]. We explain briefly how the $\mathbf{K}^0$-theoretic proof of [Paradan 2001] extends naturally to our case. Like in Definition 2.11, we have the decomposition

$$\mathcal{Q}^{\Phi_1}_{K_1\times K_2}(N)=\sum_{\beta\in\mathcal{B}_1}\mathcal{Q}^{\beta_1}_{K_1\times K_2}(N),$$

and Theorem 2.13 implies $[\,\mathcal{Q}^{\beta_1}_{K_1\times K_2}(N)]^{K_1}=0$ if $\beta_1\neq 0$. This proves the first step:

$$[\,\mathcal{Q}^{\Phi_1}_{K_1\times K_2}(N)]^{K_1}=[\,\mathcal{Q}^{0}_{K_1\times K_2}(N)]^{K_1}.$$

The analysis of the term $[\,\mathcal{Q}^{0}_{K_1\times K_2}(N)]^{K_1}$ is undertaken in [Paradan 2001] when $K_2=\{e\}$: this term is equal either to $\mathcal{Q}(N/\!\!/_0K_1)$ when 0 is a regular value (see [Paradan 2001, Section 6.2]), or to $\mathcal{Q}(N/\!\!/_aK_1)$ with a generic (see [Paradan 2001, Section 7.4]). It works similarly with an action of a compact Lie group K_2. □

Definition 2.16. The geometric quantization of $N/\!\!/_0K_1:=\Phi_1^{-1}(0)/K_1$ is taken as the K_1-invariant part of $\mathcal{Q}^{\Phi_1}_{K_1\times K_2}(N)$. Denote it by $\mathcal{Q}_{K_2}(N/\!\!/_0K_1)\in R(K_2)$.

2D. ***Quantization of points.*** Let (M,Ω,Φ) be a proper Hamiltonian K-manifold prequantized by a Kostant–Souriau line bundle L. Let $\mu\in\widehat{K}$ be a dominant weight such that $\Phi^{-1}(K\cdot\mu)$ is a K-orbit in M. Let $m^o\in\Phi^{-1}(\mu)$ so that

$$\Phi^{-1}(K\cdot\mu)=K\cdot m^o.$$

Then the reduced space $M_\mu:=\Phi^{-1}(K\cdot\mu)/K$ is a point. The aim of this section is to compute the quantization of M_μ: $\mathcal{Q}(M_\mu)\in\mathbb{Z}$.

The stabilizer subgroup H of m^o is contained in the subgroup $K_\mu\subset K$ that fixes $\mu\in\mathfrak{t}^*$. We have a linear action of H on the 1-dimensional vector space $L_{m^o}\subset L$.

Let $\mathfrak{k}_\mu$ be the Lie algebra of K_μ. We recall why the Lie algebra morphism $i\mu:\mathfrak{k}_\mu\to i\mathbb{R}$ integrates in a character χ_μ of K_μ. The group K_μ, which is connected, decomposes as $K_\mu=[K_\mu,K_\mu]Z_\mu$, where Z_μ is the connected component of the center of K_μ. For the maximal torus T, we have $T=T_\mu Z_\mu$ with $T_\mu=T\cap[K_\mu,K_\mu]=\exp(\mathfrak{t}\cap[\mathfrak{k}_\mu,\mathfrak{k}_\mu])$. The morphism $i\mu:\mathfrak{t}\to i\mathbb{R}$ integrates in a character χ^T_μ of T which is trivial on T_μ since $\langle\mu,[\mathfrak{k}_\mu,\mathfrak{k}_\mu]\rangle=0$. Hence we can define the character χ_μ as being trivial on $[K_\mu,K_\mu]$, and equal to χ^T_μ on Z_μ.

Let $\mathbb{C}_{-\mu}$ be the 1-dimensional representation of K_μ associated to the character χ_μ^{-1}. Denote by χ the character of H defined by the 1-dimensional representation $\mathbb{C}_\chi:=L_{m^o}\otimes\mathbb{C}_{-\mu}$. We know from the Kostant formula (1) that $\chi=1$ on the identity component $H^o\subset H$.

Theorem 2.17. *We have*

$$\mathcal{Q}(M_\mu)=\begin{cases}1 & \text{if } \chi=1 \text{ on } H,\\ 0 & \text{otherwise.}\end{cases}\tag{23}$$

This theorem tells us in particular that $\mathcal{Q}(M_\mu) = 1$ when the stabilizer subgroup $H \subset K$ of a point $m^o \in \Phi^{-1}(\mu)$ is *connected.*

Proof. Let $N = M \times \overline{K \cdot \mu}$ be the proper Hamiltonian K-manifold which is prequantized by the line bundle $L_N := L \otimes [\mathbb{C}_{-\mu}]$. Denote by Φ_N the moment map on N. Since $\Phi^{-1}(K \cdot \mu)$ is a K-orbit in M, we see that $\Phi_N^{-1}(0)$ is the K-orbit through $n^o := (m^o, \mu)$ where $m^o \in \Phi^{-1}(\mu)$. Note that H is the stabilizer subgroup of n^o.

Let $\mathcal{Q}_K^{\Phi_N}(N) \in R^{-\infty}(K)$ be the formal quantization of N through the proper map Φ_N. We know by Theorem 2.15 and Definition 2.16 that

$$\begin{aligned}\mathcal{Q}(M_\mu) &= [\mathcal{Q}_K^{\Phi_N}(N)]^K \\ &= [\mathcal{Q}_K^0(N)]^K,\end{aligned}$$

where $\mathcal{Q}_K^0(N)$ depends only of a neighborhood of $\Phi_N^{-1}(0)$.

The orbit $K \cdot n^o \hookrightarrow N$ is an isotropic embedding since it is the 0-level of the moment map Φ_N. To describe a K-invariant neighborhood of $K \cdot n^o$ in N we can use the normal-form recipe of Marle, Guillemin and Sternberg.

First consider, following [Weinstein 1979], the symplectic normal bundle

$$\mathcal{V} := \mathrm{T}(K \cdot n^o)^\perp / \mathrm{T}(K \cdot n^o), \tag{24}$$

where the orthogonal $^\perp$ is taken relative to the symplectic 2-form. We have

$$\mathcal{V} = K \times_H V,$$

where the vector space $V := \mathrm{T}_{n^o}(K \cdot n^o)^\perp / \mathrm{T}_{n^o}(K \cdot n^o)$ inherits a canonical symplectic structure Ω_V and a Hamiltonian action of the group H. Let $\Phi_V : V \to \mathfrak{h}^*$ be the corresponding moment map.

Consider now the symplectic manifold

$$\widetilde{N} := \mathcal{V} \oplus \mathrm{T}^*(K/H) = K \times_H \big((\mathfrak{k}/\mathfrak{h})^* \oplus V\big). \tag{25}$$

The action of K on $\widetilde{N}$ is Hamiltonian with moment map $\Phi_{\widetilde{N}} : \widetilde{N} \to \mathfrak{k}^*$ given by

$$\Phi_{\widetilde{N}}([k; \xi, v]) = k \cdot (\xi + \Phi_V(v)) \quad \text{for } k \in K,\ \xi \in (\mathfrak{k}/\mathfrak{h})^*,\ v \in V. \tag{26}$$

The Hamiltonian K-manifold $\widetilde{N}$ is prequantized by the line bundle $L_{\widetilde{N}} := K \times_H \mathbb{C}_\chi$.

The *local normal form* theorem (see [Guillemin and Sternberg 1984; Sjamaar and Lerman 1991, Proposition 2.5]) tells us that there exists a K-Hamiltonian isomorphism $\Upsilon : \mathcal{U}_1 \to \mathcal{U}_2$ between a K-invariant neighborhood $\mathcal{U}_1$ of $K \cdot n^o$ in N, and a K-invariant neighborhood $\mathcal{U}_2$ of K/H in $\widetilde{N}$. This isomorphism Υ, when restricted to $K \cdot n^o$, corresponds to the natural isomorphism $K \cdot n^o \to K/H$.

We check that

$$\{\Phi_{\widetilde{N}} = 0\} = K \times_H \{\Phi_V = 0\} \quad \text{and} \quad \mathrm{Cr}(\|\Phi_{\widetilde{N}}\|^2) = K \times_H \mathrm{Cr}(\|\Phi_V\|^2).$$

See (30). Let κ_V be the Kirwan vector field associated to the Hamiltonian action of H on (V, Ω_V). A simple computation gives

$$\Omega(\kappa_V(v), v) = -2\|\Phi_V(v)\|^2,$$

which implies that $\mathrm{Cr}(\|\Phi_V\|^2) = \{\Phi_V = 0\}$ and then $\mathrm{Cr}(\|\Phi_{\widetilde{N}}\|^2) = \{\Phi_{\widetilde{N}} = 0\}$. Note that $\{\Phi_V = 0\}$ is a cone in V since the map Φ_V is quadratic. The map Υ sends $\{\Phi_N = 0\} \cap \mathcal{U}_1$ onto $\{\Phi_{\widetilde{N}} = 0\} \cap \mathcal{U}_2$. Our hypothesis imposes that $\{\Phi_N = 0\}$ is reduced to a K-orbit, therefore the cone $\{\Phi_V = 0\}$ is reduced to $\{0\}$; this last point is equivalent to the fact that Φ_V (and then $\Phi_{\widetilde{N}}$) is proper map (see [Paradan 2009, Lemma 5.2]).

We get the equalities

$$\mathcal{Q}^0_K(N) = \mathcal{Q}^0_K(\widetilde{N}) = \mathcal{Q}^{\Phi_{\widetilde{N}}}_K(\widetilde{N}). \tag{27}$$

The first equality follows from Proposition 2.6 (applied to the isomorphism Υ), and the second one is due to the fact that $\mathrm{Cr}(\|\Phi_{\widetilde{N}}\|^2) = \Phi^{-1}_{\widetilde{N}}(0)$.

Let $\mathrm{Ind}^K_H : R^{-\infty}(H) \to R^{-\infty}(K)$ be the induction map that is defined by the relation $\langle \mathrm{Ind}^K_H(\varphi), E\rangle = \langle \varphi, E|_H\rangle$ for any $\varphi \in R^{-\infty}(H)$ and $E \in R(K)$. Note that

$$[\mathrm{Ind}^K_H(\varphi)]^K = \langle \mathrm{Ind}^K_H(\varphi), \mathbb{C}\rangle = \langle \varphi, \mathbb{C}\rangle = [\varphi]^H.$$

Since $\Phi_V : V \to \mathfrak{h}^*$ is proper, one can consider the quantization of the vector space V through the moment map Φ_V: $\mathcal{Q}^{\Phi_V}_H(V) \in R^{-\infty}(H)$. In the next proposition we consider an H-invariant complex structure J_V on V which is compatible with the symplectic structure Ω_V, and V^* denotes the complex H-module $\hom_{\mathbb{C}}(V, \mathbb{C})$.

Proposition 2.18. • *We have*

$$\mathcal{Q}^{\Phi_{\widetilde{N}}}_K(\widetilde{N}) = \mathrm{Ind}^K_H(\mathcal{Q}^{\Phi_V}_H(V) \otimes \mathbb{C}_\chi). \tag{28}$$

- *The formal quantization $\mathcal{Q}^{\Phi_V}_H(V)$ coincides, as a generalized H-module, to the H-module $S(V^*)$ of complex polynomial function on V.*
- *The set $[S(V^*)]^{H^o}$ of polynomials invariant by the connected component H^o is reduced to the scalars.*

With Proposition 2.18, we can finish proving Theorem 2.17 with a calculation:

$$\begin{aligned}\mathcal{Q}(M_\mu) &= [\,\mathcal{Q}^{\Phi}_K(N)]^K\\ &= [\,\mathcal{Q}^{\Phi_{\widetilde{N}}}_K(\widetilde{N})]^K\\ &= [\,\mathcal{Q}^{\Phi_V}_H(V) \otimes \mathbb{C}_\chi]^H\\ &= [S(V^*) \otimes \mathbb{C}_\chi]^H = [\mathbb{C}_\chi]^H.\end{aligned}$$

Proof of Proposition 2.18. The first point is implied by the induction property defined by Atiyah (see [Paradan 2001, Section 3.4]) by the following argument: We work with the H-manifold[5] $\mathcal{Y}=\mathfrak{k}/\mathfrak{h}\oplus V$ and the H-equivariant map $j:\mathcal{Y}\hookrightarrow\widetilde{N}:=K\times_H\mathcal{Y}$, $y\mapsto[e,y]$.

