THE PRINCIPLE OF STATIONARY PHASE
FOR THE FOURIER TRANSFORM OF $D$-MODULES

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We show that the formal germ at the infinity of the Fourier transform of a holonomic $D$-module depends only on the formal germ of the $D$-module at its singular points and at the infinity.

1. Introduction

The stationary phase approximation is a basic principle of asymptotic analysis, exemplified by the oscillatory integral

$$I(t') = \int g(t) e^{it'f(t)} dt.$$  

If the derivative of $f(t)$ does not vanish at any point in Supp$(f)$, then $I(t')$ is rapidly decreasing at $\infty$. If $f(t)$ has only finitely many critical points in Supp$(f)$, the major contribution to the value of the integral $I(t')$ for large $t'$ comes from neighborhoods of those critical points. More generally, consider the integral

$$I(t') = \int_{a(t')}^{b(t')} g(t, t') e^{if(t, t')} dt',$$

where all the functions are real-valued. Under certain conditions, for $t' \to \infty$,

$$I(t') = \sum_{f(t, t') = 0} \left( g(t, t') \sqrt{\frac{2\pi}{|f_{tt}(t, t')|}} e^{if(t, t')} + \frac{i\pi}{2} \text{sgn} f_{tt}(t, t') + o \left( \frac{g(t, t')}{\sqrt{|f_{tt}(t, t')|}} \right) \right).$$

The classical principle of stationary phase outlined above relates to the real Fourier transform. To study Deligne’s $\ell$-adic Fourier transform, Gérard Laumon [1987] introduced a corresponding principle of stationary phase and the local $\ell$-adic Fourier transform. (See [Katz 1988] for a good exposition.) We are interested in the $D$-module case.

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We fix a field $k$ of characteristic 0 and use the following notations:

(1) Let $p_1$, $p_2$ be the projections $\text{Spec } k[t, t'] = \mathbb{A}^1_k \times_k \mathbb{A}^1_k \to \mathbb{A}^1_k$, and let $\tilde{p}_1$, $\tilde{p}_2$ be the projections $\mathbb{P}^1_k \times_k \mathbb{P}^1_k \to \mathbb{P}^1_k$. Let $\alpha : \mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$ and $\mu : \mathbb{A}^1_k \times_k \mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k \times_k \mathbb{P}^1_k$ be the inclusions.

(2) For any $x \in k$, let $t_x = t - x$ and $t'_x = t' - x$. Let $t_\infty = 1/t = z$, $t'_\infty = 1/t' = z'$ and $\eta = \text{Spec } k(t')$. For any $x \in k \cup \{\infty\}$, let $\eta_x = \text{Spec } k((t_x))$, $\eta'_x = \text{Spec } k((t'_x))$.

(3) For any $x$, $y \in k \cup \{\infty\}$, let $k((t_x, t'_y))$ be the field of the formal Laurent series $\sum_{i,j \gg -\infty} a_{ij} t_x^i t_y^j$, $a_{ij} \in k$. For any $k((t_x))$-vector space $M$, let $M((t'_y)) = M \otimes_{k((t_x))} k((t_x, t'_y))$.

(4) Denote by $\mathcal{L}$ the rank-one connection $(\mathcal{O}_{\mathbb{A}^1_k}, d+dt)$ on $\mathbb{A}^1_k$. Then $\mathcal{L}$ corresponds to the $D$-module $\mathcal{O}_{\mathbb{A}^1_k} \cdot e^t$ on $\mathbb{A}^1_k$. So $\mathcal{L}$ is a substitute of $e^{it}$ in classical Fourier analysis. Let $X$ be a scheme. Any section $f \in \mathcal{O}(X)$ defines a morphism $\phi : X \to \mathbb{A}^1_k$ and let $\mathcal{L}_f = \phi^* \mathcal{L}$.

Let $\mathcal{M}$ be a vector bundle with a connection $\nabla$ on a nonempty open subscheme $U$ of $\mathbb{A}^1_k$ and let $i : U \hookrightarrow \mathbb{A}^1_k$ and $j : U \to \mathbb{P}^1_k$ be the inclusions. The connection $\nabla$ on $\mathcal{M}$ can be extended to a connection $i_* \nabla$ on $i_* \mathcal{M}$ and a connection $j_* \nabla$ on $j_* \mathcal{M}$. The global (geometric) Fourier transform of the $D$-module $i_* \mathcal{M}$ on $\mathbb{A}^1_k$ is defined to be

$$\mathcal{F}(i_* \mathcal{M}) = p_{2+}(p_1^* i_* \mathcal{M} \otimes_{\mathcal{O}_{\mathbb{A}^1_k} \times \mathcal{O}_{\mathbb{A}^1_k}} \mathcal{L}_f)[1],$$

where $\otimes$ and $p_{2+}$ are derived functors of $D$-modules. This definition is analogous to

$$\hat{f}(t') = \int f(t)e^{i t'} dt.$$

More precisely, we have

$$\mathcal{F}(i_* \mathcal{M}) \cong R^1 p_{2+}(p_1^* i_* \mathcal{M} \otimes_{\mathcal{O}_{\mathbb{A}^1_k} \otimes \mathcal{O}_{\mathbb{A}^1_k}} \mathcal{L}_f)[1],$$

where

$$\mathcal{F}(i_* \mathcal{M}) \cong R^1 p_{2+}(p_1^* i_* \mathcal{M} \otimes_{\mathcal{O}_{\mathbb{A}^1_k} \otimes \mathcal{O}_{\mathbb{A}^1_k}} \mathcal{L}_f)[1].$$

