A SYSTEM OF THIRD-ORDER DIFFERENTIAL OPERATORS
CONFORMALLY INARIANT UNDER $\mathfrak{sl}(3, \mathbb{C})$ AND $\mathfrak{so}(8, \mathbb{C})$

TOSHIHISA KUBO
A SYSTEM OF THIRD-ORDER DIFFERENTIAL OPERATORS
CONFORMALLY IN Variant UNDER $\mathfrak{sl}(3, \mathbb{C})$ AND $\mathfrak{so}(8, \mathbb{C})$

TOSHIHISA KUBO

Barchini, Kable, and Zierau constructed a number of conformally invariant systems of differential operators associated to Heisenberg parabolic subalgebras in simple Lie algebras. The construction was systematic, but the existence of such a system was left open in two cases, namely, the $\Omega_3$ system for type $A_2$ and type $D_4$. Here, such a system is shown to exist for both cases. The construction of the system may also be interpreted as giving an explicit homomorphism between generalized Verma modules.

1. Introduction

Conformally invariant systems of differential operators on a smooth manifold $M$ on which a Lie algebra $\mathfrak{g}$ acts by first order differential operators were studied by Barchini, Kable, and Zierau in [BKZ08] and [BKZ09]. To recall the definition of the conformally invariant systems from [BKZ09], let $\mathfrak{g}_0$ be a real Lie algebra. A smooth manifold $M$ is a $\mathfrak{g}_0$-manifold if there exists a $\mathfrak{g}_0$-homomorphism $\Pi_M : \mathfrak{g}_0 \to C^\infty(M) \oplus \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the space of smooth vector fields on $M$. Given a $\mathfrak{g}_0$-manifold $M$, write $\Pi_M(X) = \Pi_0(X) + \Pi_1(X)$ with $\Pi_0(X) \in C^\infty(M)$ and $\Pi_1(X) \in \mathfrak{X}(M)$. Let $\mathbb{D}(\mathcal{V})$ denote the space of differential operators on a vector bundle $\mathcal{V} \to M$. A vector bundle $\mathcal{V} \to M$ is a $\mathfrak{g}_0$-bundle if there exists a $\mathfrak{g}_0$-homomorphism $\Pi_{\mathcal{V}} : \mathfrak{g}_0 \to \mathbb{D}(\mathcal{V})$ so that in $\mathbb{D}(\mathcal{V})[\Pi_{\mathcal{V}}(X), f] = \Pi_1(X) \cdot f$ for all $X \in \mathfrak{g}_0$ and all $f \in C^\infty(M)$, where the dot $\cdot$ denotes the action of the differential operator $\Pi_1(X)$. We regard any smooth functions $f$ on $M$ as elements in $\mathbb{D}(\mathcal{V})$ by identifying them with the multiplication operator they induce. Then, given a $\mathfrak{g}_0$-bundle $\mathcal{V} \to M$, a list of differential operators $D_1, \ldots, D_m \in \mathbb{D}(\mathcal{V})$ is said to be a conformally invariant system on $\mathcal{V}$ with respect to $\Pi_{\mathcal{V}}$ if the following two conditions are satisfied:

MSC2010: primary 22E46; secondary 17B10, 22E47.
Keywords: intertwining differential operator, generalized Verma module, real flag manifold.
(S1) The list $D_1, \ldots, D_m$ is linearly independent at each point of $M$.

(S2) For each $Y \in \mathfrak{g}_0$ there is an $m \times m$ matrix $C(Y)$ of smooth functions on $M$ so that in $\mathbb{D}(V)$,

$$[\Pi_Y(Y), D_j] = \sum_{i=1}^{m} C_{ij}(Y) D_i.$$ 

By extending the $\mathfrak{g}_0$-homomorphisms $\Pi_M$ and $\Pi_Y$ $\mathbb{C}$-linearly, the definitions of a $\mathfrak{g}_0$-manifold, a $\mathfrak{g}_0$-bundle, and a conformally invariant system can be applied equally well to the complexified Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{R} \mathbb{C}$.

A general theory of conformally invariant systems is developed in [BKZ09], and examples of such systems of differential operators associated to the Heisenberg parabolic subalgebras of any complex simple Lie algebras are constructed in [BKZ08]. The purpose of this paper is to answer a question left open in [BKZ08] concerning the existence of certain conformally invariant systems of third-order differential operators. This is done by constructing the required systems.

This result may be interpreted as giving an explicit homomorphism between two generalized Verma modules, one of which is nonscalar. See [BKZ09, Section 6] for the general theory. In this paper we describe it explicitly in a less general setting (see the discussion after Lemma 3.4). The problem of constructing and classifying homomorphisms between scalar generalized Verma modules has received a lot of attention. For example, Matumoto [2006] classifies the nonzero $\mathfrak{u}(\mathfrak{g})$-homomorphisms between scalar generalized Verma modules associated to maximal parabolics of non-Hermitian symmetric type. For the Hermitian symmetric cases, Boe [1985] solved the existence problem for scalar generalized Verma modules of maximal parabolic type. However, much less is known about maps between generalized Verma modules that are not necessarily scalar.

To explain our main results, we briefly review the results of [BKZ08]. To begin, let $\mathfrak{g}$ be a complex simple Lie algebra and $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ be the parabolic subalgebra of Heisenberg type, that is, $\mathfrak{n}$ is a two-step nilpotent algebra with one-dimensional center. Denote by $\gamma$ the highest root of $\mathfrak{g}$. For each root $\alpha$ let $\{X_{-\alpha}, H_\alpha, X_\alpha\}$ be a corresponding $\mathfrak{sl}(2)$-triple, normalized as in [BKZ08, Section 2]. Then $\text{ad}(H_\gamma)$ on $\mathfrak{g}$ has eigenvalues $-2, -1, 0, 1, 2$, and the corresponding eigenspace decomposition of $\mathfrak{g}$ is denoted by

$$\mathfrak{g} = \mathfrak{z}(\mathfrak{n}) \oplus \mathfrak{V}^- \oplus \mathfrak{l} \oplus \mathfrak{V}^+ \oplus \mathfrak{z}(\mathfrak{n}).$$

Let $\mathbb{D}[:n:]$ be the Weyl algebra of $\mathfrak{n}$, that is, the algebra of partial differential operators on $\mathfrak{n}$ with polynomial coefficients. Then each system of $k$-th order differential operators constructed in [BKZ08] derives from a $\mathbb{C}$-linear map $\Omega_k : \mathfrak{g}(2-k) \to \mathbb{D}[:n:]$ with $1 \leq k \leq 4$ and $\mathfrak{g}(2-k)$ the $2-k$ eigenspace of $\text{ad}(H_\gamma)$. Let $\Pi_s : \mathfrak{g} \to \mathbb{D}[:n:]$
be the Lie algebra homomorphism constructed in [BKZ08, Section 4]. Here $s$ is a complex parameter, which indexes line bundles $\mathcal{L}_{-s}$ over a real flag manifold $G_0/\mathcal{Q}_0$, where $G_0$ is a real Lie group with Lie algebra $\mathfrak{g}_0$ and $\mathcal{Q}_0$ is a parabolic subgroup of $G_0$ with complexified Lie algebra $\mathcal{Q}$ opposite to $\mathfrak{q}$. We say that the $\Omega_k$ system has special value $s_0$ when the system is conformally invariant under $\Pi_{s_0}$.

In [BKZ08] the special values of $s$ are determined for the $\Omega_k$ systems with $k = 1, 2, 4$ for all complex simple Lie algebras, but only exceptional cases are considered for the $\Omega_3$ system. A table in [BKZ08, Section 8.10] lists the special values of $s$. (Beware that the entries in the columns for the systems $2_\text{big}$ and $2_\text{small}$ should be transposed.) [BKZ09, Theorem 21] then shows that the $\Omega_3$ system does not exist for $A_r$ with $r \geq 3$, $B_r$ with $r \geq 3$, and $D_r$ with $r \geq 5$.

