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We focus on the uniqueness problem of a 3D transonic shock solution in a conic nozzle when the variable end pressure in the diverging part of the nozzle lies in an appropriate scope. By establishing the monotonicity of the position of shock surface relative to the end pressure, we remove the nonphysical assumptions on the transonic shock past a fixed point made in previous studies and further obtain uniqueness.

1. Introduction and the main results

We study the uniqueness of a 3D transonic shock in a conic nozzle when the variable end pressure of the diverging part lies in an appropriate scope. The transonic shock problem in a nozzle is a fundamental one in fluid dynamics and has been extensively studied by many authors under various assumptions, for example, that either the transonic flow is quasi-one-dimensional or that the transonic shock goes through some fixed point in advance; see [Liu 1982; Embid et al. 1984; Chen et al. 2007; Chen 2008; Chen and Yuan 2008; Xin and Yin 2008a; 2008b; Xin et al. 2009] and so on. However, Courant and Friedrichs [1948, p. 386] indicated that transonic shock in a nozzle can be formulated as follows: Given appropriately large end pressure $p_e(x)$, if the upstream flow is still supersonic behind the throat of the three-dimensional de Laval nozzle, then at a certain place in the diverging part of the nozzle, a shock front intervenes and the gas is compressed and slowed down to subsonic speed. The position and the strength of the shock front are automatically adjusted so that the end pressure at the exit becomes $p_e(x)$. This statement indicates that the position of the transonic shock should be completely

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free. More importantly, the assumption of shock going through some fixed point in advance will lead in general to the transonic shock problem not being well-posed [Xin and Yin 2008a; Xin et al. 2009]. On the other hand, Courant and Friedrichs [1948, pp. 372, 375] pointed out that it is a question of great importance to know under what circumstances a steady flow involving shocks is uniquely determined and stable by the boundary conditions and by the conditions at the entrance, and when further conditions at the exit are appropriate. Motivated by these two basic problems, in this paper, we will establish the uniqueness result on a 3D transonic shock solution for the 3D Euler system when the variable end pressure $p_e(x)$ of the conic part of the nozzle lies in an appropriate scope without the assumption that the shock goes through a fixed point in advance. The existence of a 3D transonic shock solution under suitable restrictions on the end pressures was given in [Li et al. 2010].

We will consider only the isentropic gas for simplicity. By a slight modification, our discussions also apply to the nonisentropic case. The steady isentropic Euler system in three-dimensional spaces is

$$\begin{cases} \operatorname{div}(\rho u) = 0, \\ \operatorname{div}(\rho u \otimes u) + \nabla p = 0, \end{cases} \tag{1-1}$$

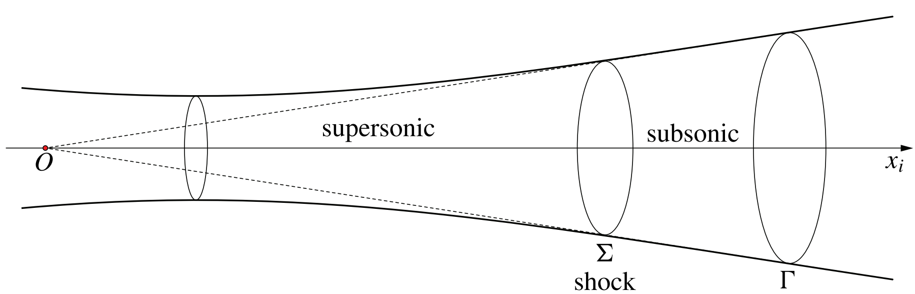
where $u = (u_1, u_2, u_3)$, ρ and P are the velocity, density and pressure, respectively. Moreover, the pressure function $P = P(\rho)$ is smooth with $P'(\rho) > 0$ for $\rho > 0$, and $c(\rho) = \sqrt{P'(\rho)}$ is called the local sound speed.

For ideal polytropic gases, the equation of state is given by

$$P = A\rho^{\gamma},$$

where A and γ are positive constants and $1 < \gamma < 3$.

It will be assumed that the nozzle wall Γ is $C^{4,\alpha}$-regular for $X_0 - 1 \le r = \sqrt{x_1^2 + x_2^2 + x_3^2} \le X_0 + 1$, where $X_0 > 1$ is a fixed constant and $\alpha \in (0, 1)$, and the wall Γ consists of two curved surfaces Π_1 and Π_2, where Π_1 includes the converging part of the nozzle and Π_2 is the conic diverging part of the nozzle (see figure). More precisely, the equation of Π_2 is represented by $x_2^2 + x_3^2 = x_1^2 \tan^2 \theta_0$ with $x_1 > 0$ and $X_0 < r < X_0 + 1$, where $0 < \theta_0 < \pi/2$ is sufficiently small. For



simplicity, we suppose that the $C^{4,\alpha}$-smooth supersonic incoming flow

$$(\rho_0^-(x), u_{1,0}^-(x), u_{2,0}^-(x), u_{3,0}^-(x))$$

is symmetric near $r = X_0$, where

$$\rho_0^-(x) = \rho_0^-(r) \quad \text{and} \quad u_{i,0}^-(x) = \frac{U_0^-(r)x_i}{r} \quad \text{for } i = 1, 2, 3$$

(this assumption can be easily realized by the hyperbolicity of the supersonic incoming flow and the symmetry of the nozzle wall for $X_0 < r < X_0 + 1$).

Denote the equation of the possible multidimensional shock Σ in the nozzle by $x_1 = \eta(x_2, x_3)$ and the flow field behind the shock by

$$(\rho^+(x), u_1^+(x), u_2^+(x), u_3^+(x)).$$

Then the Rankine–Hugoniot conditions on Σ are

$$\begin{cases} [\rho u_1] - \partial_2\eta(x_2, x_3)[\rho u_2] - \partial_3\eta(x_2, x_3)[\rho u_3] = 0, \\ [P + \rho u_1^2] - \partial_2\eta(x_2, x_3)[\rho u_1 u_2] - \partial_3\eta(x_2, x_3)[\rho u_1 u_3] = 0, \\ [\rho u_1 u_2] - \partial_2\eta(x_2, x_3)[P + \rho u_2^2] - \partial_3\eta(x_2, x_3)[\rho u_2 u_3] = 0, \\ [\rho u_1 u_3] - \partial_2\eta(x_2, x_3)[\rho u_2 u_3] - \partial_3\eta(x_2, x_3)[P + \rho u_3^2] = 0. \end{cases} \tag{1-2}$$

In addition, $P^+(x)$ should satisfy the physical entropy condition (see [Courant and Friedrichs 1948])

$$P^+(x) > P^-(x) \quad \text{on } x_1 = \eta(x_2, x_3). \tag{1-3}$$

On the exit of the nozzle, we place the end pressure condition

$$P^+(x) = P_e + \varepsilon P_0(x_2, x_3) \quad \text{on } r = X_0 + 1, \tag{1-4}$$

where $\varepsilon > 0$ is sufficiently small and

$$P_0(x_2, x_3) \in C^{3,\alpha}\{(x_2, x_3) : x_2^2 + x_3^2 \le (X_0 + 1)^2 \sin^2\theta_0\}.$$

The positive constant P_e stands for the end pressure when a symmetric shock lies at the position $r = r_0$ with $r_0 \in (X_0, X_0 + 1)$ and the supersonic incoming flow admits the state $(\rho_0^-(r), U_0^-(r))$. For detailed information on P_e, see Theorem A.1 in Appendix A.

The flow is assumed to be tangent to the nozzle wall Γ, thus,

$$x_1 u_1^+ \tau^2 - x_2 u_2^+ - x_3 u_3^+ = 0 \quad \text{on } x_2^2 + x_3^2 = x_1^2 \tan^2\theta_0. \tag{1-5}$$

Finally, X_0 and θ_0 are assumed to satisfy

$$X_0\theta_0 = 1 \quad \text{and} \quad \frac{\eta_0}{2} < \theta_0 < \eta_0, \tag{1-6}$$

where $\eta_0 > 0$ is a suitably small constant. This assumption means that the nozzle wall Γ is close to the cylindrical surface $x_2^2+x_3^2=1$ for $X_0 \le r \le X_0+1$.

Theorem 1.1 (uniqueness). *Under the assumptions above and*

$$M_0^-(X_0) \equiv \frac{U_0^-(X_0)}{c(\rho_0^-(X_0))} > \sqrt{\frac{2^{\gamma+1}-2}{\gamma}},$$

then for large X_0 and $0<\varepsilon<1/X_0^2$, Equation (1-1) with the boundary conditions (1-2)–(1-5) has no more than one solution

$$(P^+(x), u_1^+(x), u_2^+(x), u_3^+(x); \eta(x_2, x_3))$$

with the following estimates:

(i) *$\eta(x_2,x_3) \in C^{4,\alpha}(\bar{S})$, where $S=\{(x_2,x_3):(\eta(x_2,x_3),x_2,x_3)\in\Sigma\}$ is the projection of the shock surface Σ on the x_2x_3-plane. Moreover, there exists a constant $C_0>0$ (depending only on α and the supersonic incoming flow) such that*

$$\|\eta(x_2,x_3)-\sqrt{r_0^2-x_2^2-x_3^2}\|_{L^\infty(\bar{S})} \le C_0X_0\varepsilon,$$
$$\|\nabla_{x_2,x_3}(\eta(x_2,x_3)-\sqrt{r_0^2-x_2^2-x_3^2})\|_{C^{3,\alpha}(\bar{S})} \le C_0\varepsilon.$$

(ii) *Let*

$$\Omega_+=\{(x_1,x_2,x_3):\eta(x_2,x_3)<x_1<\sqrt{(X_0+1)^2-x_2^2-x_3^2},\, x_2^2+x_3^2 \le x_1^2\tan^2\theta_0\}.$$

The solution $(P^+(x),u_1^+(x),u_2^+(x),u_3^+(x)) \in C^{3,\alpha}(\overline{\Omega}_+)$ satisfies

$$\|(P^+(x),u_1^+(x),u_2^+(x),u_3^+(x))-(\hat{P}_0^+(r),\hat{u}_{1,0}^+(x),\hat{u}_{2,0}^+(x),\hat{u}_{3,0}^+(x))\|_{C^{3,\alpha}(\bar{\Omega}_+)} \le C_0\varepsilon,$$

where

$$\hat{u}_{i,0}^+(x)=\hat{U}_0^+(r)\frac{x_i}{r} \quad \textit{for } i=1,2,3$$

and $(\hat{P}_0^+(r),\hat{U}_0^+(r))$ is the extension of the subsonic part of the background solution $(P_0^+(r),U_0^+(r))$ in Ω_+ (given in more detail in Theorem A.1 and Remark A.2).

Remark 1.1. The solution is required to have $C^{3,\alpha}$ regularity in Theorem 1.1. This is plausible, as in to [Li et al. 2009], since such a $C^{3,\alpha}$ smooth solution can be obtained as in [Li et al. 2010] under suitable assumptions on the compatibility conditions of the variable end pressure. It will be also shown that the position of the shock depends on the given end pressure monotonically. This will be given more precisely in Proposition 2.2. In addition, the order $X_0\varepsilon$ in the bound on

$$\|\eta(x_2,x_3)-\sqrt{r_0^2-x_2^2-x_3^2}\|_{L^\infty(\bar{S})}$$

comes essentially from the relation between the shock position and the end pressure (see (4-8)). As pointed out in [Li et al. 2009], this actually means that the shock position will move with order $X_0 O(\varepsilon)$ when the end pressure changes in order $O(\varepsilon)$ in (1-4).

Remark 1.2. The uniqueness result in [Xin and Yin 2008b] needs the key assumption that the transonic shock goes through a fixed point which is determined by the resulting ordinary differential equation in the case of the symmetric solutions. Using a completely different method, we remove this assumption.

Remark 1.3. If the transonic shock lies in a converging part of the symmetric nozzle, then a similar result to Theorem 1.1 still holds true. However, as shown in [Xin and Yin 2008b], an unsteady symmetric transonic shock is structurally unstable in a global-in-time sense when it lies in the symmetric converging part of the nozzle.

Remark 1.4. In Theorem 1.1, we assume that the regularity of the transonic shock surface is higher than that of the transonic shock solution $(\rho^+, u_1^+, u_2^+, u_3^+)$. The necessity of this assumption is plausible, in view of the existence result in [Li et al. 2010] under the condition of axisymmetric exit pressure. The assumption is also natural, as it comes up in the existence and stability theory of multidimensional shocks in [Majda 1983a; 1983b].

The steady transonic problem has been studied in great detail; see [Courant and Friedrichs 1948; Liu 1982; Gilbarg and Trudinger 1983; Embid et al. 1984; Morawetz 1994; Čanić et al. 2000; Kuz'min 2002; Zheng 2003; 2006; Chen et al. 2007; Chen 2008; Chen and Yuan 2008; Xin and Yin 2008a; 2008b; Xin et al. 2009; Li et al. 2010] and the references therein. However, most known results deal with 2D problems or 3D problems with special symmetries, or make additional a priori assumptions on shock positions. In this paper, we consider the uniqueness problem for general exit pressure and without stringent conditions on shock locations.

Next we comment on the proofs of the main results. Compared with previous studies, one of the main difficulties is the uncertainty of the shock position. As in the 2-dimensional case [Li et al. 2009], we overcome this difficulty by deriving the monotonic dependence of the shock position on the end pressure along the nozzle wall. Although the strategy here is somewhat similar to [Li et al. 2009], much more delicate and technical a priori estimates are needed to overcome some essential difficulties occurring in the 3-dimensional case. In particular, more complicated and careful analysis is needed for the estimates on the difference of two possible pressures P^+, $\tilde{P}^+$ and the suitable regularity arguments of the difference of two possible velocities (u_1^+, u_2^+, u_3^+), $(\tilde{u}_1^+, \tilde{u}_2^+, \tilde{u}_3^+)$ in the x_2 and x_3 directions. The pressure difference solves a second-order elliptic equation, while the velocity differences satisfy hyperbolic equations. Thus it would be plausible that the regularities of the velocity difference are lower than that of the pressure difference. This leads to the difficulty in deriving the $C^{3,\alpha}$-regularity of the difference of the shock surfaces. Our key observation to overcome this difficulty is that the difference $(u_i^+ - \tilde{u}_i^+)$ for $i = 2, 3$ satisfies a first-order elliptic system with respect to the variables x_2 and x_3 in the interior of subsonic domain Ω_+. Combining this with the transport equations for the velocity differences, we can obtain the $C^{2,\alpha}$-estimate of the velocity difference in the full variable x in Ω_+. This will yield the same regularities of the differences of the pressure and velocity simultaneously.

The rest of the paper is organized as follows. In Section 2, we reformulate the problem (1-1) with the boundary conditions (1-2)–(1-5) by suitable decompositions. To this end, first we transform the nozzle wall Π_2 into a cylindrical surface $y_2^2 + y_3^2 = 1$ and give a suitable decomposition on the velocity $u^+ = (u_1^+, u_2^+, u_3^+)$. Then we decompose the resulting 4×4 three-dimensional Euler system (1-1) into a second-order elliptic equation on the density ρ^+ with mixed boundary conditions and three first-order equations on the velocity components U_1^+, U_2^+ and U_3^+ by making use of Bernoulli's law. Furthermore, by an analysis of the R-H conditions (1-2) and the first equation in (1-1), we can show that (U_2^+, U_3^+) is governed by the Cauchy–Riemann system on the shock surface (see (2-9)–(2-10)). In Section 3, by use of the decomposition techniques in Section 2, we can establish some a priori estimates on the derivatives of the difference $(Y_1, Y_2, Y_3, Y_4, Y_5)$ of two possible solutions $(U_1^+, U_2^+, U_3^+, \rho^+, \xi_1)$ and $(V_1^+, V_2^+, V_3^+, q^+, \xi_2)$. In this process, we especially observe that Y_2 and Y_3 also satisfy a first-order elliptic system with a parameter y_1 in the interior of the nozzle so that one can obtain the same regularity of (Y_2, Y_3) as the pressure difference Y_4 and the suitable $C^{2,\alpha}$-estimates (see Lemma 3.5). With Bernoulli's law, this gives the analogous estimate on the gradients of Y_1 in Lemma 3.6. In Section 4, based on the estimates given in Section 3, we can determine the position of the shock surface and complete the proof of the uniqueness result in Theorem 1.1. Finally, for the reader's convenience, descriptions of the background solution illustrated in [Xin and Yin 2008b] are given in Appendix A. Some useful computations and estimates are given in Appendix B.