Notice[6] that $\mathrm{T}\widetilde{N}\simeq K\times_H(\mathfrak{k}/\mathfrak{h}\oplus\mathrm{T}\mathcal{Y})$ and that $\mathrm{T}_K\widetilde{N}\simeq K\times_H(\mathrm{T}_H\mathcal{Y})$. Hence the map j induces an isomorphism $j_*:\mathbf{K}^0_H(\mathrm{T}_H\mathcal{Y})\to\mathbf{K}^0_K(\mathrm{T}_K\widetilde{N})$. By [Atiyah 1974, Theorem 4.1], the diagram

$$\begin{tikzcd}
\mathbf{K}^0_H(\mathrm{T}_H\mathcal{Y}) \arrow[r, "j_*"] \arrow[d, "\mathrm{Index}^H_{\mathcal{Y}}"'] & \mathbf{K}^0_K(\mathrm{T}_K\widetilde{N}) \arrow[d, "\mathrm{Index}^K_{\widetilde{N}}"] \\
R^{-\infty}(H) \arrow[r, "\mathrm{Ind}^K_H"'] & R^{-\infty}(K)
\end{tikzcd} \tag{29}$$

is commutative. The tangent bundle $\mathrm{T}\widetilde{N}$ is equivariantly diffeomorphic to

$$K\times_H[\mathfrak{k}/\mathfrak{h}\oplus\mathrm{T}(\mathfrak{k}/\mathfrak{h})\oplus\mathrm{T}V]\simeq K\times_H[\mathcal{Y}\times((\mathfrak{k}/\mathfrak{h})_{\mathbb{C}}\oplus V)],$$

where $(\mathfrak{k}/\mathfrak{h})_{\mathbb{C}}$ is the complexification of the real vector space $\mathfrak{k}/\mathfrak{h}$. Consider on $\widetilde{N}$ the almost complex structure $J_{\widetilde{N}}=(i,J_V)$ for i the complex structure on $(\mathfrak{k}/\mathfrak{h})_{\mathbb{C}}$. Note that $J_{\widetilde{N}}$ is compatible with the symplectic structure on a neighborhood U of the 0-section of the bundle $\widetilde{N}\to K/H$.

We compute the Kirwan vector field $\kappa_{\widetilde{N}}$ on $\widetilde{N}$. If we take $Y=k\cdot X$ and $\widetilde{n}=[k;\xi\oplus v]\in\widetilde{N}$ we have the following relations in $\mathrm{T}_{\widetilde{n}}\widetilde{N}\simeq(\mathfrak{k}/\mathfrak{h})_{\mathbb{C}}\oplus V$:

- $Y_{\widetilde{N}}(\widetilde{n})=-X$ when $X\in\mathfrak{k}/\mathfrak{h}$,
- $Y_{\widetilde{N}}(\widetilde{n})=i[\xi,X]\oplus-X\cdot v$ when $X\in\mathfrak{h}$.

By taking $Y=\Phi_{\widetilde{N}}([k;\xi,v])=k\cdot(\xi+\Phi_V(v))$ we get

$$\kappa_{\widetilde{N}}([k;\xi,v])=-\xi+i\,[\xi,\Phi_V(v)]\oplus\kappa_V(v)\in(\mathfrak{k}/\mathfrak{h})_{\mathbb{C}}\oplus V. \tag{30}$$

Since κ_V vanishes only on $\{0\}\subset V$, the vector field $\kappa_{\widetilde{N}}$ vanishes exactly on the 0-section of the bundle $\widetilde{N}\to K/H$.

Let $\mathbf{c}^{\kappa_{\widetilde{N}}}$ be the symbol $\mathrm{Thom}(\widetilde{N},J_{\widetilde{N}})\otimes L_{\widetilde{N}}$ pushed by the vector field $\kappa_{\widetilde{N}}$. The generalized character $\mathcal{Q}^{\Phi_{\widetilde{N}}}_K(\widetilde{N})$ is either computed as the equivariant index of the symbols $\mathbf{c}^{\kappa_{\widetilde{N}}}$ or $\mathbf{c}^{\kappa_{\widetilde{N}}}|_U$.

Remark 2.19. The fact that $J_{\widetilde{N}}$ is not compatible on the entire manifold $\widetilde{N}$ is not problematic, since $J_{\widetilde{N}}$ is compatible in a neighborhood U of the set where $\kappa_{\widetilde{N}}$ vanishes. See the first point of Proposition 2.6.

---

[5] We have an H-equivariant identification $(\mathfrak{k}/\mathfrak{h})^*\simeq\mathfrak{k}/\mathfrak{h}$.

[6] These identities come from the following $(K\times H)$-equivariant isomorphism of vector bundles over $K\times\mathcal{Y}$: $\mathrm{T}_H(K\times\mathcal{Y})\to K\times(\mathfrak{k}/\mathfrak{h}\times\mathrm{T}\mathcal{Y})$, $(k,m;\frac{d}{dt}|_{t=0}(ke^{tX})\oplus v_m)\mapsto(k,m;\mathrm{pr}_{\mathfrak{k}/\mathfrak{h}}(X)+v_m)$, where $\mathrm{pr}_{\mathfrak{k}/\mathfrak{h}}:\mathfrak{k}\to\mathfrak{k}/\mathfrak{h}$ is the orthogonal projection.

For $(X+i\eta, w) \in \mathrm{T}_{[k;\xi,v]}\widetilde{N} \simeq (\mathfrak{k}/\mathfrak{h})_{\mathbb{C}} \times V$, the map

$$\mathbf{c}^{\kappa_{\widetilde{N}}}(X+i\eta, w) = \mathbf{c}\big(X+\xi+i\eta-i[\xi, \Phi_V(v)]\big) \odot \mathbf{c}\big(w-\kappa_V(v)\big) \tag{31}$$

acts on the vector space $\wedge_{\mathbb{C}}(\mathfrak{k}/\mathfrak{h})_{\mathbb{C}} \otimes \wedge_{\mathbb{C}} V \otimes \mathbb{C}_\chi$. We see that

$$\mathbf{c}^{\kappa_{\widetilde{N}}} = j_*(\mathbf{c}^{\mathcal{Y}}),$$

where $\mathbf{c}^{\mathcal{Y}}$ is the symbol on $\mathcal{Y}$ defined as follows. For $(\xi, v) \in \mathcal{Y} = \mathfrak{k}/\mathfrak{h} \times V$, the map $\mathbf{c}^{\mathcal{Y}}|_{(\xi,v)}(\eta, w)$ acts on $\wedge_{\mathbb{C}}(\mathfrak{k}/\mathfrak{h})_{\mathbb{C}} \otimes \wedge_{\mathbb{C}} V \otimes \mathbb{C}_\chi$ as the product

$$\mathbf{c}(\xi+i\eta-i[\xi, \Phi_V(v)]) \odot \mathbf{c}(w-\kappa_V(v)).$$

Let $\mathrm{Bott}(\mathfrak{k}/\mathfrak{h})$ be the Bott symbol on the vector space $\mathfrak{k}/\mathfrak{h}$. It is an elliptic morphism defined by

$$\mathrm{Bott}(\mathfrak{k}/\mathfrak{h})|_\xi(\eta) = \mathbf{c}(\xi+i\eta) \quad \text{acting on} \quad \wedge_{\mathbb{C}}(\mathfrak{k}/\mathfrak{h})_{\mathbb{C}},$$

for $\eta \in \mathrm{T}_\xi(\mathfrak{k}/\mathfrak{h})$. Let $\mathbf{c}^{\kappa_V}$ be the symbol $\mathrm{Thom}(V, J_V)$ pushed by the vector field κ_V.

Lemma 2.20. *We have*

$$\mathbf{c}^{\kappa_{\widetilde{N}}} = j_*\big(\mathrm{Bott}(\mathfrak{k}/\mathfrak{h}) \odot \mathbf{c}^{\kappa_V} \otimes \mathbb{C}_\chi\big).$$

Proof. We work with the family of symbols σ^T, $T \in [0,1]$, on $\mathcal{Y} = \mathfrak{k}/\mathfrak{h} \times V$ defined for $(\eta, w) \in \mathrm{T}_{(\xi,v)}\mathcal{Y}$ as the map

$$\sigma^T|_{(\xi,v)}(\eta, w) = \mathbf{c}(\xi+i\eta-iT\,[\xi, \Phi_V(v)]) \odot \mathbf{c}(w-\kappa_V(v))$$

acting on the vector space $\wedge_{\mathbb{C}}(\mathfrak{k}/\mathfrak{h})_{\mathbb{C}} \otimes \wedge_{\mathbb{C}} V \otimes \mathbb{C}_\chi$. Note $\sigma^0 = \mathrm{Bott}(\mathfrak{k}/\mathfrak{h}) \odot \mathbf{c}^{\kappa_V} \otimes \mathbb{C}_\chi$, and $\sigma^1 = \mathbf{c}^{\mathcal{Y}}$. It is now easy to check that

$$\mathrm{Char}(\sigma^T) = \{(0,0) \in \mathrm{T}(\mathfrak{k}/\mathfrak{h})\} \times \{(v, \kappa_V(v)),\, v \in V\} \subset \mathrm{T}\mathcal{Y}$$

and that $\mathrm{Char}(\sigma^T) \cap \mathrm{T}_H\mathcal{Y} = \{(0,0) \in \mathrm{T}(\mathfrak{k}/\mathfrak{h})\} \times \{(0,0) \in \mathrm{T}V\}$ for any $T \in [0,1]$. Hence σ^T, $T \in [0,1]$, is a homotopy of H-transversally elliptic symbols on $\mathfrak{k}/\mathfrak{h} \times V$. It gives finally that $\mathbf{c}^{\kappa_{\widetilde{N}}} = j_*(\mathbf{c}^{\mathcal{Y}}) = j_*(\sigma^0)$. □

The commutative diagram (29) and the last lemma give

$$\begin{aligned}
\mathcal{Q}_K^{\Phi_{\widetilde{N}}}(\widetilde{N}) &= \mathrm{Index}_{\widetilde{N}}^K(\mathbf{c}^{\kappa_{\widetilde{N}}}) \\
&= \mathrm{Ind}_H^K\left(\mathrm{Index}_{\mathfrak{k}/\mathfrak{h}\times V}^H\left(\mathrm{Bott}(\mathfrak{k}/\mathfrak{h}) \odot \mathbf{c}^{\kappa_V}\right) \otimes \mathbb{C}_\chi\right) \\
&= \mathrm{Ind}_H^K\left(\mathrm{Index}_{\mathfrak{k}/\mathfrak{h}}^H(\mathrm{Bott}(\mathfrak{k}/\mathfrak{h})) \otimes \mathrm{Index}_V^H(\mathbf{c}^{\kappa_V}) \otimes \mathbb{C}_\chi\right) \\
&= \mathrm{Ind}_H^K(\mathcal{Q}_H^{\Phi_V}(V) \otimes \mathbb{C}_\chi).
\end{aligned}$$

We have used here that the H-equivariant index of $\mathrm{Bott}(\mathfrak{k}/\mathfrak{h})$ is equal to 1, that is, the trivial representation of H; see [Paradan and Vergne 2009, Section 2.4.1].

We now prove the second point of Proposition 2.18. Since the Kirwan vector field κ_V satisfies the relations $(\kappa_V(v), J_V v) = -\Omega(\kappa_V(v), v) = 2\|\Phi_V(v)\|^2$, we have

$$(\kappa_V(v), J_V v) > 0 \tag{32}$$

for $v \neq 0$. Consider on V the family of symbols σ^s:

$$\sigma^s|_v(w) = \mathbf{c}\big(w - s\kappa_V(v) - (1-s)J_V v\big)$$

viewed as a map from $\wedge_{\mathbb{C}}^{\text{even}} V$ to $\wedge_{\mathbb{C}}^{\text{odd}} V$. By (32), one sees that σ^s is a family of H-transversally elliptic symbols on V. Hence $\sigma^1 = \mathbf{c}^{\kappa_V}$ and $\sigma^0 = \mathbf{c}(w - J_V v)$ defines the same class in the group $\mathbf{K}_H^0(\mathrm{T}_H V)$. The symbol σ^0 was first studied by Atiyah [1974] when $\dim_{\mathbb{C}} V = 1$. [Paradan 2001, Proposition 5.4] considered the general case. We have

$$\mathrm{Index}_V^H(\sigma^0) = S(V^*) \quad \text{in } R^{-\infty}(H).$$

The last point of Proposition 2.18 is a consequence of the properness of the moment map Φ_V; see [Paradan 2009, Section 5]. □

This completes the proof of Theorem 2.17. □

Example 2.21 [Paradan 2009]. Consider the action of the unitary group U_n on $\mathbb{C}^n$. The symplectic form on $\mathbb{C}^n$ is defined by $\Omega(v, w) = \frac{i}{2}\sum_k v_k\overline{w_k} - \overline{v_k}w_k$. Identify the Lie algebra $\mathfrak{u}_n$ with its dual through the trace map. The moment map $\Phi : \mathbb{C}^n \to \mathfrak{u}_n$ is defined by $\Phi(v) = (1/2i)v \otimes v^*$ where $v \otimes v^* : \mathbb{C}^n \to \mathbb{C}^n$ is the linear map $w \mapsto (\sum_k \overline{v_k}w_k)v$. One checks easily that the pullback by Φ of a U_n-orbit in $\mathfrak{u}_n$ is either empty or a U_n-orbit in $\mathbb{C}^n$. We know also that the stabilizer subgroup of a nonzero vector of $\mathbb{C}^n$ is connected since it is diffeomorphic to U_{n-1}. Finally,

$$\mathcal{Q}((\mathbb{C}^n)_\mu) = \begin{cases} 1 & \text{if } \mu \in \widehat{\mathrm{U}_n} \text{ belongs to the image of } \Phi, \\ 0 & \text{if } \mu \in \widehat{\mathrm{U}_n} \text{ does not belong to the image of } \Phi. \end{cases} \tag{33}$$

Then one can check that $\mathcal{Q}_{\mathrm{U}_n}^{-\infty}(\mathbb{C}^n)$ coincides in $R^{-\infty}(\mathrm{U}_n)$ with the algebra $S((\mathbb{C}^n)^*)$ of polynomial function on $\mathbb{C}^n$.

