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A series of pairs of graphs (Γ_k, Γ'_k), $k = 0, 1, 2, \ldots$, has been considered as candidates for dual pairs of principal graphs of subfactors of small Jones index above 4 and it has recently been proved that the pair (Γ_k, Γ'_k) comes from a subfactor if and only if $k = 0$ or $k = 1$. We show that nevertheless there exists a unique fusion system compatible with this pair of graphs for all nonnegative integers k.

1. Introduction

A subfactor $N \subset M$ with finite index and finite depth generates finitely many isomorphism classes of bimodules with four different combinations of left and right coefficients. They form a bigraded fusion category. Its Grothendieck ring forms a *fusion ring* or a *fusion hypergroup*, namely a bigraded $\mathbb{Z}$-algebra $\mathscr{A}$ satisfying:

- $\mathscr{A}$ has a basis given by finitely many irreducible bimodules of four different kinds: $\mathscr{X} = {}_N\mathscr{X}_N \sqcup {}_N\mathscr{X}_M \sqcup {}_M\mathscr{X}_N \sqcup {}_M\mathscr{X}_M$ (we call the labels N and M right or left coefficients, depending on the position).
- An involution $X \in {}_P\mathscr{X}_Q \to \bar{X} \in {}_Q\mathscr{X}_P$ is defined, where $P, Q \in \{N, M\}$.
- A product is defined for a pair of bimodules with "matching" coefficients, namely, for a pair $(X, Y) \in \mathscr{X} \times \mathscr{X}$ such that the right coefficient of X and the left coefficient of Y match, XY is defined. It decomposes as
$$XY = \sum N^Z_{X,Y} Z,$$
where the sum is taken over those $Z \in \mathscr{X}$ that have the same left (respectively, right) coefficient as X (respectively, Y), and $N^Z_{X,Y} \in \mathbb{N}_0$. Moreover, Frobenius reciprocity holds:
$$N^Z_{X,Y} = N^X_{Z,\bar{Y}} = N^Y_{\bar{X},Z} = N^{\bar{Z}}_{\bar{Y},\bar{X}} = N^{\bar{Y}}_{\bar{Z},X} = N^{\bar{X}}_{Y,\bar{Z}}.$$
- There are identity objects $\mathbf{1}_N \in {}_N\mathscr{X}_N$, $\mathbf{1}_M \in {}_M\mathscr{X}_M$ that act as identity with respect to the product, whenever it is defined.

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The involution extends linearly to define an involution on $\mathcal{A}$. For a fusion ring $\mathcal{A}$, there is a unique weight function $\mu : \mathcal{A} \to \mathbb{R}_{\geq}$ satisfying

$$\mu(\mathbf{1}_N) = \mu(\mathbf{1}_M) = 1,$$
$$\mu(XY) = \mu(X)\mu(Y),$$
$$\mu(X + Z) = \mu(X) + \mu(Z),$$

where X, Y, $Z \in \mathcal{X}$ are with suitable coefficients for each equality, so that XY and $X + Z$ are defined. The (*dual*) *principal graph* of the subfactor encodes partial information of the fusion algebra: namely, the (dual) principal graph has the vertices corresponding to ${}_N\mathcal{X}_N \sqcup {}_N\mathcal{X}_M$ (respectively, ${}_M\mathcal{X}_N \sqcup {}_M\mathcal{X}_M$), with the number of the edges between vertices ${}_NX_N$ and ${}_NY_M$ (respectively, ${}_MX_M$ and ${}_MY_N$) given by $N^Y_{X,{}_NM_M}$ (respectively, $N^Y_{X,{}_MM_N}$.)

On the other hand, one may start with a pair of graphs and may consider if there is a fusion algebra compatible with the fusion constraints determined by the graphs. Such investigation may be used to exclude graphs as (dual) principal graphs of subfactors. For example, type E_7 and D_{2n+1} Dynkin diagrams are proved *not* to be (dual) principal graphs of subfactors, by showing that the fusion constraints given by the graphs give rise to inconsistency in fusion rules [Izumi 1991; Sunder and Vijayarajan 1993]. Note that the existence of a fusion algebra compatible with a given pair of graphs does not imply the existence of a subfactor with given graphs as (dual) principal graphs.

In this paper, we deal with the series of pairs of graphs shown in Figure 1.

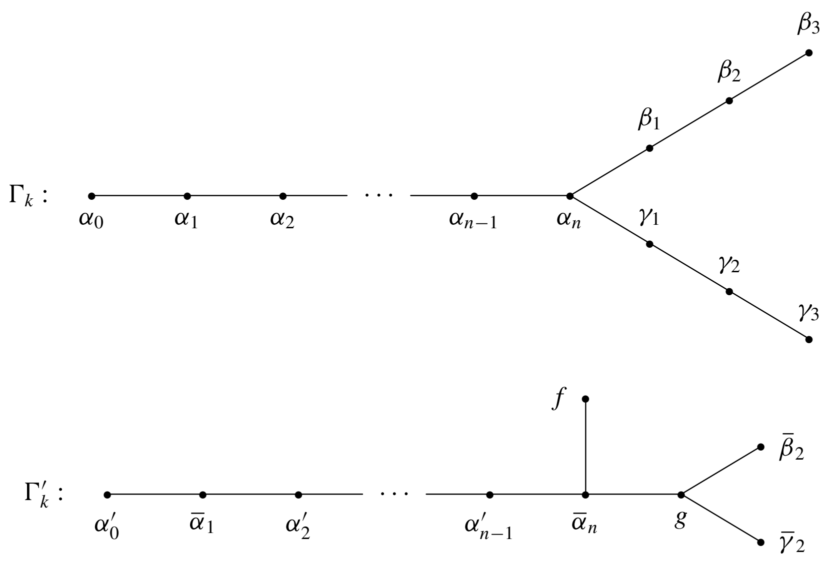

Figure 1. $n = 4k + 3$, $k = 0, 1, \ldots$

These graphs are a part of the list of the graphs that were candidates for (dual) principal graphs of a subfactor with indices between 4 and $3+\sqrt{3}$ given by [Haagerup 1994]. The notation used here is somewhat different from the one used in [Haagerup 1994]. It has been already proved that, for $k=0,1$, the graphs Γ_k (respectively, Γ'_k) are (dual) principal graphs of a subfactors [Asaeda and Haagerup 1999; Bigelow et al. 2009], and for $k>1$, they are not realized as (dual) principal graphs [Asaeda and Yasuda 2009]. In this paper, we prove that, despite that the Γ_k (respectively, Γ'_k) are not principal graphs for $k>1$, there are still fusion algebras consistent with the graphs, and moreover such fusion algebras are unique for each k.

Theorem 1.1. *Let $V_{11}:=\{\textit{even vertices of } \Gamma_k\}$, $V_{12}:=\{\textit{odd vertices of } \Gamma_k\}$, $V_{21}:=\{\textit{odd vertices of } \Gamma'_k\}$, $V_{22}:=\{\textit{even vertices of } \Gamma'_k\}$, and $V:=V_{11}\sqcup V_{12}\sqcup V_{21}\sqcup V_{22}$. For each k, there is a unique fusion algebra $\mathcal{A}=\mathbb{Z}\mathcal{X}$, where*

$$\mathcal{X}={}_N\mathcal{X}_N\sqcup{}_N\mathcal{X}_M\sqcup{}_M\mathcal{X}_N\sqcup{}_M\mathcal{X}_M$$

is compatible with the graphs Γ_k, Γ'_k. Namely,

$$\begin{aligned}{}_N\mathcal{X}_N&=V_{11},\\ {}_N\mathcal{X}_M&=V_{12},\\ {}_M\mathcal{X}_N&=V_{21},\\ {}_M\mathcal{X}_M&=V_{22}\end{aligned}$$

as sets, and

$$N^Y_{X,\alpha_1}(\textit{respectively, } N^Y_{X,\bar{\alpha}_1})=\begin{cases}1 & \textit{if } X \textit{ and } Y \textit{ are connected by an edge},\\ 0 & \textit{else,}\end{cases}$$
$$N^Y_{X,1}=\delta_{X,Y},$$

where $X,Y\in\mathcal{X}$, and 1 denotes identity objects $1_N=\alpha_0\in{}_N\mathcal{X}_N$ or $1_M=\alpha'_0\in{}_M\mathcal{X}_M$.

In Section 2 we show that if there is a fusion system compatible with the graphs Γ_k, Γ'_k, it must be unique. In Section 3 we show the existence of such a fusion system.

2. Uniqueness, positivity, and integrality of the fusion rules

In this section we prove that if there is a fusion algebra compatible with the graphs, it is unique. Positivity and integrality of fusion coefficients is derived: we do not impose them in showing uniqueness of the fusion rules.

2A. ***Fusion rules for the even vertices.*** In this subsection we show that there is a unique fusion algebra structure on $\mathcal{A}_1=\mathbb{Z}{}_N\mathcal{X}_N$ compatible with the graph Γ_k.

The main issue is to determine the fusion rule among $\beta_1, \beta_3, \gamma_1, \gamma_3$. The rest will follow easily from this.

In the following we assume there is a fusion algebra compatible with (Γ_k, Γ_k'). The involution $\gamma \in V \to \bar{\gamma} \in V$ extends linear to a map on $\mathbb{R}V$. For simplicity, we refer to the objects in $\mathscr{X}$ by corresponding vertices in V. For $X := \sum N_X^Z Z \in \mathbb{R}V$ and $Y \in V$, denote

$$\langle X, Y\rangle = \langle Y, X\rangle := N_X^Y.$$

Observe that $\langle \cdot, \cdot \rangle$ expends linearly to define a bilinear form on $\mathbb{R}V$, and

$$\langle XY, Z\rangle = \langle X, Z\bar{Y}\rangle = \langle Y, \bar{X}Z\rangle$$

holds by Frobenius reciprocity. The graph Γ_k encodes the decomposition of $X\alpha_1$ for X in V_{11} as a direct sum of vertices from V_{12} and the decomposition of $Y\bar{\alpha}_1$ as a direct sum of vertices from V_{11}. Let G be the adjacency matrix for (V_{11}, V_{12}), that is,

$$G = (G_{X,Y})_{X\in V_{11}, Y\in V_{12}},$$

where $G_{X,Y}$ is the number of the edges connecting X and Y, namely

$$G_{X,Y} = \langle X\alpha_1, Y\rangle = \langle Y\bar{\alpha}_1, X\rangle.$$

G has dimensions $\left(\frac{n+1}{2}+4\right) \times \left(\frac{n+1}{2}+2\right)$ and can be written as

$$G = \begin{array}{c} \\ \beta_3 \\ \beta_1 \\ \gamma_3 \\ \gamma_1 \\ \alpha_{n-1} \\ \vdots \\ \alpha_2 \\ \alpha_0 \end{array} \begin{array}{c} \begin{array}{ccccccc} \beta_2 & \gamma_2 & \alpha_n & \alpha_{n-2} & \cdots & \cdots & \alpha_1 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 1 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 & 1 \end{pmatrix} \end{array}. \tag{1}$$

Letting

$$\Delta := \begin{pmatrix} 0 & G \\ G^t & 0 \end{pmatrix},$$

we have

$$\Delta^2 = \begin{pmatrix} GG^t & 0 \\ 0 & G^tG \end{pmatrix}.$$

Put $\mathbb{D} := GG^t$, which acts on $\bar{\mathscr{A}}_1 := \mathbb{R}V_{11}$. We utilize certain eigenvectors of $\mathbb{D}$ to determine the fusion structure of $\mathscr{A}_1$.

Observe from the graph that

$$\begin{aligned}
\Delta\beta_1 &= \alpha_n + \beta_2, & \Delta\gamma_1 &= \alpha_n + \gamma_2,\\
\Delta\beta_2 &= \beta_1 + \beta_3, & \Delta\gamma_2 &= \gamma_1 + \gamma_2,\\
\Delta\beta_3 &= \beta_2, & \Delta\gamma_3 &= \gamma_2.
\end{aligned}$$

Put

$$\begin{aligned}
\xi &= (\beta_1 - \gamma_1) + (\beta_3 - \gamma_3),\\
\eta &= (\beta_1 - \gamma_1) - (\beta_3 - \gamma_3).
\end{aligned}$$

Then

$$\begin{aligned}
\mathbb{D}\xi &= \Delta^2\xi = \Delta(2\beta_2 - 2\gamma_2) = 2\xi,\\
\mathbb{D}\eta &= \Delta^2\eta = 0.
\end{aligned}$$

Let $E(\mathbb{D}, c)$, $c \in \mathbb{R}$, be the eigenspace of the eigenvalue c for $\mathbb{D}$ in $\mathbb{R}(V_{11})$.

Lemma 2.1. $\dim E(\mathbb{D}, 2) = E(\mathbb{D}, 0) = 2$.