In the remainder of the paper, we will use the following conventions: $O(\varepsilon)$ and $O(1)$ mean that there exists a constant $C_1 > 0$, independent of X_0 and ε, such that

$$\|O(\varepsilon)\|_{C^{1,\alpha}} \le C_1\varepsilon \quad \text{and} \quad \|O(1)\|_{C^{1,\alpha}} \le C_1,$$

respectively. $O(1/X_0^m)$ for $m > 0$ means that there exists a generic constant $C_2 > 0$ independent of X_0 and ε such that

$$\|O(1/X_0^m)\|_{C^{1,\alpha}} \le C_2/X_0^m.$$

Also we set $\tau = \tan\theta_0$.

2. Reformulation in terms of radial and angular velocities

In this section, we first decompose the velocity $u=(u_1^+, u_2^+, u_3^+)$ as (U_1^+, U_2^+, U_3^+), where U_1^+ is the radial velocity and U_2^+ and U_3^+ are the angular velocities. Then we reformulate the nonlinear problem (1-1) with (1-2)–(1-5) to obtain a second-order elliptic equation on ρ^+ and a coupled system on U_2^+, U_3^+ and the first-order equation on U_1^+. The relations between (ρ^+, U_1^+) and (U_2^+, U_3^+) on the shock Σ will also be derived.

Due to the symmetry of the nozzle in the diverging part, it is convenient to introduce a coordinate transformation where $\tau = \tan\theta_0$.

$$\begin{cases} y_1 = \sqrt{x_1^2+x_2^2+x_3^2}, \\ y_i = \dfrac{x_i}{x_1\tau}, \qquad\qquad i=2,3, \end{cases} \tag{2-1}$$

and a decomposition of (u_1^+, u_2^+, u_3^+)

$$\begin{cases} u_1^+ = \dfrac{U_1^+ - y_2\tau U_2^+ - y_3\tau U_3^+}{1+(y_2^2+y_3^2)\tau^2}, \\ u_2^+ = \dfrac{y_2\tau U_1^+ + (1+y_3^2\tau^2)U_2^+ - y_2y_3\tau^2 U_3^+}{1+(y_2^2+y_3^2)\tau^2}, \\ u_3^+ = \dfrac{y_3\tau U_1^+ - y_2y_3\tau^2 U_2^+ + (1+y_2^2\tau^2)U_3^+}{1+(y_2^2+y_3^2)\tau^2}. \end{cases} \tag{2-2}$$

The transformation (2-1) changes the domain

$$\Omega = \{(x_1,x_2,x_3) : X_0 \le \sqrt{x_1^2+x_2^2+x_3^2} \le X_0+1,\ x_2^2+x_3^2 \le x_1^2\tau^2\}$$

and

$$\Omega_+ = \{(x_1,x_2,x_3) : \eta(x_2,x_3) < x_1 < \sqrt{(X_0+1)^2 - x_2^2 - x_3^2},\ x_2^2+x_3^2 \le x_1^2\tau^2$$

into the domains

$$\omega = \{(y_1,y_2,y_3) : X_0 \le y_1 \le X_0+1,\ y_2^2+y_3^2 \le 1\}$$

and

$$\omega_+ = \{(y_1,y_2,y_3) : \xi(y_2,y_3) \le y_1 \le X_0+1,\ y_2^2+y_3^2 \le 1\},$$

respectively. Here $y_1 = \xi(y_2,y_3)$ stands for the equation of the shock surface Σ in the new coordinates $y=(y_1,y_2,y_3)$.

To simplify notation, set

$$\begin{cases} D_0 = \dfrac{1}{y_1\sqrt{1+(y_2^2+y_3^2)\tau^2}}, \\ D_1 = \dfrac{1}{\sqrt{1+(y_2^2+y_3^2)\tau^2}}\partial_{y_1}, \\ D_i = \dfrac{\sqrt{1+(y_2^2+y_3^2)\tau^2}}{y_1\tau}\partial_{y_i}, \quad i=2,3. \end{cases} \tag{2-3}$$

Then for any C^1 solution, a direct but tedious computation yields that (1-1) takes the form

$$\begin{cases} U_1^+D_1\rho^+ + U_2^+D_2\rho^+ + U_3^+D_3\rho^+ \\ \qquad\qquad + \rho^+(D_1U_1^+ + D_2U_2^+ + D_3U_3^+) = f_1, \\ \rho^+U_1^+D_1U_1^+ + \rho^+U_2^+D_2U_1^+ + \rho^+U_3^+D_3U_1^+ \\ \qquad\qquad + \big(1+(y_2^2+y_3^2)\tau^2\big)c^2(\rho^+)D_1\rho^+ = f_2, \\ \rho^+U_1^+D_1U_2^+ + \rho^+U_2^+D_2U_2^+ + \rho^+U_3^+D_3U_2^+ \\ \quad + (1+y_2^2\tau^2)c^2(\rho^+)D_2\rho^+ + y_2y_3\tau^2c^2(\rho^+)D_3\rho^+ = f_3, \\ \rho^+U_1^+D_1U_3^+ + \rho^+U_2^+D_2U_3^+ + \rho^+U_3^+D_3U_3^+ \\ \quad + y_2y_3\tau^2c^2(\rho^+)D_2\rho^+ + (1+y_3^2\tau^2)c^2(\rho^+)D_3\rho^+ = f_4, \end{cases} \tag{2-4}$$

and on the shock position $y_1 = \xi(y_2, y_3)$, Equation (1-2) becomes

$$\begin{cases} \dfrac{y_1\tau}{1+(y_2^2+y_3^2)\tau^2}[\rho U_1] - \partial_{y_2}\xi[\rho U_2] - \partial_{y_3}\xi[\rho U_3] = 0, \\ \dfrac{y_1\tau}{1+(y_2^2+y_3^2)\tau^2}[\rho U_1^2 + (1+(y_2^2+y_3^2)\tau^2)P] \\ \qquad\qquad - \partial_{y_2}\xi[\rho U_1U_2] - \partial_{y_3}\xi[\rho U_1U_3] = 0, \\ \dfrac{y_1\tau}{1+(y_2^2+y_3^2)\tau^2}[\rho U_1U_2] - \partial_{y_2}\xi[\rho U_2^2 + (1+y_2^2\tau^2)P] \\ \qquad\qquad - \partial_{y_3}\xi[\rho U_2U_3 + y_2y_3\tau^2P] = 0, \\ \dfrac{y_1\tau}{1+(y_2^2+y_3^2)\tau^2}[\rho U_1U_3] - \partial_{y_2}\xi[\rho U_2U_3 + y_2y_3\tau^2P] \\ \qquad\qquad - \partial_{y_3}\xi[\rho U_3^2 + (1+y_3^2\tau^2)P] = 0, \end{cases} \tag{2-5}$$

where

$$
\text{(2-6)}\qquad \begin{cases} f_1 = -2\rho^+ D_0\big(U_1^+ - y_2\tau U_2^+ - y_3\tau U_3^+\big), \\ f_2 = \rho^+ D_0\big(U_1^+(y_2\tau U_2^+ + y_3\tau U_3^+) + (U_2^+)^2 + (U_3^+)^2 \\ \qquad\qquad\qquad\qquad + (y_3\tau U_2^+ - y_2\tau U_3^+)^2\big), \\ f_3 = -\rho^+ D_0\big(U_1^+ U_2^+ - y_2\tau (U_2^+)^2 - y_3\tau U_2^+ U_3^+\big), \\ f_4 = -\rho^+ D_0\big(U_1^+ U_3^+ - y_2\tau U_2^+ U_3^+ - y_3\tau^2 (U_3^+)^2\big). \end{cases}
$$

Meanwhile, (1-5) is changed into

$$
\text{(2-7)}\qquad y_2 U_2^+ + y_3 U_3^+ = 0 \quad \text{on } y_2^2 + y_3^2 = 1.
$$

Since the transformation (2-1) between the coordinate systems (x_1, x_2, x_3) and (y_1, y_2, y_3) preserves the $C^{4,\alpha}$ norm, from now on, we will use (y_1, y_2, y_3) to discuss our problem instead of (x_1, x_2, x_3). In addition, we will neglect the "+" superscripts for notational simplification.

The third and the fourth equalities in (2-5) give

$$
\text{(2-8)}\qquad \partial_{y_2}\xi(y_2, y_3) = \frac{\Delta_2}{\Delta_1}, \quad \partial_{y_3}\xi(y_2, y_3) = \frac{\Delta_3}{\Delta_1},
$$

where

$$
\Delta_1 = \rho\big((1 + y_3^2\tau^2)U_2^2 - 2y_2y_3\tau^2 U_2U_3 + (1 + y_2^2\tau^2)U_3^2\big) + [P](1 + (y_2^2 + y_3^2)\tau^2),
$$

$$
\Delta_2 = \frac{\xi(y_2, y_3)\tau\rho U_1}{1 + (y_2^2 + y_3^2)\tau^2}(U_2 + y_3^2\tau^2 U_2 - y_2y_3\tau^2 U_3),
$$

$$
\Delta_3 = \frac{\xi(y_2, y_3)\tau\rho U_1}{1 + (y_2^2 + y_3^2)\tau^2}(-y_2y_3\tau^2 U_2 + U_3 + y_2^2\tau^2 U_3).
$$

It follows from the compatibility condition

$$
\partial_{y_3}(\partial_{y_2}\xi) = \partial_{y_2}(\partial_{y_3}\xi)
$$

that

$$
\text{(2-9)}\qquad \begin{aligned} &(\partial_{y_3}\xi\,\partial_{y_1} + \partial_{y_3})U_2 - (\partial_{y_2}\xi\,\partial_{y_1} + \partial_{y_2})U_3 \\ &\qquad = H_0(y_2, y_3, \rho, U_2, U_3, \xi, \nabla_{y_2,y_3}\rho, \nabla_{y_2,y_3}U_2, \nabla_{y_2,y_3}U_3, \nabla_{y_2,y_3}\xi) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } y_1 = \xi(y_2, y_3), \end{aligned}
$$

where for large X_0,

$$
\begin{aligned} H_0 &= O(|U_2|^2 + |U_3|^2) + O(|\nabla_{y_2,y_3}\rho|^2) \\ &\quad + O(|\nabla_{y_2,y_3}U_2|^2) + O(|\nabla_{y_2,y_3}U_3|^2) + O(|\nabla_{y_2,y_3}\xi|^2) \\ &\qquad + O(1/X_0)(|U_2| + |U_3| + |\nabla_{y_2,y_3}\rho| + |\nabla_{y_2,y_3}U_2| + |\nabla_{y_2,y_3}U_3| + |\nabla_{y_2,y_3}\xi|). \end{aligned}
$$

The concrete expression of H_0 is given in Lemma B.1 in Appendix B.

In addition, the first equation in (2-4) can be rewritten as

$$D_2U_2 + D_3U_3 = \frac{1}{\rho}(f_1 - \rho D_1U_1 - U_1D_1\rho - U_2D_2\rho - U_3D_3\rho). \tag{2-10}$$

It is clear that for small $|\nabla_{y_2,y_3}\xi|$, Equations (2-9) and (2-10) consist of a first-order elliptic system for (U_2, U_3) on the shock surface $y_1 = \xi(y_2, y_3)$.

Next we determine the equations of U_2, U_3 in ω_+ and their boundary conditions. By the third and fourth equations of (2-4) and (2-9), (U_2, U_3) satisfies

$$\begin{cases} \rho U_1D_1U_2 + \rho U_2D_2U_2 + \rho U_3D_3U_2 \\ \qquad\qquad + (1 + y_2^2\tau^2)c^2(\rho)D_2\rho + y_2y_3\tau^2c^2(\rho)D_3\rho = f_3, \\ \rho U_1D_1U_3 + \rho U_2D_2U_3 + \rho U_3D_3U_3 \\ \qquad\qquad + y_2y_3\tau^2c^2(\rho)D_2\rho + (1 + y_3^2\tau^2)c^2(\rho)D_3\rho = f_4, \\ (\partial_{y_3}\xi\partial_{y_1} + \partial_{y_3})U_2 - (\partial_{y_2}\xi\partial_{y_1} + \partial_{y_2})U_3 = H_0 \quad \text{on } y_1 = \xi(y_2, y_3), \\ \qquad\qquad y_2U_2 + y_3U_3 = 0 \quad \text{on } y_2^2 + y_3^2 = 1. \end{cases} \tag{2-11}$$

Next, U_1 can be obtained from the equation

$$(\rho U_1D_1 + \rho U_2D_2 + \rho U_3D_3) \times \left(\frac{U_1^2 + U_2^2 + U_3^2 + (y_3\tau U_2 - y_2\tau U_3)^2}{2\big(1 + (y_2^2 + y_3^2)\tau^2\big)} + h(\rho)\right) = 0 \tag{2-12}$$

with

$$h'(\rho) = \frac{c^2(\rho)}{\rho}.$$

Finally, we determine the equation and the boundary conditions for the density ρ. By (2-7) and the third and the fourth equations in (2-4), the corresponding boundary condition of ρ on $y_2^2 + y_3^2 = 1$ is

$$y_2\partial_{y_2}\rho + y_3\partial_{y_3}\rho = \frac{\rho(U_2^2 + U_3^2)}{(1 + \tau^2)c^2(\rho)} \quad \text{on } y_2^2 + y_3^2 = 1. \tag{2-13}$$

We now derive a Dirichlet boundary condition for ρ on the shock Σ. Substituting the expression (2-8) into the first two equations of (2-5) yields on Σ

$$\begin{cases} G_1(\rho, U) \equiv [\rho U_1]\tilde{\Delta}_1 - [\rho U_2]\tilde{\Delta}_2 - [\rho U_3]\tilde{\Delta}_3 = 0, \\ G_2(\rho, U) \equiv [P + \rho U_1^2]\tilde{\Delta}_1 - [\rho U_1U_2]\tilde{\Delta}_2 - [\rho U_1U_3]\tilde{\Delta}_3 = 0, \end{cases} \tag{2-14}$$

with

$$\begin{cases} \tilde{\Delta}_1 = \Delta_1, \\ \tilde{\Delta}_2 = \rho U_1 (U_2 + y_3^2 \tau^2 U_2 - y_2 y_3 \tau^2 U_3), \\ \tilde{\Delta}_3 = \rho U_1 (-y_2 y_3 \tau^2 U_2 + U_3 + y_2^2 \tau^2 U_3). \end{cases}$$

In terms of (2-1), the background solution

$$(P_0^{\pm}(x), u_{1,0}^{\pm}(x), u_{2,0}^{\pm}(x), u_{3,0}^{\pm}(x))$$

in Appendix A is changed into

$$\text{(2-15)} \quad (\overline{P}_0^{\pm}(y_1), \overline{U}_{1,0}^{\pm}(y), \overline{U}_{2,0}^{\pm}(y), \overline{U}_{3,0}^{\pm}(y)) = (P_0^{\pm}(y_1), \sqrt{1+(y_2^2+y_3^2)\tau^2}\, U_0^{\pm}(y_1), 0, 0).$$

Then by Remarks A.1 and A.2 of Appendix A and a direct computation, there exists a constant $C > 0$ such that

$$\text{(2-16)} \qquad \left| \frac{d^k \overline{P}_0^{\pm}(y_1)}{dy_1} \right| + \left| \partial_{y_1}^k \overline{U}_{1,0}^{\pm}(y) \right| \le \frac{C}{X_0^k}, \quad k = 1, 2, 3, 4,$$

$$\text{(2-17)} \qquad \left| \partial_{y_2}^k \overline{U}_{1,0}^{\pm}(y) \right| + \left| \partial_{y_3}^k \overline{U}_{1,0}^{\pm}(y) \right| \le \frac{C}{X_0^2}.$$

Therefore, due to (2-16), (2-14) and the implicit function theorem, a direct computation yields on Σ

$$\text{(2-18)} \quad (U_1 - \overline{U}_{1,0}^{+}(r_0), \rho - \bar{\rho}_0^{+}(r_0)) = (\tilde{g}_1, \tilde{g}_2)(U_2^2, U_3^2, \overline{P}_0^{-} - \overline{P}_0^{-}(r_0), \overline{U}_{1,0}^{-} - \overline{U}_{1,0}^{-}(r_0)),$$

where $\tilde{g}_i$ satisfies

$$\text{(2-19)} \qquad \tilde{g}_i = \big(O(\varepsilon) + O(1/X_0)\big)\big(O(U_2) + O(U_3) + O(\xi - r_0)\big).$$

Equation (2-19) implies that on the shock surface, the influence of U_2 and U_3 on $U_1 - \overline{U}_{1,0}^{+}(r_0)$ and $\rho^+ - \bar{\rho}_0^{+}(r_0)$ can be almost "neglected".