Example 2.22 [Paradan 2003]. Consider the Lie group $\mathrm{SL}_2(\mathbb{R})$ and its compact torus of dimension 1 denoted by T. The Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ is identified with its dual through the trace map, and the Lie algebra $\mathfrak{t}$ is naturally identified with $\mathfrak{sl}_2(\mathbb{R})^T$. For $l \in \mathbb{Z} \setminus \{0\}$, consider the character χ_l of T defined by

$$\chi_l \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = e^{il\theta}.$$

Its differential $\frac{1}{i}d\chi_l \in \mathfrak{t}^*$ corresponds (through the trace map) to the matrix

$$X_l = \begin{pmatrix} 0 & l/2 \\ -l/2 & 0 \end{pmatrix}.$$

Let $\mathbb{O}_l$ be the coadjoint orbit of the group $\mathrm{SL}_2(\mathbb{R})$ through the matrix X_l. It is a Hamiltonian $\mathrm{SL}_2(\mathbb{R})$-manifold prequantized by the $\mathrm{SL}_2(\mathbb{R})$-equivariant line bundle $L_l \simeq \mathrm{SL}_2(\mathbb{R}) \times_T \mathbb{C}_l$, where $\mathbb{C}_l$ is the T-module associated to the character χ_l. We look at the Hamiltonian action of T on $\mathbb{O}_l$. Let $\Phi_T : \mathbb{O}_l \to \mathfrak{t}^*$ be the corresponding moment map. This moment map Φ_T is *proper* and its image is equal to the half-line $\{aX_l, a \geq 1\} \subset \mathfrak{t}^*$.

We check that for each $\xi \in \{aX_l, a \geq 1\}$ the fiber $\Phi_T^{-1}(\xi)$ is equal to a T-orbit in $\mathbb{O}_l$. For $k \in \mathbb{Z}$, denote by $(\mathbb{O}_l)_k$ the symplectic reduction of $\mathbb{O}_l$ at the level X_k. We know that $(\mathbb{O}_l)_k = \varnothing$ if $k \notin \{al, a \geq 1\}$, and that $(\mathbb{O}_l)_k$ is a point if $k \in \{al, a \geq 1\}$.

To compute $\mathcal{Q}((\mathbb{O}_l)_k)$, we look at the stabilizer subgroup $T_m := \{t \in T \mid t \cdot m = m\}$ for each point $m \in \mathbb{O}_l$. One sees that $T_m = T$ if $m = X_l$ and T_m is equal to the center $\{\pm \mathrm{Id}\}$ of $\mathrm{SL}_2(\mathbb{R})$, when $m \neq X_l$.

Theorem 2.17 gives in this setting that, for $k \in \{al, a \geq 1\}$,

$$\mathcal{Q}((\mathbb{O}_l)_k) = \begin{cases} 1 & \text{if } l-k \text{ is even,} \\ 0 & \text{if } l-k \text{ is odd.} \end{cases} \tag{34}$$

Hence the formal geometric quantization of the proper T-manifold $\mathbb{O}_l$ is

$$\mathcal{Q}_T^{-\infty}(\mathbb{O}_l) = \begin{cases} \mathbb{C}_l \cdot \sum_{p\geq 0} \mathbb{C}_{2p} & \text{if } l > 0, \\ \mathbb{C}_l \cdot \sum_{p\geq 0} \mathbb{C}_{-2p} & \text{if } l < 0. \end{cases} \tag{35}$$

Here the quantization $\mathcal{Q}_T^{-\infty}(\mathbb{O}_l)$ coincides with the restriction of the holomorphic (respectively antiholomorphic) discrete series representation Θ_l to the group T when $l > 0$ (respectively $l < 0$).

2E. ***Wonderful compactifications and symplectic cuts.*** Another equivalent definition of the quantization $\mathcal{Q}^{-\infty}$ uses a generalization of the technique of symplectic cutting (originally due to [Lerman 1995]) that was introduced in [Paradan 2009] and was motivated by the wonderful compactifications of [De Concini and Procesi 1983; 1985]; see also [Brion 1998].

Recall that T is a maximal torus of the compact connected Lie group K, and W is the corresponding Weyl group. Define a K*-adapted* polytope in $\mathfrak{t}^*$ to be a W-invariant Delzant polytope P in $\mathfrak{t}^*$ whose vertices are regular elements of the weight lattice $\wedge^*$. If $\{\lambda_1, \ldots, \lambda_N\}$ are the dominant weights lying in the union of all the closed one-dimensional faces of P, then there is a $(G \times G)$-equivariant embedding of $G = K_{\mathbb{C}}$ into

$$\mathbb{P}\Big(\bigoplus_{i=1}^{N}(V_{\lambda_i}^{K})^{*}\otimes V_{\lambda_i}^{K}\Big)$$

associating to $g \in G$ its representation on $\bigoplus_{i=1}^{N} V_{\lambda_i}^{K}$. The closure $\mathcal{X}_P$ of the image of G in this projective space is smooth and is equipped with a $(K\times K)$-action

$$(k_1, k_2)\cdot x = k_2 \cdot x \cdot k_1^{-1}.$$

The restriction of the canonical Kähler structure on $\mathcal{X}_P$ defines a symplectic 2-form $\Omega_{\mathcal{X}_P}$. Recall briefly the different properties of $(\mathcal{X}_P, \Omega_{\mathcal{X}_P})$ — all the details can be found in [Paradan 2009].

(1) $\mathcal{X}_P$ is equipped with an Hamiltonian action of $K \times K$. Let $\Phi := (\Phi_l, \Phi_r) : \mathcal{X}_P \to \mathfrak{k}^* \times \mathfrak{k}^*$ be the corresponding moment map.

(2) The image of Φ is equal to $\{(k\cdot\xi, -k'\cdot\xi) \mid \xi \in P \text{ and } k, k' \in K\}$.

(3) The Hamiltonian $(K\times K)$-manifold $(\mathcal{X}_P, \Omega_{\mathcal{X}_P})$ has no multiplicities: the pullback by Φ of a $(K\times K)$-orbit in the image is a $(K\times K)$-orbit in $\mathcal{X}_P$.

Let $\mathcal{U}_P := K \cdot P^{\circ}$, where P° is the interior of P. Define

$$\mathcal{X}_P^{\circ} := \Phi_l^{-1}(\mathcal{U}_P),$$

which is an invariant, open and dense subset of $\mathcal{X}_P$. We have the following important properties concerning $\mathcal{X}_P^{\circ}$.

(4) There exists an equivariant diffeomorphism $\Upsilon : K \times \mathcal{U}_P \to \mathcal{X}_P^{\circ}$ such that $\Upsilon^*(\Phi_l)(k, \xi) = k\cdot\xi$ and $\Upsilon^*(\Phi_r)(k, \xi) = -\xi$.

(5) This diffeomorphism Υ is a quasisymplectomorphism in the sense that there is a homotopy of symplectic forms taking the symplectic form on the open subset $K \times \mathcal{U}_P$ of the cotangent bundle T^*K to the pullback of the symplectic form $\Omega_{\mathcal{X}_P}$ on $\mathcal{X}_P^{o}$.

(6) The symplectic manifold $(\mathcal{X}_P, \Omega_{\mathcal{X}_P})$ is prequantized by the restriction of the hyperplane line bundle $\mathbb{O}(1) \to \mathbb{P}(\bigoplus_{i=1}^{N}(V_{\lambda_i}^{K})^{*}\otimes V_{\lambda_i}^{K})$ to $\mathcal{X}_P$: denote by L_P the corresponding $(K\times K)$-equivariant line bundle.

(7) The pullback of the line bundle L_P by the map $\Upsilon : K \times \mathcal{U}_P \hookrightarrow \mathcal{X}_P$ is trivial.

Let (M, Ω_M, Φ_M) be a proper Hamiltonian K-manifold and $\mathcal{X}_P$ be the Hamiltonian $(K\times K)$-manifold associated to a K-adapted polytope P. Consider now the product $M \times \mathcal{X}_P$ with the following $K \times K$ action:

- the action $k \cdot_1 (m, x) = (k \cdot m, x \cdot k^{-1})$, with corresponding moment map $\Phi_1(m, x) = \Phi_M(m) + \Phi_r(x)$,
- the action $k \cdot_2 (m, x) = (m, k \cdot x)$, with corresponding moment map $\Phi_2(m, x) = \Phi_l(x)$.

Definition 2.23. Denote by M_P the symplectic reduction at 0 of $M \times \mathcal{X}_P$ for the action $\cdot_1$: $M_P := (\Phi_1)^{-1}(0)/(K, \cdot_1)$.

Then M_P inherits a Hamiltonian K-action with moment map $\Phi_{M_P} : M_P \to \mathfrak{k}^*$ whose image is $\Phi_M(M) \cap K \cdot P$.

In [Paradan 2009], we proved that M_P contains an open and dense subset of smooth points which is quasisymplectomorphic to the open subset $(\Phi_M)^{-1}(\mathcal{U}_P)$. If the polytope P is fixed, we can work with the dilated polytopes nP for $n \geq 1$. We have then the family of compact, perhaps singular, K-Hamiltonian manifolds M_{nP}, $n \geq 1$. In Section 2C, we explained how their geometric quantization was defined:

$$\mathcal{Q}_K(M_{nP}) := [\mathcal{Q}^{\Phi_1}_{K \times K}(M \times \mathcal{X}_{nP})]^{(K, \cdot_1)} \in R(K).$$

We have the following convenient property of $\mathcal{Q}^{-\infty}$.

Proposition 2.24 [Paradan 2009]. *The following equality in $R^{-\infty}(K)$ holds*:

$$\mathcal{Q}^{-\infty}_K(M) = \lim_{n \to \infty} \mathcal{Q}_K(M_{nP}). \tag{36}$$

Here the limit is taken using the convention of Definition 2.3.

3. Proof of Theorem 1.4

The main result of this section is:

Theorem 3.1. *Let $r_P := \inf_{\xi \in \partial P} \|\xi\| > 0$. The generalized character*

$$\mathcal{Q}^{\Phi}_K(M) - \mathcal{Q}_K(M_P) \in R^{-\infty}(K)$$

is supported outside the ball B_{r_P}.

Then, for the dilated polytope nP, $n \geq 1$, the character $\mathcal{Q}^{\Phi}_K(M) - \mathcal{Q}_K(M_{nP})$ is supported outside the ball B_{nr_P}. Taking the limit as n goes to infinity gives

$$\mathcal{Q}^{\Phi}_K(M) = \lim_{n \to \infty} \mathcal{Q}_K(M_{nP}). \tag{37}$$

Finally, identity (6) of Theorem 1.4,

$$\mathcal{Q}^{\Phi}_K(M) = \mathcal{Q}^{-\infty}_K(M),$$

is a direct consequence of (36) and (37).

Recall that $O(r) \in R^{-\infty}(K)$ denotes any generalized character supported outside the ball B_r.

Theorem 3.1 follows from the comparison of three different geometrical situations. All of them concern Hamiltonian actions of $K_1 \times K_2$, where K_1 and K_2 are two copies of K.

First setting. We work with the Hamiltonian $(K_1 \times K_2)$-manifold $M \times \mathcal{X}_P$, where K_1 acts both on M and on $\mathcal{X}_P$. Since the moment map Φ_1 (relative to the K_1-action) is proper, we may "quantize" $M \times \mathcal{X}_P$ via the map $\|\Phi_1\|^2$. Denote the corresponding generalized character by

$$\mathcal{Q}^{\Phi_1}_{K_1 \times K_2}(M \times \mathcal{X}_P) \in R^{-\infty}(K_1 \times K_2).$$

Recall that $\mathcal{Q}_{K_2}(M_P)$ is equal to $[\mathcal{Q}^{\Phi_1}_{K_1 \times K_2}(M \times \mathcal{X}_P)]^{K_1}$.

Second setting. Consider as before the Hamiltonian action of $K_1 \times K_2$ on $M \times \mathcal{X}_P$, but "quantize" $M \times \mathcal{X}_P$ through the global moment map $\Phi = (\Phi_1, \Phi_2)$. Here we have some liberty in the choice of the scalar product on $\mathfrak{k}_1^* \times \mathfrak{k}_2^*$. If $\|\xi\|^2$ is an invariant Euclidean norm on $\mathfrak{k}^*$, we take on $\mathfrak{k}_1^* \times \mathfrak{k}_2^*$ the Euclidean norm

$$\|(\xi_1, \xi_2)\|^2_\rho = \|\xi_1\|^2 + \rho\|\xi_2\|^2 \tag{38}$$

depending on a parameter $\rho > 0$. Consider the quantization of $M \times \mathcal{X}_P$ via the map $\|\Phi\|^2_\rho$:

$$\mathcal{Q}^{\Phi,\rho}_{K_1 \times K_2}(M \times \mathcal{X}_P) \in R^{-\infty}(K_1 \times K_2).$$

Third setting. Consider the cotangent bundle T^*K with the Hamiltonian action of $K_1 \times K_2$, where K_1 acts by *right* translations and K_2 by *left* translations. Consider the Hamiltonian action of $K_1 \times K_2$ on $M \times \mathrm{T}^*K$, where K_1 acts both on M and on T^*K. Let $\Phi = (\Phi_1, \Phi_2)$ be the global moment map on $M \times \mathrm{T}^*K$. Since the moment map Φ is proper we can "quantize" $M \times \mathrm{T}^*K$ via the map $\|\Phi\|^2_\rho$. Let

$$\mathcal{Q}^{\Phi,\rho}_{K_1 \times K_2}(M \times \mathrm{T}^*K) \in R^{-\infty}(K_1 \times K_2)$$

be the corresponding generalized character.