Proof. The matrix $\mathbb{D}$ is

$$\mathbb{D} = \begin{array}{cc}
 & \begin{array}{cccccccccc} \beta_3 & \beta_1 & \gamma_3 & \gamma_1 & \alpha_{n-1} & \cdots & \cdots & \cdots & \alpha_2 & \alpha_0 \end{array}\\
\begin{array}{c} \beta_3\\ \beta_1\\ \gamma_3\\ \gamma_1\\ \alpha_{n-1}\\ \alpha_{n-3}\\ \vdots\\ \vdots\\ \alpha_2\\ \alpha_0 \end{array} &
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0\\
1 & 2 & 0 & 1 & 1 & 0 & & & & \vdots\\
0 & 0 & 1 & 1 & 0 & 0 & & & & \vdots\\
0 & 1 & 1 & 2 & 1 & 0 & & & & \vdots\\
0 & 1 & 0 & 1 & 2 & 1 & 0 & & & \vdots\\
0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & & \vdots\\
\vdots & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\
\vdots & & & & & 0 & 1 & 2 & 1 & 0\\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 2 & 1\\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1
\end{pmatrix}
\end{array}.$$

Recall that $n = 4k+3$. Let $\rho_k(x) := \det(tI - \mathbb{D})$ be the characteristic polynomial of $\mathbb{D} = GG^t$. It was proved in [Asaeda 2007] that the characteristic polynomial of G^tG is equal to $(t-2)^2 q_k(t)$, where the polynomials $q_k(t)$, $k \geq 0$, can be defined recursively by

$$\begin{aligned}
q_0(t) &= t^2 - 5t + 3,\\
q_1(t) &= (t-1)(t^3 - 8t^2 + 17t - 5),\\
q_k(t) &= (t^2 - 4t + 2)q_{k-1}(t) - q_{k-2}(t), \quad k \geq 2.
\end{aligned}$$

Since the matrix G has $2k+6$ rows and $2k+4$ columns, GG^t is a unitary conjugate of $G^tG \oplus 0_2$, where 0_2 is the zero 2×2 matrix. Hence

$$\begin{aligned}\rho_k(t) &= t^2 \det(tI - G^tG)\\ &= t^2(t-2)^2 q_k(t).\end{aligned}$$

The recursion formula for $q_k(t)$ gives $q_k(0) = 2k+3$ and $q_k(2) = (-1)^{(k+1)}(2k+3)$ In particular neither 0 nor 2 is a root of q_k. Hence 0 and 2 are roots of multiplicity 2 in ρ_k. Since $\mathbb{D} = GG^t$ is a symmetric matrix, the dimensions of the eigenspaces for $\mathbb{D}$ for the eigenvalues 0 and 2 are both equal to 2.

Bases of $E(\mathbb{D}, 2)$, $E(\mathbb{D}, 0)$ may be taken as

$$\begin{aligned}E(\mathbb{D}, 2) &:= \operatorname{span}\{x_1, x_2\},\\ E(\mathbb{D}, 0) &:= \operatorname{span}\{y_1, y_2\},\end{aligned}$$

where

$$\begin{aligned}x_1 &:= 2(\alpha_0 + \alpha_2) - 2(\alpha_4 + \alpha_6) + \cdots + (-1)^k 2(\alpha_{4k} + \alpha_{4k+2})\\ &\qquad\qquad + (-1)^{k+1}(\beta_1 + \gamma_1 + \beta_3 + \gamma_3),\\ x_2 &:= \xi = (\beta_1 - \gamma_1) + (\beta_3 - \gamma_3),\\ y_1 &:= 2\alpha_0 - 2\alpha_2 + \cdots + 2\alpha_{4k} - 2\alpha_{4k+2} + (\beta_1 + \gamma_1) - (\beta_3 + \gamma_3),\\ y_2 &:= \eta = (\beta_1 - \gamma_1) - (\beta_3 - \gamma_3).\end{aligned}$$

Assume that we have a fusion algebra compatible with the pair of the graphs (Γ_k, Γ'_k), and let π and π' be the conjugate maps $\gamma \mapsto \bar{\gamma}$ on V_{11} and V_{22}. By the argument used in [Haagerup 1994, pp 28–31], the map π' fixes every element of V_{22}. For π, there are only two possibilities:

Case 1 [Haagerup 1994, Case (b), p 31].

$$\bar{\beta}_1 = \beta_1, \quad \bar{\gamma}_1 = \gamma_1, \quad \bar{\beta}_3 = \gamma_3\ (\Leftrightarrow \bar{\gamma}_3 = \beta_3).$$

Case 2 [Haagerup 1994, Case (a), p 31]. (This case will be eliminated.)

$$\bar{\beta}_1 = \gamma_1\ (\Leftrightarrow \bar{\gamma}_1 = \beta_1), \quad \bar{\beta}_3 = \beta_3, \quad \bar{\gamma}_3 = \gamma_3.$$

In both cases, $\bar{\alpha}_{2j} = \alpha_{2j}$ for $j = 0, 1, \ldots, 2k+1$. Note that π extends linearly to $\mathscr{A}_1$ and $\bar{\mathscr{A}}_1 = \mathbb{R}V_{11}$. Let $E(\mathbb{D}, c)_{\mathrm{sc}} := E(\mathbb{D}, c)^\pi$. Observe that

$$c_1\bar{x}_1 + c_2\bar{x}_2 = c_1x_1 + c_2x_2, \quad c_1, c_2 \in \mathbb{R},$$

holds if and only if $c_2 = 0$ in both Cases 1 and 2, and similarly

$$c_1c_1\bar{y}_1 + c_2\bar{y}_2 = c_1y_1 + c_2y_2, \quad c_1, c_2 \in \mathbb{R},$$

if and only if $c_2 = 0$ in both cases. Therefore

$$E(\mathbb{D}, 2)_{\mathrm{sc}} = \mathbb{R}x_1,$$
$$E(\mathbb{D}, 0)_{\mathrm{sc}} = \mathbb{R}y_1.$$

By the definition of principal graphs, the matrix $\mathbb{D} : \mathbb{R}V_{11} \to \mathbb{R}V_{11}$ corresponds to the fusion rule of the right tensor product by $\alpha\overline{\alpha}$, where $\alpha = \alpha_1$. Therefore

$$\mathbb{D}(\overline{\xi}\xi) = \overline{\xi}\mathbb{D}(\xi) = 2\overline{\xi}\xi,$$
$$\mathbb{D}(\overline{\eta}\eta) = \overline{\eta}\mathbb{D}(\eta) = 0.$$

Hence

$$\overline{\xi}\xi \in E(\mathbb{D}, 2)_{\mathrm{sc}} = \mathbb{R}x_1,$$
$$\overline{\eta}\eta \in E(\mathbb{D}, 0)_{\mathrm{sc}} = \mathbb{R}y_1.$$

Thus

$$\langle \overline{\xi}\xi, \alpha_0 \rangle = \langle \xi, \xi\alpha_0 \rangle = \langle \xi, \xi \rangle = 4.$$

Hence the coefficient of $\overline{\xi}\xi$ at α_0 is 4. Since $\overline{\xi}\xi \in \mathbb{R}x_1$, we have $\overline{\xi}\xi = 2x_1$. Likewise we obtain $\overline{\eta}\eta = 2y_1$. Noting that

$$\overline{\xi} = \begin{cases} \eta & \text{in Case 1,} \\ -\eta & \text{in Case 2,} \end{cases}$$

we have

$$\begin{cases} \xi\eta = 2y_1, & \eta\xi = 2x_1 \quad \text{in Case 1,} \\ \xi\eta = -2y_1, & \eta\xi = -2x_1 \quad \text{in Case 2,} \end{cases}$$

which completes the proof. □

Lemma 2.2. $\xi^2 = 0$ *and* $\eta^2 = 0$.

Proof. The equality $\mathbb{D}(\xi^2) = \xi\mathbb{D}(\xi) = 2\xi^2$ implies $\xi^2 = c_1x_1 + c_2x_2$ for some $c_1, c_2 \in \mathbb{R}$. Moreover, since $\langle \xi, \eta \rangle = 0$, we have

$$\langle \xi^2, \alpha_0 \rangle = \langle \xi, \overline{\xi}\alpha_0 \rangle = \pm\langle \xi, \eta \rangle = 0.$$

Together with $\langle c_1x_1 + c_2x_2, \alpha_0 \rangle = 2c_1$, $c_1, c_2 \in \mathbb{R}$, we obtain

$$\xi^2 = c_2x_2 = c_2\xi.$$

We show that $c_2 = 0$, using that $\overline{\xi}\xi = 2x_1$ and $\xi\overline{\xi} = 2y_1$ in Cases 1 and 2:

$$\begin{aligned} 4c_2 &= \langle c_2\xi, c_2\xi \rangle = \langle \xi^2, \xi^2 \rangle = \langle \overline{\xi}\xi, \xi\overline{\xi} \rangle = 4\langle x_1, y_1 \rangle \\ &= (2-2) - (2-2) + \cdots + (-1)^k(2-2) + (1+1-1-1) = 0. \end{aligned}$$

Thus $\xi^2 = 0$. Then $\overline{\xi}^2 = \eta^2 = 0$ for both cases. □

Since $\beta_3-\gamma_3=\frac{1}{2}(\xi-\eta)$, we get

$$\begin{aligned}(\beta_3-\gamma_3)^2&=\tfrac{1}{4}(\xi-\eta)^2\\&=\tfrac{1}{4}(\xi^2+\eta^2-\xi\eta-\eta\xi)\\&=-\tfrac{1}{4}(\xi\eta+\eta\xi)\\&=\begin{cases}-\frac{1}{2}(x_1+y_1) & \text{in Case 1,}\\ \frac{1}{2}(x_1+y_1) & \text{in Case 2.}\end{cases}\end{aligned}$$

Remark 2.3. For k even, that is, $n=3 \pmod 8$ and $k=2l$,

$$\tfrac{1}{2}(x_1+y_1)=2(\alpha_0-\alpha_6+\alpha_8-\alpha_{14}+\alpha_{16}-\cdots+\alpha_{8l})-(\beta_3+\gamma_3)$$

and for k odd, that is, $n=7 \pmod 8$ and $k=2l+1$,

$$\tfrac{1}{2}(x_1+y_1)=2(\alpha_0-\alpha_6+\alpha_8-\alpha_{14}+\alpha_{16}-\cdots+\alpha_{8l}-\alpha_{8l+6})+(\beta_1+\gamma_1).$$

Consider next the sequence of polynomials R_n given recursively by

$$R_0(t)=1,\ R_1(t)=t,\ R_m(t)=tR_{m-1}(t)-R_{m-2}(t),\quad n\geq 2,$$

as in [Haagerup 1994, pp 33–34]. Note that $R_m(t)=U_m(\frac{t}{2})$, where U_m is the m-th Chebyshev polynomial of second kind [Erdélyi et al. 1981, Section 10.11]. Moreover,

$$R_m(2\cos\theta)=\frac{\sin(m+1)\theta}{\sin\theta},\quad 0<\theta<\pi.$$

By the recursion formula for R_n,

$$\begin{aligned}R_j(\Delta)\alpha_0&=\alpha_j,\quad 0\leq j\leq n,\\R_{n+1}(\Delta)\alpha_0&=\beta_1+\gamma_1,\\R_{n+2}(\Delta)\alpha_0&=\alpha_n+\beta_2+\gamma_2,\\R_{n+3}(\Delta)\alpha_0&=\alpha_{n-1}+\beta_1+\gamma_1+\beta_3+\gamma_3.\end{aligned}$$

Hence

$$\begin{aligned}\beta_3+\gamma_3&=(R_{n+3}(\Delta)-R_{n+1}(\Delta)-R_{n-1}(\Delta))\alpha_0\\&=(R_{4k+6}(\Delta)-R_{4k+4}(\Delta)-R_{4k+2}(\Delta))\alpha_0.\end{aligned}$$

For m even, $R_m(t)$ is an even polynomial in t, thus there is are unique polynomials $(Q_j)_{j=0,1,2,\dots}$ with $\deg(Q_l)=l$, such that

$$Q_j(t^2)=R_{2j}(t),\quad t\in\mathbb{R},\ j=0,1,2,\dots.$$

With this notation, we have

$$\begin{aligned}\beta_3+\gamma_3&=(Q_{2k+3}(\mathbb{D})-Q_{2k+2}(\mathbb{D})-Q_{2k+1}(\mathbb{D}))\alpha_0\\&=(Q_{2k+3}-Q_{2k+2}-Q_{2k+1})(\alpha\overline{\alpha}).\end{aligned}$$

Therefore

$$\begin{aligned}(\beta_3-\gamma_3)(\beta_3+\gamma_3) &= (Q_{2k+3}-Q_{2k+2}-Q_{2k+1})(\mathbb{D})(\beta_3-\gamma_3)\\ &= \tfrac{1}{2}(Q_{2k+3}-Q_{2k+2}-Q_{2k+1})(\mathbb{D})(\xi-\eta).\end{aligned}$$

Since $\mathbb{D}\xi = 2\xi$ and

$$\begin{aligned}Q_m(2) = R_{2j}(\sqrt{2}) &= \frac{\sin(2j+1)\pi/4}{\sin\pi/4}\\ &= \begin{cases} 1 & j = 0, 1 \pmod 4,\\ -1 & j = 2, 3 \pmod 4,\end{cases}\end{aligned}$$

we have

$$Q_j(\mathbb{D})\xi = \begin{cases} \xi & j = 0, 1 \pmod 4,\\ -\xi & j = 2, 3 \pmod 4.\end{cases}$$

