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EMBEDDED CONSTANT-CURVATURE CURVES ON CONVEX SURFACES

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We prove the existence of embedded closed constant-curvature curves on convex surfaces.

1. Introduction

Let (S^2, g) be a two-dimensional oriented sphere with a smooth Riemannian metric g. We prove existence results for closed embedded curves with prescribed geodesic curvature in (S^2, g), when the Gauss curvature K_g of the metric g is positive. In particular, we study the existence of closed embedded constant-curvature curves on strictly convex spheres.

Let $c : S^2 \to \mathbb{R}$ be a smooth positive function. We consider the following equation for curves γ on S^2:

$$D_{t,g}\dot{\gamma}(t) = |\dot{\gamma}(t)|_g c(\gamma(t)) J_g(\gamma(t))\dot{\gamma}(t), \tag{1-1}$$

where $D_{t,g}$ is the covariant derivative with respect to g, and $J_g(x)$ is the rotation by $\pi/2$ in $T_x S^2$ with respect to g and the given orientation. Solutions γ to Equation (1-1) are constant-speed curves with geodesic curvature $c_g(\gamma, t)$ given by $c(\gamma(t))$. Besides the geometric interpretation, (1-1) describes the motion of a charged particle on (S^2, g) in a magnetic field with magnetic form $c\mu_g$, where μ_g denotes the volume form of g [Arnold 1986; Novikov 1982; Ginzburg 1996].

By [Schneider 2011b; Taimanov 1992], closed embedded solutions to (1-1) exist if the curvature function c is large enough, depending on the metric g. When g is $\frac{1}{4}$-pinched, that is, $\sup K_g < 4 \inf K_g$, then there are embedded closed solutions of (1-1) for every positive function c [Schneider 2011b; Rosenberg and Smith 2010]. It is conjectured [Novikov 1982, §5; Rosenberg and Smith 2010] that this remains true for an arbitrary metric g on S^2. If K_g and c are positive, then by [Rosenberg and Smith 2010; Robadey 2001; Schneider 2011a] there are always Alexandrov-embedded, closed solutions to (1-1), that is, curves that bound an immersed disc.

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We show that on strictly convex spheres, that is, $K_g > 0$, there are closed embedded solutions to (1-1) if the curvature function is small enough, depending on the metric g. In particular, we show:

Theorem 1.1. *Suppose (S^2, g) has positive Gauss curvature. Then there exists an $\varepsilon_0 > 0$ such that for all $0 < c \le \varepsilon_0$ there are two embedded closed curves with constant geodesic curvature c.*

Hence, on strictly convex spheres, there are closed, embedded constant-curvature curves for large and small values of $c > 0$. We conjecture that this remains true for all $c > 0$ and any metric on S^2.

We use the degree theory developed in [Schneider 2011b] to prove our existence result. The required compactness results are given in Section 2. The a priori estimates follow from Reilly's formula [1977]. The fact that a geodesic cannot touch itself continues to hold for solutions to (1-1) when the geodesic curvature is close to zero. This allows us to carry out the degree argument within the class of embedded curves. The existence result is given in Section 3.

2. The a priori estimate

Lemma 2.1. *Suppose (S^2, g) has positive Gauss curvature K_g and $\gamma \in C^2(S^1, S^2)$ is an (Alexandrov) embedded curve with nonnegative geodesic curvature. Then the length $L(\gamma)$ of γ is bounded by*

$$L(\gamma) \le 2\pi\sqrt{2}\big(\inf_{S^2} K_g\big)^{-1/2}.$$

Proof. As in [Choi and Wang 1983], where area bounds for embedded compact minimal surfaces in S^3 are given, we use Reilly's formula [1977]: Let (M, g) be a compact Riemannian manifold with boundary ∂M, $f \in C^\infty(M)$, $z = f|_{\partial M}$ and $u = \partial f/\partial n$ on ∂M, where n denotes the outer normal. Then

$$\int_M (\bar\Delta f)^2 - |\bar\nabla^2 f|^2 = \int_M \mathrm{Ric}(\bar\nabla f, \bar\nabla f) + \int_{\partial M} (\Delta z + Hu)u - \langle \nabla z, \nabla u\rangle + \Pi(\nabla z, \nabla z), \tag{2-1}$$

where we denote by $\bar\Delta$, Δ and $\bar\nabla$, ∇ the Laplacians and covariant derivatives on M and ∂M respectively; H is the mean curvature and Π is the second fundamental form of ∂M.