Additionally, as in [Xin and Yin 2008b, Section 5], one can combined equations (2-4) in the form

$$\begin{aligned} &D_1(\text{the second equation}) + D_2(\text{the third equation}) + D_3(\text{the fourth equation}) \\ &\quad - D_1(U_1 \times \text{the first equation}) - D_2(U_2 \times \text{the first equation}) \\ &\quad - D_3(U_3 \times \text{the first equation}) + (D_1U_1 + D_2U_2 + D_3U_3) f_1, \end{aligned}$$

obtaining a second-order equation on ρ with mixed boundary value conditions (by (2-18), (2-13) and (1-4)) as follows:

$$
\text{(2-20)}\quad
\begin{cases}
D_1\Big(\big(c^2(\rho)-U_1^2+(y_2^2+y_3^2)\tau^2c^2(\rho)\big)D_1\rho \\
\qquad -U_1U_2D_2\rho-U_1U_3D_3\rho\Big) \\
+D_2\Big(-U_1U_2D_1\rho+\big(c^2(\rho)-U_2^2+y_2\tau^2c^2(\rho)\big)D_2\rho \\
\qquad +(y_2y_3\tau^2c^2(\rho)-U_2U_3)D_3\rho\Big) \\
+D_3\Big(-U_1U_3D_1\rho+\big(y_2y_3\tau^2c^2(\rho)-U_2U_3\big)D_2\rho \\
\qquad +\big(c^2(\rho)-U_3^2+y_3^2\tau^2c^2(\rho)\big)D_3\rho\Big) \\
\qquad\qquad =H_1(y_2,y_3,\rho,U,\nabla\rho,\nabla U) & \text{in } \omega_+, \\
\rho-\bar{\rho}_0^+(r_0)=\tilde{g}_2 & \text{on } y_1=\xi(y_2,y_3), \\
y_2\partial_{y_2}\rho+y_3\partial_{y_3}\rho=\dfrac{\rho(U_2^2+U_3^2)}{(1+\tau^2)c^2(\rho)} & \text{on } y_2^2+y_3^2=1, \\
P(\rho)=P_e+\varepsilon\tilde{P}_0(y_2,y_3) & \text{on } y_1=X_0+1,
\end{cases}
$$

where $\tilde{P}_0(y_2, y_3)$ is the function $P_0(x_2, x_3)$ under the transformation (2-1) and

$$
\begin{aligned}
&H_1(y_2,y_3,\rho,U,\nabla\rho,\nabla U)\\
&\quad=D_1(\rho U_1)D_2U_2+D_1(\rho U_1)D_3U_3-D_1(\rho U_2)D_2U_1-D_1(\rho U_3)D_3U_1\\
&\qquad+D_2(\rho U_2)D_1U_1+D_2(\rho U_2)D_3U_3-D_2(\rho U_1)D_1U_2-D_2(\rho U_3)D_3U_2\\
&\qquad+D_3(\rho U_3)D_1U_1+D_3(\rho U_3)D_2U_2-D_3(\rho U_1)D_1U_3-D_3(\rho U_2)D_2U_3\\
&\qquad+\rho U_1([D_1,D_2]U_2+[D_1,D_3]U_3)+\rho U_2([D_2,D_1]U_1+[D_2,D_3]U_3)\\
&\qquad\qquad+\rho U_3([D_3,D_1]U_1+[D_3,D_2]U_2)\\
&\qquad+D_1\Big(\rho D_0\big(U_1(y_2\tau U_2+y_3\tau U_3)+(1+y_3^2\tau^2)U_2^2-2y_2y_3\tau^2U_2U_3+(1+y_2^2\tau^2)U_3^2\\
&\qquad\qquad+2U_1(U_1-y_2\tau U_2-y_3\tau U_3)\big)\Big)\\
&\qquad+D_2\big(\rho D_0(U_1U_2-y_2\tau U_2^2-y_3\tau U_2U_3)\big)+D_3\big(\rho D_0(U_1U_3-y_2\tau U_2U_3-y_3\tau U_3^2)\big),
\end{aligned}
$$

where $[D_i, D_j] = D_iD_j - D_jD_i$.

Therefore, we only need to prove the next result to show Theorem 1.1.

Theorem 2.1. *Let the assumptions of Theorem 1.1 hold. Then the problem (2-9)–(2-12), (2-18) and (2-20) has no more than one solution*

$$
(P(y),U_1(y),U_2(y),U_3(y);\xi(y_2,y_3))
$$

with the following estimates.

(1) *$\xi(y_2, y_3) \in C^{4,\alpha}(\overline{B_1(0)})$ with $B_1(0)$ a unit circle centered at $(0,0)$, and there exists a constant $C > 0$ (depending on α and the supersonic incoming flow) such that*

$$\|\xi(y_2,y_3)-r_0\|_{L^\infty(\overline{B_1(0)})} \le CX_0\varepsilon, \quad \|\nabla_{y_2,y_3}(\xi(y_2,y_3)-r_0)\|_{C^{3,\alpha}(\overline{B_1(0)})} \le C\varepsilon.$$

(2) *If $\omega_+ = \{(y_1,y_2,y_3) : \xi(y_2,y_3) < y_1 < X_0+1,\ y_2^2+y_3^2 < 1\}$, then*

$$(P(y), U_1(y), U_2(y), U_3(y)) \in C^{3,\alpha}(\overline{\omega_+})$$

satisfies

$$\|(P(y),U_1(y),U_2(y),U_3(y)) - (\bar{P}_0^+(y_1), \bar{U}_{1,0}^+(y), 0, 0)\|_{C^{3,\alpha}(\overline{\omega_+})} \le C\varepsilon.$$

To prove Theorem 2.1, as in [Xin and Yin 2008b], we first reduce the free boundary problem (2-9)–(2-12), (2-18) and (2-20) into a fixed boundary problem by the transformation

$$\begin{cases} z_1 = \dfrac{y_1-\xi(y_2,y_3)}{X_0+1-\xi(y_2,y_3)}, & \\ z_i = y_i & i=2,3. \end{cases} \tag{2-21}$$

Under (2-21), the region ω_+ is changed into

$$E_+ = \{(z_1,z_2,z_3) : 0 < z_1 < 1,\ z_2^2+z_3^2 < 1\}. \tag{2-22}$$

Correspondingly,

$$\begin{cases} D_0 = \dfrac{1}{\big(\xi(z_2,z_3)+z_1(X_0+1-\xi(z_2,z_3))\big)\sqrt{1+(z_2^2+z_3^2)\tau^2}}, \\ D_1 = \dfrac{1}{\sqrt{1+(z_2^2+z_3^2)\tau^2}}\dfrac{1}{X_0+1-\xi(z_2,z_3)}\partial_{z_1}, \\ D_i = \dfrac{\sqrt{1+(z_2^2+z_3^2)\tau^2}}{\big(\xi(z_2,z_3)+z_1(X_0+1-\xi(z_2,z_3))\big)\tau} \\ \qquad\qquad \times\left(\dfrac{(z_1-1)\partial z_i\xi}{X_0+1-\xi(z_2,z_3)}\partial_{z_1}+\partial_{z_2}\right), \quad i=2,3. \end{cases} \tag{2-23}$$

In next section, we will establish some basic estimates on the problem (2-9)–(2-12), (2-18) and (2-20) in the coordinate $z = (z_1, z_2, z_3)$, which are crucial in the proof of Theorem 2.1.

A further by-product of the analysis for Theorems 1.1 and 2.1 is estimates on the location of the shock and its monotonic dependence on the end pressure.

Proposition 2.2. *Let the assumptions of Theorem 1.1 hold. Suppose the problem (2-4) with (2-5), (2-7) has two $C^{3,\alpha}$ solutions*

$$(\rho, U_1, U_2, U_3; \xi_1(y_2, y_3)) \quad \textit{and} \quad (q, V_1, V_2, V_3; \xi_2(y_2, y_3))$$

which satisfy the exit pressure conditions

$$P_e + \varepsilon(P_0(x_2, x_3) + C_{0,1}) \quad \textit{and} \quad P_e + \varepsilon(P_0(x_2, x_3) + C_{0,2})$$

at $r = X_0 + 1$, respectively, and which admit the estimates in Theorem 2.1, with the two constants satisfying $C_{0,1} < C_{0,2}$. Then

$$\xi_1(y_2, y_3) > \xi_2(y_2, y_3). \tag{2-24}$$

3. A priori estimates

In this section, we will derive some elementary estimates on the difference of two possible solutions to the problem (2-9)–(2-12), (2-18) and (2-20). Based on these estimates, we can show the monotonicity of the end pressure on the position of the shock along the nozzle wall. Assume that the problem (2-9)–(2-12), (2-18) and (2-20) has two solutions $(\rho, U_1, U_2, U_3; \xi_1(z_2, z_3))$ and $(q, V_1, V_2, V_3; \xi_2(z_2, z_3))$, which satisfy the assumptions in Theorem 2.1. Denote by $Q = P(q)$ the pressure for the density q. In addition, (D_0, D_1, D_2, D_3) and $(\widetilde{D_0}, \widetilde{D_1}, \widetilde{D_2}, \widetilde{D_3})$ satisfy (2-23) with $(q, V_1, V_2, V_3; \xi_2(z_2, z_3))$ instead of $(\rho, U_1, U_2, U_3; \xi(z_2, z_3))$ in the $(\widetilde{D_0}, \widetilde{D_1}, \widetilde{D_2}, \widetilde{D_3})$ case.

Set

$$\begin{cases} (Y_i, Y_4)(z_1, z_2, z_3) \\ \qquad = (U_i, \rho)(\xi_1(z_2, z_3) + z_1(X_0 + 1 - \xi_1(z_2, z_3)), z_2, z_3) \\ \qquad\quad - (V_i, q)(\xi_2(z_2, z_3) + z_1(X_0 + 1 - \xi_2(z_2, z_3)), z_2, z_3), \quad i = 1, 2, 3, \\ Y_5(z_2, z_3) = \xi_1(z_2, z_3) - \xi_2(z_2, z_3). \end{cases}$$

We estimate the derivatives of Y_i for $i = 1, 2, 3, 4, 5$ in a series of lemmas.

Lemma 3.1. *Under the assumptions of Theorem 2.1, the following estimates hold*:

$$\begin{cases} D_0 - \widetilde{D_0} = O(1/X_0^2) Y_5, \\ D_1 - \widetilde{D_1} = O(1) Y_5 \partial_{z_1}, \\ D_i - \widetilde{D_i} = O(\varepsilon) Y_5 \partial_{z_1} + O(1) \partial_{z_2} Y_5 \partial_{z_1} + O(1/X_0) Y_5 \partial_{z_2}, \quad i = 2, 3. \end{cases} \tag{3-1}$$

Proof. We estimate $D_1 - \widetilde{D_1}$ only since the other terms can be treated analogously.

By (2-23), one has

$$D_1 - \widetilde{D_1} = \frac{Y_5}{(X_0 + 1 - \xi_1(z_2, z_3))(X_0 + 1 - \xi_2(z_2, z_3)) \sqrt{1 + (z_2^2 + z_3^2)\tau^2}} \partial_{z_1},$$

where

$$\left\| \frac{1}{(X_0+1-\xi_1(z_2,z_3))(X_0+1-\xi_2(z_2,z_3))\sqrt{1+(z_2^2+z_3^2)\tau^2}} \right\|_{C^{1,\alpha}} \le C.$$

This immediately implies $D_1 - \widetilde{D_1} = O(1)Y_5\partial_{z_1}$. □

Lemma 3.2 (estimates of $\nabla_{z_2,z_3}Y_5$). *Under the assumptions of Theorem 2.1, we have*

$$\text{(3-2)}\quad \begin{aligned}\|(\partial_{z_2}Y_5,\partial_{z_3}Y_5)\|_{C^{2,\alpha}} &\le C\varepsilon\|(Y_1,(\varepsilon X_0^2)^{-1}Y_2,(\varepsilon X_0^2)^{-1}Y_3,Y_4,Y_5)\|_{C^{1,\alpha}}\\ &\quad+\frac{C}{X_0^2}\|\nabla_{z_2,z_3}(\varepsilon Y_1,\varepsilon X_0^2Y_4)\|_{C^{1,\alpha}}\\ &\quad+C\|(\partial_{z_2}Y_2,\partial_{z_2}Y_3)\|_{C^{1,\alpha}}+\frac{C}{X_0^2}\|(\partial_{z_3}Y_2,\partial_{z_2}Y_3)\|_{C^{1,\alpha}}.\end{aligned}$$

Remark 3.1. It follows from (3-2) that the term $\|(\partial_{z_2}Y_5,\partial_{z_3}Y_5)\|_{C^{2,\alpha}}$ is controlled mainly by $\|\partial_{z_2}Y_2\|_{C^{1,\alpha}}+\|\partial_{z_3}Y_3\|_{C^{1,\alpha}}$.

Proof of Lemma 3.2. Equation (2-8) yields

$$\begin{cases}\partial_{z_2}\xi_1(z_2,z_3)=\dfrac{\Delta_2}{\Delta_1}, & \partial_{z_3}\xi_1(z_2,z_3)=\dfrac{\Delta_3}{\Delta_1},\\ \partial_{z_2}\xi_2(z_2,z_3)=\dfrac{\widetilde{\Delta_2}}{\widetilde{\Delta_1}}, & \partial_{z_3}\xi_2(z_2,z_3)=\dfrac{\widetilde{\Delta_3}}{\widetilde{\Delta_1}},\\ z_2\partial_{z_2}Y_5+z_3\partial_{z_3}Y_5=0 \quad \text{on } l,\end{cases}$$

where $\tilde{\Delta}_i$ for $i=1,2,3$ has a similar expression to Δ_i with $(q,V_1,V_2,V_3;\xi_2(z_2,z_3))$ instead of $(\rho,U_1,U_2,U_3;\xi(z_2,z_3))$, and l denotes the circle $\{z: z_1=0,\ z_2^2+z_3^2=1\}$.