There remain two open cases, namely, the $\Omega_3$ system for type $A_2$ and type $D_4$. The aim of this paper is to show that the $\Omega_3$ system does exist for both cases (see Theorem 4.1 and Theorem 5.6). To achieve the result we use several facts from [BKZ08] and [BKZ09]. By using these facts, we significantly reduce the amount of computation to show the existence of the system.

There are two differences between our conventions and those used in [BKZ08]. One is that we choose the parabolic $Q_0 = L_0N_0$ for the real flag manifold, while the opposite parabolic $\mathcal{Q}_0 = L_0\overline{N}_0$ is chosen in [BKZ08]. Because of this, our special values of $s$ are of the form $s = -s_0$, where $s_0$ are the special values found in [BKZ08, Section 8.10]. The other is that we identify $(V^+)^*$ with $V^-$ by using the Killing form, while $(V^+)^*$ in [BKZ08] is identified with $V^+$ by using the nondegenerate alternating form $\langle \cdot, \cdot \rangle$ on $V^+$ defined by $[X_1, X_2] = \langle X_1, X_2 \rangle X_\gamma$ for $X_1, X_2 \in V^+$. Because of this difference the right action $R$, which will be defined in Section 2, will play the role played by $\Omega_1$ in [BKZ08]. In addition to these notational differences, there are also some methodological differences between [BKZ08] and here. These stem from the fact that we make systematic use of the results of [BKZ09] to streamline the process of proving conformal invariance.

We now outline the remainder of this paper. In Section 2, we review the setting and results of [BKZ09, Section 5], simultaneously specializing them to the situation considered here. In Section 3, we specialize further by taking $\mathfrak{g}$ to be simply laced. We fix a suitable Chevalley basis and define the $\Omega_3'$ system by $\Omega_3' = \tilde{\Omega}_3 + tC_3$ for $t \in \mathbb{C}$. A remark on notation might be helpful here. In [BKZ08], a system $\Omega_3'$ is initially defined. It is then shown to decompose as a sum of a leading term $\tilde{\Omega}_3$ and a correction term $C_3$. These two are recomposed with different coefficients to give $\Omega_3$, which is finally shown to be conformally invariant for exceptional algebras. Thus, the $\Omega_3$ system is defined to exist if there exists $t_0 \in \mathbb{C}$ so that the $\Omega_3^{t_0}$ system is conformally invariant.

In Section 4, we take $\mathfrak{g}$ to be of type $A_2$ and show that the $\Omega_3$ system(s) exists over the line bundle $\mathcal{L}_0$. The Heisenberg parabolic subalgebra coincides with the
Borel subalgebra in this case. Thus $V^-$ decomposes as the direct sum of two one-
dimensional $l$-submodules. This implies that there will be two $\Omega_3$ systems, each of
the operators will be conformally invariant all by itself. The conformal invariance
of these operators is shown in Theorem 4.1.

In Section 5, we take $\mathfrak{g}$ to be of type $D_4$. For type $D_4$, the data on p. 831 and
Theorem 6.1 of [BKZ08] suggest that the complex parameter $t_0$ for the $\Omega_3^t$ system
to be conformally invariant is $t_0 = 0$, so that the correction term $C_3$ is discarded
completely. For this reason, we simply proceed to show that $\hat{\Omega}_3$ is conformally
invariant. This is done in Theorem 5.6.

2. A specialization of the theory

The purpose of this section is to introduce the $\mathfrak{g}$-manifold and the $\mathfrak{g}$-bundle studied
in this paper. Let $G_0$ be a connected real semisimple Lie group with Lie algebra $\mathfrak{g}_0$
and complexified Lie algebra $\mathfrak{g}$. Let $Q_0$ be a parabolic subgroup of $G_0$ and $Q_0 =
L_0 N_0$ a Levi decomposition of $Q_0$. By the Bruhat decomposition, the subset $\overline{N}_0 Q_0$
of $G_0$ is open and dense in $G_0$, where $\overline{N}_0$ is the nilpotent subgroup of $G_0$ opposite
to $N_0$. Let $\bar{n}$ and $q$ be the complexifications of the Lie algebras of $\overline{N}_0$ and $Q_0$,
respectively; we have the direct sum $\mathfrak{g} = \bar{n} \oplus q$. For $Y \in \mathfrak{g}$, write $Y = Y_{\bar{n}} + Y_q$ for the
decomposition of $Y$ in this direct sum. Similarly, write the Bruhat decomposition
of $g \in \overline{N}_0 Q_0$ as $g = \bar{n}(g) q(g)$ with $\bar{n}(g) \in \overline{N}_0$ and $q(g) \in Q_0$. For $Y \in \mathfrak{g}_0$, we have

$$Y_{\bar{n}} = \frac{d}{dt} \bar{n}(\exp(tY)) \bigg|_{t=0},$$

and a similar equality holds for $Y_q$.

Consider the homogeneous space $G_0/Q_0$. Let $\mathbb{C}_{\chi^{-s}}$ be the one-dimensional
representation of $L_0$ with character $\chi^{-s}$ with $s \in \mathbb{C}$, where $\chi$ is a real-valued
character of $L_0$. The representation $\chi^{-s}$ is extended to a representation of $Q_0$
by making it trivial on $N_0$. For any manifold $M$, denote by $C^\infty(M, \mathbb{C}_{\chi^{-s}})$ the smooth
functions from $M$ to $\mathbb{C}_{\chi^{-s}}$. The group $G_0$ acts on the space

$$C^\infty_\chi(G_0/Q_0, \mathbb{C}_{\chi^{-s}})$$

$$= \{ F \in C^\infty(G_0, \mathbb{C}_{\chi^{-s}}) \mid F(gq) = \chi^{-s}(q^{-1}) F(g) \text{ for all } q \in Q_0 \text{ and } g \in G_0 \}$$

by left translation, and the action $\Pi_s$ of $\mathfrak{g}$ on $C^\infty_\chi(G_0/Q_0, \mathbb{C}_{\chi^{-s}})$ arising from this
action is given by

$$(\Pi_s(Y) \cdot F)(g) = \frac{d}{dt} F(\exp(-tY)g) \bigg|_{t=0}$$

for $Y \in \mathfrak{g}_0$, where the dot $\cdot$ denotes the action of $\Pi_s(Y)$. This action is extended
$\mathbb{C}$-linearly to $\mathfrak{g}$ and then naturally to the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. We
use the same symbols for the extended actions.
The restriction map $C^\infty_\chi(G_0/Q_0, \mathbb{C}_\chi^{-s}) \to C^\infty_\chi(N_0, \mathbb{C}_\chi^{-s})$ is an injection. Define the action of $\mathcal{U}(g)$ on the image of the restriction map by $\Pi_s(u) \cdot f = (\Pi_s(u) \cdot F)|_{\bar{N}_0}$ for $u \in \mathcal{U}(g)$ and $F \in C^\infty_\chi(G_0/Q_0, \mathbb{C}_\chi^{-s})$ with $f = F|_{\bar{N}_0}$. Define a right action $R$ of $\mathcal{U}(\bar{n})$ on $C^\infty_\chi(N_0, \mathbb{C}_\chi^{-s})$ by

$$(R(X) \cdot f)(\bar{n}) = \frac{d}{dt} f(\bar{n} \exp(tX))|_{t=0}$$

for $X \in \bar{n}_0$ and $f \in C^\infty_\chi(N_0, \mathbb{C}_\chi^{-s})$. A direct computation shows that

$$(\Pi_s(Y) \cdot f)(\bar{n}) = -s \, d\chi((\text{Ad}(\bar{n}^{-1})Y)_q) \, f(\bar{n}) - (R((\text{Ad}(\bar{n}^{-1})Y)\bar{n}) \cdot f)(\bar{n})$$

for $Y \in g$ and $f$ in the image of the restriction map

$C^\infty_\chi(G_0/Q_0, \mathbb{C}_\chi^{-s}) \to C^\infty_\chi(N_0, \mathbb{C}_\chi^{-s})$.