Similarly, since $\mathbb{D}\eta = 0$ and

$$Q_j(0) = R_{2j}(0) = \frac{\sin(2j+1)\pi/2}{\sin\pi/2} = (-1)^j,$$

we have

$$Q_j(\mathbb{D})\eta = (-1)^j\eta, \quad j = 0, 1, 2 \ldots.$$

Therefore,

$$\begin{aligned}&(Q_{2k+3}(\mathbb{D})-Q_{2k+2}(\mathbb{D})-Q_{2k+1}(\mathbb{D}))\xi\\ &\quad= \begin{cases}(Q_{4l+3}(\mathbb{D})-Q_{4l+2}(\mathbb{D})-Q_{4l+1}(\mathbb{D}))\xi = -\xi & \text{for } k = 2l,\ l \in \mathbb{N}_0,\\ (Q_{4l+5}(\mathbb{D})-Q_{4l+4}(\mathbb{D})-Q_{4l+3}(\mathbb{D}))\xi = \xi & \text{for } k = 2l+1,\ l \in \mathbb{N}_0,\end{cases}\end{aligned}$$

and in both cases

$$(Q_{2k+3}(\mathbb{D})-Q_{2k+2}(\mathbb{D})-Q_{2k+1}(\mathbb{D}))\eta = -\eta.$$

Hence

$$\begin{aligned}(\beta_3-\gamma_3)(\beta_3+\gamma_3) &= \tfrac{1}{2}(Q_{2k+3}-Q_{2k+2}-Q_{2k+1})(\mathbb{D})(\xi-\eta)\\ &= \begin{cases}\tfrac{1}{2}(-\xi+\eta) = \gamma_3-\beta_3 & k \text{ even},\\ \tfrac{1}{2}(\xi+\eta) = \beta_1-\gamma_1 & k \text{ odd}.\end{cases}\end{aligned}$$

Using the contragredient map we get in Case 1 that

$$\begin{aligned}(\beta_3+\gamma_3)(\beta_3-\gamma_3) &= \overline{(\bar\beta_3-\bar\gamma_3)(\bar\beta_3+\bar\gamma_3)}\\ &= \overline{(\gamma_3-\beta_3)(\gamma_3+\beta_3)}\\ &= -\overline{(\beta_3-\gamma_3)(\beta_3+\gamma_3)}\\ &= \begin{cases}-(\bar\gamma_3-\bar\beta_3) = -(\beta_3-\gamma_3) & k \text{ even},\\ -(\bar\beta_1-\bar\gamma_1) = -(\beta_1-\gamma_1) & k \text{ odd},\end{cases}\end{aligned}$$

and in Case 2 (to be eliminated) that

$$\begin{aligned}(\beta_3+\gamma_3)(\beta_3-\gamma_3) &= \overline{(\bar\beta_3-\bar\gamma_3)(\bar\beta_3+\bar\gamma_3)}\\ &= \overline{(\beta_3-\gamma_3)(\beta_3+\gamma_3)}\\ &= \begin{cases}\bar\gamma_3-\bar\beta_3=\gamma_3-\beta_3 & k \text{ even},\\ \bar\beta_1-\bar\gamma_1=\gamma_1-\beta_1 & k \text{ odd}.\end{cases}\end{aligned}$$

Thus in both cases,

$$(\beta_3+\gamma_3)(\beta_3-\gamma_3)=\begin{cases}\gamma_3-\beta_3 & k \text{ even},\\ \gamma_1-\beta_1 & k \text{ odd}.\end{cases}$$

So far, we have obtained the three formulae

(A) $$(\beta_3-\gamma_3)^2=\begin{cases}-\frac{1}{2}(x_1-y_1) & \text{in Case 1},\\ \frac{1}{2}(x_1-y_1) & \text{in Case 2},\end{cases}$$

(B) $$(\beta_3-\gamma_3)(\beta_3+\gamma_3)=\begin{cases}\frac{1}{2}(-\xi+\eta)=\gamma_3-\beta_3 & k \text{ even},\\ \frac{1}{2}(\xi+\eta)=\beta_1-\gamma_1 & k \text{ odd},\end{cases}$$

(C) $$(\beta_3+\gamma_3)(\beta_3-\gamma_3)=\begin{cases}\gamma_3-\beta_3 & k \text{ even},\\ \gamma_1-\beta_1 & k \text{ odd}.\end{cases}$$

Next we compute $(\beta_3+\gamma_3)^2$, in order to find β_3^2, γ_3^2, $\beta_3\gamma_3$ and $\gamma_3\beta_3$.

Claim 2.4. *We have*

(D) $$(\beta_3+\gamma_3)^2=2(c_0\alpha_0+c_1\alpha_2+\cdots+c_{2k+1}\alpha_{4k+2})+c_{2k+2}(\beta_1+\gamma_1)+c_{2k}(\beta_3+\gamma_3),$$

where the c_j are defined by

$$\begin{aligned}&c_0=1,\\ &c_1=c_2=0,\\ &c_j=c_{j-1}+c_{j-2}+c_{j-3} \quad \textit{for } j\ge 3.\end{aligned}$$

Proof. Recall that

$$\begin{aligned}(\beta_3+\gamma_3) &= (Q_{2k+3}-Q_{2k+2}-Q_{2k+1})(\mathbb{D})\alpha_0\\ &= (R_{4k+6}(\Delta)-R_{4k+4}(\Delta)-R_{4k+2}(\Delta))\alpha_0;\end{aligned}$$

thus

($\sharp$) $$(\beta_3+\gamma_3)^2=(R_{4k+6}(\Delta)-R_{4k+4}(\Delta)-R_{4k+2}(\Delta))(\beta_3+\gamma_3).$$

Our strategy of the proof is as follows: First we find a sequence of polynomials (S_j) such that $S_j(\Delta)(\beta_3+\gamma_3)$ is given by a simple formula. Next we rewrite the right-hand side of ($\sharp$) using the S_j.

From the graph, we obtain

$$\begin{aligned}
R_0(\Delta)(\beta_3+\gamma_3) &= (\beta_3+\gamma_3),\\
R_1(\Delta)(\beta_3+\gamma_3) &= (\beta_2+\gamma_2),\\
R_2(\Delta)(\beta_3+\gamma_3) &= \Delta(\beta_2+\gamma_2)-(\beta_3+\gamma_3) = \beta_1+\gamma_1,\\
R_3(\Delta)(\beta_3+\gamma_3) &= \Delta(\beta_1+\gamma_1)-(\beta_2+\gamma_2) = 2\alpha_n,\\
R_4(\Delta)(\beta_3+\gamma_3) &= 2\Delta\alpha_n-(\beta_1+\gamma_1) = 2\alpha_{n-1}+\beta_1+\gamma_1.
\end{aligned}$$

Define the polynomials $(S_j(t))_{j\ge 3}$ by the recursive formula

$$\begin{aligned}
S_3(t) &= R_3(t),\\
S_4(t) &= R_4(t)-R_2(t),\\
S_j(t) &= tS_{j-1}(t)-S_{j-2}(t), \quad j\ge 5.
\end{aligned}$$

By definition $S_3(\Delta)(\beta_3+\gamma_3)=2\alpha_n$ and $S_4(\Delta)(\beta_3+\gamma_3)=2\alpha_{n-1}$. Since $\alpha_{l-1}=\Delta\alpha_l-\alpha_{l+1}$ for $l=1,2,\dots,n-1$, we easily obtain

$$S_j(\Delta)(\beta_3+\gamma_3)=2\alpha_{n-j+3}$$

for $j=3,4,\dots,n+3$. Next we express the R_j in terms of the S_j.

Lemma 2.5. *For $j\ge 2$,*

$$\begin{aligned}
R_{2j-1} &= d_0S_{2j-1}+d_1S_{2j-3}+\cdots+d_{j-2}S_3+(d_{j-1}-d_{j-2})R_1,\\
R_{2j} &= d_0S_{2j}+d_1S_{2j-2}+\cdots+d_{j-2}S_4+d_{j-1}R_2+d_{j-3}R_0,
\end{aligned}$$

where the d_j satisfy

$$d_{-1}=0, \quad d_0=d_1=1, \quad d_j=d_{j-1}+d_{j-2}+d_{j-3}.$$

Proof. For $j=2$ this is obvious by the definition of the S_j. We proceed with induction. Assume the statement is true for $j\ge 2$. Using the recursion formulae for the R_j and S_j, we have

$$\begin{aligned}
R_{2j+1}(t) &= tR_{2j}(t)-R_{2j-1}(t)\\
&= t(d_0S_{2j}+d_1S_{2j-2}+\cdots+d_{j-2}S_4+d_{j-1}R_2+d_{j-3})\\
&\qquad -(d_0S_{2j-1}+d_1S_{2j-3}+\cdots+d_{j-2}S_3+(d_{j-1}-d_{j-2})R_1)\\
&= d_0S_{2j+1}+d_1S_{2j-1}+\cdots+d_{j-2}S_5+t(d_{j-1}R_2+d_{j-3})-(d_{j-1}-d_{j-2})R_1\\
&= d_0S_{2j+1}+d_1S_{2j-1}+\cdots+d_{j-2}S_5+d_{j-1}(tR_2-R_1)+td_{j-3}-d_{j-2}R_1\\
&= d_0S_{2j+1}+d_1S_{2j-1}+\cdots+d_{j-2}S_5+d_{j-1}S_3+(d_{j-3}-d_{j-2})R_1.
\end{aligned}$$

The last equality was obtained using $S_3 = R_3$, $R_1 = t$, and $d_{j-2}+d_{j-3} = d_j - d_{j-1}$. Likewise we have

$$\begin{aligned}
R_{2j+2}(t) &= tR_{2j+1}(t) - R_{2j}(t) \\
&= d_0 S_{2j+2} + d_1 S_{2j} + \cdots + d_{j-2} S_6 \\
&\qquad\qquad + t(d_{j-1} S_3 + (d_j - d_{j-1}) R_1) - (d_{j-1} R_2 + d_{j-3} R_0) \\
&= d_0 S_{2j+2} + d_1 S_{2j} + \cdots + d_{j-2} S_6 + d_{j-1} R_4 \\
&\qquad\qquad + (d_j - d_{j-1})(R_2 + R_0) - d_{j-3} R_0 \\
&= d_0 S_{2j+2} + d_1 S_{2j} + \cdots + d_{j-2} S_6 + d_{j-1} S_4 \\
&\qquad\qquad + d_j R_2 + (d_j - d_{j-1} - d_{j-3}) R_0 \\
&= d_0 S_{2j+2} + d_1 S_{2j} + \cdots + d_{j-2} S_6 + d_{j-1} S_4 + d_j R_2 + d_{j-2} R_0,
\end{aligned}$$

which completes the proof of Lemma 2.5. □

We return to (♯). Using Lemma 2.5,

$$\begin{aligned}
&R_{4k+6} - R_{4k+4} - R_{4k+2} \\
&\quad = d_0 S_{4k+6} + (d_1 - d_0) S_{4k+4} + d_{-1} S_{4k+2} + d_0 S_{4k} + d_1 S_{4k-2} \\
&\qquad\qquad + \cdots + d_{2k-2} S_4 + d_{2k-1} R_2 + d_{2k-3} R_0 \\
&\quad = S_{4k+6} + d_0 S_{4k} + d_1 S_{4k-2} + \cdots + d_{2k-2} S_4 + d_{2k-1} R_2 + d_{2k-3} R_0.
\end{aligned}$$

Recall

$$\begin{aligned}
S_j(\Delta)(\beta_3 + \gamma_3) &= 2\alpha_{n-j+3}, \\
R_2(\beta_3 + \gamma_3) &= \beta_1 + \gamma_1.
\end{aligned}$$

Letting $c_0 := 1$, $c_1 = c_2 = 0$ and $c_j := d_{j-3}$ for $j \geq 3$, we obtain Equation (D), which concludes the proof of Claim 2.4. □

Thus far we have obtained the formulae for $(\beta_3 - \gamma_3)^2$, $(\beta_3 - \gamma_3)(\beta_3 + \gamma_3)$, $(\beta_3 + \gamma_3)(\beta_3 - \gamma_3)$ and $(\beta_3 + \gamma_3)^2$ in Equations (A), (B), (C) and (D). This enables us to understand the fusion rules among β_3, γ_3 and their conjugates.

Proposition 2.6. *Case 2 does not occur. Namely, β_1 and γ_1 are self conjugate and $\bar{\beta}_3 = \gamma_3$ if there is a fusion algebra compatible with the graphs Γ_k and Γ'_k.*

Proof. First observe that, by the definition of c_j, $j \geq 0$, in Claim 2.4, it follows that c_j (mod 4) is periodic in j with period 8. The values are:

j (mod 8)	0	1	2	3	4	5	6	7
c_j (mod 4)	1	0	0	1	1	2	0	0

In particular,

$$(*)\qquad \begin{cases} c_{2j} = 1 \pmod 4 & \text{for j even,} \\ c_{2j} = 0 \pmod 4 & \text{for j odd.} \end{cases}$$

In the following we assume Case 2 and derive a contradiction.

First consider the case when k is even. By (B) and (C), we have

$$(\beta_3 - \gamma_3)(\beta_3 + \gamma_3) = (\beta_3 + \gamma_3)(\beta_3 - \gamma_3),$$

hence

$$\begin{aligned} \beta_3\gamma_3 = \gamma_3\beta_3 &= \tfrac{1}{2}(\beta_3\gamma_3 + \gamma_3\beta_3) \\ &= \tfrac{1}{4}((\beta_3+\gamma_3)^2 - (\beta_3-\gamma_3)^2). \end{aligned}$$

From (A) for Case 2, (D) and Remark 2.3, the coefficient of β_3 in the expansion of $\beta_3\gamma_3$ in irreducible objects is equal to

$$\frac{c_{2k}+1}{4}.$$

Since k is even, $c_{2k} = 1 \bmod 4$ by ($\star$), so $(c_{2k}+1)/4$ is not an integer. This implies that Case 2 does not occur if k is even.