This shows that on $z_1=0$,

$$\text{(3-3)}\quad \begin{cases}\partial_{z_2}Y_5=O(\varepsilon)\cdot(Y_1,Y_4,X_0^{-1}Y_5)+O(1)Y_2+O(1/X_0^2)Y_3,\\ \partial_{z_3}Y_5=O(\varepsilon)\cdot(Y_1,Y_4,X_0^{-1}Y_5)+O(1/X_0^2)Y_2+O(1)Y_3,\\ z_2\partial_{z_2}Y_5+z_3\partial_{z_3}Y_5=0 \quad \text{on } l,\end{cases}$$

From this, one can obtain a first-order elliptic system on $(\partial_{z_2}Y_5,\partial_{z_3}Y_5)$ as

$$\text{(3-4)}\quad \begin{cases}\partial_{z_2}(\partial_{z_2}Y_5)+\partial_{z_3}(\partial_{z_3}Y_5)=F_1 & \text{on } z_1=0,\\ \partial_{z_3}(\partial_{z_2}Y_5)-\partial_{z_2}(\partial_{z_3}Y_5)=0 & \text{on } z_1=0,\\ z_2\partial_{z_2}Y_5+z_3\partial_{z_3}Y_5=0 & \text{on } l,\end{cases}$$

with

$$\begin{aligned}F_1&=O(\varepsilon)\cdot(Y_1,Y_4,X_0^{-1}Y_5)\\ &\quad+O(1/X_0^2)\cdot(Y_2,Y_3,\partial_{z_3}Y_2,\partial_{z_2}Y_3)+O(1)\partial_{z_2}Y_2+O(1)\partial_{z_3}Y_3\\ &\quad+O(\varepsilon)\cdot(\partial_{z_2}Y_1,\partial_{z_2}Y_4,X_0^{-1}\partial_{z_2}Y_5,\partial_{z_3}Y_1,\partial_{z_3}Y_4,X_0^{-1}\partial_{z_3}Y_5).\end{aligned}$$

It follows from the Hilbert problem for first-order elliptic systems with index -2 that (see [Bers 1950; 1951; Vekua 1952])

$$\|(\partial_{z_2} Y_5, \partial_{z_3} Y_5)\|_{C^{2,\alpha}} \le C\|F_1\|_{C^{1,\alpha}}. \tag{3-5}$$

This yields (3-2). □

Lemma 3.3 (estimates of $\partial_{z_1} Y_i$ for $i = 1, 2, 3, 4$). *Under the assumptions of Theorem 2.1, we have the following estimates*:

$$\begin{aligned}\|(\partial_{z_1} Y_1, \partial_{z_1} Y_4)\|_{C^{1,\alpha}} &\le \frac{C}{X_0}\|(Y_1, X_0^{-1}Y_2, X_0^{-1}Y_3, Y_4, Y_5)\|_{C^{1,\alpha}}\\ &\quad + C\varepsilon\|(\partial_{z_2}Y_1, \partial_{z_2}Y_3, \partial_{z_2}Y_4, \partial_{z_2}Y_5, \partial_{z_3}Y_1, \partial_{z_3}Y_2, \partial_{z_3}Y_4, \partial_{z_3}Y_5)\|_{C^{1,\alpha}}\\ &\quad + C\|(\partial_{z_2}Y_2, \partial_{z_3}Y_3)\|_{C^{1,\alpha}},\end{aligned} \tag{3-6}$$

$$\begin{aligned}\|(\partial_{z_1} Y_2, \partial_{z_1} Y_3)\|_{C^{1,\alpha}} &\le C\varepsilon\|(Y_1, (\varepsilon X_0)^{-1}Y_2, (\varepsilon X_0)^{-1}Y_3, Y_4, Y_5)\|_{C^{1,\alpha}}\\ &\quad + C\varepsilon\|(\partial_{z_2}Y_1, \partial_{z_2}Y_2, \partial_{z_2}Y_3, \partial_{z_3}Y_1, \partial_{z_3}Y_2, \partial_{z_3}Y_3)\|_{C^{1,\alpha}}\\ &\quad + \frac{C}{X_0}\|(\partial_{z_2}Y_5, \partial_{z_3}Y_5)\|_{C^{1,\alpha}} + C\|(\partial_{z_2}Y_4, \partial_{z_3}Y_4)\|_{C^{1,\alpha}},\end{aligned} \tag{3-7}$$

$$\begin{aligned}\|(\partial_{z_1}^2 Y_1, \partial_{z_1}^2 Y_4)\|_{C^{\alpha}} &\le \frac{C}{X_0^2}\|(Y_1, X_0Y_2, X_0Y_3, Y_4, Y_5)\|_{C^{1,\alpha}}\\ &\quad + C\varepsilon\|(\partial_{z_2}Y_1, \partial_{z_2}Y_3, \partial_{z_3}Y_1, \partial_{z_3}Y_2)\|_{C^{1,\alpha}}\\ &\quad + \frac{C}{X_0}\|(\partial_{z_2}Y_5, \partial_{z_3}Y_5, \partial_{z_2}Y_2, \partial_{z_3}Y_3)\|_{C^{1,\alpha}} + C\|(\partial_{z_2}Y_4, \partial_{z_3}Y_4)\|_{C^{1,\alpha}}.\end{aligned} \tag{3-8}$$

Remark 3.2. Equations (3-6) and (3-7) imply the terms $\|(\partial_{z_1} Y_1, \partial_{z_1} Y_4)\|_{C^{1,\alpha}}$ and $\|(\partial_{z_1} Y_2, \partial_{z_1} Y_3)\|_{C^{1,\alpha}}$ are controlled mainly by

$$\frac{C}{X_0}\|Y_5\|_{C^{1,\alpha}} + C(\|\partial_{z_2}Y_2\|_{C^{1,\alpha}} + \|\partial_{z_3}Y_3\|_{C^{1,\alpha}}) \quad \text{and} \quad C(\|\partial_{z_2}Y_4\|_{C^{1,\alpha}} + \|\partial_{z_3}Y_4\|_{C^{1,\alpha}}),$$

respectively. In fact, $(C/X_0)\|Y_5\|_{C^{1,\alpha}}$ is not a "good" term (see Remark 4.1). To overcome this difficulty and for more applications (see Remark 3.4), we must treat the term $\|(\partial_{z_1}^2 Y_1, \partial_{z_1}^2 Y_4)\|_{C^{\alpha}}$ instead of $\|(\partial_{z_1} Y_1, \partial_{z_1} Y_4)\|_{C^{1,\alpha}}$. Fortunately, the term $\|(\partial_{z_1}^2 Y_1, \partial_{z_1}^2 Y_4)\|_{C^{\alpha}}$ can be controlled mainly by

$$\frac{C}{X_0^2}\|Y_5\|_{C^{1,\alpha}}, \quad \frac{C}{X_0}\|(\partial_{z_1}Y_2, \partial_{z_1}Y_3)\|_{C^{1,\alpha}} \quad \text{and} \quad C\|(\partial_{z_2}Y_4, \partial_{z_3}Y_4)\|_{C^{1,\alpha}},$$

which are all "good" (roughly speaking, a "good" term can be directly absorbed by the left hand side in the related a priori estimates).

Proof of Lemma 3.3. It follows from (2-4), Lemma 3.1 and the assumptions in Theorem 2.1 that $\partial_{z_1} Y_i$ for $i = 1, 2, 3, 4$ satisfy

$$
\text{(3-9)}\quad
\begin{cases}
\rho\partial_{z_1}Y_1 + U_1\partial_{z_1}Y_4 \\
\qquad = O(1/X_0)\cdot(Y_1, X_0^{-1}Y_2, X_0^{-1}Y_3, Y_4, Y_5) + O(1)\cdot(\partial_{z_2}Y_2, \partial_{z_3}Y_3) \\
\qquad\quad + O(\varepsilon)\cdot(\partial_{z_1}Y_2, \partial_{z_1}Y_3, \partial_{z_2}Y_4, \partial_{z_2}Y_5, \partial_{z_3}Y_4, Y_5, \varepsilon\partial_{z_1}Y_4), \\
\rho U_1\partial_{z_1}Y_1 + (1 + (z_2^2 + z_3^2)\tau^2)c^2(\rho)\partial_{z_1}Y_4 \\
\qquad = O(1/X_0)\cdot(Y_1, X_0^{-1}Y_2, X_0^{-1}Y_3, Y_4, Y_5) \\
\qquad\quad + O(\varepsilon)\cdot(\varepsilon\partial_{z_1}Y_1, \partial_{z_2}Y_1, \partial_{z_3}Y_1, X_0^{-1}\partial_{z_2}Y_5, X_0^{-1}\partial_{z_3}Y_5), \\
\partial_{z_1}Y_2 = O(\varepsilon)\cdot(Y_1, (\varepsilon X_0)^{-1}Y_2, Y_3, Y_4, Y_5) \\
\qquad\quad + O(\varepsilon)\cdot(\varepsilon\partial_{z_1}Y_2, \partial_{z_1}Y_4, \partial_{z_2}Y_2, \partial_{z_3}Y_2, (\varepsilon X_0^2)^{-1}\partial_{z_3}Y_4) \\
\qquad\quad + O(1/X_0)(\partial_{z_2}Y_5, X_0^{-2}\partial_{z_3}Y_5) + O(1)\partial_{z_2}Y_4, \\
\partial_{z_1}Y_3 = O(\varepsilon)\cdot(Y_1, Y_2, (\varepsilon X_0)^{-1}Y_3, Y_4, Y_5) \\
\qquad\quad + O(\varepsilon)\cdot(\varepsilon\partial_{z_1}Y_3, \partial_{z_1}Y_4, \partial_{z_2}Y_3, (\varepsilon X_0^2)^{-1}\partial_{z_2}Y_4, \partial_{z_3}Y_3) \\
\qquad\quad + O(1/X_0)\cdot(X_0^{-2}\partial_{z_2}Y_5, \partial_{z_3}Y_5) + O(1)\partial_{z_3}Y_4.
\end{cases}
$$

So a direct computation yields (3-6) and (3-7).

From the expressions of $\partial_{z_1}Y_1$ and $\partial_{z_1}Y_4$ obtained by solving the first and second equations in (3-9), one has again for $i = 1, 4$,

$$
\begin{aligned}
\text{(3-10)}\quad \partial_{z_1}^2 Y_i &= O(1/X_0^2)\cdot(Y_1, Y_2, Y_3, Y_4) \\
&\quad + O(1/X_0)\cdot(\partial_{z_1}Y_1, X_0^{-1}\partial_{z_1}Y_2, X_0^{-1}\partial_{z_1}Y_3, \partial_{z_1}Y_4, Y_5) \\
&\quad + \partial_{z_1}\big(O(\varepsilon)\cdot(\partial_{z_1}Y_2, \partial_{z_1}Y_3, \partial_{z_2}Y_1, \partial_{z_2}Y_4, \partial_{z_2}Y_5, \partial_{z_3}Y_1, \partial_{z_3}Y_4, \partial_{z_3}Y_5)\big) \\
&\quad + O(1/X_0^2)\cdot(\partial_{z_2}Y_2, \partial_{z_3}Y_3) + O(1)\cdot(\partial_{z_1z_2}^2 Y_2, \partial_{z_1z_3}^2 Y_3).
\end{aligned}
$$

Equation (3-8) follows from (3-10) and a direct computation. □

Next, we estimate $\nabla_{z_2,z_3}Y_2$ and $\nabla_{z_2,z_3}Y_3$.

Lemma 3.4 (estimates of $Y_2(0, z_2, z_3)$ and $Y_3(0, z_2, z_3)$). *Under the assumptions of Theorem 2.1, we have*

$$
\begin{aligned}
\text{(3-11)}\quad &\|(Y_2(0, z_2, z_3), Y_3(0, z_2, z_3))\|_{C^{2,\alpha}(\bar{B}B_1(0))} \\
&\le \frac{C}{X_0}\|(Y_1, X_0^{-1}Y_2, X_0^{-1}Y_3, Y_4, X_0^{-1}Y_5)\|_{C^{1,\alpha}} \\
&\quad + C\varepsilon\|(\partial_{z_1}Y_1, \partial_{z_1}Y_2, \partial_{z_1}Y_3, \partial_{z_1}Y_4)\|_{C^{1,\alpha}} \\
&\quad + C\|(\partial_{z_2}Y_1, \partial_{z_2}Y_4, \partial_{z_3}Y_1, \partial_{z_3}Y_4)\|_{C^{1,\alpha}} + \frac{C}{X_0}\|(\partial_{z_2}Y_5, \partial_{z_3}Y_5)\|_{C^{1,\alpha}}.
\end{aligned}
$$

Remark 3.3. It follows from (3-11) that $\|(Y_2(0, z_2, z_3), Y_3(0, z_2, z_3))\|_{C^{2,\alpha}(\bar{B}B_1(0))}$ is controlled mainly by $(C/X_0^2)\|Y_5\|_{C^{1,\alpha}}$ and $C\|(\partial_{z_2}Y_1, \partial_{z_2}Y_4, \partial_{z_3}Y_1, \partial_{z_3}Y_4)\|_{C^{1,\alpha}}$.

Proof of Lemma 3.4. From (2-9)–(2-10), the assumptions in Theorem 2.1, and a direct computation, it follows that on $z_1 = 0$,

$$\begin{cases} \partial_{z_3} Y_2 - \partial_{z_2} Y_3 = F_2, \\ \partial_{z_2} Y_2 + \partial_{z_3} Y_3 = F_3, \\ z_2 Y_2 + z_3 Y_3 = 0 \quad \text{on } z_2^2 + z_3^2 = 1, \end{cases} \tag{3-12}$$

with

$$\begin{aligned} F_2 = {} & O(\varepsilon) \cdot (Y_1, Y_4, X_0^{-1} Y_5) + O(1/X_0^2) \cdot (Y_2, Y_3) \\ & + O(\varepsilon)(\partial_{z_2} Y_1, (\varepsilon X_0^2)^{-1} \partial_{z_2} Y_2, \varepsilon \partial_{z_2} Y_3, \partial_{z_2} Y_4, X_0^{-1} \partial_{z_2} Y_5) \\ & + O(\varepsilon) \cdot (\partial_{z_3} Y_1, (\varepsilon X_0^2)^{-1} \partial_{z_3} Y_2, (\varepsilon X_0^2)^{-1} \partial_{z_3} Y_3, \partial_{z_3} Y_4, \partial_{z_3} Y_5) \\ & + O(\varepsilon)(\varepsilon \partial_{z_1} Y_1, \partial_{z_1} Y_2, X_0^{-2} \partial_{z_1} Y_3, \varepsilon \partial_{z_1} Y_4), \\ F_3 = {} & O(1/X_0) \cdot (Y_1, X_0^{-1} Y_2, X_0^{-1} Y_3, Y_4, Y_5) \\ & + O(\varepsilon) \cdot (\partial_{z_1} Y_2, \partial_{z_1} Y_3, \partial_{z_2} Y_4, \partial_{z_2} Y_5, \partial_{z_3} Y_4, \partial_{z_3} Y_5) \\ & + O(1) \cdot (\partial_{z_1} Y_1, \partial_{z_1} Y_4), \end{aligned}$$

where F_3 is given in Lemma B.2 of Appendix B.

As in (3-5), one can obtain from (3-12) that

$$\|(Y_2(0, z_2, z_3), Y_3(0, z_2, z_3))\|_{C^{2,\alpha}(\bar{B} B_1(0))} \le C \|(F_2, F_3)\|_{C^{1,\alpha}(\bar{B} B_1(0))}. \tag{3-13}$$

On the other hand, due to the second equation and the boundary condition in (3-12),

$$\int_{B_1(0)} F_3 \, ds = \int_{B_1(0)} (\partial_{z_2} Y_2 + \partial_{z_3} Y_3) \, ds = \int_{\partial B_1(0)} (z_2 Y_2 + z_3 Y_3) \, dl = 0 \quad \text{on } z_1 = 0.$$

Since $F_3 \in C^{1,\alpha}(\Omega)$, it follows from the integral mean value theorem that there exists a point (z_2*, z_3*) such that

$$F_3(0, z_2*, z_3*) = 0.$$

This implies

$$\|F_3(0, z_2, z_3)\|_{C^{1,\alpha}} \le C \|\nabla_{z_2, z_3} F_3(0, z_2, z_3)\|_{C^{\alpha}}.$$

Combining this with (3-13) and a direct computation yields

$$\begin{aligned} & \|(Y_2(0, z_2, z_3), Y_3(0, z_2, z_3))\|_{C^{2,\alpha}(\bar{B} B_1(0))} \\ & \le \frac{C}{X_0} \left\| \left(Y_1, X_0^{-1} Y_2, X_0^{-1} Y_3, Y_4, \frac{1}{X_0} Y_5 \right) \right\|_{C^{1,\alpha}} \\ & \quad + C\varepsilon \|(\partial_{z_1} Y_1, \partial_{z_1} Y_2, \partial_{z_1} Y_3, \partial_{z_1} Y_4)\|_{C^{1,\alpha}} + C \|(\partial_{z_2} Y_1, \partial_{z_2} Y_4, \partial_{z_3} Y_1, \partial_{z_3} Y_4)\|_{C^{1,\alpha}} \\ & \quad + \frac{C}{X_0} \|(\partial_{z_2} Y_5, \partial_{z_3} Y_5)\|_{C^{1,\alpha}}, \end{aligned}$$

which completes the proof of Lemma 3.4. □

Using Lemmas 3.3–3.4 and Lemma B.3 in Appendix B, we can estimate $\nabla_{z_2,z_3} Y_2$ and $\nabla_{z_2,z_3} Y_3$ as follows:

Lemma 3.5 (estimates of $\partial_{z_2} Y_2$, $\partial_{z_3} Y_2$ and $\partial_{z_2} Y_3$, $\partial_{z_3} Y_3$). *Under the assumptions of Theorem 2.1, $\partial_{z_2} Y_2$, $\partial_{z_3} Y_2$ and $\partial_{z_2} Y_3$, $\partial_{z_3} Y_3$ satisfy*

$$\begin{aligned}\|(\partial_{z_2} Y_2, \partial_{z_3} Y_2, \partial_{z_2} Y_3, \partial_{z_3} Y_3)\|_{C^{1,\alpha}} &\le \frac{C}{X_0}\big(\|(Y_1, Y_2, Y_3, Y_4, X_0^{-1} Y_5)\|_{C^{1,\alpha}} + \|(\partial_{z_2} Y_5, \partial_{z_3} Y_5)\|_{C^{2,\alpha}}\big) \\ &\quad + C\|(\partial_{z_2} Y_1, \partial_{z_2} Y_4, \partial_{z_3} Y_1, \partial_{z_3} Y_4)\|_{C^{1,\alpha}}.\end{aligned} \tag{3-14}$$

Remark 3.4. Thanks to (3-8), the right hand side of (3-14) can be controlled by the "good" term $(C/X_0^2)\|Y_5\|_{C^{1,\alpha}}$. This can be seen in (3-16) and (3-17) below.

