Pacific Journal of Mathematics

CELL DECOMPOSITIONS OF TEICHMÜLLER SPACES OF SURFACES WITH BOUNDARY

REN GUO AND FENG LUO

Volume 253 No. 2 October 2011

PACIFIC JOURNAL OF MATHEMATICS
Vol. 253, No. 2, 2011

CELL DECOMPOSITIONS OF TEICHMÜLLER SPACES OF SURFACES WITH BOUNDARY

REN GUO AND FENG LUO

A family of coordinates ψ_h for the Teichmüller space of a compact surface with boundary was introduced by the second author. Mondello showed that the coordinate ψ_0 can be used to produce a natural cell decomposition of the Teichmüller space, invariant under the action of the mapping class group. In this paper, we show that, for any $h \geq 0$, the coordinate ψ_h produces a natural cell decomposition of the Teichmüller space.

1. Introduction

In this paper, we will show that each of the coordinates ψ_h ($h \geq 0$) introduced in [Luo 2006] can be used to produce a natural cell decomposition of the Teichmüller space of a compact surface with nonempty boundary and negative Euler characteristic. We will show that the underlying point sets of the cells are the same as the ones obtained in [Ushijima 1999; Hazel 2004; Mondello 2009]. However, the coordinates ψ_h for $h \geq 0$ introduce different attaching maps for the cell decomposition. In this paper, unless mentioned otherwise, we will always assume that the surface S is compact with nonempty boundary so that the Euler characteristic of S is negative.

1.1. ***The arc complex.*** We begin by briefly recalling some related concepts. An *essential arc* a in S is an embedded arc with boundary in ∂S so that a is not homotopic into ∂S while keeping endpoints fixed. The arc complex $A(S)$ of the surface, introduced in [Harer 1986], is the simplicial complex so that each vertex is the homotopy class $[a]$ of an essential arc a, and each simplex is represented by a collection of disjoint arcs $[a_1], \ldots, [a_k]$, that is, $a_i \cap a_j = \varnothing$ for all $i \neq j$. For instance, the isotopy class of an ideal triangulation corresponds to a simplex of maximal dimension in $A(S)$. The nonfillable subcomplex $A_\infty(S)$ of $A(S)$ consists of those simplexes $([a_1], \ldots, [a_k])$ such that one component of $S - \bigcup_{i=1}^{k} a_i$ is

The work of the second author is supported in part by the NSF.
MSC2010: primary 57M50; secondary 30F60.
Keywords: Teichmüller space, cell decomposition, arc complex, Delaunay decomposition, ψ_h-coordinates.

not simply connected. The simplices in $A(S) - A_\infty(S)$ are called *fillable*. The underlying space of $A(S) - A_\infty(S)$ is denoted by $|A(S) - A_\infty(S)|$.

1.2. ***Teichmüller space.*** It is well-known that there are hyperbolic metrics with totally geodesic boundary on the surface S. Two hyperbolic metrics with geodesic boundary on S are called *isotopic* if there is an isometry isotopic to the identity between them. The space of all isotopy classes of hyperbolic metrics with geodesic boundary on S is called the *Teichmüller space* of the surface S, denoted by $\mathrm{Teich}(S)$. Topologically, $\mathrm{Teich}(S)$ is homeomorphic to a ball of dimension $6g-6+3n$, where g is the genus and $n > 0$ is the number of boundary components of S.

Note that there are several versions of the Teichmüller space of a surface with boundary (or a bordered surface). For example, it can be defined and investigated via the point of view of complex structures, the point of view of Fuchsian groups or the point of view of complete hyperbolic metrics with ends of funnel type, where a *funnel* of a hyperbolic surface is a region with infinite area bounded on one side by a simple closed geodesic. The shear coordinates were introduced in [Thurston 1976–1979; Bonahon 1996] to study the Teichmüller space via the point of view of complete hyperbolic metrics. In our case, we consider the incomplete hyperbolic metrics on a surface S such that each boundary component of S is realized as a geodesic, or, equivalently, we consider the convex core of a surface with complete hyperbolic metric. This point of view is also taken in [Ushijima 1999; Penner 2004; Hazel 2004; Mondello 2009; Andersen et al. 2009].

Theorem 1 [Ushijima 1999; Hazel 2004; Mondello 2009]. *There is a natural cell decomposition of the Teichmüller space* $\mathrm{Teich}(S)$, *invariant under the action of the mapping class group.*

Ushijima [1999] proved this theorem by following the convex hull construction in [Penner 1987]. Following the approach of [Bowditch and Epstein 1988], Hazel [2004] obtained a cell decomposition of the Teichmüller space of surfaces with geodesic boundary and fixed boundary lengths. In [Luo 2007], the ψ_0-coordinate was introduced to parametrize the Teichmüller space $\mathrm{Teich}(S)$ of a surface S with a fixed ideal triangulation. Mondello [2009] pointed out that the ψ_0-coordinate produces a natural cell decomposition of $\mathrm{Teich}(S)$.

In [Luo 2006], ψ_h-coordinates were introduced (for each real number h) to parametrize the space $\mathrm{Teich}(S)$ of a surface S with a fixed ideal triangulation. The ψ_0-coordinate is a special case of ψ_h-coordinates. For the definition of the ψ_h-coordinates and the meaning of the parameter h, please refer to Section 2, where it will appear as the exponent of the hyperbolic cosine function.

The main result of the paper is the following.

Theorem 2. *Suppose S is a compact surface with nonempty boundary and negative Euler characteristic. For each $h \geq 0$, there is a homeomorphism*

$$\Pi_h : \mathrm{Teich}(S) \to |A(S) - A_\infty(S)| \times \mathbb{R}_{>0}$$

equivariant under the action of the mapping class group so that the restriction of Π_h on each simplex of maximal dimension is given by the ψ_h-coordinate. In particular, this map produces a natural cell decomposition of the moduli space of surfaces with boundary.

We will show that the underlying cell-structures for various h's are the same.