Next consider the case when k is odd. From (B) and (C), we get

$$(\beta_3 - \gamma_3)(\beta_3 + \gamma_3) = -(\beta_3 + \gamma_3)(\beta_3 - \gamma_3).$$

Hence

$$\begin{aligned} \beta_3^2 = \gamma_3^2 &= \tfrac{1}{2}(\beta_3^2 + \gamma_3^2) \\ &= \tfrac{1}{4}((\beta_3+\gamma_3)^2 + (\beta_3-\gamma_3)^2). \end{aligned}$$

From (A) for Case 2, (D) and Remark 2.3, it follows that the coefficient of β_1 in the expansion of β_3^2 in irreducible objects is equal to

$$\frac{c_{2k+2}+1}{4}.$$

Since k is odd, $c_{2k+2} = 1 \bmod 4$ by ($\star$), so $(c_{2k}+1)/4$ is not an integer. This excludes Case 2 for k odd as well. □

In the following we determine all the irreducible decompositions for the products of any two objects in V and show that the coefficients are nonnegative integers. Since we excluded Case 2, we rewrite (A) as

$$(\mathrm{A}')\quad (\beta_3-\gamma_3)^2 = \begin{cases} -2(\alpha_0 - \alpha_6 + \alpha_8 - \alpha_{14} + \alpha_{16} - \cdots + \alpha_{8l}) - (\beta_3+\gamma_3) \\ \qquad\qquad\qquad\qquad\qquad k = 2l,\ l = 0, 1, 2, \ldots, \\ -2(\alpha_0 - \alpha_6 + \alpha_8 - \alpha_{14} + \alpha_{16} - \cdots + \alpha_{8l} - \alpha_{8l+6}) + (\beta_1+\gamma_1) \\ \qquad\qquad\qquad\qquad\qquad k = 2l+1,\ l = 0, 1, 2, \ldots. \end{cases}$$

Put

$$A := (\beta_3 - \gamma_3)^2, \qquad B := (\beta_3 - \gamma_3)(\beta_3 + \gamma_3),$$
$$C := (\beta_3 + \gamma_3)(\beta_3 - \gamma_3), \quad D := (\beta_3 + \gamma_3)^2.$$

Then

$$\beta_3\gamma_3 = \frac{(D-A)+(B-C)}{4}, \quad \beta_3^2 = \frac{(D+A)+(B+C)}{4},$$
$$\gamma_3\beta_3 = \frac{(D-A)-(B-C)}{4}, \quad \gamma_3^2 = \frac{(D+A)-(B+C)}{4}.$$

We introduce new constants $(f_j)_{j\geq 0}$, $(g_j)_{j\geq 0}$ by

$$\begin{cases} f_j = \frac{1}{2}(c_j+1),\ g_j = \frac{1}{2}(c_j-1) & \text{for } j = 0 \text{ (mod 4)}, \\ f_j = \frac{1}{2}(c_j-1),\ g_j = \frac{1}{2}(c_j+1) & \text{for } j = 3 \text{ (mod 4)}, \\ f_j = g_j = \frac{1}{2}c_j & \text{for } j = 1, 2 \text{ (mod 4)}. \end{cases}$$

Note that $f_j + g_j = c_j$ for all j. Further, from the table on page 268, observe that f_j, g_j is an nonnegative integer for all $j \geq 0$. Here are some values of f_j and g_j:

j	0	1	2	3	4	5	6	7	8	9	10	11	12
f_j	1	0	0	0	1	1	2	3	7	12	22	40	75
g_j	0	0	0	1	0	1	2	4	6	12	22	41	74

For k even, using (A′), (B), (C), (D), we have

$$\frac{D-A}{4} = f_0\alpha_0 + f_1\alpha_2 + \cdots + f_{2k+1}\alpha_{4k+2} + \tfrac{1}{4}c_{2k+2}(\beta_1+\gamma_1) + \tfrac{1}{4}(c_{2k}-1)(\beta_3+\gamma_3),$$
$$\frac{D+A}{4} = g_0\alpha_0 + g_1\alpha_2 + \cdots + g_{2k+1}\alpha_{4k+2} + \tfrac{1}{4}c_{2k+2}(\beta_1+\gamma_1) + \tfrac{1}{4}(c_{2k}+1)(\beta_3+\gamma_3),$$
$$\frac{B-C}{4} = 0,$$
$$\frac{B+C}{4} = \tfrac{1}{2}(\gamma_3 - \beta_3).$$

Since k is even, $c_{2k+2} = 2f_{2k+2} = 2g_{2k+2}$, $c_{2k}+1 = 2f_{2k}$ and $c_{2k}-1 = 2g_{2k}$. Hence we obtain the following theorem:

Theorem 2.7. *For k even,*

$$\begin{aligned} \beta_3\gamma_3 = \gamma_3\beta_3 = f_0\alpha_0 + f_1\alpha_2 + \cdots + f_{2k+1}\alpha_{4k+2} \\ + \tfrac{1}{2}f_{2k+2}(\beta_1+\gamma_1) + \tfrac{1}{2}(f_{2k}-1)(\beta_3+\gamma_3), \end{aligned}$$
$$\beta_3^2 = g_0\alpha_0 + g_1\alpha_2 + \cdots + g_{2k+1}\alpha_{4k+2} + \tfrac{1}{2}g_{2k+2}(\beta_1+\gamma_1) + \tfrac{1}{2}g_{2k}\beta_3 + \tfrac{1}{2}(g_{2k}+2)\gamma_3,$$
$$\gamma_3^2 = g_0\alpha_0 + g_1\alpha_2 + \cdots + g_{2k+1}\alpha_{2k+2} + \tfrac{1}{2}g_{2k+2}(\beta_1+\gamma_1) + \tfrac{1}{2}(g_{2k}+2)\beta_3 + \tfrac{1}{2}g_{2k}\gamma_3.$$

All the coefficients of irreducible elements are nonnegative integers.

Proof. The only remaining thing to prove is that f_{2k+2} is even, f_{2k} is odd and g_{2j} is even for any j. Since k is even, $c_{2k+2} = 0 \pmod 4$. Thus $f_{2k+2} = \frac{1}{2}c_{2k+2}$ is even. Likewise $c_{2k} = 1 \pmod 4$, thus $f_{2k} = \frac{1}{2}(c_{2k}+1)$ is odd. Now,

$$g_{2j} = \begin{cases} \frac{1}{2}(c_{2j}-1) & \text{for } j \text{ even,} \\ \frac{1}{2}c_{2j} & \text{for } j \text{ odd.} \end{cases}$$

Since $c_{2j}-1 = 0 \pmod 4$ for j even and $c_{2j} = 0 \pmod 4$ for j odd, we have that g_{2j} is even for any j. $\square$

In the same way, we get for k odd,

$$\begin{aligned}
\frac{D-A}{4} &= f_0\alpha_0 + f_1\alpha_2 + \cdots + f_{2k+1}\alpha_{4k+2} + \tfrac{1}{4}(c_{2k+2}+1)(\beta_1+\gamma_1) + \tfrac{1}{4}c_{2k}(\beta_3+\gamma_3), \\
\frac{D+A}{4} &= g_0\alpha_0 + g_1\alpha_2 + \cdots + g_{2k+1}\alpha_{2k+2} + \tfrac{1}{4}(c_{2k+2}-1)(\beta_1+\gamma_1) + \tfrac{1}{4}c_{2k}(\beta_3+\gamma_3), \\
\frac{B-C}{4} &= \tfrac{1}{2}(\beta_1-\gamma_1), \\
\frac{B+C}{4} &= 0.
\end{aligned}$$

Since k is odd, $c_{2k+2}+1 = 2f_{2k+2}$, $c_{2k+2}-1 = 2g_{2k+2}$ and $c_{2k} = 2f_{2k} = 2g_{2k}$. Hence we get:

Theorem 2.8. *For k odd,*

$$\begin{aligned}
\beta_3\gamma_3 &= f_0\alpha_0 + f_1\alpha_2 + \cdots + f_{2k+1}\alpha_{4k+2} \\
&\qquad + \tfrac{1}{2}(f_{2k+2}+1)\beta_1 + \tfrac{1}{2}(f_{2k+2}-1)\gamma_1 + \tfrac{1}{2}f_{2k}(\beta_3+\gamma_3), \\
\gamma_3\beta_3 &= f_0\alpha_0 + f_1\alpha_2 + \cdots + f_{2k+1}\alpha_{4k+2} \\
&\qquad + \tfrac{1}{2}(f_{2k+2}-1)\beta_1 + \tfrac{1}{2}(f_{2k+2}+1)\gamma_1 + \tfrac{1}{2}f_{2k}(\beta_3+\gamma_3), \\
\beta_3^2 = \gamma_3^2 &= g_0\alpha_0 + g_1\alpha_2 + \cdots + g_{2k+1}\alpha_{4k+2} + \tfrac{1}{2}g_{2k+2}(\beta_1+\gamma_1) + \tfrac{1}{2}g_{2k}(\beta_3+\gamma_3).
\end{aligned}$$

All the coefficients of irreducible elements are nonnegative integers.

Proof. It remains to show that f_{2k+2} is odd and f_{2k} is even. In the proof of Theorem 2.7, it has been already proved that g_{2j} is even for any j.

Since k is odd, $c_{2k+2} = 1 \pmod 4$. Thus $f_{2k+2} - 1 = \frac{1}{2}(c_{2k+2}-1)$ is even, that is, f_{2k+2} is odd. Likewise $c_{2k} = 0 \pmod 4$, thus $f_{2k} = \frac{1}{2}c_{2k}$ is even. $\square$

Thus far we determined that β_1 and γ_1 are self-conjugate and computed the full irreducible decompositions of β_3 and γ_3, in particular, $\overline{\beta_3} = \gamma_3$. This determines the rest of the fusion rule. Note that the conjugate map π on $\mathbb{Z}V_{11}$ is now determined.

First, for α_{2j}, $j = 0, 1, \ldots, 2k+1$, the right and left multiplication of α_{2j} on any other object from V_{11} is represented by the matrices $Q_j(\mathbb{D})$ and $Q_j(\pi\mathbb{D}\pi)$ respectively.

Claim 2.9. *The entries of the matrices $R_i(\Delta)$ for $i = 0, 1, \ldots, 4k+3$ are nonnegative integers. In particular, the entries of the matrices $Q_j(\mathbb{D})$ for $j = 0, 1, \ldots, 2k+1$ are nonnegative integers.*

Proof. This immediate from the result in [de la Harpe and Wenzl 1987], which states that when Δ is an adjacency matrix of a graph with norm greater than 2, the matrix $R_i(\Delta)$ has nonnegative integer entries for any i. □

It remains to determine the decomposition of tensor product of β_1 and γ_1 with themselves and β_3 and γ_3.

Since by the graph $\beta_1 = \beta_3\alpha_2$ and $\gamma_1 = \gamma_3\alpha_2$, the fusion among β_3 and γ_3 together with the fusion of α_2 with all the objects determine $\beta_3\beta_1$, $\gamma_3\gamma_1$, $\beta_3\gamma_1$, $\gamma_3\beta_1$ by imposing associativity. Taking the conjugate, we obtain $\beta_1\beta_3$, $\gamma_1\gamma_3$, $\beta_1\gamma_3$, $\gamma_1\beta_3$ as well. Thus $\beta_1^2 = \beta_1\gamma_3\alpha_2$, $\gamma_1^2 = \gamma_1\gamma_3\alpha_2$, $\beta_1\gamma_1 = \beta_1\gamma_3\alpha_2$, $\gamma_1\beta_1 = \gamma_1\beta_3\alpha_2$ are all determined. Since there is no division, subtraction of objects are involved in the process of determining each desired fusion rule, the coefficients are all nonnegative integers.

2B. ***Fusion rules on $_N\mathscr{X}_N \times {}_N\mathscr{X}_M$.*** We identify $_N\mathscr{X}_N$ with V_{11} and $_N\mathscr{X}_M$ with V_{12}. Claim 2.9 implies that $\alpha_i Y$ for i even and any $Y \in V_{12}$ are determined, and so are $X\alpha_i$ for $X \in V_{11}$ and i odd. Thus it remains to obtain $\beta_i Y$ and $\gamma_i Y$, where $i = 1, 3$, $Y = \beta_2$ or γ_2. They are easily determined, since $\beta_2 = \beta_3\alpha_1$, $\gamma_2 = \gamma_3\alpha_1$, and the fusion among $\beta_i, \gamma_j, i, j = 1, 3$ are already determined. (Here we used associativity again.) Since the fusion coefficients among the β_i and the γ_j are nonnegative integers and the product of α_1 from the right gives fusion with nonnegative integers, the fusion coefficients of $\beta_i Y$ and $\gamma_i Y$ are nonnegative integers as well.