Consider the complex

$$\mathcal{F}(i_* \mathcal{M})|_{\eta_{\infty'}} = R^1 \tilde{p}_{2+}(\mathcal{L}_f)[1].$$

We have

$$\mathcal{F}(i_* \mathcal{M})|_{\eta_{\infty'}} = R^1 \tilde{p}_{2+}(\mathcal{L}_f)[1].$$
To study $\mathcal{F}(i_*\mathcal{M})|_{\eta^\infty}$, one needs to study $R^1\bar{\rho}_{2*}(\ast)|_{\text{Spf} k[[z]]}$. The complex ($\ast$) involves quasicoherent sheaves that may not be coherent sheaves. To study the localization of ($\ast$) on $\text{Spf} k[[z]]$, we need to transform them into coherent sheaves. For this reason, Bloch and Esnault [2004] rewrote ($\ast$) in terms of the cohomology of a complex of coherent modules. They found a good lattice pair $\mathcal{V}, \mathcal{W}$ of the connection $j_*\mathcal{M}$ such that $(\bar{\rho}^*_1 j_*\nabla + t'dt)(\bar{\rho}^*_1\mathcal{V}) \subset \bar{\rho}^*_1(\Omega^1_{\mathbb{P}^1_k}(T) \otimes \mathcal{W})$ and the inclusion of complexes

$$
(\bar{\rho}^*_1\nabla + t'dt)(\bar{\rho}^*_1\mathcal{V}) \subset \bar{\rho}^*_1(\Omega^1_{\mathbb{P}^1_k}(T) \otimes (\mathcal{W} + \mathcal{V}(|\infty|)))(\mathbb{P}^1_k \times \{\infty\})
$$

and a subcomplex

$$
(1-1) \quad (\bar{\rho}^*_1\nabla + t'dt)(\bar{\rho}^*_1\mathcal{V}) \subset \bar{\rho}^*_1(\Omega^1_{\mathbb{P}^1_k}(T) \otimes (\mathcal{W} + \mathcal{V}(|\infty|)))(\mathbb{P}^1_k \times \{\infty\})
$$

of ($\ast$). This inclusion of complexes (1-1) $\subset$ ($\ast$) is still not a quasi-isomorphism. Using Deligne’s construction of good lattice pairs, we find a good lattice pair $\mathcal{V}, \mathcal{W}$ of $j_*\mathcal{M}$ in Lemma 2.3 such that (1-1) $|\mathbb{P}^1_k \otimes k(t') \subset$ ($\ast$) $|\mathbb{P}^1_k \otimes k(t')$ is a quasi-isomorphism. From this, we get the following stationary phase formula.

**Theorem 1.1.** Let $\mathcal{M}$ be a vector bundle with a connection $\nabla$ on a nonempty open subscheme $U$ of $\mathbb{A}^1_k$, and let $i : U \hookrightarrow \mathbb{A}^1$ be the inclusion. Suppose all points in $\mathbb{A}^1_k \setminus U$ are $k$-rational. Then the natural map

$$
(1-2) \quad \mathcal{F}(i_*\mathcal{M})|_{\eta^\infty} \to \bigoplus_{x \in \mathbb{A}^1_k \setminus U} \text{coker}((\mathcal{M}|_{\eta_x})(z')^\prime \partial_{\eta_x} + 1) \to (\mathcal{M}|_{\eta_x})(z'))
$$

is an isomorphism of formal connections on $k((z'))$.

The direct summands on the right side of (1-2) induce the definition of local Fourier transforms for formal connections.

The paper is organized as follows. In Section 2, we discuss the good lattice pairs of connections on a smooth curve. Passing to the stalks, we discuss the good lattice pairs of connections on a discrete valuation field. In Section 3, we prove the stationary phase formula using proper base change theorem between formal schemes.
2. Good lattice pairs

Let $X$ be a smooth algebraic curve over $k$ and $j: X \hookrightarrow \bar{X}$ the smooth compactification. Let $\mathcal{F}$ be a vector bundle on $X$ with a connection $\nabla$. Set $\Sigma = \bar{X} - X$. A pair of good lattices $\mathcal{V}, \mathcal{W}$ of $j_*\mathcal{F}$ is a pair of vector bundles on $\bar{X}$ which extends $\mathcal{F}$ and satisfies the following conditions:

1. $\mathcal{V} \subset \mathcal{W} \subset j_*\mathcal{F}$.
2. $\nabla(\mathcal{V}) \subset \Omega^1_X(\Sigma) \otimes \mathcal{W}$.
3. For any effective divisor $D$ supported on $\Sigma$, the inclusion of complexes

$$(\mathcal{V} \xrightarrow{\nabla} \Omega^1_X(\Sigma) \otimes \mathcal{W}) \to (\mathcal{V}(D) \xrightarrow{\nabla} \Omega^1_X(\Sigma) \otimes \mathcal{W}(D))$$

is a quasi-isomorphism. Taking the direct limit with respect to $D$, we get a quasi-isomorphism:

$$(\mathcal{V} \xrightarrow{\nabla} \Omega^1_X(\Sigma) \otimes \mathcal{W}) \to (j_*\mathcal{F} \xrightarrow{\nabla} \Omega^1_X \otimes j_*\mathcal{F})$$.

The existence of good lattice pairs can be passed to the stalks. So we only need to consider the local case: good lattice pairs of connections on a discrete valuation field.

Let $K$ be a discrete valuation field with the valuation $v$. Let $A$ be the valuation ring, $t$ a uniformizer, and $\partial$ a continuous derivation on $K$ such that $\partial(t) = 1$ and $\partial(A) \subseteq A$.

**Definition 2.1.** A connection on $K$ (of rank $k$, where $k$ is finite) is a $k$-dimensional vector space $M$ over $K$ with an additive map $\partial: M \to M$ satisfying $\partial(fm) = f\partial(m) + \partial(f)m$ for any $f \in K$ and $m \in M$.

Let $r$ be the rank of the connection $M$. Set $\tau = t\partial$. There exists a cyclic element $v \in M$, in the sense that the elements $\tau^i v$, for $0 \leq i \leq r - 1$, form a basis of $M$ over $K$. We have

$$\tau^r v = \sum_{0 \leq i \leq r-1} a_i \tau^i v$$

for some $a_i \in K$. The Newton polygon $N(M)$ of $M$ is the convex hull of

$$\{(u, v) \mid u \leq i, v \geq v(a_i)\}$$

in the plane $\mathbb{R}^2$. The slopes of $M$ are the slopes of nonvertical edges of $N(M)$, and we eliminate the slope 0 if the horizontal edge is situated in $u \leq 0$. The slopes are independent of the choice of the cyclic elements. The sum of all the slopes of $M$ is called the irregularity of $M$, and is denoted by $i(M)$. Then

$$i(M) = \max_{0 \leq i \leq r} (0, -v(a_i)).$$
A lattice of $M$ is a finitely generated $A$-submodule $V$ of $M$ that spans $M$. For any artinian $A$-module $V$, the length of $V$ is denoted by $\ell(V)$.

**Definition 2.2.** A pair of lattices $V, W$ of $(M, \partial)$ is called good if the following conditions are satisfied

1. $V \subset W \subset M$.
2. $\partial V \subset (1/t)W$.
3. For any $i \in \mathbb{N}$, the natural inclusion of complexes
   $$
   (V \xrightarrow{\partial} \frac{1}{t}W) \to (\frac{1}{t^i}V \xrightarrow{\partial} \frac{1}{t^{i+1}}W)
   $$
   is a quasi-isomorphism.

Note that if $V, W$ is a good lattice pair, so is $(1/t^i)W, (1/t^i)V$ for any $i \in \mathbb{N}$.

Condition (3) above is equivalent to the following:

$(3')$ For any $i \in \mathbb{N}$, the map

$$
\frac{1}{t^i}V / \frac{1}{t^{i-1}}V \xrightarrow{\text{gr,} \partial} \frac{1}{t^{i+1}}W / \frac{1}{t^i}W
$$

induced by $\partial$ is an isomorphism.

One can show that $i(M) = \ell(W/V)$.

**Lemma 2.3.** Let $k \hookrightarrow k'$ be an extension of fields of characteristic 0. Let $\partial_t$ be the natural derivation on $k(t)$ and on $k'(t)$. The variable $t$ defines a discrete valuation $v$ on $k(t)$ and $k'(t)$. Let $A$ and $A'$ be their discrete valuation rings, respectively. Suppose $c$ is an element in $k'$ which is not algebraic over $k$. Let $M$ be a connection on $k(t)$, and let $M_c$ be the connection on $k'(t)$ whose underlying space is the $k'(t)$-vector space $M \otimes_{k(t)} k'(t)$, and with the operation $\partial_t$ defined by

$$
\partial_t(m \otimes f) = \partial_t(m) \otimes f + m \otimes \partial_t(f) - m \otimes \frac{c}{t^2}
$$

for any $m \in M$ and any $f \in k'(t)$. Then there exists a good lattice pair $V, W$ of $M$, such that $V \otimes_A A', (W + (1/t)V) \otimes_A A'$ is also a good lattice pair of the connection $M_c$ on $k'(t)$.