Theorem 3.1 is a consequence of the following propositions.

First we compare $\mathcal{Q}^{\Phi}_{K_2}(M)$ with the K_1-invariant part of $\mathcal{Q}^{\Phi,\rho}_{K_1 \times K_2}(M \times \mathrm{T}^*K)$.

Proposition 3.2. *For any* $\rho \in]0, 1]$, *we have*

$$[\mathcal{Q}^{\Phi,\rho}_{K_1 \times K_2}(M \times \mathrm{T}^*K)]^{K_1} = \mathcal{Q}^{\Phi}_{K_2}(M) \quad \text{in } R^{-\infty}(K_2). \tag{39}$$

Next we compare the K_1-invariant parts of the generalized characters

$$\mathcal{Q}^{\Phi,\rho}_{K_1 \times K_2}(M \times \mathrm{T}^*K) \quad \text{and} \quad \mathcal{Q}^{\Phi,\rho}_{K_1 \times K_2}(M \times \mathcal{X}_P).$$

Proposition 3.3. *For any* $\rho \in]0, 1]$, *we have the following relation in* $R^{-\infty}(K_2)$

$$[\mathcal{Q}^{\Phi,\rho}_{K_1 \times K_2}(M \times \mathcal{X}_P)]^{K_1} - [\mathcal{Q}^{\Phi,\rho}_{K_1 \times K_2}(M \times \mathrm{T}^*K)]^{K_1} = O(r_P), \tag{40}$$

where $r_P := \inf_{\xi \in \partial P} \|\xi\| > 0$.

Finally we compare the K_1-invariant parts of the generalized characters

$$\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M\times\mathcal{X}_P)\quad\text{and}\quad\mathcal{Q}^{\Phi_1}_{K_1\times K_2}(M\times\mathcal{X}_P).$$

Proposition 3.4. *There exists* $\epsilon>0$ *such that*

$$\mathcal{Q}_{K_2}(M_P)-[\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M\times\mathcal{X}_P)]^{K_1}=O((\epsilon/\rho)^{1/2})\quad\text{in } R^{-\infty}(K_2),\tag{41}$$

if $\rho>0$ *is small enough.*

If we sum the relations (39), (40) and (41) we get

$$\mathcal{Q}^{\Phi}_{K_2}(M)=\mathcal{Q}_{K_2}(M_P)+O(r_P)+O((\epsilon/\rho)^{1/2})$$

if ρ is small enough. So Theorem 3.1 follows by taking $(\epsilon/\rho)^{1/2}\geq r_P$.

3A. ***Proof of Proposition 3.2.*** The cotangent bundle T^*K is identified with $K\times\mathfrak{k}^*$. The data is then (see Section 5A):

- the Liouville 1-form $\lambda=\sum_j\omega_j\otimes E_j$, where (E_j) is a basis of $\mathfrak{k}$ with dual basis (E_j^*) and ω_j is the left invariant 1-form on K defined by $\omega_j\big(\frac{d}{dt}a\,e^{tX}|_0\big)=\langle E_j^*,X\rangle$.
- the symplectic form $\Omega:=-d\lambda$,
- the action of $K_1\times K_2$ on $K\times\mathfrak{k}^*$ given by $(k_1,k_2)\cdot(a,\xi)=(k_2ak_1^{-1},k_1\cdot\xi)$,
- the moment map relative to the K_1-action $\Phi_r(a,\xi)=-\xi$,
- the moment map relative to the K_2-action $\Phi_l(a,\xi)=a\cdot\xi$.

We work now with the Hamiltonian action of $K_1\times K_2$ on $M\times\mathrm{T}^*K$ given by

$$(k_1,k_2)\cdot(m,a,\xi)=(k_1\cdot m,k_2ak_1^{-1},k_1\cdot\xi).$$

The corresponding moment map is $\Phi=(\Phi_1,\Phi_2)$: $\Phi_1(m,a,\xi)=\Phi_M(m)-\xi$ and $\Phi_2(m,a,\xi)=a\cdot\xi$.

Let $\mathbf{c}_1$ be a symbol $\mathrm{Thom}(M,J_1)\otimes L$ attached to the prequantized Hamiltonian K_1-manifold (M,Ω). The cotangent bundle T^*K is prequantized by the trivial line bundle. Let $\mathbf{c}_2$ be the symbol $\mathrm{Thom}(\mathrm{T}^*K,J_2)$ attached to the prequantized Hamiltonian $(K_1\times K_2)$-manifold T^*K. The product $\mathbf{c}=\mathbf{c}_1\odot\mathbf{c}_2$ corresponds to the symbol $\mathrm{Thom}(N,J)\otimes L$ on $N=M\times\mathrm{T}^*K$.

For the rest of this section we fix $\rho>0$. Let κ_ρ be the Kirwan vector field associated to the map $\|\Phi\|^2_\rho: M\times\mathrm{T}^*K\to\mathbb{R}$. We check that $\|\Phi\|^2_\rho(m,k,\xi)=\|\Phi_M(m)-\xi\|^2+\rho\|\xi\|^2$, and

$$\kappa_\rho(m,k,\xi)=\Big(\underbrace{(\widetilde{\Phi_M(m)}-\tilde{\xi})\cdot m}_{\kappa_I};\ \underbrace{\widetilde{\Phi_M(m)}-(1+\rho)\tilde{\xi}}_{\kappa_{II,\rho}};\ \underbrace{-[\widetilde{\Phi_M(m)},\tilde{\xi}]}_{\kappa_{III}}\Big).$$

Here $\mathrm{T}_{(m,k,\xi)}(M \times \mathrm{T}^*K) \simeq \mathrm{T}_m M \times \mathfrak{k} \times \mathfrak{k}$. We have

$$\begin{aligned}\mathrm{Cr}(\|\Phi\|^2_\rho) &= \{\kappa_\rho = 0\}\\ &= \bigcup_{\beta\in\mathcal{B}} \underbrace{K_1 \times K_2 \cdot \Big[M^{\widetilde{\beta}} \cap \Phi_M^{-1}(\beta) \times \{1\} \times \Big\{\frac{\widetilde{\beta}}{\rho+1}\Big\}\Big]}_{Z_\beta},\end{aligned}$$

where $\mathcal{B} \subset \mathfrak{t}^*_+$ parametrizes $\mathrm{Cr}(\|\Phi_M\|^2)$. One can check that

$$\|\Phi\|^2_\rho(Z_\beta) = \Big(\frac{\rho}{\rho+1}\Big)\|\beta\|^2$$

and $\|\Phi_M\|^2(Z_\beta) = \|\beta\|^2$ for $\beta \in \mathcal{B}$.

Let $\mathbf{c}^{\kappa_\rho}$ be the symbol $\mathbf{c}$ pushed by the vector field κ_ρ. We have

$$\mathbf{c}^{\kappa_\rho}(v;\, X;\, Y) = \mathbf{c}_1(v - \kappa_I) \odot \mathbf{c}_2(X - \kappa_{II,\rho}\,;\; Y - \kappa_{III})$$

for $(v;\, X;\, Y) \in \mathrm{T}_{(m,k,\xi)}(M \times \mathrm{T}^*K) \simeq \mathrm{T}_m M \times \mathfrak{k} \times \mathfrak{k}$.

For a real $R > 0$, define the open invariant subsets of $M \times \mathrm{T}^*K$

$$\begin{aligned}U_R &:= \{\|\Phi\|^2_\rho < R\},\\ V_R &:= \{\|\Phi_M\|^2 < R\} \times \mathrm{T}^*K.\end{aligned}$$

We see that $Z_\beta \subset U_R$ if and only if $(\rho/(\rho+1))\|\beta\|^2 < R$ and $Z_\beta \subset V_R$ if and only if $\|\beta\|^2 < R$. By Definition 2.11, the generalized index $\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M \times \mathrm{T}^*K)$ is defined as the limit of the equivariant index

$$\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(U_R) := \mathrm{Index}^{K_1\times K_2}_{U_R}(\mathbf{c}^{\kappa_\rho}|_{U_R}) = \sum_{(\rho/(\rho+1))\|\beta\|^2<R} \mathcal{Q}^{\beta,\rho}_{K_1\times K_2}(M \times \mathrm{T}^*K)$$

when R goes to infinity (and stays outside the critical values of $\|\Phi\|^2_\rho$).

On the other hand, when R' is a regular value of $\|\Phi_M\|^2$, the symbol $\mathbf{c}^{\kappa_\rho}|_{V_{R'}}$ is $(K_1\times K_2)$-transversally elliptic since

$$\mathrm{Cr}(\|\Phi\|^2_\rho) \cap \overline{V_{R'}} = \bigcup_{\|\beta\|^2<R'} Z_\beta \tag{42}$$

is compact. The index map is well-defined on $V_{R'} = \{\|\Phi_M\|^2 < R'\} \times \mathrm{T}^*K$ since T^*K can be seen as an invariant open subset of a compact $(K_1\times K_2)$-manifold.

Lemma 3.5. *The character* $\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M \times \mathrm{T}^*K)$ *is equal to the limit of*

$$\mathrm{Index}^{K_1\times K_2}_{V_{R'}}(\mathbf{c}^{\kappa_\rho}|_{V_{R'}})$$

when R' *goes to infinity* (*and stays outside the critical values of* $\|\Phi_M\|^2$).

Proof. Thanks to (42) and to the excision property we have

$$\mathrm{Index}^{K_1\times K_2}_{V_{R'}}(\mathbf{c}^{\kappa_\rho}|_{V_{R'}}) = \sum_{\|\beta\|^2<R'} \mathcal{Q}^{\beta,\rho}_{K_1\times K_2}(M\times \mathrm{T}^*K),$$

and then

$$\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M\times \mathrm{T}^*K) - \mathrm{Index}^{K_1\times K_2}_{V_{R'}}(\mathbf{c}^{\kappa_\rho}|_{V_{R'}}) = \sum_{\|\beta\|^2\geq R'} \mathcal{Q}^{\beta,\rho}_{K_1\times K_2}(M\times \mathrm{T}^*K).$$

By Definition 2.11, the support of $\mathcal{Q}^{\beta,\rho}_{K_1\times K_2}(M\times \mathrm{T}^*K)$ is contained in

$$\left\{(\gamma_1,\gamma_2)\in\widehat{K}\times\widehat{K}\;\middle|\;\|\gamma_1\|^2+\rho\|\gamma_2\|^2\geq\frac{\rho}{\rho+1}\|\beta\|^2\right\}$$
$$\subset\left\{(\gamma_1,\gamma_2)\in\widehat{K}\times\widehat{K}\;\middle|\;\|\gamma_1\|^2+\|\gamma_2\|^2\geq\frac{\rho}{(\rho+1)^2}\|\beta\|^2\right\}.$$

Finally we have proved that

$$\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M\times \mathrm{T}^*K) - \mathrm{Index}^{K_1\times K_2}_{V_{R'}}(\mathbf{c}^{\kappa_\rho}|_{V_{R'}}) = \sum_{(\gamma_1,\gamma_2)} m^{R'}_{(\gamma_1,\gamma_2)} V^{K_1}_{\gamma_1}\otimes V^{K_2}_{\gamma_2}$$

with $m^{R'}_{(\gamma_1,\gamma_2)}=0$ if $\|\gamma_1\|^2+\|\gamma_2\|^2\leq(\rho/(\rho+1)^2)R'$. Hence the right hand side of the last equation tends to 0 in $R^{-\infty}(K_1\times K_2)$ when $R'\to\infty$. ☐

Look now to the deformation $\kappa_\rho(s)=(\kappa^s_{I};\kappa^s_{II,\rho};s\kappa_{III})$, $s\in[0,1]$, where

$$\kappa^s_I(m,\xi)=(\widetilde{\Phi_M(m)}-s\tilde{\xi})\cdot m \quad\text{and}\quad \kappa^s_{II,\rho}(m,\xi)=s\widetilde{\Phi_M(m)}-(1+s\rho)\tilde{\xi}.$$

Let $\mathbf{c}^{\kappa_\rho(s)}$ be the symbol $\mathbf{c}$ pushed by the vector field $\kappa_\rho(s)$.