Equation (2.2) implies that the representation $\Pi_s$ extends to a representation of $\mathcal{U}(g)$ on the whole space $C^\infty_\chi(N_0, \mathbb{C}_\chi^{-s})$. For all $Y \in g$, the linear map $\Pi_s(Y)$ is in $C^\infty_\chi(N_0) \oplus \mathfrak{X}(N_0)$. This property of $\Pi_s(Y)$ makes $N_0$ a $g$-manifold.

Let $\mathscr{L}_{-s}$ be the trivial bundle of $N_0$ with fiber $\mathbb{C}_\chi^{-s}$. Then the space of smooth sections of $\mathscr{L}_{-s}$ is identified with $C^\infty(N_0, \mathbb{C}_\chi^{-s})$. For $Y \in g$ and $f \in C^\infty(N_0)$, a computation shows that in $\mathfrak{D}(\mathscr{L}_{-s})$,

$$([\Pi_s(Y), f])(\bar{n}) = -(R((\text{Ad}(\bar{n}^{-1})Y)\bar{n}) \cdot f)(\bar{n}).$$

This verifies that $\Pi_s$ gives $\mathscr{L}_{-s}$ the structure of a $g$-bundle.

Now define

$$\mathfrak{D}(\mathscr{L}_{-s}) = \{ D \in \mathfrak{D}(\mathscr{L}_{-s}) \mid [\Pi_s(X), D] = 0 \text{ for all } X \in \bar{n} \}.$$

**Proposition 2.1** [BKZ09, Proposition 13]. Let $D_1, \ldots, D_m$ be a list of operators in $\mathfrak{D}(\mathscr{L}_{-s})^{\bar{n}}$. Suppose that the list is linearly independent at $e$ and that there is a map $b : g \to \mathfrak{gl}(m, \mathbb{C})$ such that

$$([\Pi_s(Y), D_i] \cdot f)(e) = \sum_{j=1}^m b(Y)_{ji}(D_j \cdot f)(e)$$

for all $Y \in g$, $f \in C^\infty(N_0, \mathbb{C}_\chi^{-s})$, and $1 \leq i \leq m$. Then $D_1, \ldots, D_m$ is a conformally invariant system on $\mathscr{L}_{-s}$. The structure operator of the system is given by $C(Y)(\bar{n}) = b(\text{Ad}(\bar{n}^{-1})Y)$ for all $\bar{n} \in N_0$ and $Y \in g$.

As shown in [BKZ09, pp. 801–802] the differential operators in $\mathfrak{D}(\mathscr{L}_{-s})^{\bar{n}}$ can be described in terms of elements of the generalized Verma module

$$\mathcal{M}_q(C_s d\chi) = \mathcal{U}(g) \otimes_{\mathcal{U}(q)} C_s d\chi,$$
where $C_{s\chi}$ is the $q$-module derived from the $Q_0$-representation $(\chi^s, C)$. By identifying $M_q(C_{s\chi})$ as $\mathfrak{u}(\bar{n}) \otimes C_{s\chi}$, the map $M_q(C_{s\chi}) \rightarrow \mathfrak{u}(\bar{n})$ given by $u \otimes 1 \mapsto u$ is an isomorphism. The composition

\[(2.3) \quad M_q(C_{s\chi}) \rightarrow \mathfrak{u}(\bar{n}) \rightarrow \mathbb{D}(\mathcal{L}_-)^{\bar{n}}\]

is then a vector-space isomorphism, where the map $\mathfrak{u}(\bar{n}) \rightarrow \mathbb{D}(\mathcal{L}_-)^{\bar{n}}$ is given by $u \mapsto R(u)$.

Suppose that $f \in C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-1}})$ and $l \in L_0$. Then define an action of $L_0$ on $C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-1}})$ by

\[(l \cdot f)(\bar{n}) = \chi^{-s}(l)f(l^{-1}\bar{n}l).\]

This action agrees with the action of $L_0$ by left translation on the image of the restriction map $C^\infty(G_0/Q_0, \mathbb{C}_{\chi^{-1}}) \rightarrow C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-1}})$. In terms of this action, define an action of $L_0$ on $\mathbb{D}(\mathcal{L}_-)$ by

\[(l \cdot D) \cdot f = l \cdot (D \cdot (l^{-1} \cdot f)).\]

One can check that $l \cdot R(u) = R(\text{Ad}(l)u)$ for $l \in L_0$ and $u \in \mathfrak{u}(\bar{n})$; in particular, this $L_0$-action stabilizes the subspace $\mathbb{D}(\mathcal{L}_-)^{\bar{n}}$. Define an action of $L_0$ on $M_q(C_{s\chi})$ by $l \cdot (u \otimes z) = \text{Ad}(l)u \otimes z$. With these actions, the isomorphism (2.3) is $L_0$-equivariant. For $D \in \mathbb{D}(\mathcal{L}_-)$, denote by $D_{\bar{n}}$ the linear functional $f \mapsto (D \cdot f)(\bar{n})$ for $f \in C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-1}})$. The following result is the specialization of [BKZ09, Theorem 15] to the present situation.

**Theorem 2.2.** Suppose that $F$ is a finite-dimensional $q$-submodule of the generalized Verma module $M_q(C_{s\chi})$. Let $f_1, \ldots, f_k$ be a basis of $F$ and define constants $a_{ri}(Y)$ by

\[Yf_i = \sum_{r=1}^{k} a_{ri}(Y) f_r\]

for $1 \leq i \leq k$ and $Y \in q$. Let $D_1, \ldots, D_k \in \mathbb{D}(\mathcal{L}_-)^{\bar{n}}$ correspond to the elements $f_1, \ldots, f_k \in F$. Then for all $Y \in q$, $1 \leq i \leq k$, and $\bar{n} \in \bar{N}_0$,

\[[\Pi_s(Y), D_i]_{\bar{n}} = \sum_{r=1}^{k} a_{ri}(\text{Ad}(\bar{n}^{-1})Y)_q(D_r)_{\bar{n}} - s d\chi((\text{Ad}(\bar{n}^{-1})Y)_q(D_i)_{\bar{n}}].\]

**3. The $\Omega_3^t$ system**

Let $G$ be a complex simple Lie group with Lie algebra $q$ simply laced. In this section we specialize to the situation where $G_0$ is a real form of $G$ that contains a real parabolic subgroup of Heisenberg type. In this setting, we construct a system of differential operators over the line bundle $\mathcal{L}_-$ and show some technical facts that will be used later sections. We first introduce some notation.
Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and let $\Delta$ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Fix $\Delta^+$ a positive system and denote by $S$ the corresponding set of simple roots. Write $\rho$ for half the sum of the positive roots. Denote the highest root by $\gamma$. Let $B_{\mathfrak{g}}$ denote a positive multiple of the Killing form on $\mathfrak{g}$ and denote by $(\cdot, \cdot)$ the corresponding inner product induced on $\mathfrak{h}^*$. The normalization of $B_{\mathfrak{g}}$ will be specified below. Write $\|\alpha\|^2 = (\alpha, \alpha)$ for any $\alpha \in \Delta$. For $\alpha \in \Delta$, let $\mathfrak{g}_\alpha$ be the root space of $\mathfrak{g}$ corresponding to $\alpha$. For any $\text{ad}(\mathfrak{h})$-invariant subspace $V \subset \mathfrak{g}$, denote by $\Delta(V)$ the set of roots $\alpha$ so that $\mathfrak{g}_\alpha \subset V$.