**Proof.** Set $r = \text{rk} M$. Choose a cyclic element $v$ of $M$. Let $\varepsilon$ be the basis $\{\tau^i v \mid 0 \leq i \leq r - 1\}$ of $M$ over $k(t')$. We have $\tau^r v = \sum_{0 \leq i < r} a_i \tau^i v$ for some $a_i \in K$. The irregularity $i(M)$ of $M$ is $\max_{0 \leq i < r} (0, -v(a_i))$. Consider the Newton polygon of the differential operator $\tau^r - \sum_{0 \leq i \leq r-1} a_i \tau^i$. Let $j$ be the integer such that $(j, v(a_j))$ is a vertex of this Newton polygon, and such that the slopes of
this Newton polygon on the right side (respectively left side) of \((j, v(a_j))\) is \(> 1\) (respectively \(\leq 1\)). Set \(a_r = 1\). Then we have

\[
v(a_{j+i}) - v(a_j) > i \quad \text{for any } 1 \leq i \leq r - j,
\]
\[
v(a_{j-i}) - v(a_j) \geq -i \quad \text{for any } 0 \leq i \leq j.
\]

Then

\[
(2-1) \quad v(a_j) - j = \min_{0 \leq i \leq r} (v(a_i) - i).
\]

The matrix of the differential operator \(\tau\) with respect to the basis \(\varepsilon\) is

\[
\Gamma = \begin{pmatrix}
0 & a_0 \\
1 & a_1 \\
& \ddots & \ddots \\
& & 1 & a_{r-1}
\end{pmatrix}
\]

The characteristic polynomial of \(\Gamma\) is \(\lambda^r - \sum_{0 \leq i \leq r-1} a_i \lambda^i\). Let

\[
\Lambda = \text{diag}\{1, \ldots, 1, t, \ldots, t^{r-j+i(M)+v(a_j)}\},
\]

and let \(e = \varepsilon \Lambda = \{e_i \mid 0 \leq i < r\}\). Set \(l = j - v(a_j) - i(M) \geq 0\). Then the matrix of the differential operator \(\tau\) with respect to the basis \(e\) is

\[
\Gamma' = \begin{pmatrix}
0 & t^{r-l}a_0 \\
1 & t^{r-l}a_1 \\
& \ddots & \ddots \\
& & 1 & t^{r-l}a_{l-1} \\
& & & 1/t & t^{r-l-1}a_l \\
& & & & \ddots \\
& & & & & 1/t & a_{r-1}
\end{pmatrix} + \text{diag}\{0, \ldots, 0, 1, \ldots, r-l\}.
\]

Let \(P(\lambda) = \lambda^r - \sum_{0 \leq i \leq r-1} a_i' \lambda^i\) be the characteristic polynomial of \(\Gamma'\). Since

\[
\Gamma' = \Lambda^{-1} \Gamma \Lambda + \text{diag}\{0, \ldots, 0, 1, \ldots, r-l\},
\]

we have

\[
a_i' - a_i \in \sum_{i < j < r} \mathbb{Z}a_j + \mathbb{Z} \subset K.
\]

So

\[
\max\{0, -v(a_i') \mid 0 \leq i < r\} = \max\{0, -v(a_i) \mid 0 \leq i < r\} = i(M).
\]

Write \(P(\lambda) = t^{-i(M)} \sum_i b_i \lambda^i\), \(b_i \in K\). Then \(b_i \in A\) and \(v(b_i) = 0\) for at least one \(i\). The residue polynomial \(\sum_i b_i \lambda^i\) of \(\sum_i b_i \lambda^i\) is nonzero. For almost all \(n \in \mathbb{Z}\),
\[ \sum_i \bar{b}_i(-n)^i \neq 0. \] In this case, we have
\[ -v(\det(n + \Gamma')) = -v((-1)^r P(-n)) = -v(t^{-i(M)}\left(\sum_i b_i(-n)^i\right))^i = i(M). \]
Then, for almost all \( n \in \mathbb{Z} \),
\[ (2-2) \quad i(M) = -v(\det(n + \Gamma')). \]
Let \( V \) be the lattice of \( M \) generated by \( e \). Define
\[ (2-3) \quad [(n + \Gamma')V : V] = \ell((n + \Gamma')V + V/V) - \ell((n + \Gamma')V + V/(n + \Gamma')V). \]
By [Deligne 1970, p. 48, Proposition 2], we have
\[ (2-4) \quad [(n + \Gamma')V : V] = -v(\det(n + \Gamma')). \]
Let \( W \) be the lattice of \( M \) generated by
\[ e_0, \ldots, e_{l-1}, \frac{1}{l} e_l, \ldots, \frac{1}{l} e_{r-1}. \]
Then \( \ell(W/V) = r - l \). Since \((n + \Gamma')V + V)/W \) is an artinian \( A \)-module generated by the single element
\[ x = \sum_{0 \leq i \leq l-1} a_i t^{r-l} e_i + \sum_{l \leq i \leq r-1} a_i t^{r-1-i} e_i = \sum_{0 \leq i \leq l-1} a_i t^{r-l} e_i + \sum_{l \leq i \leq r-1} a_i t^{r-1-i} \frac{1}{l} e_i. \]
For any \( i \), we have \( i(M) \geq -v(a_i) \) and \( v(a_j) - j \leq v(a_i) - i \). Then
\[ v(t^{i(M)+l-r} a_i t^{r-l}) \geq 0 \quad \text{and} \quad v(t^{i(M)+l-r} a_i t^{r-j}) \geq v(t^{i(M)+l-r} a_j t^{r-j}) = 0. \]
Then the annihilator of \( x \) in \((n + \Gamma')V + V)/W \) is \( t^{i(M)+l-r} \). So
\[ \ell((n + \Gamma')V + V/W) = i(M) + l - r. \]
Then
\[ (2-5) \quad \ell((n + \Gamma')V + V/V) = \ell(W/V) + \ell((n + \Gamma')V + V/W) = i(M). \]
Comparing this equality with (2-2), (2-3), and (2-4), we get
\[ \ell((n + \Gamma')V + V/(n + \Gamma')V) = 0 \]
for almost \( n \in \mathbb{Z} \), that is, \((n + \Gamma')V \supset V \) for almost all \( n \in \mathbb{Z} \).