If the curve γ is embedded or Alexandrov-embedded, then we may assume that we are in the above situation with $\partial M = \gamma$.

We take z an eigenfunction of λ_1 the first nontrivial eigenvalue on ∂M,

$$\Delta z + \lambda_1 z = 0 \ \text{ on } \ \partial M,$$

and f its harmonic extension to M. In dimension two, (2-1) leads to

$$\int_M (\bar{\Delta} f)^2 - |\bar{\nabla}^2 f|^2 = \int_M K_g |\bar{\nabla} f|^2 + \int_{\partial M} \Delta z u + c u^2 - \langle \nabla z, \nabla u \rangle + c |\nabla z|^2,$$

where c is the geodesic curvature of ∂M and K_g denotes the Gauss curvature of M. Using the facts that the geodesic curvature c of ∂M is nonnegative, f is harmonic, and z is an eigenfunction, we obtain

$$0 \geq \left(\inf_M K_g\right) \int_M |\bar{\nabla} f|^2 - 2\lambda_1 \int_{\partial M} z u.$$

Integrating by parts again, we see

$$\int_{\partial M} z u = \int_M |\bar{\nabla} f|^2 + f \bar{\Delta} f = \int_M |\bar{\nabla} f|^2.$$

Since z is a nontrivial eigenfunction, f is nonconstant and we arrive at

$$\left(\inf_M K_g\right) \leq 2\lambda_1.$$

The first nontrivial eigenvalue λ_1 depends only on the length $L(\partial\Omega)$ of ∂M and is given by

$$\lambda_1 = \frac{4\pi^2}{L(\partial\Omega)^2}. \qquad \square$$

Lemma 2.2. *Let (γ_n) be a sequence of simple closed curves that converge in $C^2(S^1, S^2)$ to a nonconstant closed geodesic γ in (S^2, g). Then γ is also simple.*

Proof. To obtain a contradiction, assume that there are $\theta_1 \neq \theta_2$ in $S^1 = \mathbb{R}/\mathbb{Z}$ such that $\gamma(\theta_1) = \gamma(\theta_2)$. Since γ is a limit of simple curves and $|\dot{\gamma}| \equiv \text{const}$, we have

$$\dot{\gamma}(\theta_1) = \pm\dot{\gamma}(\theta_2).$$

From the uniqueness of geodesics, we have for $t \in S^1$

$$\gamma(t) = \gamma(\pm(t - \theta_1) + \theta_2).$$

Setting $t = (\theta_1 + \theta_2)/2$, we find that

$$\gamma(t) = \gamma(t - \theta_1 + \theta_2).$$

Consequently, γ is an n-fold covering of a simple geodesic for some $n \geq 2$. From the stability of the winding number, we get a contradiction. $\square$

We denote by g_{can} the standard round metric on S^2 with curvature $K_{g_{\text{can}}} \equiv 1$. We fix a function $\varphi \in C^\infty(S^2, \mathbb{R})$ and a conformal metric

$$g = e^{\varphi} g_{\text{can}}$$

on S^2 with positive Gauss curvature $K_g > 0$. We consider the family of metrics $\{g_t : t \in [0,1]\}$ defined by

$$g_t := e^{t\varphi} g_{\mathrm{can}}.$$

Then the Gauss curvature K_{g_t} of the metric g_t satisfies, for some $K_0 > 0$,

$$\begin{aligned} K_{g_t} &= e^{-t\varphi}\bigl(-t\Delta_{g_{\mathrm{can}}}(\varphi)+2\bigr) \\ &= e^{-t\varphi}\bigl(-t(2-K_g e^{\varphi})+2\bigr) \geq K_0, \end{aligned}$$

because K_g is positive.