It is known that we can choose $X_\alpha \in \mathfrak{g}_\alpha$ and $H_\alpha \in \mathfrak{h}$ for each $\alpha \in \Delta$ in such a way that the following conditions (C1)–(C5) hold. Our normalizations are special cases of those used in [BKZ08].

(C1) For any $\alpha \in \Delta^+$, $\{X_{-\alpha}, H_\alpha, X_\alpha\}$ is an $\mathfrak{sl}(2)$-triple. In particular, we have $[X_\alpha, X_{-\alpha}] = H_\alpha$.

(C2) For each $\alpha, \beta \in \Delta$, $[H_\alpha, X_\beta] = (H_\alpha)X_\beta$.

(C3) For $\alpha \in \Delta$ we have $B_{\mathfrak{g}}(X_\alpha, X_{-\alpha}) = 1$. In particular, $(\alpha, \alpha) = 2$.

(C4) For $\alpha, \beta \in \Delta$, we have $B_{\mathfrak{g}}(H_\alpha, H_\beta) = (\beta, \alpha)$.

(C5) If $\alpha, \beta, \alpha + \beta \in \Delta$ then there is a nonzero integer $N_{\alpha, \beta}$ so that $[X_\alpha, X_\beta] = N_{\alpha, \beta}X_{\alpha + \beta}$. For $Z \in \mathfrak{l}$ and $\alpha \in \Delta(V^+)$, define a scalar $M_{\alpha, \beta}(Z)$ by $[Z, X_\alpha] = \sum_{\beta \in \Delta(V^+)} M_{\alpha, \beta}(Z)X_\beta$.

Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ be the standard parabolic subalgebra of $\mathfrak{g}$ of Heisenberg type with $\mathfrak{l}$ its Levi factor and $\mathfrak{n}$ its nilpotent radical. Then, the action of $\text{ad}(H_\gamma)$ on $\mathfrak{g}$ induces the eigenspace decomposition (1.1) of $\mathfrak{g}$, where $\gamma$ is the highest root of $\mathfrak{g}$. Since $\mathfrak{z}(\mathfrak{n}) = \mathfrak{g}_\gamma$ is one-dimensional, there is a character $\chi$ of $L_0$ so that $\text{Ad}(l)X_\gamma = \chi(l)X_\gamma$ for all $l \in L_0$. Note that $\text{Ad}(l)X_{-\gamma} = \chi(l)^{-1}X_{-\gamma}$ for all $l \in L_0$, as $\mathfrak{g}_{-\gamma}$ is the $B_{\mathfrak{g}}$-dual space of $\mathfrak{g}_\gamma$. For the rest of this paper, fix $\chi$ so that its differential $d\chi$ is $d\chi = \gamma$.

Let $\mathbb{D}_\gamma(\mathfrak{g}, \mathfrak{h})$ be the deleted Dynkin diagram associated to the Heisenberg parabolic $\mathfrak{q}$, that is, the subdiagram of the Dynkin diagram of $(\mathfrak{g}, \mathfrak{h})$ obtained by deleting the node corresponding to the simple root that is not orthogonal to $\gamma$, and the edges that involve it.

As in [BKZ08, p. 789] the operator $\Omega_2$ is given in terms of $R$ by

$$
\Omega_2(Z) = -\frac{1}{2} \sum_{\alpha, \beta \in \Delta(V^+)} N_{\beta, \beta'} M_{\alpha, \beta'}(Z) R(X_{-\alpha}) R(X_{-\beta})
$$

for $Z \in \mathfrak{l}$. One can check that $\Omega_2(\text{Ad}(l)Z) = \chi(l)l \cdot \Omega_2(Z)$ for all $l \in L_0$. This is different from the $\text{Ad}(l)$ transformation law of $\Omega_2$ that appears in [BKZ08], because the parabolic $\mathfrak{q}$ is chosen in this paper, while the opposite parabolic $\mathfrak{q}$...
is chosen in [BKZ08]. We extend the \( \mathbb{C} \)-linear maps \( d \chi, R, \) and \( \Omega_2 \) to be left \( C^\infty(\tilde{N}_0) \)-linear so that certain relationships can be expressed more easily.

Now for \( t \in \mathbb{C} \) define an operator \( \Omega_3^t : V^- \to \mathbb{D}(\mathcal{L}_{-\gamma}) \tilde{\chi} \) by

\[
\Omega_3^t(Y) = \widetilde{\Omega}_3(Y) + tC_3(Y),
\]

where the operators \( \widetilde{\Omega}_3(Y) \) and \( C_3(Y) \) are defined in terms of \( R \) and \( \Omega_2 \) by

\[
\widetilde{\Omega}_3(Y) = \sum_{\epsilon \in \Delta(V^+)} R(X_{-\epsilon})\Omega_2([X_\epsilon, Y]),
\]

\[
C_3(Y) = R(Y)R(X_{-\gamma})
\]
as in [BKZ08, p. 801].

**Lemma 3.1.** Let \( W_1, \ldots, W_m \) be a basis for \( V^+ \) and \( W_1^*, \ldots, W_m^* \) be the \( B_3 \)-dual basis of \( V^- \). Then

\[
\widetilde{\Omega}_3(Y) = \sum_{i=1}^m R(W_i^*)\Omega_2([W_i, Y]).
\]

**Proof.** Suppose that \( \Delta(V^+) = \{ \epsilon_1, \ldots, \epsilon_m \} \). Each \( W_i \) then may be expressed by \( W_i = \sum_{j=1}^m a_{ij}X_{\epsilon_j} \) for \( a_{ij} \in \mathbb{C} \). Let \( [a_{ij}] \) be the change of basis matrix and set \( [b_{ij}] = [a_{ij}]^{-1} \). Then define \( W_i^* = \sum_{k=1}^m b_{ki}X_{-\epsilon_k} \) for \( i = 1, \ldots, m \). Note that \( \sum_{j=1}^m a_{is}b_{sj} = \delta_{ij} \) with \( \delta_{ij} \) the Kronecker delta. Since \( B_3(X_{\epsilon_i}, X_{-\epsilon_j}) = \delta_{ij} \), it follows that \( B_3(W_i, W_j^*) = \delta_{ij} \). So \( \{W_1^*, \ldots, W_m^*\} \) is the dual basis of \( \{W_1, \ldots, W_m\} \).

Hence,

\[
\sum_{i=1}^m R(W_i^*)\Omega_2([W_i, Y]) = \sum_{j,k=1}^m \left( \sum_{i=1}^m b_{ki}a_{ij} \right)R(X_{-\epsilon_k})\Omega_2([X_{\epsilon_j}, Y]) = \sum_{j=1}^m R(X_{-\epsilon_j})\Omega_2([X_{\epsilon_j}, Y]). \]

\[
\square
\]

**Lemma 3.2.** For all \( l \in L_0, Z \in \mathfrak{l}, \) and \( Y \in V^- \), we have

\[
\Omega_3^l(\text{Ad}(l)Y) = \chi(l)l \cdot \Omega_3^l(Y),
\]

\[
\Omega_3^l([Z, Y]) = d\chi(Z)\Omega_3^l(Y) + [\Pi_s(Z), \Omega_3^l(Y)].
\]