Lemma 3.6. *Let R' be a regular value of $\|\Phi_M\|^2$.*

- *The family $\mathbf{c}^{\kappa_\rho(s)}|_{V_{R'}}$, $s\in[0,1]$, defines a homotopy of $(K_1\times K_2)$-transversally elliptic symbols on $V_{R'}$.*
- *The K_1-invariant part of* $\mathrm{Index}^{K_1\times K_2}_{V_{R'}}(\mathbf{c}^{\kappa_\rho(0)}|_{V_{R'}})$ *is equal to $\mathcal{Q}^{\Phi}_{K_2}(M_{<R'})$.*

Proof. The first point follows from the fact that $\mathrm{Char}(\mathbf{c}^{\kappa_\rho(s)}|_{V_{R'}})\cap\mathrm{T}_{K_1\times K_2}(V_{R'})$, which is equal to

$$\left\{\left(m,k,\frac{s}{1+s\rho}\widetilde{\Phi_M}(m)\right)\;\middle|\;k\in K \text{ and } m\in\mathrm{Cr}(\|\Phi_M\|^2)\cap\{\|\Phi_M\|^2<R'\}\right\},$$

stays in a compact set when $s\in[0,1]$.

The symbol $\mathbf{c}^{\kappa_\rho(0)}|_{V_{R'}}$ is equal to the product of the symbol $\mathbf{c}^{\kappa}_1|_{M<R'}$, which is K_1-transversally elliptic, with the symbol

$$\mathbf{c}^{\kappa}_2(X;Y)=\mathbf{c}_2(X+\xi;Y),$$

which is a K_2-transversally elliptic on T^*K. A basic computation in Section 5A1 gives that

$$\begin{aligned}\mathrm{Index}^{K_1\times K_2}_{\mathrm{T}^*K}(\mathbf{c}^{\kappa}_2) &= \mathrm{L}^2(K)\\ &= \sum_{\mu\in\widehat{K}} (V^{K_1}_\mu)^* \otimes V^{K_2}_\mu\end{aligned}$$

in $R^{-\infty}(K_1 \times K_2)$. Finally the multiplicative property (Theorem 2.1) gives

$$\begin{aligned}\mathrm{Index}^{K_1\times K_2}_{V_{R'}}(\mathbf{c}^{\kappa_\rho(0)}|_{V_{R'}}) &= \mathrm{Index}^{K_1}_{M_{<R'}}(\mathbf{c}^{\kappa}_1|_{M_{<R'}}) \otimes \mathrm{Index}^{K_1\times K_2}_{\mathrm{T}^*K}(\mathbf{c}^{\kappa}_2)\\ &= \sum_{\mu\in\widehat{K}} \mathcal{Q}^{\Phi}_{K_1}(M_{<R'}) \otimes (V^{K_1}_\mu)^* \otimes V^{K_2}_\mu.\end{aligned}$$

Taking the K_1-invariant part completes the proof of the second point. □

Finally we have proved that the generalized character $[\mathrm{Index}^{K_1\times K_2}_{V_{R'}}(\mathbf{c}^{\kappa_\rho}|_{V_{R'}})]^{K_1}$ is equal to $\mathcal{Q}^{\Phi}_{K_2}(M_{<R'})$. Taking the limit $R' \to \infty$ gives

$$\begin{aligned}[\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M \times \mathrm{T}^*K)]^{K_1} &= \lim_{R'\to\infty} [\mathrm{Index}^{K_1\times K_2}_{V_{R'}}(\mathbf{c}^{\kappa_\rho}|_{V_{R'}})]^{K_1}\\ &= \lim_{R'\to\infty} \mathcal{Q}^{\Phi}_{K_2}(M_{<R'}) = \mathcal{Q}^{\Phi}_{K_2}(M).\end{aligned}$$

3B. ***Proof of Proposition 3.3.*** We work here with the Hamiltonian action of the product $K_1 \times K_2$ on $M \times \mathcal{X}_P$. The action is $(k_1, k_2)\cdot(m, x) = (k_1\cdot m, k_2\cdot x\cdot k_1^{-1})$ and the corresponding moment map is $\Phi = (\Phi_1, \Phi_2)$ with $\Phi_1(m, x) = \Phi_M(m) + \Phi_r(x)$ and $\Phi_2(m, x) = \Phi_l(x)$. Let $\|(\xi_1, \xi_2)\|^2_\rho = \|\xi_1\|^2 + \rho\|\xi_2\|^2$ be the Euclidean norm $\mathfrak{k}_1^* \times \mathfrak{k}_2^*$ attached to $\rho > 0$.

Consider the quantization of $M \times \mathcal{X}_P$ via the map $\|\Phi\|^2_\rho$:

$$\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M \times \mathcal{X}_P) \in R^{-\infty}(K_1 \times K_2).$$

The critical set $\mathrm{Cr}(\|\Phi\|^2_\rho)$ admits the decomposition

$$\mathrm{Cr}(\|\Phi\|^2_\rho) = \bigcup_{\gamma\in\mathcal{B}_\rho} K_1 \times K_2 \cdot \mathcal{C}_\gamma, \tag{43}$$

where $(m, x) \in \mathcal{C}_\gamma$ if and only if $\gamma = (\gamma_1, \gamma_2)$ with

$$\begin{cases} \Phi_M(m) + \Phi_r(x) = \gamma_1,\\ \Phi_l(x) = \gamma_2,\\ \widetilde{\gamma}_1 \cdot m = 0,\\ \widetilde{\gamma}_1 \cdot_r x + \rho\, \widetilde{\gamma}_2 \cdot_l x = 0.\end{cases} \tag{44}$$

Here $\mathcal{B}_\rho \subset \mathfrak{t}^*_+ \times \mathfrak{t}^*_+$ is defined as the set of elements $\gamma = (\gamma_1, \gamma_2) \in \mathfrak{t}^*_+ \times \mathfrak{t}^*_+$ where the equations (44) have solutions in $M \times \mathcal{X}_P$.

We have

$$\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M\times\mathcal{X}_P)=\sum_{\gamma\in\mathcal{B}_\rho}\mathcal{Q}^{\gamma,\rho}_{K_1\times K_2}(M\times\mathcal{X}_P),\tag{45}$$

where the generalized character $\mathcal{Q}^{\gamma,\rho}_{K_1\times K_2}(M\times\mathcal{X}_P)$ is computed as an index of a transversally elliptic symbol in a neighborhood of

$$K_1\times K_2\cdot\mathcal{C}_\gamma\subset M\times\Phi_l^{-1}(K_2\cdot\gamma_2).$$

By Theorem 2.9, the support of the generalized character $\mathcal{Q}^{\gamma,\rho}_{K_1\times K_2}(M\times\mathcal{X}_P)$ is contained in $\{(a,b)\in\widehat{K_1}\times\widehat{K_2}\mid \|a\|^2+\rho\|b\|^2\geq\|\gamma\|^2_\rho\}$. Hence

$$\text{support}\big([\mathcal{Q}^{\gamma,\rho}_{K_1\times K_2}(M\times\mathcal{X}_P)]^{K_1}\big)\subset\{b\in\widehat{K_2}\mid\rho\|b\|^2\geq\|\gamma\|^2_\rho\}.$$

Let $r_P=\inf_{\xi\in\partial P}\|\xi\|$. We know then that

$$[\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M\times\mathcal{X}_P)]^{K_1}=\sum_{\substack{\gamma\in\mathcal{B}_\rho\\ \|\gamma\|^2_\rho<\rho r_P^2}}[\mathcal{Q}^{\gamma,\rho}_{K_1\times K_2}(M\times\mathcal{X}_P)]^{K_1}+O(r_P).$$

Let $R_P<\rho r_P^2$ be a regular value of $\|\Phi\|^2_\rho:M\times\mathcal{X}_P\to\mathbb{R}$ such that for all $\gamma\in\mathcal{B}_\rho$ we have $\|\gamma\|^2_\rho<\rho r_P^2$ if and only if $\|\gamma\|^2_\rho<R_P$. Then

$$[\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M\times\mathcal{X}_P)]^{K_1}=[\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}((M\times\mathcal{X}_P)_{<R_P})]^{K_1}+O(r_P).\tag{46}$$

For the generalized index $\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M\times \mathrm{T}^*K)$ we have also a decomposition

$$\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M\times\mathrm{T}^*K)=\sum_{\gamma\in\mathcal{B}'_\rho}\mathcal{Q}^{\gamma,\rho}_{K_1\times K_2}(M\times\mathrm{T}^*K),$$

where $\mathcal{B}'_\rho$ parametrizes the critical set of $\|\Phi\|^2_\rho:M\times\mathrm{T}^*K\to\mathbb{R}$. As before,

$$[\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M\times\mathrm{T}^*K)]^{K_1}=[\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}((M\times\mathrm{T}^*K)_{<R'_P})]^{K_1}+O(r_P).\tag{47}$$

Here $R'_P<\rho r_P^2$ is a regular value of $\|\Phi\|^2_\rho:M\times\mathrm{T}^*K\to\mathbb{R}$ such that for all $\gamma\in\mathcal{B}'_\rho$ we have $\|\gamma\|^2_\rho<\rho r_P^2$ if and only if $\|\gamma\|^2_\rho<R'_P$.

Lemma 3.7. *We have*

$$\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}((M\times\mathcal{X}_P)_{<R_P})=\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}((M\times\mathrm{T}^*K)_{<R'_P}).\tag{48}$$

Proof. The lemma follows from Proposition 2.6. We take here $V'=M\times\mathcal{X}^o_P$, $V=M\times K\times\mathcal{U}_P\subset M\times\mathrm{T}^*K$ and the equivariant diffeomorphism $\Psi:V\to V'$ equal to $\mathrm{Id}\times\Upsilon$ where Υ was introduced in Section 2E. The map Ψ satisfies points (1)–(3) of Proposition 2.6.

The inequality $\|\Phi(m,x)\|^2_\rho<\rho r_P^2$ implies that $\|\Phi_l(x)\|<r_P$ and then $x\in\mathcal{X}^o_P$. Hence the open subset $U':=(M\times\mathcal{X}_P)_{<R_P}$ is contained in $V'=M\times\mathcal{X}^o_P$. In

the same way the open subset $U := (M \times \mathrm{T}^*K)_{<R'_P}$ is contained in V. We have $\Psi(U) = U'$ if $R_P = R'_P$.

We have proved that (48) is a consequence of Proposition 2.6. □

Finally, taking the difference between (46) and (47) gives

$$[\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M \times \mathcal{X}_P)]^{K_1} - [\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M \times \mathrm{T}^*K)]^{K_1} = O(r_P),$$

which is the relation of Proposition 3.3.

3C. ***Proof of Proposition 3.4.*** Here we want to compare the K_1-invariant part of the characters $\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M \times \mathcal{X}_P)$ and $\mathcal{Q}^{\Phi_1}_{K_1\times K_2}(M \times \mathcal{X}_P)$.

By Theorem 2.15,

$$\mathcal{Q}_{K_2}(M_P) = [\mathcal{Q}^{\Phi_1}_{K_1\times K_2}(M \times \mathcal{X}_P)]^{K_1} = [\mathcal{Q}^{\Phi_1}_{K_1\times K_2}(U_\epsilon)]^{K_1}$$

when $\epsilon > 0$ is any regular value of $\|\Phi_1\|^2$, and $U_\epsilon := \{\|\Phi_1\|^2 < \epsilon\} \subset M \times \mathcal{X}_P$.

In this section we fix once and for all $\epsilon > 0$ small enough so that

$$\mathrm{Cr}(\|\Phi_1\|^2) \cap \{\|\Phi_1\|^2 \le \epsilon\} = \{\Phi_1 = 0\}. \tag{49}$$

Let $\mathbf{c}_1$ be the symbol $\mathrm{Thom}(M, J_1) \otimes L$ attached to the prequantized Hamiltonian K_1-manifold (M, Ω). Let $\mathbf{c}_3$ be the symbol $\mathrm{Thom}(\mathcal{X}_P, J_3) \otimes L_P$ attached to the prequantized Hamiltonian $(K_1\times K_2)$-manifold $\mathcal{X}_P$. The product $\mathbf{c} = \mathbf{c}_1 \odot \mathbf{c}_3$ corresponds to the symbol $\mathrm{Thom}(N, J) \otimes L$ on $N = M \times \mathcal{X}_P$.

Let κ_0 and κ_ρ be the Kirwan vector fields associated to the functions $\|\Phi_1\|^2$ and $\|\Phi\|^2_\rho$ on $M \times \mathcal{X}_P$:

$$\kappa_0(m,x) = \big(\underbrace{\widetilde{\Phi_1}(m,x)\cdot m}_{\kappa_I}\,;\ \underbrace{\widetilde{\Phi_1}(m,x)\cdot_r x}_{\kappa_{II}}\big),$$
$$\kappa_\rho(m,x) = \kappa_0(m,x) + \rho\,\big(0, \underbrace{\widetilde{\Phi_l}(x)\cdot_l x}_{\kappa_{III}}\big).$$

Let $\mathbf{c}^{\kappa_\rho}$ be the symbol $\mathbf{c}$ pushed by the vector field κ_ρ. Then

$$\mathbf{c}^{\kappa_\rho}(v;\eta) = \mathbf{c}_1(v - \kappa_I) \odot \mathbf{c}_3(\eta - \kappa_{II} - \rho\kappa_{III})$$

for $(v;\eta) \in \mathrm{T}_{(m,x)}(M \times \mathcal{X}_P)$.

The character $\mathcal{Q}^{\Phi_1}_{K_1\times K_2}(U_\epsilon)$ is given by the index of the K_1-transversally elliptic symbol $\mathbf{c}^{\kappa_0}|_{U_\epsilon}$. The character $\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M \times \mathcal{X}_P)$ is given by the index of the $(K_1\times K_2)$-transversally elliptic symbol $\mathbf{c}^{\kappa_\rho}$.