1.3. ***Related results.*** For a punctured surface S with weights on each puncture, the classical Teichmüller space of S admits cell decompositions, invariant under the action of the mapping class groups. This was first proved by Harer [1986] and Thurston (unpublished) using Strebel's work on quadratic differentials and flat cone metrics. The corresponding result in the context of hyperbolic geometry was proved by Bowditch and Epstein [1988] and Penner [1987] using complete hyperbolic metrics of finite area on S so that each cusp has an assigned horocycle. The constructions in [Bowditch and Epstein 1988] and [Penner 1987] are more geometrically oriented. Indeed, in [Bowditch and Epstein 1988] the construction of spines and Delaunay decompositions based on a given set of points and horocycles are used. Our approach is the same as that of [Bowditch and Epstein 1988] using Delaunay decompositions. The existence of such Delaunay decompositions for compact hyperbolic manifolds with geodesic boundary was established in the work of Kojima [1992] for 3-manifolds. However, the same method of proof in [Kojima 1992] also works for compact hyperbolic surfaces. Our main observation in this paper is that those ψ_h-coordinates introduced in [Luo 2006] capture the characterization of Delaunay decomposition well.

1.4. ***Plan of the paper.*** In Section 2, we recall the definition and properties of ψ_h-coordinates which will be used in the proof of Theorem 2. In Section 3, we prove a simple lemma which clarifies the geometric meaning of ψ_h-coordinates. In Section 4, we review the Delaunay decomposition associated to a hyperbolic metric following Bowditch and Epstein [1988] and Kojima [1992]. Theorem 2 is proved in Section 5.

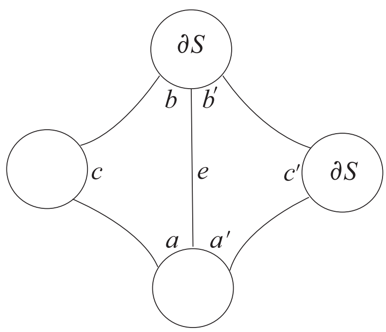
2. ψ_h-coordinates

An ideal triangulated compact surface with boundary (S, T) is obtained by removing a small open regular neighborhood of the vertices of a triangulation of a closed surface. The edges of an ideal triangulation T correspond bijectively to the edges of the triangulation of the closed surface. Given a hyperbolic metric d with geodesic boundary on an ideal triangulated surface (S, T), there is a unique geometric ideal triangulation T^* isotopic to T so that all edges are geodesics orthogonal to the boundary. The edges in T^* decompose the surface into hyperbolic right-angled hexagons.

Let E be the set of edges in T. For any real number h, the ψ_h-coordinate of a hyperbolic metric introduced in [Luo 2006] is defined as $\psi_h : E \to \mathbb{R}$,

$$\psi_h(e) = \int_0^{\frac{a+b-c}{2}} \cosh^h(t)\,dt + \int_0^{\frac{a'+b'-c'}{2}} \cosh^h(t)\,dt,$$

where e is an edge shared by two hyperbolic right-angled hexagons and c, c' are lengths of arcs in the boundary of S facing e and a, a', b, b' are the lengths of arcs in the boundary of S adjacent to e so that a, b, c lie in a hexagon, as shown here:



Now consider the map $\Psi_h : \text{Teich}(S) \to \mathbb{R}^E$ sending a hyperbolic metric d to its ψ_h-coordinate. The following two theorems are proved in [Luo 2006]. They will be used in the proof of the main result Theorem 2.

Theorem 3 [Luo 2006]. *Fix an ideal triangulation of S. For each $h \in \mathbb{R}$, the map $\Psi_h : \text{Teich}(S) \to \mathbb{R}^E$ is a smooth embedding.*

An *edge cycle* $(e_1, H_1, \ldots, e_n, H_n)$ is a collection of hexagons and edges in an ideal triangulation so that two adjacent hexagons H_{i-1} and H_i share the edge e_i for $i = 1, \ldots, n$ where $H_0 = H_n$.

Theorem 4 [Luo 2006]. *Fix an ideal triangulation of S. For each $h \geq 0$, we have*

$$\Psi_h(\text{Teich}(S)) = \Big\{ z \in \mathbb{R}^E \mid \text{for each edge cycle } (e_1, H_1, \ldots, e_n, H_n),\ \sum_{i=1}^{n} z(e_i) > 0 \Big\},$$

and the image $\Psi_h(\text{Teich}(S))$ is a convex polytope.

3. Hyperbolic right-angled hexagons

We will use the following notations and conventions. Given two points P, Q in the hyperbolic plane $\mathbb{H}$, the distance between P and Q will be denoted by

$|PQ|$. If $P \neq Q$, the complete geodesic in $\mathbb{H}$ containing P and Q will be denoted by $\overline{PQ}$. Suppose H is a hyperbolic right-angled hexagon whose vertices are $A_1, B_1, A_2, B_2, A_3, B_3$ labeled cyclically (see figure below). Let C be the circle tangent to the three geodesics $\overline{A_1B_1}$, $\overline{A_2B_2}$ and $\overline{A_3B_3}$. The hyperbolic center of C is denoted by O. Let $X_i = C \cap \overline{A_iB_i}$ be the tangent point for $i = 1, 2, 3$. The geodesic $\overline{B_iA_{i+1}}$ decomposes the hyperbolic plane into two sides. The indices are counted modulo 3, that is, $A_4 = A_1$, etc.

Lemma 5. *For $i = 1, 2, 3$, the value of $|A_iB_i| + |A_{i+1}B_{i+1}| - |A_{i+2}B_{i+2}|$ is* $2|X_iB_i|, -2|X_iB_i|$, *or* 0, *according to whether O lies on the same side of $\overline{B_iA_{i+1}}$ as H, on the opposite side, or on $\overline{B_iA_{i+1}}$ itself.*

Proof. Since X_j is the tangent point for $j = 1, 2, 3$, we have

$$|X_jB_j| = |X_{j+1}A_{j+1}|. \tag{1}$$

According to the location of O with respect to the hexagon, we have three cases.