2C. ***Fusion rules on $_N\mathscr{X}_M \times {}_M\mathscr{X}_N$.*** Let $X \in {}_N\mathscr{X}_M$. Then for j odd,

$$X\bar{\alpha}_j = R_j(\Delta)X.$$

Claim 2.9 implies that $R_j(\Delta)X$ is a linear combination of the objects in $_N\mathscr{X}_N$ with nonnegative integer coefficients. It remains to show that $\beta_2\bar{\beta}_2$, $\beta_2\bar{\gamma}_2$, $\gamma_2\bar{\beta}_2$ and $\gamma_2\bar{\gamma}_2$ also have this property. It is immediate, since $\bar{\beta}_2 = \bar{\alpha}_1\bar{\beta}_3$, $\bar{\gamma}_2 = \bar{\alpha}_1\bar{\gamma}_3$, $\beta_2\bar{\alpha} = \beta_1 + \beta_3$, $\gamma_2\bar{\alpha} = \gamma_1 + \gamma_3$, and all the fusion rules involved have decompositions into simple objects with $\mathbb{Z}_{\geq 0}$-coefficients.

2D. ***Fusion rules on $_M\mathscr{X}_M \times {}_M\mathscr{X}_M$ and $_M\mathscr{X}_M \times {}_M\mathscr{X}_N$.*** Recall that we have identification $_M\mathscr{X}_M = V_{22}$ and $_M\mathscr{X}_N = V_{21}$. Let Δ' be the adjacency matrix for Γ'. Then the fusion rules of the tensor products of the α'_j for $j = 0, 2, \ldots, n-1$, as well as the $\bar{\alpha}_k$ for $k = 1, 3, \ldots, n-1$ with any objects in $V_{21} \sqcup V_{22}$ are given by the matrices $R_l(\Delta')$, where $l = 0, 1, \ldots, n$. Similarly to Claim 2.9, the entries of $R_l(\Delta')$ are all nonnegative integers. Furthermore, using Frobenius reciprocity, this also takes care of the coefficients of the α'_j and $\overline{\alpha}_k$ in the tensor product of two bimodules.

2E. ***Fusion rules on*** $_M\mathscr{X}_M \times {}_M\mathscr{X}_M$. The remaining issue is to determine the fusion rule among f and g. Observing the Perron–Frobenius weights shows that $\overline{f} = f$, $\overline{g} = g$. Since for j even, each α'_j is self-conjugate as well, $fg = gf$.

Theorem 2.10. *We have*

$$\langle f^2, f\rangle = d_{2k-1}, \quad \langle fg, f\rangle = d_{2k},$$
$$\langle fg, g\rangle = d_{2k+1}, \quad \langle g^2, g\rangle = d_{2k+2},$$

where the d_k are defined as in the proof of Claim 2.4 by

$$d_j = d_{j-1} + d_{j-2} + d_{j-3}, \quad d_{-1} = 0, \quad d_0 = d_1 = 1.$$

Lemma 2.11. *We have*

$$\langle f^2, f\rangle - \langle fg, g\rangle = d_{2k-1} - d_{2k+1},$$
$$\langle fg, f\rangle - \langle g^2, g\rangle = d_{2k} - d_{2k+2},$$
$$\langle fg, g\rangle - \langle g^2, g\rangle = d_{2k+1} - d_{2k+2}.$$

Proof of Lemma 2.11. We use a similar strategy to the proof of Claim 2.4. Let G' be the adjacency matrix for (V_{22}, V_{21}) corresponding to the graph Γ'_k (see Figure 1), and let

$$\Delta' := \begin{pmatrix} 0 & G' \\ G'^t & 0 \end{pmatrix}.$$

Observe that

$$R_0(\Delta')(g-f) = (g-f),$$
$$R_1(\Delta')(g-f) = \overline{\gamma}_2 + \overline{\beta}_2,$$
$$R_2(\Delta')(g-f) = g+f,$$
$$R_3(\Delta')(g-f) = 2\alpha'_n,$$
$$R_4(\Delta')(g-f) = 2\alpha'_{n-1} + f + g,$$

where $\alpha'_j = \overline{\alpha}_j$ for j odd. Then we have

$$S_j(\Delta')(g-f) = 2\alpha'_{n-j+3}$$

for $j = 3, 4, \ldots, n+3$, where the polynomial S_j is defined in the proof of Claim 2.4. On the other hand,

$$g + f = R_{n+1}(\mathbb{D}')\alpha'_0 = R_{4k+4}(\mathbb{D}')\alpha'_0 = Q_{2k+2}(\overline{\alpha}_1\alpha_1).$$

Using Lemma 2.5,

$$
\begin{aligned}
&(g+f)(g-f)\\
&\quad = (d_0 S_{2(2k+2)} + d_1 S_{2(2k+1)} + \cdots + d_{2k} S_4 + d_{2k+1} R_2 + d_{2k-1} R_0)(\Delta')(g-f)\\
&\quad = (\text{linear combination of the } \alpha'_*) + d_{2k+1}(g+f) + d_{2k-1}(g-f)\\
&\quad = (\text{linear combination of the } \alpha'_*) + (d_{2k+1}+d_{2k-1})g + (d_{2k+1}-d_{2k-1})f.
\end{aligned}
$$

Therefore we have

$$
(\flat 1)\qquad
\begin{aligned}
\langle (g-f)(g+f), g\rangle &= \langle g^2, g\rangle - \langle f^2, g\rangle = d_{2k+1} + d_{2k-1} = d_{2k+2} - d_{2k},\\
\langle (g-f)(g+f), f\rangle &= \langle g^2, f\rangle - \langle f^2, f\rangle = d_{2k+1} - d_{2k-1}.
\end{aligned}
$$

We obtain further information by investigating $R_2(\Delta')(g+f)(g-f)$. Note that $R_2(\Delta')(g+f) = 2\alpha'_{n-1} + f + 3g$. Therefore

$$
\begin{aligned}
(\sharp 1)\quad & R_2(\Delta')(g+f)(g-f)\\
&\quad = (2\alpha'_{n-1} + f + 3g)(g-f)\\
&\quad = 2\alpha'_{n-1}(g-f) + 3g^2 - f^2 - 2fg\\
&\quad = (\alpha'_*\text{'s}) + 2(d_{2k}(g+f) + d_{2k-2}(g-f)) + 3g^2 - f^2 - 2fg\\
&\quad = (\alpha'_*\text{'s}) + 2(d_{2k}+d_{2k-2})g + 2(d_{2k}-d_{2k-2})f + 3g^2 - f^2 - 2fg.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
(\sharp 2)\quad & R_2(\Delta')(g+f)(g-f)\\
&\quad = R_2(\Delta')(2(d_0\alpha'_2 + d_1\alpha'_4 + \cdots + d_{2k}\alpha'_{4k+2})) + (d_{2k+1}+d_{2k-1})R_2(\Delta')g\\
&\qquad\qquad + (d_{2k+1}-d_{2k-1})R_2(\Delta')f\\
&\quad = (\alpha'_*\text{'s}) + 2d_{2k}(f+g) + (d_{2k+1}+d_{2k-1})(\alpha'_{n-1} + f + 2g)\\
&\qquad\qquad + (d_{2k+1}-d_{2k-1})(\alpha'_{n-1} + g)\\
&\quad = (\alpha'_*\text{'s}) + (2d_{2k} + d_{2k+1} + d_{2k-1})f + (2d_{2k} + 3d_{2k+1} + d_{2k-1})g.
\end{aligned}
$$

Comparing ($\sharp$1) and ($\sharp$2) we obtain

$$
(\flat 2)\qquad
\begin{aligned}
3\langle g^2, g\rangle - \langle f^2, g\rangle - 2\langle fg, g\rangle &= 3d_{2k+1} + d_{2k-1} - 2d_{2k-2},\\
3\langle g^2, f\rangle - \langle f^2, f\rangle - 2\langle fg, f\rangle &= d_{2k+1} + d_{2k-1} + 2d_{2k-2}.
\end{aligned}
$$

Combining Equations ($\flat$1) and ($\flat$2), we obtain the statement of the lemma. Note that we use Frobenius reciprocity such as $\langle fg, f\rangle = \langle f^2, g\rangle$, etc. □

The next lemma, together with Lemma 2.11, implies Theorem 2.10.

Lemma 2.12. $\langle g^2, g\rangle = d_{2k+2}$.

Proof. Since $g = \bar{\beta}_2\alpha_1 = \bar{\gamma}_2\alpha_1$,

$$2g = (\bar{\beta}_2 + \bar{\gamma}_2)\alpha_1 = \overline{(\beta_3+\gamma_3)\alpha_1}\alpha_1 = \bar{\alpha}_1(\beta_3+\gamma_3)\alpha_1.$$

Also, $\bar{\gamma}_2 = \bar{\gamma}_3\alpha_1 = \bar{\alpha}_1\beta_3$. Therefore

$$\begin{aligned} 4\langle g^2, g\rangle &= \langle \bar{\alpha}_1(\beta_3+\gamma_3)\alpha_1\bar{\alpha}_1(\beta_3+\gamma_3)\alpha_1, \bar{\alpha}_1\beta_3\alpha_1\rangle \\ &= \langle \alpha_1\bar{\alpha}_1(\beta_3+\gamma_3)\alpha_1\bar{\alpha}_1(\beta_3+\gamma_3)\alpha_1\bar{\alpha}_1, \beta_3\rangle \\ &= \langle (\beta_3+\gamma_3)^2(\alpha_1\bar{\alpha}_1)^3, \beta_3\rangle = \langle (\beta_3+\gamma_3)^2, \beta_3(\alpha_1\bar{\alpha}_1)^3\rangle, \end{aligned}$$

where we used

$$\begin{aligned} \alpha_1\bar{\alpha}_1(\beta_3+\gamma_3) &= \beta_1+\beta_3+\gamma_1+\gamma_3 \\ &= \overline{\beta_1+\beta_3+\gamma_1+\gamma_3} = \overline{(\beta_3+\gamma_3)}\alpha_1\bar{\alpha}_1 = (\beta_3+\gamma_3)\alpha_1\bar{\alpha}_1. \end{aligned}$$

A computation using the graph Γ_k gives

$$\beta_3(\alpha_1\bar{\alpha}_1)^3 = 5\beta_3 + 10\beta_1 + 6\alpha_{n-1} + 6\gamma_1 + \alpha_{n-3} + \gamma_3.$$

Using the formula for $(\beta_3+\gamma_3)^2$ given in Claim 2.4, we obtain

$$\begin{aligned} \langle (\beta_3+\gamma_3)^2, \beta_3(\alpha_1\bar{\alpha}_1)^3\rangle &= 8c_{2k} + 12c_{2k+1} + 16c_{2k+2} = 4c_{2k+1} + 8c_{2k+2} + 8c_{2k+3} \\ &= 4c_{2k+2} + 4c_{2k+3} + 4c_{2k+4} = 4c_{2k+5} = 4d_{2k+2}. \end{aligned}$$

Therefore $\langle g^2, g\rangle = d_{2k+2}$. □

2F. ***Fusion rules on*** $_M\mathscr{X}_M \times {}_M\mathscr{X}_N$**.** The remaining problem is to determine the fusion rule on $\{f, g\} \times \{\bar{\beta}_2, \bar{\gamma}_2\}$:

$$\begin{aligned} \langle f\bar{\beta}_2, \bar{\beta}_2\rangle = \langle f, \bar{\beta}_2\beta_2\rangle = \langle f, \bar{\alpha}_1\beta_3^2\alpha_1\rangle = \langle \alpha_1 f\bar{\alpha}_1, \beta_3^2\rangle = \langle \alpha_n\bar{\alpha}_1, \beta_3^2\rangle \\ = \langle \beta_3^2, \beta_1\rangle + \langle \beta_3^2, \gamma_1\rangle + \langle \beta_3^2, \alpha_{n-1}\rangle. \end{aligned}$$

Theorems 2.7 and 2.8 imply that

$$\langle f\bar{\beta}_2, \bar{\beta}_2\rangle = g_{2k+2} + g_{2k+1}.$$

Both values are nonnegative integers. Similarly we obtain

$$\begin{aligned} \langle f\bar{\beta}_2, \bar{\gamma}_2\rangle &= \langle f\bar{\gamma}_2, \bar{\beta}_2\rangle = f_{2k+2} + f_{2k+1}, \\ \langle f\bar{\gamma}_2, \bar{\gamma}_2\rangle &= g_{2k+2} + g_{2k+1}, \\ \langle g\bar{\beta}_2, \bar{\beta}_2\rangle &= \langle \bar{\beta}_2\alpha_1\bar{\beta}_2, \bar{\beta}_2\rangle = \langle \bar{\alpha}_1\bar{\beta}_3\alpha_1\bar{\alpha}_1\bar{\beta}_3, \bar{\alpha}_1\bar{\beta}_3\rangle = \langle \alpha_1\bar{\alpha}_1\gamma_3\alpha_1\bar{\alpha}_1, \gamma_3\beta_3\rangle \\ &= \langle \overline{\overline{(\gamma_1+\gamma_3)}\alpha_1\bar{\alpha}_1}, \gamma_3\beta_3\rangle, \end{aligned}$$

$$\begin{aligned} \overline{(\gamma_1+\gamma_3)}\alpha_1\bar{\alpha}_1 &= (\gamma_1+\beta_3)\alpha_1\bar{\alpha}_1 = (\alpha_{n-1}+\beta_1+2\gamma_1+\gamma_3)+\beta_1+\beta_3 \\ &= \alpha_{n-1}+2(\beta_1+\gamma_1)+\gamma_3+\beta_3 = \overline{\alpha_{n-1}+2(\beta_1+\gamma_1)+\gamma_3+\beta_3}. \end{aligned}$$

Thus, using Theorems 2.7 and 2.8 we obtain

$$\langle g\bar\beta_2, \bar\beta_2\rangle = \begin{cases} f_{2k+1} + 2f_{2k+2} + f_{2k} - 1 & \text{for } k \text{ even,} \\ f_{2k+1} + 2f_{2k+2} + f_{2k} & \text{for } k \text{ odd.} \end{cases}$$

Similarly,

$$\begin{aligned} \langle g\bar\beta_2, \bar\gamma_2\rangle &= \langle g\bar\gamma_2, \bar\beta_2\rangle \\ &= \begin{cases} g_{2k+1} + 2g_{2k+2} + g_{2k} + 2 & \text{for } k \text{ even,} \\ g_{2k+1} + 2g_{2k+2} + g_{2k} & \text{for } k \text{ odd,} \end{cases} \quad \langle g\bar\gamma_2, \bar\gamma_2\rangle = \langle g\bar\beta_2, \bar\beta_2\rangle. \end{aligned}$$

3. Existence of the fusion algebra

Let $k \in \mathbb{N}_0$, and put $n = 4k+3$ as before. In this section we will reserve the symbols

$$(\alpha_j)_{0\le k\le n}, \quad (\beta_j)_{1\le j\le 3}, \quad (\gamma_j)_{1\le j\le 3}$$

for elements in a certain bigraded $\mathbb{Z}$-algebra $\mathcal{A}$ which we define later. Therefore we relabel the vertices of the graph Γ_k as in Figure 2.