**Proof.** To obtain the first equality it suffices to show that \( \widetilde{\Omega}_3 \) and \( C_3 \) have the proposed transformation law. Recall that \( l \cdot R(u) = R(\text{Ad}(l)u) \) for \( l \in L_0 \) and \( u \in \mathfrak{u}l(\tilde{n}) \); in particular, we have \( l \cdot R(X_{-\gamma}) = \chi(l)^{-1}R(X_{-\gamma}) \). Therefore \( \chi(l)l \cdot C_3(Y) = R(\text{Ad}(l)Y)R(X_{-\gamma}) \), which is \( C_3(\text{Ad}(l)Y) \). Since \( \Omega_2(\text{Ad}(l)W) = \chi(l)l \cdot \Omega_2(W) \) for \( l \in L_0 \) and \( W \in \mathfrak{l} \), it follows that

\[
\chi(l)l \cdot \widetilde{\Omega}_3(Y) = \sum_{\epsilon \in \Delta(V^+)} R(\text{Ad}(l)X_{-\epsilon})\Omega_2([\text{Ad}(l)X_\epsilon, \text{Ad}(l)Y]). \]
By Lemma 3.1, the value of $\widetilde{\Omega}_3(Y)$ is independent from a choice of a basis for $V^+$. Thus the righthand side of (3.2) equals $\sum_{e \in \Delta(Y^+)} R(X_\epsilon) \Omega_2([X_\epsilon, \text{Ad}(l)Y])$, which is $\widetilde{\Omega}_3(\text{Ad}(l)Y)$. The second equality is obtained by differentiating the first. □

Let $\omega'_3(Y)$ denote the element in $\mathfrak{u}(\tilde{n}) \otimes \mathbb{C}_{s dX}$ that corresponds to $\Omega'_3(Y)$ under $R$.

**Lemma 3.3.** For $Z \in \mathfrak{l}$ and $Y \in V^-$, we have

$$\omega'_3([Z, Y]) = Z\omega'_3(Y) + (1 - s) d\chi(Z)\omega'_3(Y).$$

**Proof.** Lemma 3.2 shows that $\Omega'_3(\text{Ad}(l)Y) = \chi(l) l \cdot \Omega'_3(Y)$ for $l \in L_0$. Thus by [BKZ09, Lemma 18], $\omega'_3(\text{Ad}(l)Y) = \chi(l) \text{Ad}(l)\omega'_3(Y)$. The formula is then obtained by replacing $l$ by $\exp(tZ)$ with $Z \in \mathfrak{l}_0$, differentiating, and setting at $t = 0$. □

Let $E$ be an irreducible $L_0$-submodule of $V^-$. We say that the $\Omega_3|_E$ system exists if there exist $t_0, s_0 \in \mathbb{C}$ so that the list of differential operators $\Omega^{t_0}_{3}|_E = \Omega^{t_0}_3(X_{\beta_1}, \ldots, \Omega^{t_0}_3(X_{\beta_m})$ with $\Delta(E) = \{\beta_1, \ldots, \beta_m\}$ is conformally invariant over the line bundle $\mathcal{L}_{-s_0}$. Set $F_t(E) = \text{span}_{\mathbb{C}}\{\omega'^3_t(Y) | Y \in E\}$.

**Lemma 3.4.** If the $\Omega^t_3|_E$ system is conformally invariant for $t = t_0$ over $\mathcal{L}_{-s_0}$, then $n$ acts on $F_{t_0}(E)$ trivially.

**Proof.** Since the $\Omega^{t_0}_3|_E$ system is conformally invariant over the line bundle $\mathcal{L}_{-s_0}$, the space $F_{t_0}(E)$ is a q-submodule of $M_q(\mathbb{C}_{s_0 dX})$. By applying Lemma 3.3 with $Z = H_Y$, we obtain $H_Y \omega'^3_{t_0}(Y) = (2s_0 - 3) \omega'^3_{t_0}(Y)$ for all $Y \in E$. For $U \in V^+$ we have $H_Y U \omega'^3_{t_0}(Y) = (2s_0 - 2) U \omega'^3_{t_0}(Y)$, and $H_Y X_Y \omega'^3_{t_0}(Y) = (2s_0 - 1) X_Y \omega_2(Y)$ for all $Y \in E$. Therefore if $U \in n$ then $U \omega'^3_{t_0}(Y) = 0$ for all $Y \in E$, because otherwise $U \omega'^3_{t_0}(Y)$ would have the wrong $H_Y$-eigenvalue to lie in $F_{t_0}(E)$. □

By using the transformation law $\omega'_3(\text{Ad}(l)Y) = \chi(l) \text{Ad}(l)\omega'_3(Y)$ for $l \in L_0$ and $Y \in V^-$, one can check that for any $s \in \mathbb{C}$ the vector space isomorphism

$$E \otimes \mathbb{C}_{(s-1) dX} \rightarrow F_t(E),$$

given by $Y \otimes 1 \mapsto \omega'_3(Y)$, is $L_0$-equivariant with respect to the standard action of $L_0$ on the tensor products $E \otimes \mathbb{C}_{(s-1) dX}$ and $F_t(E) \subset \mathfrak{u}(\tilde{n}) \otimes \mathbb{C}_{s dX}$. In particular, the $L_0$-module $F_{t_0}(E)$ is irreducible. The $L_0$-action on $F_t(E)$ is given by $l \cdot (u \otimes 1) = \chi^s(l)(\text{Ad}(l)u \otimes 1)$, which is different from the one that is used to establish the $L_0$-equivariant isomorphism (2.3).

Now suppose that the $\Omega^t_3|_E$ system is conformally invariant for $t = t_0$ over $\mathcal{L}_{-s_0}$. Then $F_{t_0}(E)$ is a q-submodule of $\mathfrak{u}(\tilde{n}) \otimes \mathbb{C}_{s_0 dX}$. Since $F_{t_0}(E)$ is an irreducible $L_0$-module and $n$ acts on it trivially by Lemma 3.4, the inclusion map $F_{t_0}(E) \hookrightarrow M_q(\mathbb{C}_{s_0 dX})$ induces a nonzero $\mathfrak{u}(\mathfrak{g})$-homomorphism of generalized Verma modules

$$M_q(F_{t_0}(E)) \rightarrow M_q(\mathbb{C}_{s_0 dX}).$$
that is given by $u \otimes \omega^0_3(Y) \mapsto u \cdot \omega^0_3(Y)$. In particular, the two Verma modules $\mathcal{M}_q(F_{t_0}(E))$ and $\mathcal{M}_q(\mathbb{C}_{s_0,dX})$ have the same infinitesimal characters. Since we choose a character $\chi$ so that $d\chi = \gamma$, this implies that if $\sigma\gamma$ is the highest weight for $E$ then

\[(3.4) \quad \|\sigma\gamma + (s_0 - 1)\gamma + \rho\|^2 = \|s_0\gamma + \rho\|^2,\]

This will restrict the possibility of $s_0$ for which the $\Omega^t_3$ is conformally invariant.

4. The $\Omega_3$ system on $\mathfrak{sl}(3, \mathbb{C})$

We take the complex Lie group $G$ from Section 3 to be $\text{SL}(3, \mathbb{C})$ and show that the $\Omega_3$ system(s) exists over the line bundle $\mathcal{L}_0$. Since the generalized Verma module $\mathcal{M}_q(\mathbb{C}_{s,dX})$ is a (ordinary) Verma module in this case, we simply write $\mathcal{M}(\mathbb{C}_{s,dX}) = \mathcal{M}_q(\mathbb{C}_{s,dX})$ throughout this section.