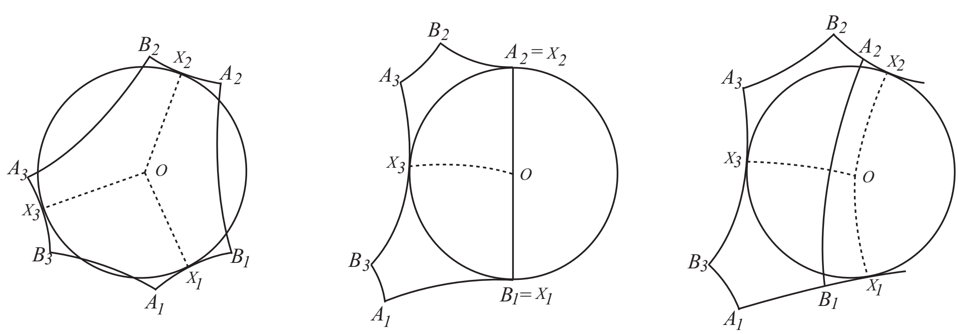
Case 1: O is in the interior of the hexagon (see left diagram below). We have

$$|A_jB_j| = |X_jA_j| + |X_jB_j|,$$

for $j = 1, 2, 3$. Combining with (1), we obtain

$$|A_jB_j| + |A_{j+1}B_{j+1}| - |A_{j+2}B_{j+2}| = 2|X_jB_j|.$$

Thus we have verified the lemma in this case since O and H are in the same side of $\overline{B_jA_{j+1}}$ for each $j = 1, 2, 3$.



Case 2: O is in the boundary of the hexagon. Without loss of generality, we assume $O \in \overline{B_1A_2}$. (See middle diagram above.) We have

$$|A_1B_1| = |X_1A_1|, \quad |A_2B_2| = |X_2B_2|, \quad |A_3B_3| = |X_3A_3| + |X_3B_3|.$$

Combining with (1), we obtain

$$|A_1B_1| + |A_2B_2| - |A_3B_3| = 0,$$
$$|A_2B_2| + |A_3B_3| - |A_1B_1| = 2|X_2B_2|,$$
$$|A_3B_3| + |A_1B_1| - |A_2B_2| = 2|X_3B_3|.$$

Thus we have verified the lemma in this case since $O \in \overline{B_1A_2}$, O and H are in the same side of $\overline{B_2A_3}$ and in the same side of $\overline{B_3A_1}$.

Case 3: O is outside of the hexagon H. Without loss of generality, we may assume O and H are in the same side of $\overline{B_2A_3}$ and in the same side of $\overline{B_3A_1}$, but in different sides of $\overline{B_1A_2}$. (See the rightmost diagram on the previous page). Then

$$|A_1B_1| = |X_1A_1| - |X_1B_1|,$$
$$|A_2B_2| = |X_2B_2| - |X_2A_2|,$$
$$|A_3B_3| = |X_3A_3| + |X_3B_3|.$$

Combining with (1), we obtain

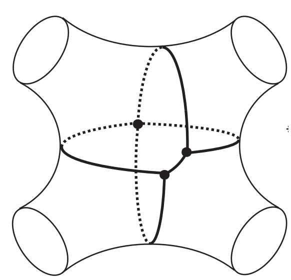
$$|A_1B_1| + |A_2B_2| - |A_3B_3| = -2|X_1B_1|,$$
$$|A_2B_2| + |A_3B_3| - |A_1B_1| = 2|X_2B_2|,$$
$$|A_3B_3| + |A_1B_1| - |A_2B_2| = 2|X_3B_3|.$$

Thus we have verified the lemma in this case. □

4. Delaunay decompositions

Let's recall the construction of the Delaunay decomposition associated to a hyperbolic metric following Bowditch and Epstein [1988]. For higher-dimensional hyperbolic manifolds, see [Epstein and Penner 1988] and [Kojima 1992].

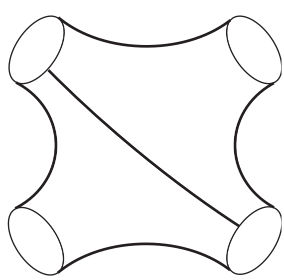
Let S be a surface with boundary equipped with a hyperbolic metric d such that the boundary is totally geodesic. The Delaunay decomposition of (S, d) produces a graph Σ_d^*, called the *spine* of the surface S, consisting of all the points in S which have two or more distinct shortest geodesics to ∂S. For example, the figure shows the spine (thick lines) of a four-holed sphere:



To be more precise, let $n(p)$ be the number of shortest geodesics arcs from p to ∂S. The spine Σ_d^* of (S, d) is the set $\{p \in S \mid n(p) \geq 2\}$. The vertex set of Σ_d^* is the set $\{p \in S \mid n(p) \geq 3\}$. The set Σ_d^* is shown to be a graph whose edges are geodesic arcs in S [Bowditch and Epstein 1988; Kojima 1992]. The edges of Σ_d^* are denoted by $e_1^*, \ldots, e_N^*$. By the construction, each point in the interior of an edge e_i^*, $i = 1, \ldots, N$, has precisely two distinct shortest geodesics to ∂S. Each edge e_i^* connects the two vertices which are the points having three or more distinct shortest geodesics to ∂S. From [Kojima 1992] or [Bowditch and Epstein 1988], it is known that Σ_d^* is a strong deformation retract of the surface S.

Associated with the spine Σ_d^* is the so-called *Delaunay decomposition* of the hyperbolic surface. Here is the construction (see figure below).