Proof of Lemma 3.5. This lemma is proved by the characteristic method.

Under the coordinate $z = (z_1, z_2, z_3)$, the characteristics curves

$$(z_2^1(s; z), z_3^1(s; z)) \quad \text{and} \quad (z_2^2(s; z), z_3^2(s; z))$$

of the first-order differential operators

$$U_1 D_1 + U_2 D_2 + U_3 D_3 \quad \text{and} \quad V_1 \widetilde{D_1} + V_2 \widetilde{D_2} + V_3 \widetilde{D_3},$$

respectively, through the point $z = (z_1, z_2, z_3)$, can be defined as

$$\begin{cases} \dfrac{dz_i^1(s; z)}{ds} = \dfrac{U_i\big(\xi_1(z_2^1, z_3^1) + s(X_0 + 1 - \xi_1(z_2^1, z_3^1)), z_2^1, z_3^1\big)}{\big(\xi_1(z_2^1, z_3^1) + s(X_0 + 1 - \xi_1(z_2^1, z_3^1))\big) A_1 \tau}, \\ z_i^1(z_1; z) = z_i, \quad i = 2, 3, \end{cases}$$

$$\begin{cases} \dfrac{dz_i^2(s; z)}{ds} = \dfrac{V_i\big(\xi_2(z_2^2, z_3^2) + s(X_0 + 1 - \xi_2(z_2^2, z_3^2)), z_2^2, z_3^2\big)}{\big(\xi_2(z_2^2, z_3^2) + s(X_0 + 1 - \xi_2(z_2^2, z_3^2))\big) A_2 \tau}, \\ z_i^2(z_1; z) = z_i, \quad i = 2, 3, \end{cases}$$

where

$$\begin{aligned} A_1 = \frac{1}{X_0 + 1 - \xi_1(z_2^1, z_3^1)} \Bigg(& \frac{U_1}{1 + (z_2^1)^2 \tau^2 + (z_3^1)^2 \tau^2} \\ & + \frac{(s-1)\partial_{z_2} \xi_1(z_2^1, z_3^1) U_2 + (s-1)\partial_{z_3} \xi_1(z_2^1, z_3^1) U_3}{\big(\xi_1(z_2^1, z_3^1) + s(X_0 + 1 - \xi_1(z_2^1, z_3^1))\big)\tau} \Bigg), \end{aligned}$$

and A_2 can be defined similarly by replacing (ξ_1, U_1, U_2, U_3) with (ξ_2, V_1, V_2, V_3).

Denote by $z_2^1(0;z)=\beta_1$, $z_3^1(0;z)=\beta_2$ and $z_2^2(0;z)=\widetilde{\beta}_1$, $z_3^2(0;z)=\widetilde{\beta}_2$. Then for $i=2,3$,

$$z_i^1(s;z)=\int_0^s \frac{U_i\big(\xi_1(z_2^1,z_3^1)+t(X_0+1-\xi_1(z_2^1,z_3^1)),z_2^1,z_3^1\big)}{\big(\xi_1(z_2^1,z_3^1)+t(X_0+1-\xi_1(z_2^1,z_3^1))\big)A_1\tau}\,dt+\beta_{i-1},$$
$$z_i=\int_0^{z_1} \frac{U_i\big(\xi_1(z_2^1,z_3^1)+t(X_0+1-\xi_1(z_2^1,z_3^1)),z_2^1,z_3^1\big)}{\big(\xi_1(z_2^1,z_3^1)+t(X_0+1-\xi_1(z_2^1,z_3^1))\big)A_1\tau}\,dt+\beta_{i-1}.$$

Similarly, $z_i^2(s,z)$ and z_i have the same expressions with (β_{i-1},ξ_1,V_i) replaced by $(\tilde{\beta}_{i-1},\xi_2,V_i)$.

From this, we can obtain immediately that for $i=2,3$,

$$\|\beta_{i-1}-z_i\|_{C^{2,\alpha}}\le C\|U_i\|_{C^{2,\alpha}},\quad \|\widetilde{\beta_{i-1}}-z_i\|_{C^{2,\alpha}}\le C\|V_i\|_{C^{2,\alpha}}.$$

Next define $l^1(s;z)=(z_2^1-z_2^2)(s;z)$ and $l^2(s;z)=(z_3^1-z_3^2)(s;z)$. Then by direct computation,

$$\begin{cases} \dfrac{dl^1(s;z)}{ds}=O(\varepsilon)\cdot(l^1,l^2)(s;z) \\ \qquad\qquad +O(\varepsilon)\cdot(Y_1,Y_3,Y_5,\varepsilon\partial_{z_2}Y_5,\varepsilon\partial_{z_3}Y_5)(s,z_2^1,z_3^1) \\ \qquad\qquad +O(1)Y_2(s,z_2^1,z_3^1), \\ l^1(0;z)=\beta_1-\widetilde{\beta}_1,\quad l^1(z_1;z)=0, \end{cases}$$

and similarly for $l^2(s;z)$.

Therefore

$$\begin{cases} \|l^1\|_{C^{2,\alpha}}+\|\beta_1-\widetilde{\beta}_1\|_{C^{2,\alpha}} \\ \qquad\le C\|Y_2\|_{C^{2,\alpha}}+C\varepsilon\|(Y_1,Y_3,Y_5,\varepsilon\partial_{z_2}Y_5,\varepsilon\partial_{z_3}Y_5)\|_{C^{2,\alpha}}, \\ \|l^2\|_{C^{2,\alpha}}+\|\beta_2-\widetilde{\beta}_2\|_{C^{2,\alpha}} \\ \qquad\le C\|Y_3\|_{C^{2,\alpha}}+C\varepsilon\|(Y_1,Y_2,Y_5,\varepsilon\partial_{z_2}Y_5,\varepsilon\partial_{z_3}Y_5)\|_{C^{2,\alpha}}. \end{cases}\tag{3-15}$$

By Lemma B.2 in Appendix B, (Y_2,Y_3) satisfies

$$\begin{cases} \partial_{z_2}Y_2+\partial_{z_3}Y_3=F_3 & \text{in } E_+, \\ \partial_{z_3}Y_2-\partial_{z_2}Y_3=F_4 & \text{in } E_+, \\ z_2Y_2+z_3Y_3=0 & \text{on } z_2^2+z_3^2=1, \end{cases}\tag{3-16}$$

where F_3 and F_4 are given in Lemma B.2.

A direct computation yields

$$\begin{cases} \partial_{z_1}F_3=O(1)(\partial_{z_1}^2Y_1,\partial_{z_1}^2Y_4)+\text{some "good" terms}, \\ \nabla_{z_2,z_3}F_3 \text{ consists of "good" terms}. \end{cases}\tag{3-17}$$

Therefore, it follows from Lemma B.3 of Appendix B and Lemmas 3.3–3.4 that

$$\begin{aligned}
&\|(\partial_{z_2}Y_2,\partial_{z_3}Y_2,\partial_{z_2}Y_3,\partial_{z_3}Y_3)\|_{C^{1,\alpha}}\\
&\quad\le C\Big(\sum_{i=2}^{3}\|\partial_{z_1}Y_i\|_{C^{1,\alpha}}+\|\nabla F_3\|_{C^{1,\alpha}}+\|F_4\|_{C^{1,\alpha}}\Big)\\
&\quad\le \frac{C}{X_0}\big(\|(Y_1,Y_2,Y_3,Y_4,X_0^{-1}Y_5)\|_{C^{1,\alpha}}+\|(\partial_{z_2}Y_5,\partial_{z_3}Y_5)\|_{C^{2,\alpha}}\big)\\
&\qquad\qquad+C\|(\partial_{z_2}Y_1,\partial_{z_2}Y_4,\partial_{z_3}Y_1,\partial_{z_3}Y_4)\|_{C^{1,\alpha}},
\end{aligned}$$

which completes the proof of Lemma 3.5. □

Lemma 3.6 (estimates of $\partial_{z_2}Y_1,\ \partial_{z_3}Y_1$). *Under the assumptions of Theorem 2.1, Y_1 satisfies*

$$\begin{aligned}
\|(\partial_{z_2}Y_1,\partial_{z_3}Y_1)\|_{C^{1,\alpha}}&\\
\le \frac{C}{X_0^2}\|(\varepsilon Y_1,Y_2,Y_3,Y_4,Y_5,\partial_{z_1}Y_4,X_0\partial_{z_2}Y_5,X_0\partial_{z_3}Y_5)\|_{C^{2,\alpha}}&\\
+C\|(\partial_{z_2}Y_4,\partial_{z_3}Y_4)\|_{C^{1,\alpha}}.&
\end{aligned}\tag{3-18}$$

Proof. Applying the characteristic method to (2-12) as in the proof of Lemma 3.5, we arrive at

$$\begin{aligned}
Y_1&=O(1/X_0^2)\cdot(l^1,l^2)+O(\varepsilon)\cdot(Y_2,Y_3)\\
&\qquad+O(1)Y_4+O(1)\cdot(Y_1,\varepsilon Y_2,\varepsilon Y_3,Y_4)(0,\beta_1(z),\beta_2(z)).
\end{aligned}$$

It follows from (2-18) that on $z_1=0$,

$$Y_i=O(\varepsilon)\cdot(Y_2,Y_3)+O(1/X_0)Y_5,\quad i=1,4.\tag{3-19}$$

By the assumptions of Theorem 2.1 and Equations (2-16)–(2-17), a direct computation yields

$$\begin{aligned}
\partial_{z_i}Y_1&\\
=\partial_{z_i}\big(O(1/X_0^2)\cdot(l^1,l^2)+O(\varepsilon)\cdot(Y_2,Y_3)+O(\varepsilon)\cdot(Y_2,Y_3)(0,\beta_1(z),\beta_2(z))\big)&\\
+O(1/X_0^2)Y_4+O(1/X_0^2)\cdot(Y_1,Y_4)(0,\beta_1(z),\beta_2(z))+O(1)\partial_{z_i}Y_4&\\
+O(1)\cdot(\partial_{z_i}Y_1,\partial_{z_i}Y_4)(0,\beta_1(z),\beta_2(z)),\quad i=2,3,&
\end{aligned}\tag{3-20}$$

and on $z_1=0$,

$$\begin{aligned}
\partial_{z_i}Y_j&\\
=\partial_{z_i}(O(\varepsilon)\cdot(Y_2,Y_3))+o(1/X_0^2)Y_5+O(1/X_0)\partial_{z_i}Y_5,\quad i=2,3,\ j=1,4.&
\end{aligned}\tag{3-21}$$

So, combining (3-20) and (3-21) with (3-14) and (3-15) yields (3-18). □

Lemmas 3.2–3.6 essentially convert the estimates on $\|\nabla_{z_2,z_3}Y_5\|_{C^{2,\alpha}}$, $\|\nabla_z Y_1\|_{C^{1,\alpha}}$, $\|\nabla_z(Y_2, Y_3)\|_{C^{1,\alpha}}$ and $\|\partial_{z_1}Y_4\|_{C^{1,\alpha}}$ into an estimate on $\|\nabla_{z_2,z_3}Y_4\|_{C^{1,\alpha}}$, so we now focus on of $\|\nabla_{z_2,z_3}Y_4\|_{C^{1,\alpha}}$. First, we derive from (2-20) some second-order elliptic equations with corresponding boundary conditions for $z_2\partial_{z_2}Y_4 + z_3\partial_{z_3}Y_4$ and $z_3\partial_{z_2}Y_4 - z_2\partial_{z_3}Y_4$. This will enable one to obtain their $C^{1,\alpha}$ boundary estimates on the nozzle wall by the theory of second-order elliptic equations with mixed boundary conditions (in this process, one cannot obtain the global $C^{1,\alpha}$ estimates directly in the whole domain due to the appearance of a singularity in the equation for $z_2\partial_{z_2}Y_4 + z_3\partial_{z_3}Y_4$; see (3-24)). This and a simple computation yield the $C^{1,\alpha}$ estimates of $\partial_{z_2}Y_4$ and $\partial_{z_3}Y_4$ on the boundary $z_2^2 + z_3^3 = 1$. Subsequently, we can use the second-order elliptic equations and the corresponding boundary conditions for $\partial_{z_2}Y_4$ and $\partial_{z_3}Y_4$ to obtain $\|(\partial_{z_2}Y_4, \partial_{z_3}Y_4)\|_{L^\infty}$ and further $C^{1,\alpha}$ estimates.

Lemma 3.7 (estimates of $\partial_{z_2}Y_4$, $\partial_{z_3}Y_4$). *Under the assumptions of Theorem 2.1, $\partial_{z_2}Y_4$, and $\partial_{z_3}Y_4$ satisfy*

$$\text{(3-22)}\quad \begin{aligned}\|(\partial_{z_2}Y_4, \partial_{z_3}Y_4)\|_{C^{1,\alpha}} \le{}& \frac{C}{X_0}\|(Y_1, Y_2, Y_3, Y_4, X_0^{-1}Y_5)\|_{C^{1,\alpha}}\\ &+\frac{C}{X_0}\|(\partial_{z_1}Y_1, \partial_{z_2}Y_2, \partial_{z_3}Y_3, \partial_{z_1}Y_4, \partial_{z_2}Y_5, \partial_{z_3}Y_5)\|_{C^{2,\alpha}}\\ &+C\varepsilon\|(\partial_{z_1}Y_2, \partial_{z_1}Y_3, \partial_{z_3}Y_2, \partial_{z_2}Y_3)\|_{C^{1,\alpha}}.\end{aligned}$$

Remark 3.5. By (3-22), the norm $\|(\partial_{z_2}Y_4, \partial_{z_3}Y_4)\|_{C^{1,\alpha}}$ has been controlled by "good" terms, in particular, $(C/X_0^2)\|Y_5\|_{C^{1,\alpha}}$.