As in Section 2A, let G be the adjacency matrix for $(\Gamma_k^{\text{even}}, \Gamma_k^{\text{odd}})$, where

$$\begin{aligned} \Gamma_k^{\text{even}} &= \{a_0, a_2, \dots, a_{n-1}, b_1, c_1, b_3, c_3\}, \\ \Gamma_k^{\text{odd}} &= \{a_1, a_3, \dots, a_n, b_2, c_2\}. \end{aligned}$$

Set $\mathbb{D} = GG^t$ and

$$\Delta := \begin{pmatrix} 0 & G \\ G^t & 0 \end{pmatrix}.$$

Let $(q_k)_{k=0}^{\infty}$ be the sequence of polynomials defined by

$$\begin{aligned} q_0(t) &= t^2 - 5t + 3, \\ q_1(t) &= (t-1)(t^3 - 8t^2 + 17t - 5), \\ q_k(t) &= (t^2 - 4t + 2)q_{k-1}(t) - q_{k-2}(t), \quad k \ge 2, \end{aligned}$$

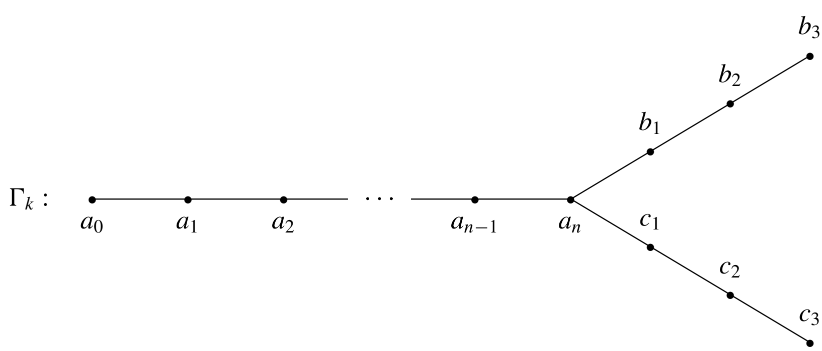

Figure 2

as in Section 2A. Then the characteristic polynomial for $\mathbb{D}$ is

$$\chi_k(t) = t^2(t-2)^2 q_k(t)$$

(see Section 2A). Moreover $q_k(t)$ is a polynomial of degree $2k+2$ with $2k+2$ distinct roots, because by [Asaeda and Yasuda 2009], either $q_k(t)$ or $q_k(t)/(t-1)$ is an irreducible polynomial. The recursion formula for the q_k-polynomials implies

$$\begin{aligned} q_k(0) &= 2k+3, \\ q_k(2) &= (-1)^{k+1}(2k+3). \end{aligned}$$

In particular, 0 and 2 are not roots of q_k. Let $k \in \mathbb{N}_0$ be fixed. Then $\chi_k(t)$ has exactly $2k+4$ distinct roots $(t_j)_{k=1}^{2k+4}$, where $t_1 = 0$, $t_2 = 2$ and $t_3, \dots, t_{2k+4}$ are the roots of $q_k(t)$. Since $\mathbb{D} = GG^t$ is a positive operator, $t_j \geq= 0$ for $1 \leq j \leq 2k+4$.

Lemma 3.1. *Let E_j be the orthogonal projection on the eigenspace of $\mathbb{D}$ corresponding to the eigenvalue t_j, $1 \leq j \leq 2k+4$, and put*

$$\mu_j = \langle E_j a_0, a_0 \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $l^2(\Gamma_k^{\text{even}})$. Then

(a) $\sum_{j=1}^{2k+4} \mu_j = 1$,

(b) $\mu_j > 0$ *for* $1 \leq j \leq 2k+4$,

(c) $\mu_1 = \mu_2 = 1/(2k+3)$.

Proof. (a) Since $\mathbb{D}$ is a symmetric matrix, $\sum_{j=1}^{2k+4} E_j = I$, thus $\sum_{j=1}^{2k+4} \mu_j = 1$.

(b) From Section 2A, we have

$$\begin{aligned} Q_j(\mathbb{D})a_0 &= R_{2j}(\Delta)a_0 = a_{2j}, \quad 0 \leq j \leq 2k+1, \\ Q_{2k+2}(\mathbb{D})a_0 &= R_{4k+4}(\Delta)a_0 = b_1 + c_1, \\ Q_{2k+3}(\mathbb{D})a_0 &= R_{4k+6}(\Delta)a_0 = b_1 + c_1 + b_3 + c_3. \end{aligned}$$

Since $\{a_0, a_2, \dots, a_{4k+2}, b_1+c_1, b_1+c_1+b_3+c_3\}$ is a set of $2k+4$ linearly independent vectors in $l^2(\Gamma_k^{\text{even}})$, and since $(Q_j)_{0 \leq j \leq 2k+3}$ spans the set of polynomials of degree less or equal to $2k+3$, we have

$$P(\mathbb{D})a_0 \neq 0$$

for every nonzero polynomial $P \in \mathbb{R}[x]$ with $\deg(P) \leq 2k+3$. On the other hand, $\mathbb{D}$ is diagonalizable with eigenvalues $(t_j)_{j=1}^{2k+4}$, so

$$E_j = P_j(\mathbb{D}),$$

where

$$P_j(t) = \prod_{i \neq j} \frac{t - t_i}{t_j - t_i}, \quad t \in \mathbb{R},$$

is a polynomial of degree $2k+3$. Hence

$$\mu_j = \langle E_k a_0, a_0\rangle = \|E_j a_0\|^2 > 0, \quad 1 \le j \le 2k+4.$$

(c) From Section 2A, we have

$$\begin{aligned} \operatorname{rg}(E_1) &= E(\mathbb{D}, 0) = \operatorname{span}\{y_1, y_2\}, \\ \operatorname{rg}(E_2) &= E(\mathbb{D}, 2) = \operatorname{span}\{x_1, x_2\}, \end{aligned}$$

where

$$\begin{aligned} x_1 &:= 2(a_0+a_2) - 2(a_4+a_6) + \cdots + (-1)^k 2(a_{4k}+a_{4k+2}), \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad + (-1)^{k+1}(b_1+c_1+b_3+c_3), \\ x_2 &:= (b_1-c_1)+(b_3-c_3), \\ y_1 &:= 2a_0 - 2a_2 + \cdots + 2a_{4k} - 2a_{4k+2} + (b_1+c_1) - (b_3+c_3), \\ y_2 &:= (b_1-c_1)-(b_3-c_3). \end{aligned}$$

Since $y_1 \perp y_2$ and $y_2 \perp a_0$, we get

$$\mu_1 = \langle E_1 a_0, a_0\rangle = \frac{|\langle y_1, a_0\rangle|^2}{\|y_1\|^2} = \frac{1}{2k+3},$$

and similarly,

$$\mu_2 = \langle E_2 a_0, a_0\rangle = \frac{|\langle x_1, a_0\rangle|^2}{\|x_1\|^2} = \frac{1}{2k+3}. \qquad \square$$

Corollary 3.2. *Let $(e_{ij})_{i,j=1}^{2k+4}$ be the matrix units of $M_{2k+4}(\mathbb{R})$. Put*

$$\begin{aligned} \mathcal{B} &= \operatorname{span}_{\mathbb{R}}\{e_{11}, e_{12}, e_{21}, e_{22}, e_{33}, e_{44}, \ldots, e_{2k+4,2k+4}\} \\ &\cong M_2(\mathbb{R}) \oplus l^\infty(\{3, 4, \ldots, 2k+4\}, \mathbb{R}). \end{aligned}$$

Then $\mathcal{B}$ is a finite dimensional real C^*-algebra and the map $\mu : \mathcal{B} \to \mathbb{R}$ given by

$$\mu(b) := \sum_{j=1}^{2k+4} \mu_j b_{jj}, \quad b = (b_{ij})_{i,j=1}^{2k+4} \in \mathcal{B},$$

is a faithful trace state on $\mathcal{B}$.

Proof. It is clear from Lemma 3.1(a), (b) that μ is a faithful state on $\mathcal{B}$. The trace property

$$\mu(bc) = \mu(cb), \quad b, c \in \mathcal{B},$$

follows from Lemma 3.1(c). $\square$

Lemma 3.3. *Fix $k \in \mathbb{N}_0$, let $\mu : \mathcal{B} \to \mathbb{R}$ be the trace in Corollary 3.2, and put*

$$A := \operatorname{diag}(0, \sqrt{2}, \sqrt{t_3}, \ldots \sqrt{t_{2k+4}})),$$

where $t_3, \ldots, t_{2k+4}$ are the roots of q_k.

(a) *For every even polynomial $P \in \mathbb{R}[x]$,*

$$\mu(P(A)) = \langle P(\Delta)a_0, a_0\rangle.$$

(b) *Let P, $Q \in \mathbb{R}[x]$ be two polynomials, which are either both even or both odd. Then*

$$\mu(P(A)Q(A)) = \langle P(\Delta)a_0, Q(\Delta)a_0\rangle.$$

(c) *Let $n = 4k+3$ (as usual). Then*

$$R_{n+4}(A) - R_{n+2}(A) - R_n(A) - R_{n-2}(A) = 0.$$

Proof. (a) Choose $Q \in \mathbb{R}[x]$ so that $P(t) = Q(t^2)$. Then

$$\langle P(\Delta)a_0, a_0\rangle = \langle Q(\mathbb{D})a_0, a_0\rangle.$$

Let E_j denote the spectral projection of $\mathbb{D}$ corresponding to the eigenvalue t_j, $1 \le j \le 2k+4$, as before, where $t_1 = 0$ and $t_2 = 2$. Then

$$Q(\mathbb{D}) = \sum_{j=1}^{2k+4} Q(t_j)E_j.$$

Hence

$$\langle Q(\mathbb{D})a_0, a_0\rangle = \sum_{j=1}^{2k+4} Q(t_j)\langle E_j a_0, a_0\rangle = \sum_{j=1}^{2k+4} \mu_j Q(t_j) = \mu(Q(A^2)) = \mu(P(A)).$$

(b) Under the assumption on P and Q, the product PQ is an even polynomial. Hence by (a) we have

$$\begin{aligned}\mu(P(A)Q(A)) &= \langle P(\Delta)Q(\Delta)a_0, a_0\rangle\\ &= \langle P(\Delta)a_0, Q(\Delta)a_0\rangle.\end{aligned}$$

(c) Put $P = Q = R_{n+4} - R_{n+2} - R_n - R_{n-2}$, which is an odd polynomial. By (b),

$$\mu(P(A)^2) = \|P(\Delta)a_0\|_2^2.$$

From the recursive formula for the polynomials R_j,

$$\begin{aligned}R_{n-2}(\Delta)a_0 &= a_{n-2},\\ R_n(\Delta)a_0 &= a_n,\\ R_{n+2}(\Delta)a_0 &= a_n + b_2 + c_2,\\ R_{n+4}(\Delta)a_0 &= a_{n-2} + 2a_n + b_2 + c_2\\ &= (R_{n+2}(A) + R_n(A) + R_{n-2}(A))a_0.\end{aligned}$$

Hence $\mu(P(A)^2) = \|P(\Delta)a_0\|_2^2 = 0$, and since μ is a faithful trace on $\mathcal{B}$, we have $P(A) = 0$. □

Remark 3.4. Since $P = R_{n+4} - R_{n+2} - R_n - R_{n-2}$ is an odd polynomial and $P(A) = 0$, we know that $P(t)$ has at least $n + 4 = 4k + 7$ roots

$$0, \pm\sqrt{2}, \pm\sqrt{t_3}, \ldots, \sqrt{t_{2k+4}},$$

which are exactly the distinct roots of $t(t^2 - 2)q_k(t^2)$. Since P and $t(t^2 - 2)q_k(t^2)$ are both monic polynomial of degree $4k + 7$, it follows that

$$(R_{n+4} - R_{n+2} - R_n - R_{n-2})(t) = t(t^2 - 2)q_k(t^2).$$

It is not hard to prove this identity directly by using the recursion formulas for the polynomials $\{q_k\}$ and $\{R_j\}$.