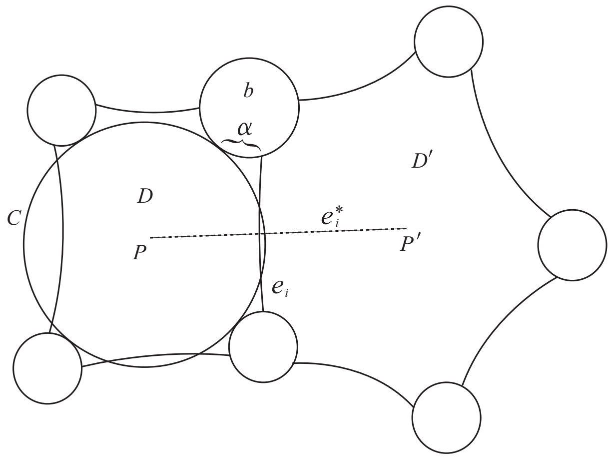
For each edge e^* of the spine, there are two boundary components B_1 and B_2 (which may coincide) of the surface so that points in the interior of e^* have exactly two shortest geodesic arcs a_1 and a_2 to B_1 and B_2. Let e be the shortest geodesic from B_1 to B_2. It is known that e is homotopic to $a_1 \cup a_2$. Furthermore, these edges are pairwise disjoint. The collection of all such e's decompose the surface S into a collection of right-angled polygons. These are the 2-cells, or the Delaunay domains. We use Σ_d to denote the cell decomposition of the surface S whose 2-cells are the Delaunay domains, whose 1-cells consist of these e's and the arcs in the boundary of S and whose "missing" 0-cells correspond to the boundary components. One can think of Σ_d^* as a combinatorial dual to Σ_d as follows. For each 2-cell D in Σ_d, by the construction, there is exactly one vertex v of Σ_d^* so that v is of equal distance to all edges of $D \cap \partial S$. We say that the vertex v is dual to the 2-cell D. Note that the vertex v can be in the interior of D, in one of the edges of D or out of D. Consider the hyperbolic circle in S centered at v so that it is tangent to all edges in $D \cap \partial S$. We call this the *inscribed circle* of the Delaunay domain D.



5. Proof of the main theorem

5.1. ***Construction of the homeomorphism.*** To prove Theorem 2, for each $h \geq 0$, we construct the map $\Pi_h : \mathrm{Teich}(S) \to |A(S) - A_\infty(S)| \times \mathbb{R}_{>0}$ as follows. Given a hyperbolic metric d with geodesic boundary, we obtain the spine Σ_d^* and the

Delaunay decomposition Σ_d of S in the metric d. Let $(e_1^*, \dots, e_N^*)$ be the edges of the spine and $(e_1, \dots, e_N)$ the edges of the Delaunay decomposition, where e_i is dual to e_i^*. Suppose e_i is shared by two 2-cells D and D', as in the figure:



The inscribed circle of D is denoted by C. Let b be one of the boundary components of S incident to the edge e_i.

Definition 6. We define a real number $\alpha(e_i, D)$ associated to e_i and the cell D as follows. We use α instead of $\alpha(e_i, D)$ if there is no confusion from the context.

- If the tangent point $C \cap b$ and the cell D are in the same side of e_i, then α is defined to be the length of the arc contained in b with end points $C \cap b$ and $e_i \cap b$.
- If the tangent point $C \cap b$ is in e_i, then α is defined to be 0.
- If the tangent point $C \cap b$ and the cell D are in different sides of e_i, then α is defined to be the negative of the length of the arc contained in b with end points $C \cap b$ and $e_i \cap b$.

According to the above definition, Lemma 5 says that for a hyperbolic right-angled hexagon, the number α can be calculated in a unified way via a linear combination of the edge lengths of the hexagon.

Similarly, we find the inscribed circle C' of D' and define the corresponding real number α' associated to e_i and the cell D'.

Now we define a function for each $h \geq 0$:

$$\pi_h(e_i) = \int_0^{\alpha} \cosh^h(t)\,dt + \int_0^{\alpha'} \cosh^h(t)\,dt. \tag{2}$$

Lemma 7. $\pi_h(e_i) > 0$ *for any* $h \geq 0$ *and* $i = 1, \dots, N$.

Proof. By the definition of α and α', the sum $\alpha+\alpha'$ is the length of the arc contained in b with end points $C\cap b$ and $C'\cap b$. Hence $\alpha+\alpha'>0$. The result is independent of the location of the edge e_i.

We show that $\pi_h(e_i)>0$ as follows.

Case 1: $\alpha \geq 0$ ***and*** $\alpha' \geq 0$. Since $\alpha+\alpha'>0$, one of the numbers α and α' is positive. Then $\pi_h(e_i)>0$.

Case 2: $\alpha<0$. Then, since $\alpha'>-\alpha$,

$$\begin{aligned}\pi_h(e_i) &= \int_0^{\alpha}\cosh^h(t)\,dt+\int_0^{\alpha'}\cosh^h(t)\,dt\\ &= -\int_0^{-\alpha}\cosh^h(t)\,dt+\int_0^{\alpha'}\cosh^h(t)\,dt>0, \qquad \square\end{aligned}$$

It is clear from the definition that the Delaunay decomposition and the coordinates $\pi_h(e_i)$ depend only on the isotopy class of the hyperbolic metric. In other words, they are independent of the choice of a representative of a point of the Teichmüller spaces $\mathrm{Teich}(S)$. For what follows, let $[d]$ denote a point of $\mathrm{Teich}(S)$. We obtain a well-defined map

$$\begin{aligned}\Pi_h : \mathrm{Teich}(S) &\to |A(S)-A_\infty(S)|\times\mathbb{R}_{>0}\\ [d] &\mapsto \Big(\sum_{i=1}^{N}\frac{\pi_h(e_i)}{\sum_{i=1}^{N}\pi_h(e_i)}\cdot[e_i],\ \sum_{i=1}^{N}\pi_h(e_i)\Big),\end{aligned}$$

where $(e_1,\dots,e_N)$ are the edges of the Delaunay decomposition of (S,d) and $[e_i]$ is the isotopy class of e_i. Note that

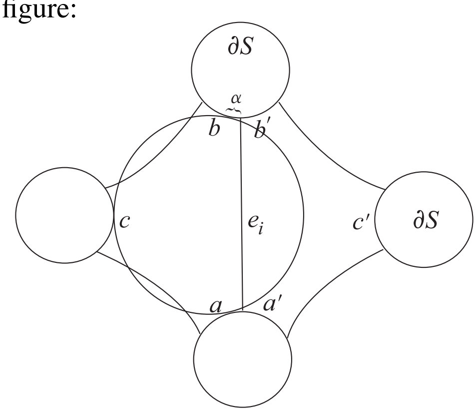
$$\sum_{i=1}^{N}\frac{\pi_h(e_i)}{\sum_{i=1}^{N}\pi_h(e_i)}\cdot[e_i]$$

is a point in the fillable simplex with vertices $[e_1],\dots,[e_N]$ of the arc complex, since the sum of the coefficient of the vertices is 1 and $\pi_h(e_i)>0$ for all i.