Proof of Lemma 3.7. It follows from (2-20), (3-19), Lemma 3.1 and a direct computation that

$$\text{(3-23)}\quad \begin{cases}\widetilde{D_1}\big((c^2(\rho)-U_1^2)\widetilde{D_1}Y_4 + c^2(\rho)(z_2^2\tau^2+z_3^2\tau^2)\widetilde{D_1}Y_4\\ \qquad -U_1U_2\widetilde{D_2}Y_4 - U_1U_3\widetilde{D_3}Y_4\big)\\ +\widetilde{D_2}\big(-U_1U_2\widetilde{D_1}Y_4 + (c^2(\rho)-U_2^2)\widetilde{D_2}Y_4 + z_2^2\tau^2c^2(\rho)\widetilde{D_2}Y_4\\ \qquad -U_2U_3\widetilde{D_3}Y_4 + z_2z_3\tau^2c^2(\rho)\widetilde{D_3}Y_4\big)\\ +\widetilde{D_3}\big(-U_1U_3\widetilde{D_1}Y_4 - U_2U_3\widetilde{D_2}Y_4 + z_2z_3\tau^2c^2(\rho)\widetilde{D_2}Y_4\\ \qquad +(c^2(\rho)-U_3^2)\widetilde{D_3}Y_4 + z_3^2\tau^2c^2(\rho)\widetilde{D_3}Y_4\big)\\ \quad = H_2(Y, \nabla Y) & \text{in } E_+,\\ Y_4 = O(\varepsilon)Y_2 + O(\varepsilon)Y_3 + O(1/X_0)Y_5 & \text{on } z_1 = 0,\\ Y_4 = 0 & \text{on } z_1 = 1,\\ z_2\partial_{z_2}Y_4 + z_3\partial_{z_3}Y_4 = O(\varepsilon)Y_2 + O(\varepsilon)Y_3 + O(\varepsilon^2)Y_4 & \text{on } z_2^2+z_3^2 = 1,\end{cases}$$

with

$$
\begin{aligned}
&H_2(Y,\nabla Y)\\
&\quad=\widetilde{D_1}\big(O(1/X_0)\cdot(Y_1,X_0^{-1}Y_2,X_0^{-1}Y_3,Y_4)\big)\\
&\qquad+\widetilde{D_2}\big(O(\varepsilon/X_0)\cdot(Y_1,\varepsilon^{-1}Y_2,X_0^{-1}Y_3,X_0Y_4,Y_5,\varepsilon^{-1}\partial_{z_2}Y_5,(\varepsilon X_0^2)^{-1}\partial_{z_3}Y_5)\big)\\
&\qquad+\widetilde{D_3}\big(O(\varepsilon/X_0)\cdot(Y_1,X_0^{-1}Y_2,\varepsilon^{-1}Y_3,X_0Y_4,Y_5,(\varepsilon X_0^2)^{-1}\partial_{z_2}Y_5,\varepsilon^{-1}\partial_{z_3}Y_5)\big)\\
&\qquad\quad+O(1/X_0)\cdot\big(\varepsilon Y_1,X_0^{-2}Y_2,X_0^{-2}Y_3,\varepsilon Y_4,X_0^{-1}Y_5\big)\\
&\qquad\quad+O(1/X_0^2)\cdot\big(\varepsilon X_0^2\partial_{z_1}Y_1,\partial_{z_1}Y_2,\partial_{z_1}Y_3,\varepsilon X_0^2\partial_{z_1}Y_4\big)\\
&\qquad\quad+O(\varepsilon)\cdot\big(\partial_{z_2}Y_1,(\varepsilon X_0)^{-1}\partial_{z_2}Y_2,\partial_{z_2}Y_3,\partial_{z_2}Y_4,\partial_{z_2}Y_5\big)\\
&\qquad\quad+O(\varepsilon)\cdot\big(\partial_{z_3}Y_1,\partial_{z_3}Y_2,(\varepsilon X_0^{-1})\partial_{z_3}Y_3,\partial_{z_3}Y_4,\partial_{z_3}Y_5\big),
\end{aligned}
$$

where we use the formula of H_1 on page 140 and the assumptions in Theorem 2.1.

Next, define

$$M_1=z_2\partial_{z_2}Y_4+z_3\partial_{z_3}Y_4\quad\text{and}\quad M_2=z_3\partial_{z_2}Y_4-z_2\partial_{z_3}Y_4$$

Applying $z_2\partial_{z_2}+z_3\partial_{z_3}$ to the first three equalities of (3-23) yields

$$
\begin{cases}
\widetilde{D_1}\big((c^2(\rho)-U_1^2)\widetilde{D_1}M_1+c^2(\rho)(z_2^2\tau^2+z_3^2\tau^2)\widetilde{D_1}M_1\\
\qquad-U_1U_2\widetilde{D_2}M_1-U_1U_3\widetilde{D_3}M_1\big)\\
+\widetilde{D_2}\Big(-U_1U_2\widetilde{D_1}M_1+(c^2(\rho)-U_2^2)\widetilde{D_2}M_1\\
\qquad+z_2^2\tau^2c^2(\rho)\widetilde{D_2}M_1-U_2U_3\widetilde{D_3}M_1+z_2z_3\tau^2c^2(\rho)\widetilde{D_3}M_1\\
\qquad+O(1)\dfrac{z_2M_1+z_3M_2}{z_2^2+z_3^2}+O(1)\dfrac{z_3M_1-z_2M_2}{z_2^2+z_3^2}\Big)\\
+\widetilde{D_3}\Big(-U_1U_3\widetilde{D_1}M_1-U_2U_3\widetilde{D_2}M_1\\
\qquad+z_2z_3\tau^2c^2(\rho)\widetilde{D_2}M_1+(c^2(\rho)-U_3^2)\widetilde{D_3}M_1+z_3^2\tau^2c^2(\rho)\widetilde{D_3}M_1\\
\qquad+O(1)\dfrac{z_2M_1+z_3M_2}{z_2^2+z_3^2}+O(1)\dfrac{z_3M_1-z_2M_2}{z_2^2+z_3^2}\Big)\\
\quad=(z_2\partial_{z_2}+z_3\partial_{z_3})H_2(Y,\nabla Y)+H_3(Y,\nabla Y)\qquad\text{in } E_+,\\
M_1=O(\varepsilon)\cdot(Y_2,Y_3,\partial_{z_2}Y_2,\partial_{z_3}Y_2,\partial_{z_2}Y_3,\partial_{z_3}Y_3)\\
\qquad+O(1/X_0)(X_0^{-1}Y_5,\partial_{z_2}Y_5,\partial_{z_3}Y_5)\qquad\text{on } z_1=0,\\
M_1=0\qquad\text{on } z_1=1,\\
M_1=O(\varepsilon)\cdot(Y_2,Y_3,\varepsilon Y_4)\qquad\text{on } z_2^2+z_3^2=1,
\end{cases}
\tag{3-24}
$$

where

$$
\begin{aligned}
&H_3(Y, \nabla Y)\\
&= O(1/X_0^2)\cdot(Y_5, \partial_{z_2}Y_5, \partial_{z_3}Y_5)+O(1/X_0^2)\partial_{z_1}\big(O(1)\partial_{z_1}Y_4+O(\varepsilon)\partial_{z_2}Y_4+O(\varepsilon)\partial_{z_3}Y_4\big)\\
&\quad+\big(O(\varepsilon)\partial_{z_1}+O(1/X_0^2)\partial_{z_2}\big)\big(O(\varepsilon)\partial_{z_1}Y_4+O(1)\partial_{z_2}Y_4+O(1/X_0^2)\partial_{z_3}Y_4\big)\\
&\quad+\big(O(\varepsilon)\partial_{z_1}+O(1/X_0^2)\partial_{z_3}\big)\big(O(\varepsilon)\partial_{z_1}Y_4+O(1/X_0^2)\partial_{z_2}Y_4+O(1)\partial_{z_3}Y_4\big)\\
&\quad+O(1)\partial_{z_1}\big(O(1/X_0^2)\partial_{z_1}Y_4+O(\varepsilon)\partial_{z_2}Y_4+O(\varepsilon)\partial_{z_3}Y_4\big)\\
&\quad+\big(O(\varepsilon)\partial_{z_1}+O(1)\partial_{z_2}\big)\big(O(\varepsilon)\partial_{z_1}Y_4+O(1/X_0^2)\partial_{z_2}Y_4+O(1/X_0^2)\partial_{z_3}Y_4\big)\\
&\quad+\big(O(\varepsilon)\partial_{z_1}+O(1)\partial_{z_3}\big)\big(O(\varepsilon)\partial_{z_1}Y_4+O(1/X_0^2)\partial_{z_2}Y_4+O(1/X_0^2)\partial_{z_3}Y_4\big),
\end{aligned}
$$

and the singular terms

$$
O(1)\frac{z_2M_1+z_3M_2}{z_2^2+z_3^2} \quad\text{and}\quad O(1)\frac{z_3M_1-z_2M_2}{z_2^2+z_3^2}
$$

in (3-24) arise essentially from the computation

$$
\begin{aligned}
&(z_2\partial_{z_2}+z_3\partial_{z_3})\big(\widetilde{D}_2(c^2(\rho)\widetilde{D}_2Y_4)+\widetilde{D}_3(c^2(\rho)\widetilde{D}_3Y_4)\big)\\
&\quad=\big(O(\varepsilon)\partial_{z_1}+O(1/X_0^2)\partial_{z_2}\big)\big(O(\varepsilon)\partial_{z_1}Y_4+O(1)\partial_{z_2}Y_4\big)\\
&\qquad+\big(O(\varepsilon)\partial_{z_1}+O(1/X_0^2)\partial_{z_3}\big)\big(O(\varepsilon)\partial_{z_1}Y_4+O(1)\partial_{z_3}Y_4\big)\\
&\qquad+\big(O(\varepsilon)\partial_{z_1}+O(1)\partial_{z_2}\big)\big(O(\varepsilon)\partial_{z_1}Y_4+O(1/X_0^2)\partial_{z_2}Y_4)\big)\\
&\qquad+\big(O(\varepsilon)\partial_{z_1}+O(1)\partial_{z_3}\big)\big(O(\varepsilon)\partial_{z_1}Y_4+O(1/X_0^2)\partial_{z_3}Y_4)\big)\\
&\qquad+\widetilde{D}_2\big(c^2(\rho)\widetilde{D}_2M_1-2c^2(\rho)\partial_{z_2}Y_4\big)+\widetilde{D}_3\big(c^2(\rho)\widetilde{D}_3M_1-2c^2(\rho)\partial_{z_3}Y_4\big)\\
&\quad=\big(O(\varepsilon)\partial_{z_1}+O(1/X_0^2)\partial_{z_2}\big)\big(O(\varepsilon)\partial_{z_1}Y_4+O(1)\partial_{z_2}Y_4\big)\\
&\qquad+\big(O(\varepsilon)\partial_{z_1}+O(1/X_0^2)\partial_{z_3}\big)\big(O(\varepsilon)\partial_{z_1}Y_4+O(1)\partial_{z_3}Y_4\big)\\
&\qquad+\big(O(\varepsilon)\partial_{z_1}+O(1)\partial_{z_2}\big)\big(O(\varepsilon)\partial_{z_1}Y_4+O(1/X_0^2)\partial_{z_2}Y_4)\big)\\
&\qquad+\big(O(\varepsilon)\partial_{z_1}+O(1)\partial_{z_3}\big)\big(O(\varepsilon)\partial_{z_1}Y_4+O(1/X_0^2)\partial_{z_3}Y_4)\big)\\
&\qquad+\widetilde{D}_2\bigg(c^2(\rho)\widetilde{D}_2M_1+O(1)\frac{z_2M_1+z_3M_2}{z_2^2+z_3^2}\bigg)\\
&\qquad+\widetilde{D}_3\bigg(c^2(\rho)\widetilde{D}_3M_1+O(1)\frac{z_3M_1-z_2M_2}{z_2^2+z_3^2}\bigg).
\end{aligned}
$$

The factors

$$\frac{z_2}{z_2^2+z_3^2} \quad \text{and} \quad \frac{z_3}{z_2^2+z_3^2}$$

in the second-order elliptic Equation (3-24) have a strong singularity on $z_2^2+z_3^2=0$. Thus it is difficult to use the standard theory on second-order elliptic equations to derive directly the global $C^{1,\alpha}$ estimate on M_1 in E_+. To overcome this difficulty, we first establish the boundary $C^{1,\alpha}$ estimate of M_1. In fact, the compatibility conditions on the intersection curve between the shock surface Σ and the nozzle wall Π_2 (see [Xin and Yin 2008b, Appendix B]) as well as the natural compatibility conditions on the intersection curve between the end $r = X_0 + 1$ and Π_2 due to the $C^{3,\alpha}$ regularity assumption of the solution have the following implication: From the estimates on the boundary of the second-order elliptic equations with the divergence form and the Dirichlet boundary values on the cornered domain (see [Azzam 1980; 1981; Lieberman 1986; 1988]), we have

$$\begin{aligned}
&\|M_1\|_{C^{1,\alpha}(\bar{B}E_+^0)} \\
&\le C\big(\|M_1\|_{L^\infty}+\|M_2\|_{C^\alpha}+\|H_2\|_{C^\alpha}+\|H_3\|_{C^\alpha} \\
&\qquad\qquad +\|M_1|_{z_1=0}\|_{C^{1,\alpha}}+\|M_1|_{z_2^2+z_3^2=1}\|_{C^{1,\alpha}}\big) \\
&\le C(\|(\partial_{z_2}Y_4,\partial_{z_3}Y_4)\|_{L^\infty}+\|M_2\|_{C^{1,\alpha}})+C\varepsilon\|(\partial_{z_3}Y_2,\partial_{z_2}Y_3)\|_{C^{1,\alpha}} \\
&\quad +\frac{C}{X_0}\|(Y_1,Y_2,Y_3,X_0^{-1}Y_5,\partial_{z_1}Y_1,X_0^{-1}\partial_{z_1}Y_2,X_0^{-1}\partial_{z_1}Y_3,\partial_{z_2}Y_2,\partial_{z_3}Y_3)\|_{C^{1,\alpha}} \\
&\qquad +\frac{C}{X_0}\|(Y_4,\partial_{z_1}Y_4,\partial_{z_2}Y_4,\partial_{z_3}Y_4)\|_{C^{1,\alpha}}+\frac{C}{X_0}\|(\partial_{z_2}Y_5,\partial_{z_3}Y_5)\|_{C^{1,\alpha}},
\end{aligned} \tag{3-25}$$

where the subdomain E_+^0 of E_+ contains the nozzle wall $\{z : 0<z_1<1,\ z_2^2+z_3^2=1\}$.

Similar analysis gives a second-order elliptic equation for M_2 with suitable boundary conditions. In fact, by the fourth equality in (3-23), one has

$$\begin{aligned}
&(z_2\partial_{z_2}+z_3\partial_{z_3})M_2 \\
&\qquad = O(\varepsilon)\cdot(Y_2,\ Y_3,\ \varepsilon Y_4,\ \partial_{z_2}Y_2,\ \partial_{z_3}Y_2,\ \partial_{z_2}Y_3,\ \partial_{z_3}Y_3,\ \varepsilon M_2) \quad \text{on } z_2^2+z_3^2=1.
\end{aligned}$$

Note that

$$\begin{aligned}
&(z_3\partial_{z_2}-z_2\partial_{z_3})\big(\widetilde{D}_2(c^2(\rho)\widetilde{D}_2Y_4)+\widetilde{D}_3(c^2(\rho)\widetilde{D}_3Y_4)\big) \\
&\quad = \big(O(\varepsilon)\partial_{z_1}+O(1/X_0^2)\partial_{z_2}\big)\big(O(\varepsilon)\partial_{z_1}Y_4+O(1)\partial_{z_2}Y_4\big) \\
&\qquad + \big(O(\varepsilon)\partial_{z_1}+O(1/X_0^2)\partial_{z_3}\big)\big(O(\varepsilon)\partial_{z_1}Y_4+O(1)\partial_{z_3}Y_4\big) \\
&\qquad + \big(O(1)\partial_{z_2}+O(1)\partial_{z_3}\big)\big(O(\varepsilon)\cdot(\partial_{z_1}Y_4,\ \partial_{z_2}Y_4)\big) \\
&\qquad + \widetilde{D}_2(c^2(\rho)\widetilde{D}_2M_2)+\widetilde{D}_3(c^2(\rho)\widetilde{D}_3M_2).
\end{aligned}$$

Then we can show that M_2 solves

$$
\begin{cases}
\widetilde{D_1}\big((c^2(\rho)-U_1^2)\widetilde{D_1}M_2+c^2(\rho)(z_2^2\tau^2+z_3^2\tau^2)\widetilde{D_1}M_2 \\
\qquad -U_1U_2\widetilde{D_2}M_2-U_1U_3\widetilde{D_3}M_2\big) \\
+\widetilde{D_2}\big(-U_1U_2\widetilde{D_1}M_2+(c^2(\rho)-U_2^2)\widetilde{D_2}M_2 \\
\qquad +z_2^2\tau^2c^2(\rho)\widetilde{D_2}M_2-U_2U_3\widetilde{D_3}M_2 \\
\qquad +z_2z_3\tau^2c^2(\rho)\widetilde{D_3}M_2\big) \\
+\widetilde{D_3}\big(-U_1U_3\widetilde{D_1}M_2-U_2U_3\widetilde{D_2}M_2 \\
\qquad +z_2z_3\tau^2c^2(\rho)\widetilde{D_2}M_2+(c^2(\rho)-U_3^2)\widetilde{D_3}M_2 \\
\qquad +z_3^2\tau^2c^2(\rho)\widetilde{D_3}M_2\big) \\
\quad =(z_3\partial_{z_2}-z_2\partial_{z_3})H_2(Y,\nabla Y)+\widetilde{H_3}(Y,\nabla Y) & \text{in } E_+, \\
M_2=O(\varepsilon)\cdot\big(Y_2,Y_3,\partial_{z_2}Y_2,\partial_{z_3}Y_2,\partial_{z_2}Y_3,\partial_{z_3}Y_3\big) \\
\qquad +O(1/X_0)\cdot\big(X_0^{-1}Y_5,\partial_{z_2}Y_5,\partial_{z_3}Y_5\big) & \text{on } z_1=0, \\
M_2=0 & \text{on } z_1=1, \\
(z_2\partial_{z_2}+z_3\partial_{z_3})M_2 \\
\quad =O(\varepsilon) \\
\qquad \cdot\big(Y_2,Y_3,\varepsilon Y_4,\partial_{z_2}Y_2,\partial_{z_3}Y_2,\partial_{z_2}Y_3,\partial_{z_3}Y_3,\varepsilon M_2\big) & \text{on } z_2^2+z_3^2=1,
\end{cases}
\tag{3-26}
$$

where $\widetilde{H_3}(Y,\nabla Y)$ has the same property as $H_3(Y,\nabla Y)$ in (3-24).