Let $\alpha_1$ and $\alpha_2$ be the two simple roots for $\mathfrak{sl}(3, \mathbb{C})$. Then $V^- = \mathbb{C}X_{-\alpha_1} \oplus \mathbb{C}X_{-\alpha_2}$; each of $\mathbb{C}X_{-\alpha_i}$ for $i = 1, 2$ is an $L_0$-submodule of $V^-$. A direct computation shows that $\Omega^t_3(X_{-\alpha_i}) = \tilde{\Omega}_3(X_{-\alpha_i}) + tC_3(X_{-\alpha_i})$ is not identically zero for $i = 1, 2$ and for any $t \in \mathbb{C}$. Then solving (3.4) with $\sigma = -\alpha_i$ for $i = 1, 2$ gives that $\Omega^t_3(X_{-\alpha_i})$ is conformally invariant over $\mathcal{L}_{-s_0}$ then the special value $s_0$ of $s$ must be $s_0 = 0$. Now we show that there exists a unique $t_i \in \mathbb{C}$ so that $\Omega^t_i(X_{-\alpha_i})$ is conformally invariant over $\mathcal{L}_0$.

**Theorem 4.1.** Let $\mathfrak{g}$ be the complex simple Lie algebra of type $A_2$, and $\mathfrak{q}$ be the parabolic subalgebra of Heisenberg type. Then for each $i = 1, 2$ the operator $\Omega^t_i(X_{-\alpha_i})$ is conformally invariant over $\mathcal{L}_0$ if and only if $t = \frac{3}{4}$.

**Proof.** Fix $\alpha_i$ and denote by $\alpha_k$ the other simple root so that $S = \{\alpha_i, \alpha_k\}$. Observe that $\omega^t_3(X_{-\alpha_i})$ is the element in $\mathcal{M}(\mathbb{C}_0)$ that corresponds to $\Omega^t_3(X_{-\alpha_i})$ in $\mathbb{D}(\mathcal{L}_0)^\pi$ under the map (2.3). By Theorem 2.2 and Lemma 3.4, it suffices to show that $\mathbb{C}\omega^t_3(X_{-\alpha_i})$ is a $\mathfrak{q}$-submodule of $\mathcal{M}(\mathbb{C}_0)$ with trivial $\mathfrak{n}$-action if and only if $t = \frac{3}{4}$.

A direct computation shows that the element in $\mathcal{M}(\mathbb{C}_0)$ that corresponds to $\tilde{\Omega}_3(X_{-\alpha_i})$ may be written as

$$-\frac{3}{2}N_{\alpha_i, \alpha_k}X^2_{-\alpha_i}X_{-\alpha_k} \otimes 1 - \frac{3}{4}X_{-\alpha_i}X_{-\gamma} \otimes 1.$$  

As $C_3(X_{-\alpha_i}) = R(X_{-\alpha_i})R(X_{-\gamma})$, the element in $\mathcal{M}(\mathbb{C}_0)$ corresponding to $C_3(X_{-\alpha_i})$ is $X_{-\alpha_i}X_{-\gamma} \otimes 1$. Thus $\omega^t_3(X_{-\alpha_i})$ is given by

$$\omega^t_3(X_{-\alpha_i}) = -\frac{3}{2}N_{\alpha_i, \alpha_k}X^2_{-\alpha_i}X_{-\alpha_k} \otimes 1 + (t - \frac{3}{4})X_{-\alpha_i}X_{-\gamma} \otimes 1.$$  

One can easily check that $\mathfrak{n}$ acts trivially on $\mathbb{C}X^2_{-\alpha_i}X_{-\alpha_2} \otimes 1$ and $\mathbb{C}X^2_{-\alpha_2}X_{-\alpha_1} \otimes 1$ and thus both of them are one-dimensional $\mathfrak{q}$-submodules of $\mathcal{M}(\mathbb{C}_0)$, while it acts nontrivially on $X_{-\alpha_1}X_{-\gamma} \otimes 1$ and $X_{-\alpha_2}X_{-\gamma} \otimes 1$ in $\mathcal{M}(\mathbb{C}_0)$. Therefore $\mathbb{C}\omega^t_3(X_{-\alpha_i})$ is a $\mathfrak{q}$-submodule with trivial $\mathfrak{n}$-action if and only if $t = \frac{3}{4}$. 

□
5. The $\Omega_3$ system on $\mathfrak{so}(8, \mathbb{C})$

We take the complex Lie group $G$ from Section 3 to be $\text{SO}(8, \mathbb{C})$ and show that the $\tilde{\Omega}_3$ system is conformally invariant over the line bundle $\mathcal{L}_1$.

Since in this case the parabolic $q$ is maximal, the $l$-module $V^-$ is irreducible with highest weight $-\alpha_γ$, where $\alpha_γ$ is the simple root that is not orthogonal to $γ$. Then by solving (3.4) with $σ = -\alpha_γ$, one can see that if the $Ω_3$ system exists then the special value $s_0$ of $s$ must be $s_0 = -1$. Thus in the rest of this paper the line bundle $\mathcal{L}_{-s}$ is assumed to be $\mathcal{L}_1$, and for simplicity, write $Π$ for the Lie algebra action $Π_s$ defined in (2.1) for $s = -1$. As stated in Section 2, for $D \in \mathbb{D}(\mathcal{L}_{-s})$, denote by $D_π$ the linear functional $f \mapsto (D \cdot f)(\bar{n})$ for $f \in C^∞(\bar{N}_0, \mathbb{C}_χ^{-})$.

Proposition 5.1. For all $X \in \mathfrak{g}$, $Y \in V^-$, and $\bar{n} \in \bar{N}_0$, we have

$$[Π(X), R(Y)]_{\bar{n}} = R([Ad(\bar{n}^{-1})X, Y]_{V^-})_{\bar{n}} - dχ ([Ad(\bar{n}^{-1})X, Y]_{l}).$$

Proof. Let $F$ be the subspace of $\mathcal{M}_q(\mathbb{C}_{-dχ})$ spanned by $X_{-\alpha} \otimes 1$ and $1 \otimes 1$ with $\alpha \in Δ(V^+)$. A direct computation shows that $F$ is a $q$-submodule of $\mathcal{M}_q(\mathbb{C}_{-dχ})$ and that for $Z \in l$ and $U \in n$ we have

$$Z(X_{-\alpha} \otimes 1) = [Z, X_{-\alpha}] \otimes 1 - dχ (Z)X_{-\alpha} \otimes 1,$$

$$U(X_{-\alpha} \otimes 1) = -dχ ([U, X_{-\alpha}]_{l}) \otimes 1.$$

Then it follows from Theorem 2.2 that if $X \in \mathfrak{g}$ and $(Ad(\bar{n}^{-1})X)_q = Z + U$ with $Z \in l$ and $U \in n$ then for $Y \in V^-$,

$$[Π(X), R(Y)]_{\bar{n}} = R([Z, Y])_{\bar{n}} - dχ ([U, Y]).$$

Since $[Z, Y] = [Ad(\bar{n}^{-1})X, Y]_{V^-}$ and $[U, X_{-\alpha}]_{l} = [Ad(\bar{n}^{-1})X, Y]_{l}$, this completes the proof. □

Let $ω_2(W)$ denote the element in $\mathfrak{u}(\bar{n}) \otimes \mathbb{C}_{-dχ}$ that corresponds to $Ω_2(W)$ under $R$. Observe that $Ω_2(Ad(l)W) = χ(l)l \cdot Ω_2(W)$ for all $l \in L_0$; this is the same $Ad(l)$ transformation law as $Ω'_3$ (see Lemma 3.2). Then Lemma 3.3 with $s = -1$ implies that for $W, Z \in l$, we have

$$ω_2([Z, W]) = Zω_2(W) + 2dχ(Z)ω_2(W).$$

\begin{equation}
(5.1) \quad ω_2([Z, W]) = Zω_2(W) + 2dχ(Z)ω_2(W).
\end{equation}