In the rest of the section we will show that Π_h is injective, onto, and is a homeomorphism. This proves the main result Theorem 2, that is, it produces a cell decomposition of the Teichmüller space.

5.2. ***Injectivity.*** We claim that the map Π_h is one-to-one. Suppose there are two hyperbolic metrics d, d' such that $\Pi_h([d])=\Pi_h([d'])$. Then their associated Delaunay decompositions are the same by definition. Say $\{e_1,\dots,e_N\}$ is the set of edges in $\Sigma_d=\Sigma_{d'}$. If $N=6g-6+3n$ where g is the genus and n is the number of boundary components of S, then $(e_1,\dots,e_N)$ is an ideal triangulation. In this case each 2-cell is a right-angled hexagon. Suppose edge e_i is shared by hexagons

D and D', as in the figure:



Let c be the length of the boundary arc opposite to e_i and a, b be lengths of boundary arcs adjacent to e_i in D. By comparing Lemma 5 with Definition 6, we have $a+b-c=2\alpha$. Similarly, for hexagon D', we have $a'+b'-c'=2\alpha'$. Thus

$$
\begin{aligned}
\pi_h(e_i) &= \int_0^{\alpha} \cosh^h(t)\,dt + \int_0^{\alpha'} \cosh^h(t)\,dt \\
&= \int_0^{\frac{a+b-c}{2}} \cosh^h(t)\,dt + \int_0^{\frac{a'+b'-c'}{2}} \cosh^h(t)\,dt \\
&= \psi_h(e_i),
\end{aligned} \tag{3}
$$

where $\psi_h(e_i)$ is exactly the ψ_h-coordinate of a hyperbolic metric evaluated at e_i. Thus from $\Pi_h([d]) = \Pi_h([d'])$ we obtain $\Psi_h([d]) = \Psi_h([d'])$ for the ideal triangulation $(e_1, \ldots, e_N)$, $N = 6g-6+3n$. By Theorem 3,

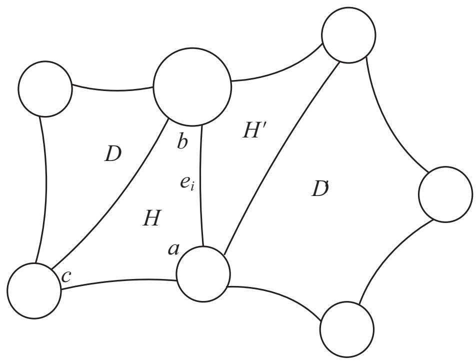
$$[d] = [d'] \in \mathrm{Teich}(S).$$

If $N < 6g-6+3n$, we add edges $e_{N+1}, \ldots, e_{6g-6+3n}$ such that

$$(e_1, \ldots, e_N, e_{N+1}, \ldots, e_{6g-6+3n})$$

is an ideal triangulation. More precisely, in a 2-cell of the Delaunay decomposition which is not a hexagon, we arbitrarily add geodesic arcs perpendicular to boundary components bounding the 2-cell, so as to decompose the 2-cell into a union of hexagons.

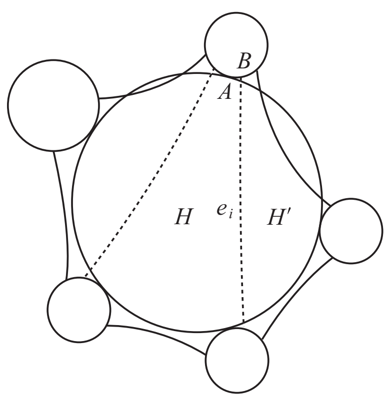
We now consider two cases. First suppose edge e_i, $i \le N$, is shared by two 2-cells D, D', as in the figure at the top of the next page. There is a hyperbolic right-angled hexagon H contained in D having e_i as an edge. Note that H is a component of $S - \bigcup_{i=1}^{6g-6+3n} e_i$. Recall that the inscribed circle C of D is also the inscribed circle of H. Let c be the length of the boundary arc opposite to e_i and a, b be lengths of boundary arcs adjacent to e_i in H. By comparing Lemma 5 with



Definition 6, we have $a + b - c = 2\alpha$, where α is the length in the definition of $\pi_h(e_i)$. From the 2-cell D', we obtain the hexagon H' and $a'+b'-c' = 2\alpha'$. Thus,

$$\pi_h(e_i) = \int_0^{\alpha} \cosh^h(t)\,dt + \int_0^{\alpha'} \cosh^h(t)\,dt \tag{4}$$
$$= \int_0^{\frac{a+b-c}{2}} \cosh^h(t)\,dt + \int_0^{\frac{a'+b'-c'}{2}} \cosh^h(t)\,dt = \psi_h(e_i).$$

Now suppose instead that the edge e_i, $i > N$, is shared by two hexagons H, H' in the ideal triangulation $(e_1, \ldots, e_N, e_{N+1}, \ldots, e_{6g-6+3n})$, where H and H' are obtained from the same 2-cell. Therefore H and H' have the same inscribed circle C which is also the inscribed circle of the 2-cell containing H, H', as shown here:



In hexagon H, let c be the length of boundary arc opposite to e_i and a, b be lengths of boundary arcs adjacent to e_i. In hexagon H', we define a', b', c'. There are two possibilities to consider. If the center of C is in e_i, then by Lemma 5, $a + b - c = 0 = a' + b' - c'$. If the center of C is not in e_i, without loss of generality, we assume the center and H are in the same side of e_i. Denote by A the tangent point of C at a boundary component. Denote by B the intersection point of e_i with the same boundary component. By Lemma 5, we have $a+b-c = 2|AB|$

and $a' + b' - c' = -2|AB|$. The two possibilities give the same conclusion: for $i > N$,

$$\psi_h(e_i) = \int_0^x \cosh^h(t)\,dt + \int_0^{-x} \cosh^h(t)\,dt = 0.$$

Thus from $\Pi_h([d]) = \Pi_h([d'])$ we obtain $\Psi_h([d]) = \Psi_h([d'])$ for the ideal triangulation $(e_1, \ldots, e_N, e_{N+1}, \ldots, e_{6g-6+3n})$. In fact, the i-th entry of

$$\Psi_h([d]) = \Psi_h([d'])$$

is zero as $N+1 \le i \le 6g-6+3n$. By Theorem 3, we see that $[d] = [d'] \in \mathrm{Teich}(S)$.

5.3. ***Surjectivity.*** We claim the map

$$\Pi_h : \mathrm{Teich}(S) \to |A(S) - A_\infty(S)| \times \mathbb{R}_{>0}$$

is onto. Fix a point

$$\Bigl(\sum_{i=1}^{N} z_i \cdot [e_i], x\Bigr) \in |A(S) - A_\infty(S)| \times \mathbb{R}_{>0}$$

with $z_i \neq 0$ for all i. If $N = 6g-6+3n$, then $(e_1, \ldots, e_N)$ is an ideal triangulation of S. The vector $(xz_1, \ldots, xz_N)$ satisfies the condition in Theorem 4 since each entry is positive; by the theorem, there is a hyperbolic metric d whose ψ_h-coordinate is $(xz_1, \ldots, xz_N)$, that is, $\psi_h(e_i) = xz_i$. We have shown in the last subsection that $\pi_h(e_i) = \psi_h(e_i)$ in this case. Therefore

$$\Pi_h([d]) = \Bigl(\sum_{i=1}^{N} z_i \cdot [e_i], x\Bigr).$$

If $N < 6g - 6 + 3n$, then $e_1, \ldots, e_N$ is a cell decomposition of S. Let T be an ideal triangulation $(e_1, \ldots, e_N, e_{N+1}, \ldots, e_{6g-6+3n})$ obtained from the cell decomposition. Then the vector $(xz_1, \ldots, xz_N, 0, \ldots, 0)$ (there are $6g-6+3n-N$ zeros) satisfies the condition in Theorem 4 since there does not exists an edge cycle consisting of only the "new" edges e_i, $i > N$; by the theorem, there is a hyperbolic metric d whose ψ_h-coordinate is $(xz_1, \ldots, xz_N, 0, \ldots, 0)$, that is, $\psi_h(e_i) = xz_i$, $i \le N$ and $\psi_h(e_i) = 0$, $i > N$.

Suppose edge e_i, $i > N$ is shared by two hexagons H, H'. By the discussion of the last subsection, from $\psi_h(e_i) = 0$ we conclude that the inscribed circles of H and H' have the same tangent points at the two boundary components intersecting e_i. Therefore the two circles have the same center. Thus they coincide. If a 2-cell is decomposed into several hexagons, then the inscribed circles of all the hexagons coincide. This shows that the 2-cell has an inscribed circle. Thus the cell decomposition $(e_1, \ldots, e_N)$ is the Delaunay decomposition of (S, h).

For edge e_i, $i \le N$, from the discussion of the last subsection, we see that $\pi_h(e_i) = \psi_h(e_i)$. Therefore,

$$\Pi_h([d]) = \left(\sum_{i=1}^{N} z_i \cdot [e_i], x \right).$$

5.4. ***Continuity of Π_h.*** To prove continuity, we follow the idea in [Bowditch and Epstein 1988, Sections 8 and 9].

Let $\{d^s\}_{s=1}^{\infty}$ be a sequence of hyperbolic metrics on S with geodesic boundary converging to a hyperbolic metric d with geodesic boundary. We claim that the sequence of points $\{\Pi_h([d^s])\}_{s=1}^{\infty}$ converges to the point $\Pi_h([d])$.

Case 1: For s sufficiently large, the Delaunay decomposition associated to d has the same combinatorial type as the Delaunay decomposition associated to d^s. Assume that the Delaunay decomposition associated to d has the edges $e_1, \dots, e_N$ with $N \le 6g - 6 + 3n$ and the Delaunay decomposition associated to d^s has the edges $e_1^s, \dots, e_N^s$ so that e_i^s is isotopic to e_i for $1 \le i \le N$. Since the metrics $\{d^s\}$ converge to the metric d, the geodesic length of edges $\{e_i^s\}$ converge to the geodesic length of the edge e_i.

Assume that the edge e_i is shared by two 2-cells D and D' of (S, d). Correspondingly, the edge e_i^s is shared by two 2-cells D^s and D'^s of (S, d^s). As in the construction at the beginning of this section (see also figure on page 430), let C be the inscribed circle of D and b one of the two edges of D adjacent to e_i. Let α be the length of the arc contained in b with end points $C \cap b$ and $e_i \cap b$. Let α^s be the length of the corresponding arc in D^s. Assume $e_{D1}, \dots, e_{Dt} \in \{e_1, \dots, e_N\}$ are the edges of D in the interior of S. By elementary hyperbolic geometry, the radius of C is a continuous function of the lengths of $e_{D1}, \dots, e_{Dt}$. Therefore α is a continuous function of the lengths of $e_{D1}, \dots, e_{Dt}$. Thus the sequence $\{\alpha^s\}$ converges to α. By the same argument, for the 2-cell D', we have the length α' and α'^s so that the sequence $\{\alpha'^s\}$ converges to α'. By the definition (2), the sequence $\{\pi_h(e_i^s)\}$ converges to $\pi_h(e_i)$. By the definition 5.1, the sequence of points $\{\Pi_h([d^s])\}_{s=1}^{\infty}$ converges to the point $\Pi_h([d])$. Geometrically, this is a sequence of interior points in a simplex of the arc complex converging to an interior point in the same simplex.