Lemma 3.8. *There exists $\rho(\epsilon) > 0$ such that*

$$\mathrm{Cr}(\|\Phi\|^2_\rho) \cap \big\{\|\Phi_1\|^2 \le \epsilon\big\} \subset \big\{\|\Phi_1\|^2 \le \tfrac{\epsilon}{2}\big\}$$

for any $0 \le \rho \le \rho(\epsilon)$.

Proof. With the help of Riemannian metrics on M and $\mathcal{X}_P$, define

$$a(\epsilon) := \inf_{\epsilon/2 \leq \|\Phi_1(m,x)\| \leq \epsilon} \|\kappa_0(m,x)\|,$$
$$b := \sup_{x \in \mathcal{X}_P} \|\Phi_l(x) \cdot_l x\|.$$

We have $a(\epsilon) > 0$ thanks to (49), and $b < \infty$ since $\mathcal{X}_P$ is compact. It is now easy to check that $\{\kappa_\rho = 0\} \cap \{\epsilon/2 \leq \|\Phi_1\|^2 \leq \epsilon\} = \varnothing$ if $0 \leq \rho < a(\epsilon)/b$. ☐

The symbols $\mathbf{c}^{\kappa_\rho}|_{U_\epsilon}$, $\rho \in [0, \rho(\epsilon)]$, are $(K_1 \times K_2)$-transversally elliptic, and they define the same class in $\mathbf{K}^0_{K_1 \times K_2}(\mathrm{T}_{K_1 \times K_2} U_\epsilon)$. Hence $\mathcal{Q}_{K_2}(M_P)$ can be computed as the K_1-invariant part of

$$\mathcal{Q}^{\Phi,\rho}_{K_1 \times K_2}(U_\epsilon) := \mathrm{Index}^{K_1 \times K_2}_{U_\epsilon}(\mathbf{c}^{\kappa_\rho}|_{U_\epsilon}) \in R^{-\infty}(K_1 \times K_2)$$

for any $\rho \in [0, \rho(\epsilon)]$.

Let $\rho \in]0, \rho(\epsilon)]$. A component $K_1 \times K_2 \cdot \mathcal{C}_\gamma$ of $\mathrm{Cr}(\|\Phi\|^2_\rho)$ is contained in U_ϵ if and only if $\|\gamma_1\|^2 < \epsilon$, so the decomposition (45) for the character $\mathcal{Q}^{\Phi,\rho}_{K_1 \times K_2}(M \times \mathcal{X}_P)$ gives

$$\mathcal{Q}^{\Phi,\rho}_{K_1 \times K_2}(M \times \mathcal{X}_P) = \mathcal{Q}^{\Phi,\rho}_{K_1 \times K_2}(U_\epsilon) + \sum_{\substack{\gamma \in \mathcal{B}_\rho \\ \|\gamma_1\|^2 \geq \epsilon}} \mathcal{Q}^{\gamma,\rho}_{K_1 \times K_2}(M \times \mathcal{X}_P),$$

where

$$\mathcal{Q}^{\Phi,\rho}_{K_1 \times K_2}(U_\epsilon) = \sum_{\substack{\gamma \in \mathcal{B}_\rho \\ \|\gamma_1\|^2 < \epsilon}} \mathcal{Q}^{\gamma,\rho}_{K_1 \times K_2}(M \times \mathcal{X}_P).$$

Taking the K_1-invariant gives

$$[\mathcal{Q}^{\Phi,\rho}_{K_1 \times K_2}(M \times \mathcal{X}_P)]^{K_1} = \mathcal{Q}_{K_2}(M_P) + \sum_{\substack{\gamma \in \mathcal{B}_\rho \\ \|\gamma_1\|^2 \geq \epsilon}} [\mathcal{Q}^{\gamma,\rho}_{K_1 \times K_2}(M \times \mathcal{X}_P)]^{K_1}. \tag{50}$$

By Theorem 2.9 the support of the generalized character $[\mathcal{Q}^{\gamma,\rho}_{K_1 \times K_2}(M \times \mathcal{X}_P))]^{K_1} \in R^{-\infty}(K_2)$ is included in $\{b \in \widehat{K_2} \mid \rho\|b\|^2 \geq \|(\gamma_1, \gamma_2)\|^2_\rho\}$. When $\|\gamma_1\|^2 \geq \epsilon$ we have then that the support of $[\mathcal{Q}^{\gamma,\rho}_{K_1 \times K_2}(M \times \mathcal{X}_P))]^{K_1}$ is contained outside the ball $B_{(\epsilon/\rho)^{1/2}}$.

Finally (50) imposes that

$$[\mathcal{Q}^{\Phi,\rho}_{K_1 \times K_2}(M \times \mathcal{X}_P)]^{K_1} = \mathcal{Q}_{K_2}(M_P) + O((\epsilon/\rho)^{1/2})$$

when $0 < \rho \leq \rho(\epsilon)$, which is the precise content of Proposition 3.4.

4. Other properties of $\mathcal{Q}^\Phi$

Let (M, ω, Φ) be a proper Hamiltonian K-manifold that is prequantized by a line bundle L. The character $\mathcal{Q}^\Phi_K(M)$ is computed by means of a scalar product on $\mathfrak{k}^*$. The fact that $\mathcal{Q}^\Phi_K(M) = \mathcal{Q}^{-\infty}_K(M)$ gives the following:

Proposition 4.1. *The character $\mathcal{Q}^\Phi_K(M)$ does not depend of the choice of an invariant scalar product on $\mathfrak{k}^*$.*

In this section we work in the setting where[7] $K = K_1 \times K_2$. Let Φ_1 be the moment map relative to the K_1-action.

4A. Φ_1 *is proper.* In this subsection, suppose that the moment map Φ_1 relative to the K_1-action is *proper*. Fix an invariant Euclidean norm $\|\cdot\|^2$ on $\mathfrak{k}$ in such a way that $\mathfrak{k}_1 = \mathfrak{k}_2^\perp$.

To "quantize" (M, Ω) via the invariant proper function $\|\Phi_1\|^2$, let

$$\mathcal{Q}^{\Phi_1}_{K_1\times K_2}(M) \in R^{-\infty}(K_1 \times K_2)$$

be the corresponding generalized character.

Theorem 4.2. *We have*

$$\mathcal{Q}^{\Phi}_{K_1\times K_2}(M) = \mathcal{Q}^{\Phi_1}_{K_1\times K_2}(M) \quad \text{in } R^{-\infty}(K_1 \times K_2). \tag{51}$$

Proof. On $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ we may consider the family of invariant Euclidean norms: $\|X_1 \oplus X_2\|^2_\rho = \|X_1\|^2 + \rho\|X_2\|^2$ for $X_j \in \mathfrak{k}_j$. Let

$$\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M) \in R^{-\infty}(K_1 \times K_2)$$

be the quantization of M computed via the map $\|\Phi\|^2_\rho = \|\Phi_1\|^2 + \rho\|\Phi_2\|^2$. By definition, $\mathcal{Q}^{\Phi_1}_{K_1\times K_2}(M)$ is equal to $\mathcal{Q}^{\Phi,0}_{K_1\times K_2}(M)$, and Proposition 4.1 implies that $\mathcal{Q}^{\Phi}_{K_1\times K_2}(M)$ coincides with the generalized character $\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M) \in R^{-\infty}(K)$ for any $\rho > 0$.

Denote by $O(r) \in R^{-\infty}(K_1 \times K_2)$ any generalized character supported outside the ball $\{\xi \in \mathfrak{t}_1^* \times \mathfrak{t}_2^* \mid \|\xi_1\|^2 + \|\xi_2\|^2 < r^2\}$. Also, denote by $O_1(r) \in R^{-\infty}(K_1 \times K_2)$ any generalized character supported outside the $\{\xi \in \mathfrak{t}_1^* \times \mathfrak{t}_2^* \mid \|\xi_1\| < r\}$.

Let $R_1 > 0$ be a regular value of $\|\Phi_1\|^2$, and let $M_{<R_1}$ be the open subset $\{\|\Phi_1\|^2 < R_1\}$, which is relatively compact. Theorem 2.13 tells us that

$$\mathcal{Q}^{\Phi_1}_{K_1\times K_2}(M) = \mathcal{Q}^{\Phi_1}_{K_1\times K_2}(M_{<R_1}) + O_1(\sqrt{R_1}).$$

As in Lemma 3.8, there exists $\rho(R_1) \in]0, 1[$ small enough such that

$$\mathrm{Cr}(\|\Phi\|^2_\rho) \cap \{\|\Phi_1\|^2 = R_1\} = \varnothing \quad \text{for } \rho \in [0, \rho(R_1)]. \tag{52}$$

---

[7] In this section the Lie groups K_1 and K_2 are not identical.

Let $\rho \in]0, \rho(R_1)]$. The identity (52) first implies that

$$\begin{aligned}\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M) &= \sum_{\substack{\gamma\in\mathcal{B}_\rho\\ \|\gamma_1\|^2<R_1}} \mathcal{Q}^{\gamma,\rho}_{K_1\times K_2}(M) + \sum_{\substack{\gamma\in\mathcal{B}_\rho\\ \|\gamma_1\|^2>R_1}} \mathcal{Q}^{\gamma,\rho}_{K_1\times K_2}(M)\\ &= \mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M_{<R_1}) + O(\sqrt{R_1}),\end{aligned}$$

where the second equality uses that $\mathcal{Q}^{\gamma,\rho}_{K_1\times K_2}(M) = O(\sqrt{R_1})$ when $\|\gamma_1\|^2 > R_1$, since the ball $\{(\xi_1,\xi_2)\in\mathfrak{t}_1^*\times\mathfrak{t}_2^* \mid \|\xi_1\|^2+\|\xi_2\|^2 < R_1\}$ is contained in

$$\left\{(\xi_1,\xi_2)\in\mathfrak{t}_1^*\times\mathfrak{t}_2^* \mid \|(\xi_1,\xi_2)\|^2_\rho < \|(\gamma_1,\gamma_2)\|^2_\rho\right\}.$$

The identity (52) shows also that the symbols $\mathbf{c}^{\kappa_\rho}|_{M_{<R_1}}$ are homotopic for $\rho \in [0, \rho(R_1)]$. Hence

$$\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M_{<R_1}) = \mathcal{Q}^{\Phi_1}_{K_1\times K_2}(M_{<R_1})$$

if $\rho > 0$ is small enough. Finally,

$$\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M) - \mathcal{Q}^{\Phi_1}_{K_1\times K_2}(M) = O(\sqrt{R_1}) + O_1(\sqrt{R_1})$$

for any regular value R_1 of $\|\Phi_1\|^2$, when $\rho \in]0, \rho(R_1)]$. Since the generalized character $\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M)$ does not depend of $\rho > 0$ (see Proposition 4.1),

$$\mathcal{Q}^{\Phi}_{K_1\times K_2}(M) = \mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M) = \mathcal{Q}^{\Phi_1}_{K_1\times K_2}(M). \qquad \square$$

We explain how Theorem 4.2 contains the identity that we called "*quantization commutes with reduction in the singular setting*" in [Paradan 2009]. By definition the K_1-invariant part of the right hand side of (51) is equal to the geometric quantization of the (possibly singular) compact Hamiltonian K_2-manifold

$$M/\!\!/_0 K_1 := \Phi_1^{-1}(0)/K_1.$$

Using now the fact that the left hand side of (51) is equal to $\mathcal{Q}^{-\infty}_{K_1\times K_2}(M)$, we see that the multiplicity of $V^{K_2}_\mu$ in $\mathcal{Q}_{K_2}(M/\!\!/_0 K_1)$ is equal to the geometric quantization of the (possibly singular) compact manifold

$$M \times \overline{K_2\cdot\mu}/\!\!/_{(0,0)} K_1 \times K_2.$$

4B. ***The symplectic reduction $M/\!\!/_0 K_1$ is smooth.*** Let (M,Ω) be an Hamiltonian $(K_1\times K_2)$-manifold with proper moment map $\Phi = (\Phi_1, \Phi_2)$. In this section, suppose that 0 is a regular value of Φ_1 and that K_1 acts freely on $\Phi_1^{-1}(0)$. We work then with the smooth Hamiltonian K_2-manifold

$$N := \Phi_1^{-1}(0)/K_1.$$

Continue to denote by $\Phi_2 : N \to \mathfrak{k}_2^*$ the moment map relative to the K_2-action; note that this map is proper. Hence we can quantize the K_2-action on N via the map Φ_2. Let $\mathcal{Q}_{K_2}^{\Phi_2}(N) \in R^{-\infty}(K_2)$ be the corresponding character.

Proposition 4.3. *We have*

$$[\mathcal{Q}_{K_1\times K_2}^{\Phi}(M)]^{K_1} = \mathcal{Q}_{K_2}^{\Phi_2}(N) \quad \text{in } R^{-\infty}(K_2). \tag{53}$$

Proof. When Φ_1 is proper, the manifold N is compact. Then the right hand side of (53) is equal to $\mathcal{Q}_{K_2}(N)$, and we know from Theorem 4.2 that the left hand side of (53) is equal to $[\mathcal{Q}_{K_1\times K_2}^{\Phi_1}(M)]^{K_1}$. In this case (53) becomes $[\mathcal{Q}_{K_1\times K_2}^{\Phi_1}(M)]^{K_1} = \mathcal{Q}_{K_2}(M/\!\!/_0 K_1)$ which is the content of Theorem 2.14.