Proposition 5.2. For all $X \in \mathfrak{g}$, $W \in l$, and $\bar{n} \in \bar{N}_0$, we have

$$[Π(X), Ω_2(W)]_{\bar{n}} = Ω_2([Ad(\bar{n}^{-1})X, W]_{l})_{\bar{n}} - dχ ((Ad(\bar{n}^{-1})X)_{l})Ω_2(W)_{\bar{n}}.$$

Proof. It follows from [BKZ08, Theorem 5.2] and the data tabulated in [BKZ08, Section 8.10] that each $Ω_2$ system associated to a singleton component of $\mathcal{B}_γ (\mathfrak{g}, \mathfrak{h})$ is conformally invariant on the line bundle $\mathcal{L}_1$. Note here that the special values of our $Ω_2$ system are of the form $-s_0$ with $s_0$ the special values of the $Ω_2$ system.
given in [BKZ08], as the parabolic $q$ is chosen in this paper, while the opposite parabolic $\tilde{q}$ is chosen in [BKZ08]. Therefore $F \equiv \text{span}_C\{\omega_2(W) \mid W \in I\}$ is a $q$-submodule of $M_q(\mathbb{C}_{-d_H})$. The same argument for the proof for Lemma 3.4 shows that $n$ acts on $F$ trivially. By (5.1), we have

$$Z\omega_2(W) = \omega_2([Z, W]) - 2d\chi(Z)\omega_2(W)$$

for $Z, W \in I$. The proposed formula now follows from Theorem 2.2.

Lemma 5.3. For $X \in V^+$ and $Y \in V^-$, we have

$$\sum_{\epsilon \in \Delta(V^+)} \Omega_2([X, X_\epsilon], [X_\epsilon, Y]) = 2\Omega_2([X, Y]).$$

Proof. Since $\|\epsilon\|^2 = 2$ for all $\epsilon \in \Delta(V^+)$, it follows from [BKZ08, Proposition 2.2] that

$$\sum_{\epsilon \in \Delta(V^+)} \Omega_2([X, X_\epsilon], [X_\epsilon, Y]) = \frac{1}{2} \sum_{\epsilon} p(D_4, \epsilon) \Omega_2(\text{pr}_\epsilon([X, Y])),$$

where $\epsilon$ are the connected components of $D_\gamma(g, h)$ as in [BKZ08] and $\text{pr}_\epsilon([X, Y])$ is the projection of $[X, Y]$ onto $I(\epsilon)$, the ideal of $[I, I]$ corresponding to $\epsilon$. (See [BKZ08, Section 2] for further discussion.) From [BKZ08, Section 8.4] we have $p(D_4, \epsilon) = 4$ for all the components $\epsilon$. Then $\Omega_2(H_\gamma) = 0$ shows that

$$\sum_{\epsilon \in \Delta(V^+)} \Omega_2([X, X_\epsilon], [X_\epsilon, Y]) = 2\Omega_2([X, Y]).$$

Now with the above lemmas and propositions we are ready to show the following key theorem.

Theorem 5.4. We have $[\Pi(X), \tilde{\Omega}_3(Y)]_e = 0$ for all $X \in V^+$ and all $Y \in V^-$. 

Proof. Observe that $\tilde{\Omega}_3(Y) = \sum_{\epsilon \in \Delta(V^+)} R(X_{-\epsilon})\Omega_2([X_\epsilon, Y])$. Then the commutator $[\Pi(X), \tilde{\Omega}_3(Y)]$ is a sum of two terms. One of them is given by

$$\sum_{\epsilon \in \Delta(V^+)} [\Pi(X), R(X_{-\epsilon})]\Omega_2([X_\epsilon, Y])$$

$$= \sum_{\epsilon \in \Delta(V^+)} R([\text{Ad}(\epsilon X, X_{-\epsilon}]_{V^-})\Omega_2([X_\epsilon, Y])$$

$$- \sum_{\epsilon \in \Delta(V^+)} d\chi([\text{Ad}(\epsilon X, X_{-\epsilon}]_I)\Omega_2([X_\epsilon, Y]),$$

by Proposition 5.1. At $e$, the first term is zero, since $[X, X_\epsilon]_{V^-} = 0$ for all $\epsilon \in \Delta(V^+)$. By writing out $X$ as a linear combination of $X_\alpha$ with $\alpha \in \Delta(V^+)$,
at the identity the second term in (5.2) evaluates to
\[- \sum_{\epsilon \in \Delta(V^+)} d\chi([X, X_\epsilon]) \Omega_2([X_\epsilon, Y])_e = -\Omega_2([X, Y])_e,\]
since \(d\chi(H_\alpha) = 1\) for \(\alpha \in \Delta(V^+)\). The other term is given by
\[
(5.3) \quad \sum_{\epsilon \in \Delta(V^+)} R(X_{-\epsilon})[\Pi(X), \Omega_2([X_\epsilon, Y])]_l
= \sum_{\epsilon \in \Delta(V^+)} R(X_{-\epsilon}) \Omega_2([\text{Ad}(\cdot^{-1})X, [X_\epsilon, Y]]_l)
- \sum_{\epsilon \in \Delta(V^+)} R(X_{-\epsilon}) d\chi((\text{Ad}(\cdot^{-1})X)_l) \Omega_2([X_\epsilon, Y]),
\]
by Proposition 5.2. To further evaluate this expression, we make use of a simple general observation. Namely, if \(D\) is a first order differential operator, \(\phi\) and \(\psi\) are smooth functions, and \(\phi(e) = 0\), then \(D_\epsilon(\phi \psi) = D_\epsilon(\phi)\psi(e)\). The map \(\bar{n} \mapsto \text{ad}(\text{Ad}(\bar{n}^{-1})X)\) is a smooth function on \(\bar{N}_0\). The left \(C^\infty(\bar{N}_0)\)-linear extension of \(\Omega_2\) implies that the first term of the righthand side of (5.3) can be expressed as
\[
\sum_{\epsilon \in \Delta(V^+)} R(X_{-\epsilon}) (\text{ad}(\text{Ad}(\cdot^{-1})X)_l \cdot \Omega_2([X_\epsilon, Y])),
\]
where \(\text{ad}(\text{Ad}(\cdot^{-1})X)_l\) denotes the map \(Z \mapsto [\text{Ad}(\cdot^{-1})X, Z]_l\) for \(Z \in g\). Since
\[
(R(X_{-\epsilon}) \bullet (\text{Ad}(\cdot^{-1})X))(e) = [X, X_\epsilon],
\]
\([X, [X_\epsilon, Y]]_l = 0\), and \(X_1 = 0\), the righthand side of (5.3) then evaluates at the identity to
\[
\sum_{\epsilon \in \Delta(V^+)} \Omega_2([[X, X_\epsilon], [X_\epsilon, Y]])_e
- \sum_{\epsilon \in \Delta(V^+)} d\chi([[X, X_\epsilon]]) \Omega_2([X_\epsilon, Y])_e,
\]
which is equivalent to
\[
\sum_{\epsilon \in \Delta(V^+)} \Omega_2([[X, X_\epsilon], [X_\epsilon, Y]])_e
- \Omega_2([X, Y])_e.
\]
Therefore,
\[
[\Pi(X), \tilde{\Omega}_3(Y)]_e = \sum_{\epsilon \in \Delta(V^+)} \Omega_2([[X, X_\epsilon], [X_\epsilon, Y]])_e
- 2\Omega_2([X, Y])_e.
\]
Now it follows from Lemma 5.3 that \([\Pi(X), \tilde{\Omega}_3(Y)]_e = 0\). □