Case 2: For s sufficiently large, the Delaunay decompositions associated to d^s have the same combinatorial type with each other but different from that associated to d. Assume that the Delaunay decomposition associated to d has edges $e_1, \dots, e_N$ with $N < 6g - 6 + 3n$ and the Delaunay decomposition associated to d^s has edges $e_1^s, \dots, e_N^s, e_{N+1}^s, \dots, e_{N+M}^s$ with $N + M \le 6g - 6 + 3n$ so that e_i^s is isotopic to e_i for $1 \le i \le N$.

Since e_j^s is isotopic to $e_j^{s'}$ for $N + 1 \le j \le N + M$ and s, s' sufficiently large, we can add an edge e_j on (S, d) which is isotopic to e_j^s for $N + 1 \le j \le N + M$. Now the edges $e_1, \dots, e_N, e_{N+1}, \dots, e_{N+M}$ produce a cell decomposition of S which has the same combinatorial type as the cell decomposition obtained from the edges $e^s_1, \dots, e^s_N, e^s_{N+1}, \dots, e^s_{N+M}$.

We get the same situation as in Case 1. The convergence of metrics implies the convergence of the edge lengths which implies the convergence of the π_h-coordinates. In Case 2, since the edges $e_{N+1}, \dots, e_{N+M}$ are added to a Delaunay decomposition, we know from Section 5.2 that $\pi_h(e_j) = 0$ as $N+1 \le j \le N+M$. Geometrically, this is a sequence of interior points in a simplex of the arc complex converging to a point on the boundary of the simplex.

Case 3: For** s **sufficiently large, the Delaunay decompositions associated to** d^s **oscillate among several combinatorial types, and do not stabilize. These oscillating Delaunay decompositions must all be subdivisions of the Delaunay decomposition associated to d. Hence there are only finitely many combinatorial types. Therefore the sequence $\{d^s\}_{s=1}^{\infty}$ is decomposed into finitely many subsequences $\{d^{s_1}\}, \{d^{s_2}\}, \dots, \{d^{s_k}\}$ so that for each $i = 1, \dots, k$ and for s_i sufficiently large, the Delaunay decompositions associated to d^{s_i} have the same combinatorial type as each other but different from that associated to d. Hence for each subsequence d^{s_i}, $i = 1, \dots, k$, we have the situation of Case 2. Therefore the subsequence of points $\{\Pi_h([d^{s_i}])\}$ converges to the point $\Pi_h([d])$. Since there are only finitely many subsequences and each of them converges to the same point, the whole sequence of points $\{\Pi_h([d^s])\}$ converges to the same point.

5.5. *Continuity of* Π_h^{-1}. Let $\{p^s\}_{s=1}^{\infty}$ be a sequence of points in

$$|A(S) - A_\infty(S)| \times \mathbb{R}_{>0}$$

converging to a point p. We claim that the sequence of hyperbolic metrics

$$\{\Pi_h^{-1}(p^s)\}$$

converges to the hyperbolic metric $\Pi_h^{-1}(p)$.

Case 1: For** s **sufficiently large,** $\{p^s\}$ **and** p **are in the same simplex. Then the Delaunay decomposition associated to $\Pi_h^{-1}(p^s)$ and $\Pi_h^{-1}(p)$ have the same combinatorial type. If it is needed, by adding edges in the 2-cells which are not hexagons, we obtain a fixed topological ideal triangulation of the surface S. Note that $\pi_h(e) = 0$ if e is an edge being added. For an edge e_i on $(S, \Pi_h^{-1}(p))$, denote by e^s_i the corresponding edge on $(S, \Pi_h^{-1}(p^s))$. Now we have a fixed ideal triangulation and the sequence of coordinates $\{\pi_h(e^s_i)\}$ converges to the coordinate $\pi_h(e_i)$ for each edge e_i. By (3) and (4), $\pi_h(e^s_i) = \psi_h(e^s_i)$ and $\pi_h(e_i) = \psi_h(e_i)$. Therefore the sequence of coordinates $\{\psi_h(e^s_i)\}$ converges to the coordinate $\psi_h(e_i)$ for each edge e_i. By Theorem 3, the sequence of hyperbolic metrics $\{\Pi_h^{-1}(p^s)\}$ converges to the hyperbolic metric $\Pi_h^{-1}(p)$.

Case 2: For s sufficiently large, $\{p^s\}$ are in the interior of a simplex and p is on the boundary of the simplex. Assume that the Delaunay decomposition associated to $\Pi_h^{-1}(p)$ has the edges $e_1, \ldots, e_N$ with $N < 6g - 6 + 3n$ and the Delaunay decomposition associated to $\Pi_h^{-1}(p^s)$ has the edges $e_1^s, \ldots, e_N^s, e_{N+1}^s, \ldots, e_{N+M}^s$ with $N + M \leq 6g - 6 + 3n$ so that e_i^s is isotopic to e_i for $1 \leq i \leq N$. We can add an edge e_j on $(S, \Pi_h^{-1}(p))$ which is isotopic to the edge e_j^s for $N + 1 \leq j \leq N + M$. Since $\{p^s\}$ converges to p, we have that $\{\pi_h(e_i^s)\}$ converges to $\pi_h(e_i)$ for $1 \leq i \leq N$ and $\{\pi_h(e_j^s)\}$ converges to 0 for $N + 1 \leq j \leq N + M$. Since e_j is added to the Delaunay decomposition of $\Pi_h^{-1}(p)$, $\pi_h(e_j) = 0$ as $N + 1 \leq j \leq N + M$. We get the situation of Case 1. We may add more edges to obtain a fixed ideal triangulation. The same arguments of Case 1 can be used to establish the claim.