Consider the general case where Φ_1 is not proper. By Theorem 1.4, the multiplicities of $V_\mu^{K_2}$ in $[\mathcal{Q}_{K_1\times K_2}^{\Phi}(M)]^{K_1}$ and in $\mathcal{Q}_{K_2}^{\Phi_2}(N)$ are respectively equal to the quantization of the (possibly singular) symplectic reductions

$$\begin{aligned} \mathcal{M}_\mu &:= M \times \overline{K_2\cdot\mu}/\!\!/_{(0,0)} K_1 \times K_2, \\ \mathcal{M}'_\mu &:= N \times \overline{K_2\cdot\mu}/\!\!/_0 K_2 \quad \text{with } N = M/\!\!/_0 K_1. \end{aligned}$$

Note that $\mathcal{M}_\mu$ and $\mathcal{M}'_\mu$ coincide as symplectic reduced space. Their geometric quantizations are identical also. The proof will be done for $\mu = 0$: the other cases follow from the shifting trick.

Let $\mathbf{c}$ be the $(K_1\times K_2)$-equivariant symbol $\mathrm{Thom}(M, J) \otimes L_M$. Let κ be the Kirwan vector field attached to the moment map $\Phi = (\Phi_1, \Phi_2)$. Let $\mathbf{c}^\kappa$ be the symbol $\mathbf{c}$ pushed by κ. Denote by $M_{<\epsilon}$ the open subset $\{\|\Phi\|^2 < \epsilon\}$. For $\epsilon > 0$ small enough, the symbol $\mathbf{c}^\kappa|_{M_{<\epsilon}}$ is $(K_1\times K_2)$-transversally elliptic, and $\mathcal{Q}(\mathcal{M}_0)$ is the $(K_1\times K_2)$-invariant part of $\mathrm{Index}_{M_{<\epsilon}}^{K_1\times K_2}(\mathbf{c}^\kappa|_{M_{<\epsilon}})$.

Let $\mathbf{c}_2$ be the K_2-equivariant symbol $\mathrm{Thom}(N, J) \otimes L_N$. Let κ_2 be the Kirwan vector field attached to the moment map Φ_2. Let $\mathbf{c}_2^{\kappa_2}$ be the symbol $\mathbf{c}_2$ pushed by κ_2. Denote by $N_{<\epsilon}$ the open subset $\{\|\Phi_2\|^2 < \epsilon\}$. For $\epsilon > 0$ small enough, the symbol $\mathbf{c}_2^{\kappa_2}|_{N_{<\epsilon}}$ is K_2-transversally elliptic, and $\mathcal{Q}(\mathcal{M}'_0)$ is the K_2-invariant part of $\mathrm{Index}_{N_{<\epsilon}}^{K_2}(\mathbf{c}_2^{\kappa_2}|_{N_{<\epsilon}})$.

Our proof follows from the comparison of the classes

$$\begin{aligned} &[\mathbf{c}^\kappa|_{M_{<\epsilon}}] \in \mathbf{K}^0_{K_1\times K_2}(\mathrm{T}_{K_1\times K_2} M_{<\epsilon}), \\ &[\mathbf{c}_2^{\kappa_2}|_{N_{<\epsilon}}] \in \mathbf{K}^0_{K_2}(\mathrm{T}_{K_2} N_{<\epsilon}). \end{aligned}$$

A neighborhood of the smooth submanifold $Z := \Phi_1^{-1}(0)$ in M is diffeomorphic to a neighborhood of the 0-section of the bundle $Z \times \mathfrak{k}_1^* \to Z$. Let $Z_{<\epsilon} = Z \cap M_{<\epsilon}$ so that $N_{<\epsilon} = Z_{<\epsilon}/K_1$. Hence $[\mathbf{c}^\kappa|_{M_{<\epsilon}}]$ can be seen naturally a class in the K-group $\mathbf{K}^0_{K_1\times K_2}(\mathrm{T}_{K_1\times K_2}(Z_{<\epsilon} \times \mathfrak{k}_1^*))$.

Following [Atiyah 1974, Theorem 4.3], the inclusion map $j : Z_{<\epsilon} \hookrightarrow Z_{<\epsilon} \times \mathfrak{k}_1^*$ induces the Thom isomorphism

$$j_! : \mathbf{K}^0_{K_1\times K_2}(\mathrm{T}_{K_1\times K_2} Z_{<\epsilon}) \to \mathbf{K}^0_{K_1\times K_2}(\mathrm{T}_{K_1\times K_2}(Z_{<\epsilon} \times \mathfrak{k}_1^*)),$$

with the commutative diagram

$$\begin{tikzcd}
\mathbf{K}^0_{K_1\times K_2}(\mathrm{T}_{K_1\times K_2} Z_{<\epsilon}) \arrow[r, "j_!"] \arrow[dr, "\mathrm{Index}^{K_1\times K_2}_{Z_{<\epsilon}}"'] & \mathbf{K}^0_{K_1\times K_2}(\mathrm{T}_{K_1\times K_2}(Z_{<\epsilon} \times \mathfrak{k}_1^*)) \arrow[d, "\mathrm{Index}^{K_1\times K_2}_{Z_{<\epsilon}\times\mathfrak{k}_1^*}"] \\
 & R^{-\infty}(K_1 \times K_2).
\end{tikzcd} \tag{54}$$

Let $\pi_1 : Z_{<\epsilon} \to N_{<\epsilon}$ be the quotient relative to the free action of K_1. The corresponding isomorphism

$$\pi_1^* : \mathbf{K}^0_{K_2}(\mathrm{T}_{K_2} N_{<\epsilon}) \to \mathbf{K}^0_{K_1\times K_2}(\mathrm{T}_{K_1\times K_2} Z_{<\epsilon})$$

satisfies the rule

$$[\mathrm{Index}^{K_1\times K_2}_{Z_{<\epsilon}}(\pi_1^*\theta)]^{K_1} = \mathrm{Index}^{K_2}_{N_{<\epsilon}}(\theta) \tag{55}$$

for any $\theta \in \mathbf{K}^0_{K_2}(\mathrm{T}_{K_2} N_{<\epsilon})$.

Lemma 4.4 [Paradan 2001]. *We have*

$$j_! \circ \pi_1^*([\mathbf{c}_2^{K_2}|_{N_{<\epsilon}}]) = [\mathbf{c}^K|_{M_{<\epsilon}}]$$

in $\mathbf{K}^0_{K_1\times K_2}(\mathrm{T}_{K_1\times K_2}(Z_{<\epsilon} \times \mathfrak{k}_1^*))$.

Proof. This lemma is proven in [Paradan 2001, Section 6.2] when the group K_2 is trivial. It is easy to check that the proof extends naturally to our setting. □

Using Lemma 4.4 together with (54) and (55), we get that

$$\mathfrak{Q}(\mathcal{M}_0) = \left[\mathrm{Index}^{K_1\times K_2}_{Z_{<\epsilon}\times\mathfrak{k}_1^*}(\mathbf{c}^K|_{M_{<\epsilon}})\right]^{K_1\times K_2} = \left[\mathrm{Index}^{K_2}_{N_{<\epsilon}}(\mathbf{c}_2^{K_2}|_{N_{<\epsilon}})\right]^{K_2} = \mathfrak{Q}(\mathcal{M}'_0). \quad \square$$

5. Example: The cotangent bundle of an orbit

5A. ***The formal quantization of*** $\mathbf{T^*}\boldsymbol{K}$**.** Let K be a compact connected Lie group equipped with the action of two copies of K given by $(k_1, k_2)\cdot a = k_2 a k_1^{-1}$. Then we have a Hamiltonian action of $K_1 \times K_2$ on the cotangent bundle T^*K. In this section, we check that each formal geometric quantization of T^*K, $\mathfrak{Q}^{-\infty}_{K_1\times K_2}(\mathrm{T}^*K)$ and $\mathfrak{Q}^{\Phi}_{K_1\times K_2}(\mathrm{T}^*K)$ are both equal to the $(K_1\times K_2)$-module $L^2(K)$.

The tangent bundle $\mathrm{T}K$ is identified with $K \times \mathfrak{k}$ through the right translations: to $(a, X) \in K \times \mathfrak{k}$, associate $\frac{d}{dt} a e^{tX}|_0$. The action of $K_1 \times K_2$ on the cotangent bundle $\mathrm{T}^*K \simeq K \times \mathfrak{k}^*$ is then

$$(k_1, k_2)\cdot(a, \xi) = (k_2 a k_1^{-1}, k_1 \cdot \xi).$$

The symplectic form on T^*K is $\Omega := -d\lambda$, where λ is the Liouville 1-form. We compute these two forms in coordinates. The tangent bundle of $\mathrm{T}^*K \simeq K \times \mathfrak{k}^*$ is identified with $\mathrm{T}^*K \times \mathfrak{k} \times \mathfrak{k}^*$: for each $(a, \xi) \in \mathrm{T}^*K$, we have a two-form $\Omega_{(a,\xi)}$ on $\mathfrak{k} \times \mathfrak{k}^*$. A direct computation gives

$$\Omega_{(a,\xi)}(X, X') = \langle \xi, [X, X'] \rangle, \quad \Omega_{(a,\xi)}(\eta, \eta') = 0, \quad \Omega_{(a,\xi)}(X, \eta) = \langle \eta, X \rangle$$

for $X, X' \in \mathfrak{k}$ and $\eta, \eta' \in \mathfrak{k}^*$. So $\Omega_{(a,\xi)} = \Omega_0 + \pi_\xi$ where Ω_0 is the canonical (constant) symplectic form on $\mathfrak{k} \times \mathfrak{k}^*$ and π_ξ is the closed two-form on $\mathfrak{k}$ defined by $\pi_\xi(X, Y) = \langle \xi, [X, Y] \rangle$.

If we identify $\mathfrak{k} \simeq \mathfrak{k}^*$ through an invariant Euclidean norm, the symplectic structure on $\mathrm{T}_{(a,\xi)}(\mathrm{T}^*K) \simeq \mathfrak{k} \times \mathfrak{k}^*$ is given by a skew-symmetric matrix

$$A_\xi := \begin{pmatrix} \mathrm{ad}(\xi) & -I_n \\ I_n & 0 \end{pmatrix},$$

so that

$$\Omega_{(a,\xi)}((X, \eta), (X', \eta')) = (A_\xi(X, \eta), (X', \eta')) = (\xi, [X, X']) + (X, \eta') - (X', \eta).$$

We will work with the following compatible almost complex structure on the tangent bundle of T^*K: $J_\xi = A_\xi(-A_\xi^2)^{-1/2}$. When $\xi = 0$, the complex structure J_0 on $\mathfrak{k} \times \mathfrak{k}^*$ is defined by the matrix

$$J_0 := \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

Hence the complex K-module $(\mathfrak{k} \times \mathfrak{k}^*, J_0)$ is naturally identified with the complexification $\mathfrak{k}_\mathbb{C}$ of $\mathfrak{k}$.

It is easy to check that the moment map relative to the $(K_1 \times K_2)$-action is the *proper* map $\Phi : \mathrm{T}^*K \to \mathfrak{k}_1^* \times \mathfrak{k}_2^*$ defined by $\Phi(a, \xi) = (-\xi, a \cdot \xi)$.

Here the symplectic manifold T^*K is prequantized by the trivial line bundle.

5A1. *Computation of* $\mathcal{Q}^{-\infty}_{K_1 \times K_2}(\mathrm{T}^*K)$. Let $\mathcal{O}_1 \times \mathcal{O}_2$ be a coadjoint orbit of $K_1 \times K_2$ in $\mathfrak{k}_1^* \times \mathfrak{k}_2^*$. One checks that

$$\Phi^{-1}(\mathcal{O}_1 \times \mathcal{O}_2) = \begin{cases} \varnothing & \text{if } \mathcal{O}_1 \neq -\mathcal{O}_2, \\ \text{a } (K_1 \times K_2)\text{-orbit} & \text{if } \mathcal{O}_1 = -\mathcal{O}_2. \end{cases} \tag{56}$$

We know that the stabilizer subgroup K_ξ of an element $\xi \in \mathfrak{k}^*$ is connected. Then the stabilizer subgroup $(K_1 \times K_2)_{(a,\xi)} = \{(k_1, ak_1a^{-1}),\ k_1 \in K_\xi\}$ is also connected.