**Proposition 5.5.** For \(Y \in V^-\), we have \([\Pi(X_\gamma), \tilde{\Omega}_3(Y)]_e = 0\).
Proof. Since $3(n) = [V^+, V^+]$, it suffices to show that $[\Pi([X_1, X_2]), \widetilde{\Omega}_3(Y)]_e = 0$ for $X_1, X_2 \in V^+$. We have $\Pi([X_1, X_2]) = [\Pi(X_1), \Pi(X_2)]$, so it follows from the Jacobi identity that $[\Pi([X_1, X_2]), \widetilde{\Omega}_3(Y)]$ may be expressed as a sum of two terms. The first is

$$[\Pi(X_1), [\Pi(X_2), \widetilde{\Omega}_3(Y)]] = \Pi(X_1)[\Pi(X_2), \widetilde{\Omega}_3(Y)] - [\Pi(X_2), \widetilde{\Omega}_3(Y)]\Pi(X_1).$$

By (2.2), we have $\Pi(X)_e = 0$ for all $X \in n$. This fact and Theorem 5.4 imply $[\Pi(X_1), [\Pi(X_2), \widetilde{\Omega}_3(Y)]]_e = 0$ since $(D_1D_2)_e = (D_1)_e D_2$ for $D_1, D_2 \in \mathbb{D}(\mathcal{L}_1)$. The second term is

$$[\Pi(X_2), [\widetilde{\Omega}_3(Y), \Pi(X_1)]] = \Pi(X_2)[\widetilde{\Omega}_3(Y), \Pi(X_1)] - [\widetilde{\Omega}_3(Y), \Pi(X_1)]\Pi(X_2).$$

By the same argument for the first term, $[\Pi(X_2), [\widetilde{\Omega}_3(Y), \Pi(X_1)]]_e = 0$, which concludes the proof. \hfill \Box

Theorem 5.6. Let $\mathfrak{g}$ be the complex simple Lie algebra of type $D_4$ and $\mathfrak{q}$ be the parabolic subalgebra of Heisenberg type. Then the $\widetilde{\Omega}_3$ system is conformally invariant on the line bundle $\mathcal{L}_1$.

Proof. By Proposition 2.1, it is enough to check the conformal invariance of $[\Pi(X), \widetilde{\Omega}_3(Y)]$ at the identity for all $X \in \mathfrak{g}$ and all $Y \in V^-$. It follows from Lemma 3.2 that

$$[\Pi(Z), \widetilde{\Omega}_3(Y)]_e = \widetilde{\Omega}_3([Z, Y])_e - d\chi(Z)\widetilde{\Omega}_3(Y)_e$$

for all $Z \in \mathfrak{l}$. Also Theorem 5.4 and Proposition 5.5 show that $[\Pi(U), \widetilde{\Omega}_3(Y)]]_e = 0$ for all $U \in n$. As $\widetilde{\Omega}_3(Y)$ is an element in $\mathbb{D}(\mathcal{L}_1)^n$, it is clear that $[\Pi(\overline{U}), \widetilde{\Omega}_3(Y)]_e = 0$ for all $\overline{U} \in \overline{n}$. Since $\mathfrak{g} = \overline{n} \oplus \mathfrak{l} \oplus \mathfrak{n}$, this implies that the $\widetilde{\Omega}_3$ system is conformally invariant on $\mathcal{L}_1$. \hfill \Box

Theorem 5.6 implies that $F_0(V^-) = \text{span}_\mathbb{C}\{\omega_3^0(Y) \mid Y \in V^-\}$ is a $\mathfrak{q}$-submodule of $\mathcal{M}_q(\mathbb{C}_{-d\chi})$, where $\omega_3^0(Y)$ is the element in $\mathcal{M}_q(\mathbb{C}_{-d\chi})$ that corresponds to $\widetilde{\Omega}_3(Y) = \Omega_3^0(Y)$ under $R$. The argument after Lemma 3.4 then shows that there exists a nonzero $\mathfrak{u}(\mathfrak{g})$-homomorphism

$$\mathcal{M}_q(F_0(V^-)) \rightarrow \mathcal{M}_q(\mathbb{C}_{-d\chi}).$$

It follows from Lemma 3.3 that $H^_r$ acts on $F_0(V^-)$ by $-5$, while it acts on $\mathbb{C}_{-d\chi}$ by $-2$; in particular, $F_0(V^-)$ is not equivalent to $\mathbb{C}_{-d\chi}$.

Corollary 5.7. Let $\mathfrak{g}$ be the complex simple Lie algebra of type $D_4$, and $\mathfrak{q}$ be the parabolic subalgebra of Heisenberg type. Then the generalized Verma module $\mathcal{M}_q(\mathbb{C}_{-d\chi})$ is reducible.
Acknowledgements

The author is grateful to Anthony Kable for his valuable suggestions and comments on this paper. He would also like to thank the referee for the helpful comments.

References


Received September 26, 2010. Revised March 30, 2011.

TOSHIHISA KUBO
DEPARTMENT OF MATHEMATICS
OKLAHOMA STATE UNIVERSITY
STILLWATER OK 74078
UNITED STATES
toskubo@math.okstate.edu
See inside back cover or pacificmath.org for submission instructions.

The subscription price for 2011 is US $420/year for the electronic version, and $485/year for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. Prior back issues are obtainable from Periodicals Service Company, 11 Main Street, Germantown, NY 12526-5635. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and the Science Citation Index.

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in L\TeX
Copyright ©2011 by Pacific Journal of Mathematics
<table>
<thead>
<tr>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fusion rules on a parametrized series of graphs</td>
<td>257</td>
</tr>
<tr>
<td>MARTA ASAEDE and UFFE HAAGERUP</td>
<td></td>
</tr>
<tr>
<td>Group gradings on restricted Cartan-type Lie algebras</td>
<td>289</td>
</tr>
<tr>
<td>YURI BAHTURIN and MIKHAIL KOCHETOV</td>
<td></td>
</tr>
<tr>
<td>B2-convexity implies strong and weak lower semicontinuity of partitions of $\mathbb{R}^n$</td>
<td>321</td>
</tr>
<tr>
<td>DAVID G. CARABALLO</td>
<td></td>
</tr>
<tr>
<td>Testing the functional equation of a high-degree Euler product</td>
<td>349</td>
</tr>
<tr>
<td>DAVID W. FARMER, NATHAN C. RYAN and RALF SCHMIDT</td>
<td></td>
</tr>
<tr>
<td>Asymptotic structure of a Leray solution to the Navier–Stokes flow around a rotating body</td>
<td>367</td>
</tr>
<tr>
<td>REINHARD FARWIG, GIOVANNI P. GALDI and MADS KYED</td>
<td></td>
</tr>
<tr>
<td>Type II almost-homogeneous manifolds of cohomogeneity one</td>
<td>383</td>
</tr>
<tr>
<td>DANIEL GUAN</td>
<td></td>
</tr>
<tr>
<td>Cell decompositions of Teichmüller spaces of surfaces with boundary</td>
<td>423</td>
</tr>
<tr>
<td>REN GUO and FENG LUO</td>
<td></td>
</tr>
<tr>
<td>A system of third-order differential operators conformally invariant under $\mathfrak{su}(3, \mathbb{C})$ and $\mathfrak{so}(8, \mathbb{C})$</td>
<td>439</td>
</tr>
<tr>
<td>TOSHIHISA KUBO</td>
<td></td>
</tr>
<tr>
<td>Axial symmetry and regularity of solutions to an integral equation in a half-space</td>
<td>455</td>
</tr>
<tr>
<td>GUOZHEN LU and JIUYI ZHU</td>
<td></td>
</tr>
<tr>
<td>Braiding knots in contact 3-manifolds</td>
<td>475</td>
</tr>
<tr>
<td>ELENA PAVELESCU</td>
<td></td>
</tr>
<tr>
<td>Gradient estimates for positive solutions of the heat equation under geometric flow</td>
<td>489</td>
</tr>
<tr>
<td>JUN SUN</td>
<td></td>
</tr>
</tbody>
</table>