Case 3: For s sufficiently large, $\{p^s\}$ oscillate among several simplices and p is in the intersection of these simplices. Note that for a fixed point p, there are only finitely many simplices containing p. We can decompose the sequence $\{p^s\}$ into finitely many subsequences $\{p^{s_1}\}, \{p^{s_2}\}, \ldots, \{p^{s_k}\}$ so that for each

$$i = 1, \ldots, k$$

and for s_i sufficiently large, $\{p^{s_i}\}$ are in the interior of the same simplex and p is on the boundary of the simplex. Hence for each subsequence $\{p^{s_i}\}$, $i = 1, \ldots, k$, we have the situation of Case 2. Therefore the subsequence of hyperbolic metrics $\{\Pi_h^{-1}(p^{s_i})\}$ converges to the hyperbolic metric $\Pi_h^{-1}(p)$. Since there are only finitely many subsequences and each of them converges to the same point, the whole sequence $\{\Pi_h^{-1}(p^s)\}$ converges to the same point.

To sum up, we have proved Theorem 2:

$$\Pi_h : \mathrm{Teich}(S) \to |A(S) - A_\infty(S)| \times \mathbb{R}_{>0}$$

is a homeomorphism.

Acknowledgments

The authors would like to thank the referees for careful reading of the manuscript, for pointing out some errors in the early draft and for valuable suggestions on correction and improving the exposition.

References

[Andersen et al. 2009] J. E. Andersen, A. J. Bene, and R. C. Penner, "Groupoid extensions of mapping class representations for bordered surfaces", *Topology Appl.* **156**:17 (2009), 2713–2725. MR 2011a:20094 Zbl 1196.57013

[Bonahon 1996] F. Bonahon, "Shearing hyperbolic surfaces, bending pleated surfaces and Thurston's symplectic form", *Ann. Fac. Sci. Toulouse Math.* (6) **5**:2 (1996), 233–297. MR 97i:57011 Zbl 0880.57005

[Bowditch and Epstein 1988] B. H. Bowditch and D. B. A. Epstein, "Natural triangulations associated to a surface", *Topology* **27**:1 (1988), 91–117. MR 89e:57004 Zbl 0649.32017

[Epstein and Penner 1988] D. B. A. Epstein and R. C. Penner, "Euclidean decompositions of noncompact hyperbolic manifolds", *J. Differential Geom.* **27**:1 (1988), 67–80. MR 89a:57020 Zbl 0611.53036

[Harer 1986] J. L. Harer, "The virtual cohomological dimension of the mapping class group of an orientable surface", *Invent. Math.* **84**:1 (1986), 157–176. MR 87c:32030 Zbl 0592.57009

[Hazel 2004] G. P. Hazel, *Triangulating Teichmuller space using the Ricci flow*, ProQuest LLC, Ann Arbor, MI, 2004. MR 2706365

[Kojima 1992] S. Kojima, "Polyhedral decomposition of hyperbolic 3-manifolds with totally geodesic boundary", pp. 93–112 in *Aspects of low-dimensional manifolds*, edited by Y. Matsumoto, Adv. Stud. Pure Math. **20**, Kinokuniya, Tokyo, 1992. MR 94c:57023 Zbl 0802.57006

[Luo 2006] F. Luo, "Rigidity of polyhedral surfaces", preprint, 2006. To appear in *J. Differential Geom.* arXiv math.GT/0612714

[Luo 2007] F. Luo, "On Teichmüller spaces of surfaces with boundary", *Duke Math. J.* **139**:3 (2007), 463–482. MR 2008g:32024 Zbl 1138.32006

[Mondello 2009] G. Mondello, "Triangulated Riemann surfaces with boundary and the Weil–Petersson Poisson structure", *J. Differential Geom.* **81**:2 (2009), 391–436. MR 2010a:32027 Zbl 1165.53026

[Penner 1987] R. C. Penner, "The decorated Teichmüller space of punctured surfaces", *Comm. Math. Phys.* **113**:2 (1987), 299–339. MR 89h:32044 Zbl 0642.32012

[Penner 2004] R. C. Penner, "Decorated Teichmüller theory of bordered surfaces", *Comm. Anal. Geom.* **12**:4 (2004), 793–820. MR 2006a:32018 Zbl 1072.32008

[Thurston 1976–1979] W. P. Thurston, "The topology and geometry of three-manifolds", Lecture notes, Princeton University, 1976–1979.

[Ushijima 1999] A. Ushijima, "A canonical cellular decomposition of the Teichmüller space of compact surfaces with boundary", *Comm. Math. Phys.* **201**:2 (1999), 305–326. MR 2000a:32029

Received December 29, 2010. Revised August 2, 2011.

REN GUO
SCHOOL OF MATHEMATICS
UNIVERSITY OF MINNESOTA
206 CHURCH ST. SE
MINNEAPOLIS, MINNESOTA 55455
UNITED STATES
guoxx170@math.umn.edu

FENG LUO
DEPARTMENT OF MATHEMATICS
RUTGERS UNIVERSITY
110 FRELINGHUYSEN RD.
PISCATAWAY NEW JERSEY 08854
UNITED STATES
fluo@math.rutgers.edu

PACIFIC JOURNAL OF MATHEMATICS

http://pacificmath.org

Founded in 1951 by
E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

See inside back cover or pacificmath.org for submission instructions.

The subscription price for 2011 is US $420/year for the electronic version, and $485/year for print and electronic.
Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. Prior back issues are obtainable from Periodicals Service Company, 11 Main Street, Germantown, NY 12526-5635. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and the Science Citation Index.

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

PACIFIC JOURNAL OF MATHEMATICS

Volume 253 No. 2 October 2011

0030-8730(201110)253:2;1-D