The \( A \)-module
\[ (n + \Gamma')V + \frac{1}{l} V \bigg/ \frac{1}{l} V \]
is artinian and is generated by one element \( x \) whose annihilator is
\[ t^{i(M)+l-r} = t^{j-v(a_i)-r}. \]
By (2-1), we have

$$\ell \left( (n + \Gamma') V + \frac{1}{t} V / V \right) = \ell \left( (n + \Gamma') V + \frac{1}{t} V / \frac{1}{t} V \right) + \ell \frac{1}{t} V / V = j - v(a_j) = \sum_{\lambda: \text{slope of } M} \max(\lambda, 1).$$

Then

$$(2-6) \quad \ell \left( (n + \Gamma') V + \frac{1}{t} V / V \right) = \ell \left( (n + \Gamma') V + \frac{1}{t} V / \frac{1}{t} V \right) + \ell \frac{1}{t} V / V = j - v(a_j) = \sum_{\lambda: \text{slope of } M} \max(\lambda, 1).$$

The matrix of the differential operator $\tau$ with respect to the basis $e$ of $M_c$ is $\Gamma - c/t$. The characteristic polynomial $P'(\lambda)$ of $\Gamma - c/t$ is

$$P'(\lambda) = \left( \lambda + \frac{c}{t} \right)^r - \sum_{0 \leq i < r} a_i \left( \lambda + \frac{c}{t} \right)^i.$$

Write $P'(\lambda) = \lambda^r + \sum_{0 \leq i < r} b_i \lambda^i$ for some $b_i \in k'(t)$. Then

$$b_0 = \left( \frac{c}{t} \right)^r - \sum_{0 \leq i < r} a_i \left( \frac{c}{t} \right)^i = \frac{a_j}{t^j} \left( \frac{1}{a_j t^{r-j}} c^r - \sum_{0 \leq i < r} \frac{a_i}{a_j t^{i-j}} c^i \right).$$

By (2-1), we have

$$\frac{1}{a_j t^{r-j}} c^r - \sum_{0 \leq i < r} \frac{a_i}{a_j t^{i-j}} c^i \in A[c],$$

and its residue in $k'$ is a nonzero polynomial over $k$ of $c$. Since $c$ is not algebraic over $k$, this residue is nonzero. Then we have

$$v(b_0) = v\left( \frac{a_j}{t^j} \right) = v(a_j) - j.$$

Also by (2-1), we have $v(b_i) \geq v(b_0)$ for any $0 \leq i < r$. So

$$\max_{0 \leq i < r} (0, -v(b_i)) = j - v(a_j) = i(M_c).$$

The matrix of the differential operator $\tau$ with respect to the basis $e$ of $M_c$ is $\Gamma'' = \Gamma' - c/t$. Write the characteristic polynomial of $\Gamma''$ as $\lambda^r + \sum_{0 \leq i < r} b'_i \lambda^i$ for some $b'_i \in k'(t)$. By a similar proof as above, we have

$$\max_{0 \leq i < r} (0, -v(b'_i)) = \max_{0 \leq i < r} (0, -v(b_i)) = i(M_c).$$

For almost $n \in \mathbb{Z}$, we have

$$-v(\det(n + \Gamma'')) = i(M_c).$$

Let $V' = V \otimes_A A'$. We have $(n + \Gamma'') V' + V' \subseteq \frac{1}{t} V' + \Gamma' V'$; therefore So

$$(2-7) \quad \ell((n + \Gamma'') V' + V' / V') \leq \ell\left( \frac{1}{t} V' + \Gamma' V' / V' \right).$$