Since the equation in (3-26) has no singular terms, a global $C^{1,\alpha}$ estimate of M_2 in E_+ can easily be given as

$$
\begin{aligned}
&\|M_2\|_{C^{1,\alpha}} \\
&\le C\big(\|H_2\|_{C^\alpha}+\|\tilde{H}_3\|_{C^\alpha}+\|M_2|_{z_1=0}\|_{C^{1,\alpha}} \\
&\qquad\qquad\qquad +\|(z_2\partial_{z_2}+z_3\partial_{z_3})M_2|_{z_2^2+z_3^2=1}\|_{C^\alpha}\big) \\
&\le \frac{C}{X_0}\|(Y_1,Y_2,Y_3,Y_4,X_0^{-1}Y_5)\|_{C^{1,\alpha}} \\
&\quad +\frac{C}{X_0}\|(\partial_{z_1}Y_1,X_0^{-1}\partial_{z_1}Y_2,X_0^{-1}\partial_{z_1}Y_3,\partial_{z_2}Y_2,\partial_{z_3}Y_3,\partial_{z_1}Y_4,\partial_{z_2}Y_4,\partial_{z_3}Y_4)\|_{C^{1,\alpha}} \\
&\qquad +C\varepsilon\|(\partial_{z_3}Y_2,\partial_{z_2}Y_3)\|_{C^{1,\alpha}}+\frac{C}{X_0}\|(\partial_{z_2}Y_5,\partial_{z_3}Y_5)\|_{C^{2,\alpha}}.
\end{aligned}
\tag{3-27}
$$

Next, we treat the bounds of $\|\partial_{z_2}Y_4\|_{L^\infty}$ and $\|\partial_{z_3}Y_4\|_{L^\infty}$ in (3-25).

As with (3-24), the first three equations of (3-23) imply that $\partial_{z_2} Y_4$ satisfies

$$
\begin{cases}
\widetilde{D_1}\big((c^2(\rho)-U_1^2)\widetilde{D_1}(\partial_{z_2}Y_4)+c^2(\rho)(z_2^2\tau^2+z_3^2\tau^2)\widetilde{D_1}(\partial_{z_2}Y_4) \\
\qquad -U_1U_2\widetilde{D_2}(\partial_{z_2}Y_4)-U_1U_3\widetilde{D_3}(\partial_{z_2}Y_4)\big) \\
+\widetilde{D_2}\big(-U_1U_2\widetilde{D_1}(\partial_{z_2}Y_4)+(c^2(\rho)-U_2^2)\widetilde{D_2}(\partial_{z_2}Y_4)+z_2^2\tau^2c^2(\rho)\widetilde{D_2}(\partial_{z_2}Y_4) \\
\qquad -U_2U_3\widetilde{D_3}(\partial_{z_2}Y_4)+z_2z_3\tau^2c^2(\rho)\widetilde{D_3}(\partial_{z_2}Y_4)\big) \\
+\widetilde{D_3}\big(-U_1U_3\widetilde{D_1}(\partial_{z_2}Y_4)-U_2U_3\widetilde{D_2}(\partial_{z_2}Y_4)+z_2z_3\tau^2c^2(\rho)\widetilde{D_2}(\partial_{z_2}Y_4) \\
\qquad +(c^2(\rho)-U_3^2)\widetilde{D_3}(\partial_{z_2}Y_4)+z_3^2\tau^2c^2(\rho)\widetilde{D_3}(\partial_{z_2}Y_4)\big) \\
\qquad =\partial_{z_2}H_2(Y,\nabla Y)+\hat{H}_3(Y,\nabla Y) & \text{in } E_+, \\
\partial_{z_2}Y_4=O(\varepsilon)\cdot(Y_2,Y_3,\partial_{z_2}Y_2,\partial_{z_2}Y_3)+O(1/X_0)\cdot(X_0^{-1}Y_5,\partial_{z_2}Y_5) & \text{on } z_1=0, \\
\partial_{z_2}Y_4=0 & \text{on } z_1=1,
\end{cases}
\tag{3-28}
$$

where $\hat{H}_3(Y,\nabla Y)$ has the same property as $H_3(Y,\nabla Y)$ in (3-24).

By the maximum principle for second-order elliptic equations of divergence form with the Dirichlet boundary condition [Gilbarg and Trudinger 1983, Theorem 8.16], we have

$$
\begin{aligned}
\|\partial_{z_2}Y_4\|_{L^\infty} &\\
\le C\big(\|\partial_{z_2}Y_4|_{z_1=0}\|_{L^\infty}&+\|\partial_{z_2}Y_4|_{z_1=1}\|_{L^\infty}+\|\partial_{z_2}Y_4|_{z_2^2+z_3^2=1}\|_{L^\infty} \\
&+\|H_2\|_{C^\alpha}+\|\hat{H}_3\|_{C^\alpha}\big).
\end{aligned}
\tag{3-29}
$$

Since $M_1=O(\varepsilon)\cdot(Y_2,Y_3,\varepsilon Y_4)$ on $z_2^2+z_3^2=1$, a simple computation yields

$$
\begin{aligned}
\|\partial_{z_2}Y_4\|_{L^\infty} &\le \|M_1|_{z_2^2+z_3^2=1}\|_{L^\infty}+\|M_2|_{z_2^2+z_3^2=1}\|_{L^\infty} \\
&\le C\varepsilon\|(Y_2,Y_3,\varepsilon Y_4)\|_{L^\infty}+C\|M_2\|_{C^{1,\alpha}}.
\end{aligned}
\tag{3-30}
$$

Substituting (3-30), (3-25), (3-27) and the boundary value conditions of (3-28) into (3-29) gives

$$
\begin{aligned}
\|\partial_{z_2}Y_4\|_{L^\infty} \le \frac{C}{X_0}&\|(Y_1,Y_2,Y_3,Y_4,X_0^{-1}Y_5)\|_{C^{1,\alpha}} \\
&+\frac{C}{X_0}\big(\|(\partial_{z_1}Y_1,\partial_{z_2}Y_2,\partial_{z_3}Y_3,\partial_{z_1}Y_4,\partial_{z_2}Y_4,\partial_{z_3}Y_4)\|_{C^{1,\alpha}} \\
&\qquad+\|(\partial_{z_2}Y_5,\partial_{z_3}Y_5)\|_{C^{2,\alpha}}\big) \\
&+C\varepsilon\|(\partial_{z_1}Y_2,\partial_{z_1}Y_3,\partial_{z_3}Y_2,\partial_{z_2}Y_3)\|_{C^{1,\alpha}}.
\end{aligned}
\tag{3-31}
$$

Similarly,

$$\begin{aligned}\text{(3-32)}\quad \|\partial_{z_3}Y_4\|_{L^\infty} \le \frac{C}{X_0}\|(Y_1, Y_2, Y_3, Y_4, X_0^{-1}Y_5)\|_{C^{1,\alpha}} &\\ + \frac{C}{X_0}\big(\|(\partial_{z_1}Y_1, \partial_{z_2}Y_2, \partial_{z_3}Y_3, \partial_{z_1}Y_4, \partial_{z_2}Y_4, \partial_{z_3}Y_4)\|_{C^{1,\alpha}} &\\ + \|(\partial_{z_2}Y_5, \partial_{z_3}Y_5)\|_{C^{2,\alpha}}\big) &\\ + C\varepsilon\|(\partial_{z_1}Y_2, \partial_{z_1}Y_3, \partial_{z_3}Y_2, \partial_{z_2}Y_3)\|_{C^{1,\alpha}}. &\end{aligned}$$

So far, we have shown that the "large" term $\|(\partial_{z_2}Y_4, \partial_{z_3}Y_4)\|_{L^\infty} + \|M_2\|_{C^{1,\alpha}}$ in the right hand side of (3-25) can be controlled by the "good" terms in (3-27) and (3-31)–(3-32). This means that $\|M_1\|_{C^{1,\alpha}(\bar{B}E_+^0)}$ has the same estimate as in (3-31)–(3-32). Namely,

$$\begin{aligned}\text{(3-33)}\quad &\|(\partial_{z_2}Y_4, \partial_{z_3}Y_4)\|_{C^{1,\alpha}(\bar{B}E_+^0)}\\ &\le \frac{C}{X_0}\|(Y_1, Y_2, Y_3, Y_4, X_0^{-1}Y_5)\|_{C^{1,\alpha}}\\ &\quad + \frac{C}{X_0}\big(\|(\partial_{z_1}Y_1, \partial_{z_2}Y_2, \partial_{z_3}Y_3, \partial_{z_1}Y_4, \partial_{z_2}Y_4, \partial_{z_3}Y_4)\|_{C^{1,\alpha}}\\ &\quad + \|(\partial_{z_2}Y_5, \partial_{z_3}Y_5)\|_{C^{2,\alpha}}\big)\\ &\quad + C\varepsilon\|(\partial_{z_1}Y_2, \partial_{z_1}Y_3, \partial_{z_3}Y_2, \partial_{z_2}Y_3)\|_{C^{1,\alpha}}.\end{aligned}$$

From this and the equations on $\partial_{z_2}Y_4$ and $\partial_{z_3}Y_4$ (see (3-28)), one has

$$\begin{aligned}&\|(\partial_{z_2}Y_4, \partial_{z_3}Y_4)\|_{C^{1,\alpha}}\\ &\le C\big(\|(\partial_{z_2}Y_4, \partial_{z_3}Y_4)\|_{L^\infty} + \|(\partial_{z_2}Y_4, \partial_{z_3}Y_4)|_{\partial E^+}\|_{C^{1,\alpha}} + \|H_2\|_{C^\alpha} + \|\hat{H}_3\|_{C^\alpha}\big)\\ &\le \frac{C}{X_0}\|(Y_1, Y_2, Y_3, Y_4, X_0^{-1}Y_5)\|_{C^{1,\alpha}}\\ &\quad + \frac{C}{X_0}\big(\|(\partial_{z_1}Y_1, \partial_{z_2}Y_2, \partial_{z_3}Y_3, \partial_{z_1}Y_4, \partial_{z_2}Y_4, \partial_{z_3}Y_4)\|_{C^{1,\alpha}} + \|(\partial_{z_2}Y_5, \partial_{z_3}Y_5)\|_{C^{2,\alpha}}\big)\\ &\quad + C\varepsilon\|(\partial_{z_1}Y_2, \partial_{z_1}Y_3, \partial_{z_3}Y_2, \partial_{z_2}Y_3)\|_{C^{1,\alpha}}.\end{aligned}$$

This completes the proof of Lemma 3.7. □

Remark 3.6. We now explain the importance of deriving the $C^{2,\alpha}$-regularity estimates on Y_4 and (Y_1, Y_2, Y_3) simultaneously. The crucial estimate in (3-14) which bounds $\|(\partial_{z_2}Y_2, \partial_{z_3}Y_2, \partial_{z_2}Y_3, \partial_{z_3}Y_3)\|_{C^{1,\alpha}}$ in terms of $\|(\nabla Y_1, \nabla Y_4)\|_{C^{1,\alpha}}$ and $\|(\partial_{z_2}Y_5, \partial_{z_3}Y_5)\|_{C^{2,\alpha}}$ follows from the key observation that though the system (2-11) is hyperbolic, the lower-dimensional first-order system (3-16) is elliptic. Indeed, without (3-16), the standard characteristic method for (2-11) gives only that (Y_2, Y_3) has the same $C^{1,\alpha}$ regularity as $(\partial_{z_2}Y_4, \partial_{z_3}Y_4) \in C^{1,\alpha}$. In this case, one can estimate $\|(\partial_{z_2}Y_2, \partial_{z_3}Y_2, \partial_{z_2}Y_3, \partial_{z_3}Y_3)\|_{C^\alpha}$ in terms of the right hand side of (3-14) by the proof of Lemma 3.5. Then, from the proof of (3-6), one can estimate $\|(\partial_{z_1}Y_1, \partial_{z_1}Y_4)\|_{C^\alpha}$ which gives an estimate of $\|(Y_2, Y_3)\|_{C^{1,\alpha}}$ on $z_1 = 0$ using the proof of (3-11). Together with boundary condition on $z_1 = 0$ in (3-28), this yields the desired estimate on $\|(\partial_{z_2} Y_4, \partial_{z_3} Y_4)\|_{C^\alpha}$. However, neither $C^{1,\alpha}$ estimates on $(\nabla Y_1, \nabla Y_2, \nabla Y_3, \nabla Y_4)$ nor $C^{2,\alpha}$ estimates on $\nabla_{z_2,z_3} Y_5$ can be obtained in this way.

Remark 3.7. We have established a priori estimates for the gradients of solutions instead of solutions themselves. Trying to derive a priori estimates on a solution directly would give from (3-9) that

$$\|\partial_{z_1} Y_4\|_{C^{1,\alpha}} \le C_1 \|(\partial_{z_2} Y_2, \partial_{z_3} Y_3)\|_{C^{1,\alpha}} + \text{positive terms with “good” coefficients},$$

while (3-12) yields

$$\|(\partial_{z_2} Y_2, \partial_{z_2} Y_3)\|_{C^{1,\alpha}} \le C_2 \|\partial_{z_1} Y_4\|_{C^{1,\alpha}} + \text{positive terms with “good” coefficients.}$$

However, it seems extremely difficult to get precise estimates on C_1 and C_2 so that $C_1 \cdot C_2 < 1$. Thus the direct estimate cannot yield useful information on $\partial_{z_1} Y_4$, $\partial_{z_2} Y_2$ and $\partial_{z_3} Y_3$.

4. Proofs of Theorem 1.1 and Proposition 2.2

Due to the equivalence between Theorem 1.1 and Theorem 2.1, it suffices to prove Theorem 2.1 only.

To this end, we first show that $\xi_1(0, 1) = \xi_2(0, 1)$ by contradiction. Without loss of generality, assume that

$$\xi_1(0, 1) < \xi_2(0, 1). \tag{4-1}$$

We will show the corresponding end pressures are different, contradicting (1-4).