Lemma 2.3. *Suppose $c : S^2 \to \mathbb{R}$ is a nonnegative smooth function. For $r \in [0,1]$, we define the set of closed curves $\mathcal{M}_r$ by*

$$\begin{aligned} \mathcal{M}_r := \{\gamma \in C^2(S^1, S^2) : \ &\gamma \textit{ is embedded}, |\dot\gamma|_g \equiv \mathrm{const}, \\ &\textit{and there exists } (t,s) \in [0,1]\times[0,r] : c_{g_t}(\gamma,\theta) = sc(\gamma(\theta)) \textit{ for all } \theta \in S^1\}, \end{aligned}$$

where $c_{g_t}(\gamma,\cdot)$ denotes the geodesic curvature of γ with respect to g_t.

Then there is $\varepsilon_0 > 0$ such that $\mathcal{M}_{\varepsilon_0}$ is compact. Moreover, $\varepsilon_0 > 0$ may be chosen uniformly with respect to $\|c\|_\infty$.

Proof. Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}_r$ for some $r > 0$. By Lemma 2.1 and (1-1), we get a uniform bound in $C^3(S^1, S^2)$, and from the Gauss–Bonnet formula, the length of γ_n is bounded below; and in both cases the bounds are uniform with respect to r. Since the metrics $\{g_t : t \in [0,1]\}$ are uniformly equivalent, there is $C_0 > 0$ such that we have for all $t \in [0,1]$

$$|\dot\gamma_n|_{g_t} > (C_0)^{-1} \quad \text{and} \quad \|\gamma_n\|_{C^3(S^1,S^2),g_t} < C_0. \tag{2-2}$$

Up to a subsequence we may assume $(t_n, s_n) \to (t,s) \in [0,1]\times[0,r]$,

$$\gamma_n \to \gamma \text{ in } C^2(S^1,S^2),$$

where $|\dot\gamma|_{g_t} \equiv \mathrm{const}$ and

$$c_{g_t}(\gamma,\theta) = sc(\gamma(\theta)) \quad \text{for all } \theta \in S^1. \tag{2-3}$$

Thus, if $\mathcal{M}_r$ is not compact, there is $(t,s) \in [0,1]\times[0,r]$ and $\gamma_r \in C^2(S^1,S^2)$ satisfying $|\dot\gamma|_{g_t} \equiv \mathrm{const}$ and (2-3), which is not embedded, but a limit of embedded curves in $\mathcal{M}_r$. Thus there are $\theta_1, \theta_2 \in S^1$, such that $\theta_1 \neq \theta_2$ and $\gamma_r(\theta_1) = \gamma_r(\theta_2)$. From (2-2) we deduce that there is $\delta > 0$ independent of r, such that

$$\delta \leq |\theta_1 - \theta_2| \leq 1 - \delta. \tag{2-4}$$

Hence for any $n \in \mathbb{N}$ there is $\gamma_n \in \mathcal{M}_r$ such that

$$\mathrm{dist}(\gamma_n(\theta_1), \gamma_n(\theta_2)) \leq \frac{1}{n}. \tag{2-5}$$

To obtain a contradiction, assume there is (r_n) converging to 0 such that $\mathcal{M}_{r_n}$ is not compact. Then for any $n \in \mathbb{N}$, there are $(t_n, s_n) \in [0, 1] \times [0, r_n]$, $\theta_{1,n}, \theta_{2,n} \in S^1$, and $\gamma_n \in \mathcal{M}_{r_n}$ that satisfy (2-4) and (2-5). From the uniform bounds, going to a subsequence, we may assume that $(t_n, s_n, \gamma_n, \theta_{1,n}, \theta_{2,n})$ converge to $(t, 0, \gamma, \theta_1, \theta_2)$, where θ_1 and θ_2 satisfy (2-4) and γ is a closed nontrivial geodesic in (S^2, g_t) satisfying $\gamma(\theta_1) = \gamma(\theta_2)$. This contradicts Lemma 2.2. Since all the above bounds are uniform with respect to $\|c\|_\infty$, the constant $\varepsilon_0 > 0$ may be chosen uniform with respect to $\|c\|_\infty$ as well. □