Since $A \to A'$ is flat and $k \otimes_A A' = k'$, for any artinian $A$-module $M$, one can prove $\ell(M) = \ell(M \otimes_A A')$. Since $(1/t) V + \Gamma' V / V$ is an artinian $A$-module, by (2-6),
we have
\begin{equation}
\ell\left(\frac{1}{t}V' + \Gamma'V' \big/ V' \right) = \ell\left(\frac{1}{t}V + \Gamma'V \big/ V \right) = j - v(a_j).
\end{equation}

By (2-4), we have, for almost \( n \in \mathbb{Z} \),
\[
\ell\((n+\Gamma'')V' + V'/V'\) \geq \ell\((n+\Gamma'')V' + V'/V'\) - \ell\((n+\Gamma'' + V'/(n+\Gamma'')V'
\]
\[
= -v(\text{det}(n+\Gamma'')) = j - v(a_j).
\]

Comparing this inequality with (2-7) and (2-8), we have for almost \( n \in \mathbb{Z} \),
\[
\ell\((n+\Gamma'')V' + V'/V'\) = j - v(a_j);
\]
\[
\ell\((n+\Gamma'')V' + V'/V'\) = 0;
\]
\begin{equation}
(n+\Gamma'')V' + V' = \frac{1}{t}V' + \Gamma'V' = \left(\frac{1}{t}V + \Gamma'V\right) \otimes_A A'.
\end{equation}

So for almost \( n \in \mathbb{Z} \), \((n+\Gamma'')V' \supseteq V'\). Let \( e'= (1/tN)e \). The matrix of \( \tau \) with respect to the basis \( e' \) of \( M \) (respectively \( M' \)) is \( \Gamma_1 := \Gamma' - N \) (respectively \( \Gamma_2 := \Gamma'' - N \)).

Let \( \mathcal{V} = (1/tN) \mathcal{V} \) and let \( \mathcal{V}' = (1/tN) \mathcal{V}' \). Choose \( N \) large enough so that for any \( n \leq 0 \), we have
\[
(n + \Gamma_1)\mathcal{V} \supseteq \mathcal{V} \quad \text{and} \quad (n + \Gamma_2)\mathcal{V}' \supseteq \mathcal{V}'.
\]

Let \( \mathcal{W} = \Gamma_1 \mathcal{V} \). By (2-9), we have \( \Gamma_2 \mathcal{V}' = (\mathcal{W} + (1/t)\mathcal{V}) \otimes_A A' \). Let’s prove \( \mathcal{V}, \mathcal{W} \)

is a good lattice pair of \( M \) now. We only need to verify condition (3') for any \( i \in \mathbb{N} \).

Conjugating by \( 1/ti \), the \( A \)-linear map
\[
\text{gr}_i \tau : \left\{ \frac{1}{t^i} \mathcal{V}' \big/ \frac{1}{t^i-1} \mathcal{V}' \right\} \to \left\{ \frac{1}{t^i} \mathcal{W} \big/ \frac{1}{t^i-1} \mathcal{W} \right\}
\]

can be identified with the \( A \)-linear map
\[
\text{gr}_0 \tau - i = \Gamma_1 - i : \mathcal{V}'/t\mathcal{V}' \to \mathcal{W}/t\mathcal{W}.
\]

Since \((\Gamma_1 - i)\mathcal{V} \supseteq \mathcal{V}'\), we have
\[
(\Gamma_1 - i)\mathcal{V} = (\Gamma_1 - i)\mathcal{V} + V \supseteq \Gamma_1 \mathcal{V} = \mathcal{W}.
\]

So \( \Gamma_1 - i : \mathcal{V}'/t\mathcal{V}' \to \mathcal{W}/t\mathcal{W} \) is surjective. But the domain and the range of \( \text{gr}_i \tau \)
are artinian \( A \)-modules of the same length \( r \), so \( \text{gr}_0 \tau - i \) is an isomorphism and so is \( \text{gr}_i \tau \). This proves \( \mathcal{V}, \mathcal{W} \)
is a good lattice pair of \( M \). Repeating the proof, we conclude that \( \mathcal{V} \otimes_A A' \), \((\mathcal{W} + (1/t)\mathcal{V}) \otimes_A A' \) is a good lattice pair of \( M_e \).

**Remark 2.4.** Lemma 2.3 is the main technical lemma for the proof of the stationary phase principle in the next section. Lemma 2.3 also allows us to choose a good lattice pair \( \mathcal{V}, \mathcal{W} \) of \( M \) such that
\begin{equation}
\dim_k\left([\mathcal{W} + \frac{1}{t} \mathcal{V}]/\mathcal{V}\right) = \sum_{\lambda \text{ : slope of } M} \max(\lambda, 1).
\end{equation}
Formula (2-10) is easily seen to give a new proof of the following result:

**Lemma 2.5** [Bloch and Esnault 2004, Lemma 3.3]. Let $M$ be a connection on $K$. The slopes of $M$ are all $\leq 1$ (respectively $\geq 1$) if and only if there exists a good lattice pair $\mathcal{V}$, $\mathcal{W}$ such that $\mathcal{W} \subseteq (1/t)\mathcal{V}$ (respectively $\mathcal{W} \supseteq (1/t)\mathcal{V}$).