Definition 3.5. Let $k \in \mathbb{N}_0$, $n = 4k + 3$, and let $\mathcal{B}$ and μ be as in Corollary 3.2 and $A = \mathrm{diag}(\sqrt{t_1}, \sqrt{t_2}, \ldots, \sqrt{t_{2k+4}}) \in \mathcal{B}$ be as in Lemma 3.3. Let $(f_{ij})_{i,j=1}^2$ be the matrix units in $M_2(\mathbb{R})$, and put

$$V := V_{11} \sqcup V_{12} \sqcup V_{21} \sqcup V_{22},$$

where $V_{ij} \subset \mathcal{B} \otimes f_{ij}$, $i, j = 1, 2$, are as follows:

(a) $V_{11} = \{\alpha_0, \alpha_2, \alpha_4, \ldots, \alpha_{4k+2}, \beta_1, \gamma_1, \beta_3, \gamma_3\}$, where

$$\begin{aligned}
\alpha_{2j} &= R_{2j}(A) \otimes f_{11}, \quad 0 \le j \le 2k+1,\\
\beta_1 &= \tfrac{1}{2}(R_{n+1}(A) + \sqrt{2k+3}(e_{12} + e_{21})) \otimes f_{11},\\
\gamma_1 &= \tfrac{1}{2}(R_{n+1}(A) - \sqrt{2k+3}(e_{12} + e_{21})) \otimes f_{11},\\
\beta_3 &= \tfrac{1}{2}((R_{n+3} - R_{n+1} - R_{n-1})(A) + \sqrt{2k+3}(e_{12} - e_{21})) \otimes f_{11},\\
\gamma_3 &= \tfrac{1}{2}((R_{n+3} - R_{n+1} - R_{n-1})(A) - \sqrt{2k+3}(e_{12} - e_{21})) \otimes f_{11}.
\end{aligned}$$

(b) $V_{12} = \{\alpha_1, \alpha_3, \alpha_5, \ldots, \alpha_{4k+3}, \beta_2, \gamma_2\}$, where

$$\begin{aligned}
\alpha_{2j+1} &= R_{2j+1}(A) \otimes f_{12}, \quad 0 \le j \le 2k+1,\\
\beta_2 &= \tfrac{1}{2}((R_{n+2} - R_n)(A) + \sqrt{2(2k+3)}e_{12}) \otimes f_{12},\\
\gamma_2 &= \tfrac{1}{2}((R_{n+2} - R_n)(A) - \sqrt{2(2k+3)}e_{12}) \otimes f_{12}.
\end{aligned}$$

(c) $V_{21} = \{\bar\alpha_1, \bar\alpha_3, \bar\alpha_5, \ldots, \bar\alpha_{4k+3}, \bar\beta_2, \bar\gamma_2\}$, where

$$\begin{aligned}
\bar\alpha_{2j+1} &= R_{2j+1}(A) \otimes f_{21}, \quad 0 \le j \le 2k+1,\\
\bar\beta_2 &= \tfrac{1}{2}((R_{n+2} - R_n)(A) + \sqrt{2(2k+3)}e_{21}) \otimes f_{21},\\
\bar\gamma_2 &= \tfrac{1}{2}((R_{n+2} - R_n)(A) - \sqrt{2(2k+3)}e_{21}) \otimes f_{21}.
\end{aligned}$$

(d) $V_{22} = \{\alpha'_0, \alpha'_2, \dots, \alpha'_{4k+2}, f, g\}$, where

$$\begin{aligned} \alpha'_j &= R_{2j}(A) \otimes f_{22}, \quad 0 \le j \le 2k+1, \\ f &= \tfrac{1}{2}(R_{n-1} + 2R_{n+1} - R_{n+3})(A) \otimes f_{22}, \\ g &= \tfrac{1}{2}(R_{n+3} - R_{n-1})(A) \otimes f_{22}. \end{aligned}$$

(e) The conjugation map $V_{12} \to V_{21}$ and $V_{21} \to V_{12}$ is already defined earlier. For V_{11} and V_{22}, all the elements are defined to be self-conjugate except β_3 and γ_3 which are defined to be conjugate of each other. Note that for every $X \in V_{ij}$, the conjugate $\overline{X}$ is equal to X^* (or X^t, since all the matrices here are real).

(f) Equip $\mathbb{R}V_{ij} \subset \mathcal{B} \otimes f_{ij}$ with inner products given by

$$\langle b \otimes f_{ij}, c \otimes f_{ij} \rangle_\mu := \mu(c^t b) = \mu(bc^t)$$

for every $b, c \in \mathbb{R}V_{ij}$, $i, j = 1, 2$.

Lemma 3.6. *Let $i, j \in \{1, 2\}$. For $X, Y \in V_{ij}$,*

$$\langle X, Y \rangle_\mu = \begin{cases} 1 & \text{if } X = Y, \\ 0 & \text{if } X \neq Y. \end{cases}$$

Proof. Let $(b, c)_\mu := \mu(c^t b) = \mu(bc^t)$, $b, c \in \mathcal{B}$, be the inner product on $\mathcal{B}$ given by μ, and put $\|b\|_\mu (b, b)_\mu^{1/2}$, $b \in \mathcal{B}$.

(a) Case $(i, j) = (1, 1)$. It suffices to show that

$$S_1 := \{R_0(A), R_2(A), \dots, R_{n+1}(A), (R_{n+3} - R_{n+1} - R_{n-1})(A), e_{12} + e_{21}, e_{12} - e_{21}\}$$

is an orthogonal set in $\mathcal{B}$ and that

$$\begin{aligned} \|R_{2j}(A)\|_\mu^2 &= 1, \quad 0 \le j \le \frac{n-1}{2}, \\ \|R_{n+1}(A)\|_\mu^2 &= 2, \\ \|(R_{n+3} - R_{n+1} - R_{n-1})(A)\|_\mu^2 &= 2, \\ \|e_{12} + e_{21}\|_\mu^2 = \|e_{12} - e_{21}\|_\mu^2 &= \frac{2}{2k+3}. \end{aligned}$$

By the definition of μ in Corollary 3.2, it is clear that $e_{12} + e_{21}$ and $e_{12} - e_{21}$ are μ-orthogonal to the remaining matrices in S_1, because $R_j(A)$ is a diagonal matrix for all $j \in \mathbb{N}_0$. Moreover, by Lemma 3.1,

$$\begin{aligned} \langle e_{12} + e_{21}, e_{12} - e_{21} \rangle_\mu &= \mu(e_{11} - e_{22}) = \mu_1 - \mu_2 = 0, \\ \|e_{12} + e_{21}\|_\mu^2 = \|e_{12} - e_{21}\|_\mu^2 &= \mu(e_{11} + e_{22}) = \mu_1 + \mu_2 = \frac{2}{2k+3}. \end{aligned}$$

By Lemma 3.3(b), the remaining part of the proof in the V_{11} case reduces to showing that

$$T_1 := \{R_0(\Delta)a_0,\, R_2(\Delta)a_0, \dots, R_{n+1}(\Delta)a_0,\, (R_{n+3}(\Delta) - R_{n+1}(\Delta) - R_{n-1}(\Delta))a_0\}$$

is an orthogonal set in $l^2(\Gamma_k)$ with

$$\begin{aligned} \|R_{2j}(\Delta)a_0\|^2 &= 1, \quad 0 \le j \le n-1, \\ \|R_{n+1}(\Delta)a_0\|^2 &= 2, \\ \|(R_{n+3} - R_{n+1} - R_{n-1})(\Delta)a_0\|^2 &= 2. \end{aligned}$$

This follows from the fact that

$$T_1 = \{a_0, a_2, \dots, a_{n-1}, b_1 + c_1, b_3 + c_3\}.$$

(b) Cases $(i, j) = (1, 2)$ and $(i, j) = (2, 1)$. It suffices to show that

$$S_2 := \{R_1(A), R_3(A), \dots R_n(A), (R_{n+2} - R_n)(A), e_{12}\}$$

is an orthonormal set in $\mathcal{B}$ and that

$$\begin{aligned} \|R_{2j+1}(A)\|_\mu^2 &= 1, \quad 0 \le j \le \frac{n-1}{2}, \\ \|(R_{n+2} - R_n)(A)\|_\mu^2 &= 2, \\ \|e_{12}\|_\mu^2 &= \frac{1}{2k+3}. \end{aligned}$$

It is easy to check that e_{12} is orthogonal to the remaining elements of S_2 and that $\|e_{12}\|_\mu^2 = (2k+3)^{-1}$ by Lemma 3.3(b). The remaining statement about the set S_2 follow from the fact that

$$\begin{aligned} T_2 &= \{R_1(\Delta)a_0, R_3(\Delta)a_0, \dots, R_n(\Delta)a_0, (R_{n+2} - R_n)(\Delta)a_0\} \\ &= \{a_1, a_3, \dots, a_n, b_2 + c_2\} \end{aligned}$$

is an orthonormal set in $l^2(\Gamma_k)$, and from the equalities

$$\|b_2 + c_2\|^2 = 2, \qquad \|a_{2j+1}\|^2 = 1 \quad \text{for } 0 \le j \le \frac{n-1}{2}.$$

(c) Case $(i, j) = (2, 2)$. The statement follows in this case if we can show that

$$S_3 := \Big\{R_0(A), R_2(A), \dots, R_{n-1}(A), \\ \tfrac{1}{2}(R_{n-1} + 2R_{n+1} - R_{n+3})(A), \tfrac{1}{2}(R_{n+3} - R_{n-1})(A)\Big\}$$

is a μ-orthogonal set in $\mathcal{B}$. By Lemma 3.3(b) this reduces to showing that

$$T_3 := \Big\{ a_0, a_2, \dots, a_{n-1}, \tfrac{1}{2}(b_1 + c_1 + b_3 + c_3), \tfrac{1}{2}(b_1 + c_1 - b_3 - c_3)\Big\}$$

is an orthogonal set in $l^2(\Gamma_k)$, which is obvious. □

Theorem 3.7. *Let $V = V_{11} \sqcup V_{12} \sqcup V_{21} \sqcup V_{22}$ as in Definition 3.5. Then $\mathbb{Z}V \subset M_2(\mathcal{B})$ forms a fusion ring, with coefficients given by*

$$N_{X,Y}^{Z} = \langle XY, Z\rangle_{\mu},$$

where $X \in V_{ij}$, $Y \in V_{jk}$, $Z \in V_{ik}$, $(i, j, k) \in \{1, 2\}^3$, and with units $\alpha_0 \in V_{11}$ and $\alpha_0' \in V_{22}$. Moreover the graph with vertices $V_{11} \sqcup V_{12}$ obtained by right multiplication by $\alpha = \alpha_1$ is Γ_k and the graph with vertices $V_{21} \sqcup V_{22}$ obtained by right multiplication by $\overline{\alpha}$ is Γ_k'.

Proof. By Lemma 3.6, for all $i, j \in \{1, 2\}$, the set V_{ij} is linearly independent in $\mathcal{B} \otimes f_{ij}$. Hence

$$\begin{aligned}
&\dim(\mathbb{R}V_{11}) = |V_{11}| = 2k + 6,\\
&\dim(\mathbb{R}V_{12}) = \dim(\mathbb{R}V_{21}) = \dim(\mathbb{R}V_{22}) = 2k + 4.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\mathbb{R}V_{11} = \mathcal{B} \otimes f_{11},\\
&\mathbb{R}V_{12} = \operatorname{span}\{e_{12}, e_{22}, e_{33}, \dots, e_{2k+4,2k+4}\} \otimes f_{12},\\
&\mathbb{R}V_{21} = \operatorname{span}\{e_{21}, e_{22}, e_{33}, \dots, e_{2k+4,2k+4}\} \otimes f_{21},\\
&\mathbb{R}V_{22} = \operatorname{span}\{e_{11}, e_{22}, e_{33}, \dots, e_{2k+4,2k+4}\} \otimes f_{22},
\end{aligned}$$

because the four inclusions $\subset$ are obvious, and the right-hand sides have dimensions $2k + 6$ (respectively, $2k + 4$, $2k + 4$, $2k + 4$). Therefore

$$\mathbb{R}V = \mathbb{R}V_{11} \oplus \mathbb{R}V_{12} \oplus \mathbb{R}V_{21} \oplus \mathbb{R}V_{22}$$

forms a bigraded $\mathbb{R}$-algebra, and the conjugation $X \to \overline{X}$ extends by linearity to all of $\mathbb{R}V$ and it is given by transposition of matrices. Moreover, for $X \in V_{ij}$, $Y \in V_{jk}$, $i, j, k \in \{1, 2\}$, we have a unique decomposition

$$XY = \sum_{Z \in V_{ik}} N_{X,Y}^{Z} Z,$$

where by Lemma 3.6,

$$N_{X,Y}^{Z} = \langle XY, Z\rangle_{\mu} \in \mathbb{R}.$$

The identities

$$N_{X,Y}^{Z} = N_{Z,\overline{Y}}^{X} = N_{\overline{X},Z}^{Y} = N_{\overline{Z},X}^{\overline{Y}} = N_{Y,\overline{Z}}^{\overline{X}}$$

are now a simple consequence of the fact that μ is a trace state on the real C^*-algebra $\mathcal{B}$, so in particular

$$\begin{aligned}
\mu(b) &= \mu(b^t), \quad b \in \mathcal{B},\\
\mu(bc) &= \mu(cb), \quad b, c \in \mathcal{B}.
\end{aligned}$$

It remains to prove that $N_{X,Y}^{Z} \in \mathbb{N}_0$ and that multiplication from the right by $\alpha = \alpha_1$ (respectively, $\overline{\alpha}$) on V_{11} (respectively, V_{22}) generates the graph Γ_k (respectively, Γ'_k).