3. Existence results

We follow [Schneider 2011b] and consider solutions to (2-3) as zeros of the vector field $X_{c,g}$ defined on the Sobolev space $H^{2,2}(S^1, S^2)$ as follows: For $\gamma \in H^{2,2}(S^1, S^2)$, we let $X_{c,g}(\gamma)$ be the unique weak solution of

$$\big(-D_{t,g}^2 + 1\big)X_{c,g}(\gamma) = -D_{t,g}\dot\gamma + |\dot\gamma|_g c(\gamma) J_g(\gamma)\dot\gamma \tag{3-1}$$

in $T_\gamma H^{2,2}(S^1, S^2)$.

Solutions to (2-3), or equivalently, zeros of $X_{c,g}$, are invariant under a circle action: For $\theta \in S^1 = \mathbb{R}/\mathbb{Z}$ and $\gamma \in H^{2,2}(S^1, S^2)$, we define $\theta * \gamma \in H^{2,2}(S^1, S^2)$ by

$$\theta * \gamma(t) = \gamma(t + \theta).$$

Thus, any solution gives rise to an S^1-orbit of solutions, and we say that two solutions γ_1 and γ_2 are (geometrically) distinct if $S^1 * \gamma_1 \neq S^1 * \gamma_2$.

We denote by $M \subset H^{2,2}(S^1, S^2)$ the set

$$M := \{\gamma \in H^{2,2}(S^1, S^2) : \dot\gamma(\theta) \neq 0 \text{ for all } \theta \in S^1 \text{ and } \gamma \text{ is embedded}\}.$$

In [Schneider 2011b], an integer-valued S^1-degree $\chi_{S^1}(X_{c,g}, M)$ is introduced. The S^1-degree is defined whenever $X_{c,g}$ is proper in M, that is, the set

$$\{\gamma \in M : X_{c,g}(\gamma) = 0\}$$

is compact, and does not change under homotopies in the class of proper vector fields.

Theorem 3.1. *Suppose (S^2, g) has positive Gauss curvature. Then there is $\varepsilon_0 > 0$ such that for all smooth functions $c : S^2 \to \mathbb{R}$ satisfying $0 < c \leq \varepsilon_0$, there are two embedded geometrically distinct closed curves that solve Equation (1-1).*

Proof. From the uniformization theorem up to isometries, we may assume without loss of generality that

$$g = e^\varphi g_{\mathrm{can}},$$

where $\varphi \in C^\infty(S^2, \mathbb{R})$ and g_{can} denotes the standard round metric on S^2.

We consider the set of metrics $\{g_t : t \in [0,1]\}$ defined by

$$g_t := e^{t\varphi} g_{\mathrm{can}}.$$

From Lemma 2.3, there is $\varepsilon_0 > 0$ such that the set

$$\{\gamma \in M : X_{c,g_t}(\gamma) = 0 \text{ for some } t \in [0,1]\}$$

is compact for all functions c with $0 < c \leq \varepsilon_0$. Consequently,

$$[0,1] \ni t \mapsto X_{c,g_t}$$

is a homotopy of proper vector fields. From [Schneider 2011b], we have

$$-2 = \chi_{S^1}(X_{c,g_{\mathrm{can}}}, M),$$

such that the homotopy invariance leads to

$$\chi_{S^1}(X_{c,g}, M) = -2.$$

Since the local degree of an isolated zero orbit is greater than or equal to -1 by [Schneider 2011b, Lemma 4.1], there are at least two geometrically distinct solutions to (1-1). This gives the claim. □

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