(Note that the original proof by Bloch and Esnault needs the assumption that $K$ is complete.)

3. **Stationary phase principle**

Let $K = k(t')$. For any scheme $X$ over $k$ and any $\mathcal{O}_X$-modules $\mathcal{F}$, let $X_K = X \otimes_k K$ and $\mathcal{F}_K = \mathcal{F}|_{X_K}$. For any $k$-morphism $f : X \to Y$, let $f_K : X_K \to Y_K$ be the base change of $f$.

We keep the notation used in Section 1. In this section we prove Theorem 1.1. For any $x \in T_K = T$, $(\mathcal{V}_x)_x$, $(\mathcal{W}_x)_x$ is a good lattice pair of the connection $(j_{K*}M_K)_x$ on $K(t_x)$. Since $t'$ is not algebraic over $k$, by Lemma 2.3, we may assume that
\[ \mathcal{V}_x \otimes_{\mathcal{O}_{\mathcal{P}_x}} \mathcal{O}_{\mathcal{P}_x}, \quad (\mathcal{W}_x + \frac{1}{z} \mathcal{V}_x) \otimes_{\mathcal{O}_{\mathcal{P}_x}} \mathcal{O}_{\mathcal{P}_x} \]
is a good lattice pair of the connection
\[ \partial_{z} - \frac{t'}{z^2} : (j_{K*}M_K)_x \to (j_{K*}M_K)_x. \]

**Lemma 3.1.** The inclusion of complexes (1-1) $\subseteq$ (*) induces a quasi-isomorphism
\[ (1-1)|_{\mathcal{P}_x^K} \simeq (*)|_{\mathcal{P}_x^K}. \]

**Proof.** We have
\[ (1-1)|_{\mathcal{P}_x^K} = (\mathcal{V}_K^{j_{K*}V_K+t'dt} \mathcal{O}_{\mathcal{P}_x^K}(T_K) \otimes (W_K + \mathcal{V}_K((\infty)))) , \]
\[ (*)|_{\mathcal{P}_x^K} = (j_{K*}M_K^{j_{K*}V_K+t'dt} \mathcal{O}_{\mathcal{P}_x^K} \otimes j_{K*}M_K). \]

First we have $(1-1)|_{U_K} = (\ast)|_{U_K}$. For any $x \in S_K$, let's prove $(1-1)|_{\mathcal{P}_x^K} \subseteq (\ast)|_{\mathcal{P}_x^K}$ induces a quasi-isomorphism on the stalks at $x$. It suffices to show that
\[ \begin{pmatrix} \frac{1}{t_x}(\mathcal{V}_K)_x & \frac{1}{t_{i-1}}(\mathcal{V}_K)_x \\ \frac{1}{t_{i+1}}(W_K)_x & \frac{1}{t_x}(W_K)_x \end{pmatrix} \xrightarrow{\text{gr}(\partial_{t_x} + t')} \begin{pmatrix} \frac{1}{t_{i+1}}(W_K)_x & \frac{1}{t_x}(W_K)_x \end{pmatrix} \]
is an isomorphism for any $i \geq 1$. As $(\mathcal{V}_K)_x \subset (W_K)_x$, the map $\text{gr}(\partial_{t_x} + t')$ is equal to $\text{gr}(\partial_{t_x})$, which is an isomorphism by the definition of good lattices. The inclusion
\[ (1-1)|_{\mathcal{P}_x^K} \to (\ast)|_{\mathcal{P}_x^K} \]
can be written as

\[
\left( \mathcal{V}_\infty \otimes \mathcal{O}_{\mathbb{P}_k^1, \infty} \otimes \mathcal{O}_{\mathbb{P}_k^1, \infty}, \frac{\partial \tau}{\tau} - \frac{\partial z}{z} \right) \xrightarrow{1} \frac{1}{z} \left( \mathcal{W}_\infty + \frac{1}{z} \mathcal{V}_\infty \right) \otimes \mathcal{O}_{\mathbb{P}_k^1, \infty}
\]

\[
\subset \left( (j_{K^*}M_{K})_\infty \xrightarrow{\partial \tau - \frac{\partial z}{z}} (j_{K^*}M_{K})_\infty \right).
\]

It is a quasi-isomorphism by the assumption on \(\mathcal{V}_\infty\) and \(\mathcal{W}_\infty\). \(\square\)

**Lemma 3.2.** \( R^1 \tilde{p}_{2*}(1-1)|_{\eta'} \cong R^1 \tilde{p}_{2*}((*)|_{\eta'}). \)

**Proof.** Consider the Cartesian diagram

\[
\begin{array}{ccc}
\mathbb{P}_k^1 & \xrightarrow{=} & \eta' = \text{Spec } K \\
\downarrow & & \downarrow \\
\mathbb{P}_k^1 \times \mathbb{P}_k^1 & \xrightarrow{\tilde{p}_{2*}} & \mathbb{P}_k^1.
\end{array}
\]