Let $(\mathrm{T}^*K)_{(\mu,\lambda)}$ be the symplectic reduction of T^*K at the level $(\mu, \lambda) \in \widehat{K}^2$. For any $\mu \in \widehat{K}$, define $\mu^* \in \widehat{K}$ by the relation $-K \cdot \mu = K \cdot \mu^*$; note that $V^K_{\mu^*} \simeq (V^K_\mu)^*$. Using Theorem 2.17 gives

$$\mathcal{Q}((\mathrm{T}^*K)_{(\mu,\lambda)}) = \begin{cases} 0 & \text{if } \lambda \neq \mu^*, \\ 1 & \text{if } \lambda = \mu^*. \end{cases} \tag{57}$$

Finally

$$\begin{aligned}\mathcal{Q}^{-\infty}_{K_1\times K_2}(\mathrm{T}^*K) &= \sum_{(\mu,\lambda)\in\widehat{K}\times\widehat{K}} \mathcal{Q}\left((\mathrm{T}^*K)_{(\mu,\lambda)}\right) V_\mu^{K_1}\otimes V_\lambda^{K_2}\\ &= \sum_{\mu\in\widehat{K}} V_\mu^{K_1}\otimes (V_\mu^{K_2})^* = \mathrm{L}^2(K).\end{aligned}$$

5A1. *Computation of* $\mathcal{Q}^{\Phi}_{K_1\times K_2}(\mathrm{T}^*K)$. The Kirwan vector field on T^*K is

$$\kappa(a,\xi) = -2\xi \in \mathfrak{k}_{\mathbb{C}}.$$

Let $\mathbf{c}^\kappa$ be the symbol Thom(T^*K, J) pushed by the vector field $\frac{1}{2}\kappa$. At each (a,ξ) in T^*K, the map $\mathbf{c}^\kappa_{(a,\xi)}(X\oplus\eta)$ from $\wedge^{\mathrm{even}}_{J_\xi}(\mathfrak{k}\times\mathfrak{k}^*)$ to $\wedge^{\mathrm{odd}}_{J_\xi}(\mathfrak{k}\times\mathfrak{k}^*)$ is equal to the Clifford map $\mathbf{c}(X+\xi\oplus\eta)$. Note that $\mathbf{c}^\kappa$ is a K_2-transversally elliptic symbol on T^*K: we have $\mathrm{Char}(\mathbf{c}^\kappa)\cap \mathrm{T}_{K_2}(\mathrm{T}^*K) = \{(1,0)\}$. We will now compute the equivariant index of $\mathbf{c}^\kappa$.

First consider the homotopy $t\in[0,1]\to J_{t\xi}$ of symplectic structure on T^*K. Let $\tilde{\mathbf{c}}^\kappa$ be the symbol acting on $\wedge^\bullet_{J_0}(\mathfrak{k}\times\mathfrak{k}^*) = \wedge^\bullet_{\mathbb{C}}\mathfrak{k}_{\mathbb{C}}$. Proposition 2.6 shows that the symbols $\mathbf{c}^\kappa$ and $\tilde{\mathbf{c}}^\kappa$ define the same class in $\mathbf{K}^0_{K_1\times K_2}(\mathrm{T}_{K_2}(\mathrm{T}^*K))$.

The projection $\pi : \mathrm{T}^*K\to\mathfrak{k}^*$ corresponds to the quotient map relative to the free action of K_2. At the level of $\mathbf{K}^0$-groups we get an isomorphism

$$\pi_* : \mathbf{K}^0_{K_1\times K_2}(\mathrm{T}_{K_2}(\mathrm{T}^*K)) \to \mathbf{K}^0_{K_1}(\mathrm{T}\mathfrak{k}^*).$$

The *free action property* (see [Atiyah 1974, Theorem 3.1]) gives that

$$\mathrm{Index}^{K_1\times K_2}_{\mathrm{T}^*K}(\sigma) = \sum_{\mu\in\widehat{K}} \mathrm{Index}^{K_1}_{\mathfrak{k}^*}\left(\pi_*(\sigma\otimes V_\mu^{K_2})\right)\otimes (V_\mu^{K_2})^*$$

for any class $\sigma\in\mathbf{K}^0_{K_1\times K_2}(\mathrm{T}_{K_2}(\mathrm{T}^*K))$. In our case the symbol $\pi_*(\tilde{\mathbf{c}}^\kappa)$ is equal to the Bott symbol Bott($\mathfrak{k}^*$), and for any K_2-module E_2 we have

$$\pi_*(\tilde{\mathbf{c}}^\kappa\otimes E_2) = \mathrm{Bott}(\mathfrak{k}^*)\otimes E_1,$$

where E_1 is the module E_2 with the action of K_1. Then

$$\begin{aligned}\mathcal{Q}^{\Phi}_{K_1\times K_2}(\mathrm{T}^*K) &= \mathrm{Index}^{K_1\times K_2}_{\mathrm{T}^*K}(\tilde{\mathbf{c}}^\kappa)\\ &= \sum_{\mu\in\widehat{K}} \mathrm{Index}^{K_1}_{\mathfrak{k}^*}\left(\mathrm{Bott}(\mathfrak{k}^*)\otimes V_\mu^{K_1}\right)\otimes (V_\mu^{K_2})^*\\ &= \sum_{\mu\in\widehat{K}} V_\mu^{K_1}\otimes (V_\mu^{K_2})^* = L^2(K),\end{aligned}$$

since $\mathrm{Index}^{K_1}_{\mathfrak{k}^*}(\mathrm{Bott}(\mathfrak{k}^*)) = 1$.

5B. ***The formal quantization of*** $\mathbf{T^*(\boldsymbol{K/H})}$**.** Let H be a closed connected subgroup of K. We look at T^*K as a Hamiltonian manifold relative to the action of $H \times K \subset K_1 \times K_2$. The moment map $\Phi = (\Phi_H, \Phi_K)$ is defined by: $\Phi_H(a, \xi) = -\mathrm{pr}(\xi)$ and $\Phi_K(a, \xi) = a \cdot \xi$, where $\mathrm{pr} : \mathfrak{k}^* \to \mathfrak{h}^*$ is the projection. Note that Φ is a proper map.

The cotangent bundle $\mathrm{T}^*(K/H)$, viewed as K-manifold, is equal to the symplectic reduction of T^*K relative to the H-action: if the kernel of the projection pr is denoted $\mathfrak{h}^\perp$, we have

$$\Phi_H^{-1}(0)/H = K \times_H \mathfrak{h}^\perp = \mathrm{T}^*(K/H).$$

This is the setting of Section 4B. The reduction of the $H \times K$ proper Hamiltonian manifold T^*K relative to the H-action is smooth. Then its formal quantization is computed as

$$\begin{aligned} \mathcal{Q}_K^\Phi(\mathrm{T}^*(K/H)) = [\mathcal{Q}_{H\times K}^\Phi(\mathrm{T}^*K)]^H &= [\mathcal{Q}_{K_1\times K_2}^\Phi(\mathrm{T}^*K)|_{H\times K}]^H \\ &= [\mathrm{L}^2(K)]^H \\ &= \mathrm{L}^2(K/H). \end{aligned} \tag{58}$$

Here the fact that $\mathcal{Q}_{H\times K}^\Phi(\mathrm{T}^*K)$ is equal to the restriction of $\mathcal{Q}_{K_1\times K_2}^\Phi(\mathrm{T}^*K) = \mathrm{L}^2(K)$ to $H \times K$ is a consequence of Theorem 1.3.

Denote by $[\mathrm{T}^*(K/H)]_\mu$ the symplectic reduction at $\mu \in \widehat{K}$ of the K-Hamiltonian manifold $\mathrm{T}^*(K/H)$. Theorem 1.4 together with (58) gives:

Corollary 5.1. *For any* $\mu \in \widehat{K}$, *we have*

$$\mathcal{Q}([\mathrm{T}^*(K/H)]_\mu) = \dim[V_\mu^K]^H,$$

where $[V_\mu^K]^H$ *is the subspace of* H*-invariant vector.*

5C. ***The formal quantization of*** $\mathbf{T^*(\boldsymbol{K/H})}$ ***relative to the action of G.*** Let G be a closed connected subgroup of K. We look at the Hamiltonian action of G on $\mathrm{T}^*(K/H)$. Let $\Phi_G : \mathrm{T}^*(K/H) \to \mathfrak{g}^*$ be the moment map. Consider also the restriction of the K-module $\mathrm{L}^2(K/H)$ to the subgroup G.

Proposition 5.2. *The following statements are equivalent*:

(1) *The moment map* $\Phi_G : \mathrm{T}^*(K/H) \to \mathfrak{g}^*$ *is proper.*

(2) $\Phi_G^{-1}(0)$ *is equal to the zero section.*

(3) $k \cdot \mathfrak{g} + \mathfrak{h} = \mathfrak{k}$ *for any* $k \in K$.

(4) $\mathfrak{g} + \mathfrak{h} = \mathfrak{k}$.

(5) *G acts transitively on K/H.*

(6) $[L^2(K/H)]^G \simeq \mathbb{C}$.

(7) $\mathrm{L}^2(K/H)|_G$ *is an admissible G-representation.*

Proof. The implication (1) ⇒ (7) is a consequence of Theorem 1.3. To prove (7) ⇒ (6), suppose now that

$$\mathrm{L}^2(K/H)|_G = \sum_{\mu\in\widehat{K}} [V_\mu^K]^H \otimes (V_\mu^K)^*|_G$$

is an admissible G-representation. This means that for any $\lambda \in \widehat{G}$, the set

$$A_\lambda := \big\{\mu \in \widehat{K} \mid [V_\mu^K]^H \neq \{0\} \text{ and } [(V_\lambda^G)^* \otimes (V_\mu^K)^*|_G]^G \neq \{0\}\big\}$$

is finite. Then the vector space $[L^2(K/H)]^G$ is equal to the finite-dimensional vector space $\sum_{\mu\in A_0}[V_\mu^K]^H \otimes [(V_\mu^K)^*]^G$. For any irreducible representation V_μ^K we have, for any $k \geq 1$, a canonical K-equivariant inclusion

$$\underbrace{V_\mu^K \otimes \cdots \otimes V_\mu^K}_{k \text{ times}} \hookrightarrow V_{k\mu}^K.$$

Hence $[V_\mu^K]^H \neq 0$ gives $[V_{k\mu}^K]^H \neq 0$ for any $k \geq 1$. Then if $\mu \in A_0$, we have $k\mu \in A_0$ for $k \geq 1$. Finally the fact that A_0 is finite implies that A_0 is reduced to $\mu = 0$. Hence the only G-invariant functions on K/H are the scalars.

The equivalences (6) ⇔ (5) ⇔ (4) ⇔ (3) are a general fact concerning smooth actions of a compact connected Lie group G on a compact connected manifold M. The manifold M does not have G-invariant functions which are not scalar if and only if the action of G on M is transitive. Also, given a point $m \in M$, the orbit $G\cdot m$ is all of M if and only if tangent spaces $\mathrm{T}_m(G\cdot m)$ and $\mathrm{T}_m M$ are equal. If we take $m = \overline{k^{-1}}$ in $M = K/H$, the condition $\mathrm{T}_m(G\cdot m) = \mathrm{T}_m M$ is equivalent to $k\cdot\mathfrak{g}+\mathfrak{h} = \mathfrak{k}$.

To check the implication (3) ⇒ (2), let $[k, \xi] \in K \times_H \mathfrak{h}^\perp = \mathrm{T}^*(K/H)$. We have $\Phi_G([k, \xi]) = 0$ if and only if $k \cdot \xi \in \mathfrak{g}^\perp$. Therefore the vector ξ belongs to $k^{-1}\cdot\mathfrak{g}^\perp \cap \mathfrak{h}^\perp = (k^{-1}\cdot\mathfrak{g} + \mathfrak{h})^\perp$. Hence condition (3) imposes that $\xi = 0$.

The implication (2) ⇔ (1) comes from the fact that Φ_G is a homogeneous map of degree one between the vector bundle $\mathrm{T}^*(K/H)$ and the vector space $\mathfrak{g}^*$. □

Suppose now that the cotangent bundle $\mathrm{T}^*(K/H)$ is a *proper* Hamiltonian G-manifold. Denote by $[\mathrm{T}^*(K/H)]_{\mu,G}$ the (compact) symplectic reduction at $\mu \in \widehat{G}$ of the G-Hamiltonian manifold $\mathrm{T}^*(K/H)$.

Corollary 5.3. *The multiplicity of* V_μ^G *in* $\mathrm{L}^2(K/H)$ *is equal to the quantization of the reduced space* $[\mathrm{T}^*(K/H)]_{\mu,G}$.

Proof. Using Theorem 1.3, Equation (58) gives

$$\mathcal{Q}_G^{-\infty}(\mathrm{T}^*(K/H)) = \mathcal{Q}_K^{-\infty}(\mathrm{T}^*(K/H))|_G = \mathrm{L}^2(K/H)|_G.$$

In other words, the multiplicity of V_μ^G in $\mathrm{L}^2(K/H)$ is equal to the quantization of the reduced space $[\mathrm{T}^*(K/H)]_{\mu,G}$. □

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PAUL-ÉMILE PARADAN
INSTITUT DE MATHÉMATIQUES ET DE MODÉLISATION DE MONTPELLIER (I3M)
UNIVERSITÉ MONTPELLIER 2
PLACE EUGÈNE BATAILLON
34095 MONTPELLIER CEDEX 5
FRANCE

paradan@math.univ-montp2.fr
http://www.math.univ-montp2.fr/~paradan/

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Los Angeles, CA 90095-1555
pacific@math.ucla.edu

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Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Robert Finn
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
finn@math.stanford.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Darren Long
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
long@math.ucsb.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Alexander Merkurjev
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
merkurev@math.ucla.edu

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

Jonathan Rogawski
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
jonr@math.ucla.edu

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