Lemma 3.8. *Let* $\alpha = \alpha_1$.

(a) *For* $X \in V_{11}$, $Y \in V_{12}$,

$$\langle X\alpha, Y\rangle_\mu = \langle X, Y\overline{\alpha}\rangle_\mu \in \mathbb{N}_0,$$

and $(\langle X\alpha, Y\rangle_\mu)_{X\in V_{11}, Y\in V_{12}}$ *is the adjacency matrix* G_k *for* Γ_k.

(b) *For* $X \in V_{22}$, $Y \in V_{21}$,

$$\langle X\overline{\alpha}, Y\rangle_\mu = \langle X, Y\alpha\rangle_\mu \in \mathbb{N}_0,$$

and $(\langle X\overline{\alpha}, Y\rangle_\mu)_{X\in V_{22}, Y\in V_{21}}$ *is the adjacency matrix* G'_k *for* Γ'_k.

Proof. This follows from simple computations using Definition 3.5, Lemma 3.6, the recursion formula

$$(\star) \qquad tR_n(t) = R_{n+1}(t) + R_{n-1}(t), \quad n \geq 1,$$

and the identity from Lemma 3.3(c)

$$(\star\star) \qquad R_{n+4}(A) - R_{n+2}(A) - R_n(A) - R_{n-2}(A) = 0.$$

(a) It follows immediately from $(\star)$ that for $1 \leq j \leq 2k+1$,

$$\alpha_{2j}\alpha = \alpha_{2j+1} + \alpha_{2j-1},$$

which shows that $\alpha_{2j} \in V_{11}$ is connected to α_{2j+1} and α_{2j-1} in V_{12} (with simple edges) and not connected to any other $Y \in V_{12}$. To prove that we recover the graph Γ_k this way we just have to check that $\alpha_0\alpha = \alpha_1$, which is obvious, and that $\beta_1\alpha = \alpha_n + \beta_2$ and $\beta_3\alpha = \beta_2$. The last equality follows from

$$\begin{aligned}
\beta_3\alpha &= \tfrac{1}{2}((R_{n+3} - R_{n+1} - R_{n-1})(A) + \sqrt{2k+3}(e_{12} + e_{21}))A) \otimes f_{12} \\
&= \tfrac{1}{2}(R_{n+4} - 2R_n - R_{n-2})(A) + \sqrt{2(2k+3)}e_{12}) \otimes f_{12} \\
&= \tfrac{1}{2}((R_{n+2} - R_n)(A) + \sqrt{2(2k+3)}e_{12}) \otimes f_{12} \\
&= \beta_2,
\end{aligned}$$

where we used $(\star)$ and $(\star\star)$ and the fact that $e_{12}A = \sqrt{2}e_{12}$, $e_{21}A = 0$. The proof of $\beta_1\alpha = \alpha_n + \beta_2$ is similar.

(b) To recover the graph Γ_k from $V_{22} \sqcup V_{21}$, it suffices to prove that

$$\begin{aligned}
\alpha'_0\overline{\alpha} &= \overline{\alpha}_1,\\
\alpha'_{2j}\overline{\alpha} &= \overline{\alpha}_{2j+1} + \overline{\alpha}_{2j-1}, \quad 1 \le j \le 2k+1,\\
f\overline{\alpha} &= \overline{\alpha}_n,\\
g\overline{\alpha} &= \overline{\alpha}_n + \overline{\beta}_2 + \overline{\gamma}_2.
\end{aligned}$$

The first two are obvious. A computation proves $f\overline{\alpha} = \overline{\alpha}_n$:

$$\begin{aligned}
f\overline{\alpha} &= \tfrac{1}{2}((R_{n-1}(A) + 2R_{n+1}(A) - R_{n+3}(A))A \otimes f_{21}\\
&= \tfrac{1}{2}(R_{n-2} + 3R_n + R_{n+2} - R_{n+4})(A) \otimes f_{21}\\
&= \tfrac{1}{2} \cdot 2R_n(A) \otimes f_{21}\\
&= \overline{\alpha}_n,
\end{aligned}$$

where we again used $(\star)$ and $(\star\star)$. The formula for $g\overline{\alpha}$ is obtained similarly. □

Lemma 3.9. *Put*

$$\xi := (\beta_1 - \gamma_1) + (\beta_3 - \gamma_3).$$

Then

$$\overline{\xi} := (\beta_1 - \gamma_1) - (\beta_3 - \gamma_3),$$

and

$$\begin{aligned}
\tfrac{1}{2}\xi\overline{\xi} &= 2\alpha_0 - 2\alpha_2 + \cdots + 2\alpha_{4k} - 2\alpha_{4k+2} + (\beta_1 + \gamma_1) - (\beta_3 + \gamma_3),\\
\tfrac{1}{2}\overline{\xi}\xi &= 2(\alpha_0 + \alpha_2) - 2(\alpha_4 + \alpha_6) + \cdots + (-1)^k 2(\alpha_{4k} + \alpha_{4k+2})\\
&\qquad + (-1)^{k+1}(\beta_1 + \gamma_1 + \beta_3 + \gamma_3).
\end{aligned}$$

Proof. Clearly $\overline{\xi} = (\beta_1 - \gamma_1) - (\beta_3 - \gamma_3)$. By Lemma 3.8, the linear maps

$$\begin{aligned}
R_\alpha &: \mathbb{R}V_{11} \to \mathbb{R}V_{12},\\
R\overline{\alpha} &: \mathbb{R}V_{12} \to \mathbb{R}V_{11}
\end{aligned}$$

obtained by right multiplication by α (respectively, by $\overline{\alpha}$) have the matrices G^t (respectively, G) expressed with respect to bases V_{11} for $\mathbb{R}V_{11}$ and V_{11} for $\mathbb{R}V_{12}$. Hence

$$R_{\alpha\overline{\alpha}} := R\overline{\alpha}R_\alpha : \mathbb{R}V_{11} \to \mathbb{R}V_{12}$$

has the matrix $\mathbb{D} = GG^t$ with respect to the basis V_{11} for $\mathbb{R}V_{11}$. We can now argue exactly as in Case 1 of Section 2A to get

$$\begin{aligned}
\xi\overline{\xi} &\in E(\mathbb{D}, 0)_{sc} = \mathbb{R}y_1,\\
\overline{\xi}\xi &\in E(\mathbb{D}, 2)_{sc} = \mathbb{R}x_1,
\end{aligned}$$

where

$$\begin{aligned}y_1 &= 2\alpha_0 - 2\alpha_2 + \cdots + 2\alpha_{4k} - 2\alpha_{4k+2} + (\beta_1+\gamma_1) - (\beta_3+\gamma_3),\\ x_1 &= 2(\alpha_0+\alpha_2) - 2(\alpha_4+\alpha_6) + \cdots + (-1)^k 2(\alpha_{4k}+\alpha_{4k+2}),\\ &\qquad\qquad + (-1)^{k+1}(\beta_1+\gamma_1+\beta_3+\gamma_3).\end{aligned}$$

Since $\langle \xi\bar{\xi}, \alpha_0\rangle_\mu = \langle \bar{\xi}\xi, \alpha_0\rangle_\mu = \langle \xi, \xi\rangle_\mu = 4$ and $\langle y_1, \alpha_0\rangle_\mu = \langle x_1, \alpha_0\rangle_\mu = 2$, it follows that $\xi\bar{\xi} = 2y_1$ and $\bar{\xi}\xi = 2x_1$. □

End of proof of Theorem 3.7. It remains to prove that $N^Z_{X,Y} \in \mathbb{N}_0$ for all $X \in V_{ij}$, $Y \in V_{jk}$ and $Z \in V_{ik}$, $(i, j \in \{1, 2, 3\})$.Having established the formulas for $\xi\bar{\xi}$ and $\bar{\xi}\xi$ in Lemma 3.8, the proof that $N^Z_{X,Y} \in \mathbb{N}_0$ can be obtained from Section 2: Using

$$N^Z_{X,Y} = N^X_{Z,\bar{Y}} = N^Y_{\bar{X},Z},$$

if X, Y or Z is one of the elements $(\alpha_j)_{0\le j\le n}$, $(\alpha'_j)_{0\le j\le n}$ (where $\alpha'_{2k+1} = \bar{\alpha}_{2k+1}$), then $N^Z_{X,Y}$ is an entry of the matrix $R_j(\Delta)$ or $R_j(\Delta')$, which is a nonnegative integer by [de la Harpe and Wenzl 1987]. In the remaining cases, X, Y and Z are compatible and come from the list

$$\beta_1,\ \gamma_1,\ \beta_3,\ \gamma_3,\ \beta_2,\ \gamma_2,\ \bar{\beta}_2,\ \bar{\gamma}_2,\ f,\ g.$$

For $X, Y, Z \in \{\beta_1, \gamma_1, \beta_3, \gamma_3\}$, we have $N^Z_{X,Y} \in \mathbb{N}_0$ by Theorems 2.7 and 2.8, and the remark at the end of Section 2A. The case $X, Y, Z \in \{f, g\}$ is treated in Theorem 2.10 and the remaining cases can easily be reduced to these two cases by using $\beta_2 = \beta_3\alpha$ and $\gamma_2 = \gamma_3\alpha$ (see Sections 2B and 2F). □

Remark 3.10. From Definition 3.5, we have

$$\begin{aligned}\xi &= (\beta_1-\gamma_1) + (\beta_3-\gamma_3) = 2\sqrt{2k+3}e_{12}\otimes f_{11},\\ \bar{\xi} &= (\beta_1-\gamma_1) - (\beta_3-\gamma_3) = 2\sqrt{2k+3}e_{21}\otimes f_{11}.\end{aligned}$$

Thus

$$\begin{aligned}\xi\bar{\xi} &= 4(2k+3)e_{11}\otimes f_{11},\\ \bar{\xi}\xi &= 4(2k+3)e_{22}\otimes f_{11}.\end{aligned}$$

Since $A = \mathrm{diag}(0, \sqrt{2}, \sqrt{t_3}, \ldots, \sqrt{t_{2k+4}})$, where $t_3, \ldots, t_{2k+4}$ are the distinct roots of $q_k(t)$, and since $0, 2 \notin \{t_3, \ldots, t_{2k+4}\}$, the maps e_{11} and e_{22} are the projections on the eigenspaces for A with eigenvalues 0 and 2, respectively. Using $q_k(0) = 2k+3$ and $q_k(2) = (-1)^{k+1}(2k+3)$ gives

$$\begin{aligned}(2-A^2)q_k(A^2) &= 2(2k+3)e_{11},\\ A^2 q_k(A^2) &= (-1)^{k+1}(2k+3)e_{22},\end{aligned}$$

because the polynomial $(2-t)q_k(t)$ vanishes at $t = 2$ and $t = t_j$, $3 \le j \le 2k+4$, and has the value $2(2k+3)$ at $t = 0$. Similarly $tq_k(t)$ vanishes at $t = 0$ and

$t = t_j$, $3 \le j \le 2k+4$, and has the value $(-1)^{k+1}2(2k+3)$ at $t = 2$. Hence the two identities

$$\xi\bar{\xi} = 2(2 - A^2)q_k(A^2) \otimes f_{11} = 2(1_N - \alpha\bar{\alpha})q_k(\alpha\bar{\alpha}),$$
$$\bar{\xi}\xi = (-1)^{k+2}2A^2 q_k(A^2) \otimes f_{11} = (-1)^{k+2}2\alpha\bar{\alpha}q_k(\alpha\bar{\alpha})$$

hold, where $1_N = \alpha_0$ and $\alpha = \alpha_1$. Let Q_j denote as usual the polynomial for which $R_{2j}(t) = Q_j(t^2)$, $t \in \mathbb{R}$. Then by Definition 3.5,

$$\begin{aligned}\alpha_{2j} &= Q_j(\alpha\bar{\alpha}),\\ \beta_1 + \gamma_1 &= Q_{2k+2}(\alpha\bar{\alpha}),\\ \beta_3 + \gamma_3 &= (Q_{2k+3} - Q_{2k+2} - Q_{2k+1})(\alpha\bar{\alpha}).\end{aligned}$$

Hence a more direct proof of Lemma 3.8 can be obtained if the two polynomial identities hold:

$$\begin{aligned}r_k &= (2Q_0 - 2Q_1 + \cdots + 2Q_{2k} - 2Q_{2k+1}) + (Q_{2k+1} + 2Q_{2k+2} - Q_{2k+3}),\\ s_k &= 2(Q_0 + Q_2) - 2(Q_2 + Q_4) + \cdots + (-1)^k 2(Q_{2k} + Q_{2k+1})\\ &\qquad + (-1)^{k+1}(Q_{2k+3} - Q_{2k+1}),\end{aligned}$$

where

$$r_k(t) = (2 - t)q_k(t), \quad s_k(t) = (-1)^{k+1} t q_k(t).$$

These two polynomials identities are actually true, and they can be proved using the recursion formulas for $(q_k)_{k=0}^{\infty}$ and $(R_j)_{j=0}^{\infty}$. □

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