By Lemma 3.1, we have

\[
R^1 \tilde{p}_{2*}(1-1)|_{\eta'} \cong H^1(\mathbb{P}_k^1, (1-1)|_{\mathbb{P}_k^1}) \cong H^1(\mathbb{P}_k^1, (*)|_{\mathbb{P}_k^1}) \cong R^1 \tilde{p}_{2*}((*)|_{\eta'}). \quad \square
\]

**Corollary 3.3.** \( \mathcal{F}(i_* M)|_{\eta'_\infty} = R^1 \tilde{p}_{2*}(1-1)|_{\eta'_\infty} \).

Denote by \( \mathbb{P}_k^1[[z']] \) the formal completion of \( \mathbb{P}_k^1 \times \mathbb{P}_k^1 \) along its closed subscheme \( \mathbb{P}_k^1 \times \{\infty\} \). For any coherent sheaf \( \mathcal{H} \) on \( \mathbb{P}_k^1 \), let \( \mathcal{H}[[z']] = \mathcal{H}|_{\mathbb{P}_k^1[[z']]} \).

**Lemma 3.4** [Bloch and Esnault 2004, Corollary 2.2].

\[
R^1 \tilde{p}_{2*}(1-1) \otimes \mathcal{O}_{\mathbb{P}_k^1} k[[z']] \cong H^1(\mathbb{P}_k^1[[z']], \mathcal{V}[[z']]) \cong \mathcal{H}[[z']].
\]

**Lemma 3.5** [Bloch and Esnault 2004, Lemma 2.4 and Corollary 2.5]. Let \( \mathcal{H} \) be the complex

\[
\mathcal{V}[[z']] \xrightarrow{z' \partial_{\mathcal{V}}} \mathcal{H}[[z']].
\]

Then \( \mathcal{H}^0 \) equals \( 0 \) and \( \mathcal{H}^1 \) is supported on \( T \subset \mathbb{P}_k^1[[z']] \). For any \( x \in T \), let \( \mathcal{V}_x = \mathcal{V}_x \otimes \mathcal{O}_{\mathbb{P}_k^1} k[[t_x]] \) and \( \mathcal{W}_x = \mathcal{W}_x \otimes \mathcal{O}_{\mathbb{P}_k^1} k[[t_x]] \). We have

\[
\mathcal{H}^1_x = \text{coker}\left( \mathcal{V}_x[[z']] \xrightarrow{z' \partial_{\mathcal{V}}} \mathcal{W}_x [[z']] \right).
\]

\( \square \)

**Corollary 3.6.**

\[
H^1(\mathbb{P}_k^1[[z']]) \cong \bigoplus_{x \in S} \text{coker}\left( \mathcal{V}_x[[z']] \xrightarrow{z' \partial_{\mathcal{V}} + 1} \mathcal{W}_x [[z']] \right) \oplus \text{coker}\left( \mathcal{W}_\infty[[z']] \xrightarrow{z' \partial_{\mathcal{W}} - \frac{1}{z}} \mathcal{W}_\infty + \frac{1}{z} \mathcal{V}_\infty [[z']] \right).
\]
Combining Corollary 3.3, Lemma 3.4 and Corollary 3.6, we have

\[
\mathcal{F}(i_* \mathcal{M})|_{\eta_{\infty}} = R^1 \hat{p}_2^* (1-1) \otimes_{\mathbb{C}_{\mathbb{P}^1}} k[[z']] \otimes k[[z']] k((z')) = \bigoplus_{x \in S} \text{coker} \left( \hat{V}_x ((z')) \xrightarrow{z' \hat{a}_x + 1} \frac{1}{t_x} \hat{W}_x ((z')) \right) + \text{coker} \left( \hat{V}_\infty ((z')) \xrightarrow{z' \hat{a}_\infty + 1} \frac{1}{z} (\hat{W}_\infty + \frac{1}{z} \hat{V}_\infty) ((z')) \right).
\]

The left side of this equality is independent of the choice \( \mathcal{V} \) and \( \mathcal{W} \). For any \( i \in \mathbb{N} \), \( \mathcal{V}(iT) \) and \( \mathcal{W}(iT) \) still satisfy the condition of Lemma 3.1. Then the above equality holds if we replace \( \mathcal{V} \) and \( \mathcal{W} \) by \( \mathcal{V}(iT) \) and \( \mathcal{W}(iT) \), respectively. Taking the direct limit on \( i \), we have

\[
\mathcal{F}(i_* \mathcal{M})|_{\eta_{\infty}} = \bigoplus_{x \in S} \text{coker} \left( (\mathcal{M}|_{\eta_x}) ((z')) \xrightarrow{z' \hat{a}_x + 1} (\mathcal{M}|_{\eta_x}) ((z')) \right) + \text{coker} \left( (\mathcal{M}|_{\eta_{\infty}}) ((z')) \xrightarrow{z' \hat{a}_\infty + 1} (\mathcal{M}|_{\eta_{\infty}}) ((z')) \right).
\]

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