Lemma 4.1. *For $\varepsilon_0 < 1/X_0^2$ in Theorem 2.1, one has*

$$\begin{cases} \|(\partial_{z_1} Y_1, \partial_{z_1} Y_4)\|_{C^{1,\alpha}} \le C|Y_4(0, 0, 1)|, \\ \|(\partial_{z_1} Y_2, \partial_{z_1} Y_3)\|_{C^{1,\alpha}} \le \dfrac{C}{X_0}|Y_4(0, 0, 1)|, \\ \displaystyle\sum_{i=1}^{5}\sum_{j=2}^{3}\|\partial_{z_j} Y_i\|_{C^{1,\alpha}} \le \dfrac{C}{X_0}|Y_4(0, 0, 1)|. \end{cases} \tag{4-2}$$

Remark 4.1. Thanks to the appearance of the term $(1/X_0^2)\|Y_5\|_{C^{1,\alpha}}$ in the right hand sides of (3-11), (3-14), (3-18) and (3-22), we can obtain the desired estimates (4-2), which will be the key in deriving the monotonicity of shock position on the end pressure and further obtaining the uniqueness result. Indeed, if the dominant term on the right hand sides of (3-11), (3-14), (3-18) and (3-22) is $(1/X_0)\|Y_5\|_{C^{1,\alpha}}$, then Lemma B.4 implies that $Y_5(0, 1) \sim X_0 Y_4(0, 0, 1)$ and the third estimate in (4-7) becomes

$$\|(\partial_{z_1}Y_2, \partial_{z_1}Y_3)\|_{C^{1,\alpha}} + \sum_{i=2}^{3}\sum_{j=1}^{5}\|\partial_{z_i}Y_j\|_{C^{1,\alpha}} \le \frac{C}{X_0}|Y_5(0, 1)|.$$

In this case, by Equation (4-11) below, one can only show that $\partial_{z_1}Y_4 = O(1/X_0)Y_5$. Thus, Equation (4-13) becomes $\partial_{z_1}Y_4 = O(1)Y_4$, which yields no useful information on Y_4. It is then unclear how to proceed to obtain the monotonic dependence of the shock position on the end pressure.

Proof of Lemma 4.1. By the estimates in Lemmas 3.2–3.7 and a direct computation,

$$\begin{cases} \|(\partial_{z_1}Y_1, \partial_{z_1}Y_4)\|_{C^{1,\alpha}} \le \dfrac{C}{X_0}\displaystyle\sum_{i=1}^{5}\|Y_i\|_{C^{1,\alpha}}, \\ \|(\partial_{z_1}Y_2, \partial_{z_1}Y_3)\|_{C^{1,\alpha}} \\ \quad + \displaystyle\sum_{i=2}^{3}\sum_{j=1}^{4}\|\partial_{z_i}Y_j\|_{C^{1,\alpha}} \le \dfrac{C}{X_0}\Big(\displaystyle\sum_{i=1}^{4}\|Y_i\|_{C^{1,\alpha}} + X_0^{-1}\|Y_5\|_{C^{1,\alpha}}\Big), \\ \|(\partial_{z_2}Y_5, \partial_{z_3}Y_5)\|_{C^{2,\alpha}} \le \dfrac{C}{X_0}\Big(\displaystyle\sum_{i=1}^{4}\|Y_i\|_{C^{1,\alpha}} + X_0^{-1}\|Y_5\|_{C^{1,\alpha}}\Big). \end{cases} \tag{4-3}$$

Note that

$$\begin{cases} \|(Y_1, Y_4)\|_{C^{1,\alpha}} \le C(|(Y_1, Y_4)(0, 0, 1)| + \|\nabla(Y_1, Y_4)\|_{C^{1,\alpha}}), \\ \|Y_5\|_{C^{1,\alpha}} \le C(|Y_5(0, 1)| + \|\nabla Y_5\|_{C^{2,\alpha}}). \end{cases} \tag{4-4}$$

The nonslip condition (2-7) implies that $z_2Y_2 + z_3Y_3 = 0$ on $z_2^2 + z_3^2 = 1$ and further $Y_2(z_1, 1, 0) = Y_3(z_1, 0, 1) = 0$, so

$$\|(Y_2, Y_3)\|_{C^{1,\alpha}} \le C\|\nabla(Y_2, Y_3)\|_{C^{1,\alpha}}. \tag{4-5}$$

In addition, at the point $(0, 0, 1)$, Equation (3-19) implies

$$|Y_1(0, 0, 1)| + |Y_4(0, 0, 1)| \le \frac{C}{X_0}|Y_5(0, 1)| + C\varepsilon(\|Y_2\|_{L^\infty} + \|Y_3\|_{L^\infty}). \tag{4-6}$$

Substituting (4-4)–(4-6) into (4-3) yields

$$\begin{cases} |Y_1(0, 0, 1)| + |Y_4(0, 0, 1)| + X_0|Y_2(0, 0, 1)| \le \dfrac{C}{X_0}|Y_5(0, 1)|, \\ \|\partial_{z_1}Y_1\|_{C^{1,\alpha}} + \|\partial_{z_1}Y_4\|_{C^{1,\alpha}} \le \dfrac{C}{X_0}|Y_5(0, 1)|, \\ \|(\partial_{z_1}Y_2, \partial_{z_1}Y_3)\|_{C^{1,\alpha}} + \displaystyle\sum_{i=2}^{3}\sum_{j=1}^{5}\|\partial_{z_i}Y_j\|_{C^{1,\alpha}} \le \dfrac{C}{X_0^2}|Y_5(0, 1)|. \end{cases} \tag{4-7}$$

In addition, by Lemma B.4,

$$|Y_5(0,1)| \le C X_0 |Y_4(0,0,1)|. \tag{4-8}$$

Combining (4-8) with (4-7) yields Lemma 4.1. □

Lemma 4.2. *Suppose that (4-1) and the assumptions in Theorem 2.1 hold. If $\rho_0^+(r_0) > 2\rho_0^-(r_0)$, then*

$$Y_4(0,0,1) > 0. \tag{4-9}$$

Proof. Lemma B.4 implies that $Y_4(0,0,1)$ and $Y_5(0,1)$ satisfy

$$Y_4 = a_0 Y_5 + O(1/X_0^2) Y_5,$$

where $a_0 < 0$ and $a_0 = O(1/X_0)$.

Thus by (4-1), we have $Y_4(0,0,1) > 0$. □

Remark 4.2. If $M_0^-(X_0) > \sqrt{(2^{\gamma+1}-1)/\gamma}$, then by [Li et al. 2009, Lemma 5.1], we can show that $\rho_0^+(r_0) > 2\rho_0^-(r_0)$ in Lemma 4.2.

Based on Lemmas 4.1 and 4.2, we can now prove Theorem 2.1.

Proof of Theorem 2.1. It follows from (2-4) and a direct computation that

$$\begin{cases} U_1 \widetilde{D}_1 Y_4 + \rho \widetilde{D}_1 Y_1 \\ \quad = O(1/X_0)\cdot(Y_1, X_0^{-1}Y_2, X_0^{-1}Y_3, Y_4) + a_1 Y_5 \\ \qquad + O(\varepsilon)\cdot(\partial_{z_1}Y_2, \partial_{z_1}Y_3, \varepsilon\partial_{z_1}Y_4, \partial_{z_2}Y_4, \partial_{z_2}Y_5, \partial_{z_3}Y_4, \partial_{z_3}Y_5) \\ \qquad + O(1)\cdot(\partial_{z_2}Y_2, \partial_{z_3}Y_3), \\ \rho U_1 \widetilde{D}_1 Y_1 + c^2(\rho)\widetilde{D}_1 Y_4 \\ \quad = O(1/X_0)\cdot(Y_1, X_0^{-1}Y_2, \varepsilon X_0 Y_3, Y_4) + a_2 Y_5 \\ \qquad + O(\varepsilon)\cdot(\varepsilon\partial_{z_1}Y_1, (\varepsilon X_0^2)^{-1}\partial_{z_1}Y_4, \partial_{z_2}Y_1, X_0^{-1}\partial_{z_2}Y_5, \partial_{z_3}Y_1, X_0^{-1}\partial_{z_3}Y_5), \end{cases} \tag{4-10}$$

where, abbreviating $\xi_1(z_2,z_3)$ by ξ_1 and $\xi_2(z_2,z_3)$ by ξ_2,

$$\begin{aligned} a_1 = &-\frac{\partial_{z_1}(\rho U_1)}{\sqrt{1+(z_2^2+z_3^2\tau^2)}(X_0+1-\xi_1)(X_0+1-\xi_2)} \\ &+ \frac{2(1-z_1)\rho U_1}{\sqrt{1+(z_2^2+z_3^2\tau^2)}(\xi_1+z_1(X_0+1-\xi_1))(\xi_2+z_1(X_0+1-\xi_2))} \\ &+ O(\varepsilon/X_0), \\ a_2 = &-\frac{c^2(\rho)\partial_{z_1}\rho + \rho U_1 \partial_{z_1}U_1}{\sqrt{1+(z_2^2+z_3^2\tau^2)}(X_0+1-\xi_1)(X_0+1-\xi_2)} \\ &+ O(1/X_0^3), \end{aligned}$$

It follows from (4-10) that

$$\begin{aligned}(4\text{-}11)\quad \partial_{z_1} Y_4 = a(z)Y_5 + O(1/X_0)\cdot(Y_1, X_0^{-1}Y_2, X_0^{-1}Y_3, Y_4) + O(1)(\partial_{z_2}Y_2, \partial_{z_3}Y_3)\\ + O(\varepsilon)\cdot(\varepsilon\partial_{z_1}Y_1, \partial_{z_1}Y_2, \partial_{z_1}Y_3, (\varepsilon X_0^2)^{-1}\\ \times \partial_{z_1}Y_4, \partial_{z_2}Y_1, \partial_{z_2}Y_4, \partial_{z_2}Y_5, \partial_{z_3}Y_1, \partial_{z_3}Y_4, \partial_{z_3}Y_5),\end{aligned}$$

where, again abbreviating $\xi_1(z_2, z_3)$ by ξ_1 and $\xi_2(z_2, z_3)$ by ξ_2,

$$\begin{aligned}a(z) &= \frac{(X_0+1-\xi_2)\sqrt{1+(z_2^2+z_3^2)\tau^2}}{c^2(\rho)-U_1^2}(a_2-a_1U_1)\\ &= -\frac{\partial_{z_1}\rho}{X_0+1-\xi_1}\\ &\qquad -\frac{2(X_0+1-\xi_2)(1-z_1)\rho U_1^2}{(c^2(\rho)-U_1^2)(\xi_1+z_1(X_0+1-\xi_1))(\xi_2+z_1(X_0+1-\xi_2))}\\ &\qquad + O(1/X_0^3),\end{aligned}\tag{4-12}$$

It should be pointed out here that the "good" coefficient $O(1/X_0^2)$ in the term of $\partial_{z_1}Y_4$ on the right hand side of (4-11) can be derived from (2-17), the assumptions on the solutions, and $\varepsilon < 1/X_0^2$ in Theorem 2.1.

In addition, under the assumptions of Theorem 2.1, one has

$$\begin{cases} \partial_{z_1}\rho = \partial_r\rho_0^+(r_0) + O(\varepsilon),\\ c^2(\rho) - U_1^2 = c^2(\rho_0^+(r_0)) - (U_0^+(r_0))^2 + O(1/X_0^2),\end{cases}$$

which yields

$$\partial_{z_1}\rho > 0, \quad c^2(\rho) - U_1^2 > 0.$$

Hence, it follows from (4-12) that $a(z)$ is a negative function in subsonic domain. In addition, (4-1) implies $Y_5(0, 1) < 0$. So $a(z)Y_5(0, 1)$ is always nonnegative along the line $(z_1, 0, 1)$. Thus along the line $(z_1, 0, 1)$, by Lemma 4.1, (4-11) can be reduced into

$$\begin{cases} \partial_{z_1}Y_4 \geq b(z)Y_4(0, 0, 1),\\ Y_4(0, 0, 1) > 0,\end{cases}\tag{4-13}$$

where $\|b(z)\|_{L^\infty} \leq O(1/X_0)$. This yields

$$Y_4(z_1, 0, 1) > C_1Y_4(0, 0, 1) > 0\tag{4-14}$$

for some constant $C_1 > 0$, which contradicts the end pressure condition (1-4), so contradicts (4-1). Thus $Y_5(0, 0, 1) = 0$.

So by Lemma 4.1,

$$Y_1 = Y_2 = Y_3 = Y_4 = Y_5 = 0.$$

This completes the proof of Theorem 2.1 and thus of Theorem 1.1. □

Proof of Proposition 2.2. It follows from the assumptions in Proposition 2.2 that $C_{0,1} < C_{0,2}$ and $Y_4(1, z_2, z_3) < 0$.

We claim that

$$Y_5(0, 1) > 0. \tag{4-15}$$

Otherwise, if $Y_5(0, 1) < 0$, then (4-13)–(4-14) imply $C_{0,1} > C_{0,2}$. If $Y_5(0, 1) = 0$, then $Y_4(0, 0, 1) = 0$ by Lemma B.4 and further $Y_4 \equiv 0$ by Lemma 4.1, hence $C_{0,1} = C_{0,2}$. Both cases contradict that $C_{0,1} < C_{0,2}$.

Since $Y_5 = Y_5(0, 1) + O(1)\partial_{z_2} Y_5 + O(1)\partial_{z_3} Y_5$, the third equality in (4-7) gives

$$Y_5(z_2, z_3) = Y_5(0, 1) + O(1/X_0^2)Y_5(0, 1). \tag{4-16}$$

Combining (4-16) and (4-15) yields $Y_5(z_2, z_3) > 0$ which implies $\xi_1(y_2, y_3) > \xi_2(y_2, y_3)$. □

Appendix A: Analysis of the background solution

Under the assumptions given in Section 1, we describe the transonic solution of the problem (1-1) with (1-2)–(1-5) when the end pressure is a given suitable constant P_e. Such a solution is called the background solution and can be obtained by solving the related ordinary differential equations. In fact, the analysis of this background solution was given in [Courant and Friedrichs 1948, Section 147]; see also [Xin and Yin 2008b, Section 2]. For the reader's convenience and the requirements of our computations in this paper, we state the main facts here.

Theorem A.1 (existence of a transonic shock for the constant end pressure). *For the* 3*D nozzle and the supersonic incoming flow given in Section 1, there exist two constant pressures P_1 and P_2 with $P_1 < P_2$, determined by the incoming flow and the nozzle, such that if the end pressure $P_e \in (P_1, P_2)$, then the system (1-1) has a symmetric transonic shock solution,*

$$(P, u_1, u_2, u_3) = \begin{cases} (P_0^-(r), u_{1,0}^-(x), u_{2,0}^-(x), u_{3,0}^-(x)) & \text{for } r < r_0, \\ (P_0^+(r), u_{1,0}^+(x), u_{2,0}^+(x), u_{3,0}^+(x)) & \text{for } r > r_0, \end{cases}$$

where $u_{i,0}^\pm = U_0^\pm x_i/r$ for $i = 1, 2, 3$ and $(P_0^\pm(r), U_0^\pm(r))$ is $C^{4,\alpha}$-smooth. Moreover, the position $r = r_0$ with $X_0 < r_0 < X_0 + 1$ and the strength of the shock are determined by P_e.

Proof. See [Xin and Yin 2008b, Section 2]. □

Remark A.1. By (1-6) and the analysis of [Xin and Yin 2008b, Theorem A, Section 2], there exists a constant $C > 0$ independent of X_0 such that for $r_0 \le r \le X_0+1$,

$$\left|\frac{d^k U_0^+(r)}{dr^k}\right| + \left|\frac{d^k P_0^+(r)}{dr^k}\right| \le \frac{C}{X_0^k}, \quad k = 1, 2, 3.$$

Remark A.2. It follows from (2-1) that we can obtain an extension $(\hat{\rho}_0^+(r), \hat{U}_0^+(r))$ of $(\rho_0^+(r), U_0^+(r))$ for $r \in (X_0, X_0+1)$ and large X_0.

Appendix B

We first give a detailed computation for H_0 in (2-9), and then derive a first-order elliptic system on (U_2, U_3) in the interior of the nozzle. Next, we discuss the regularity problem of solutions to a class of first-order elliptic system which includes a parameter. Finally, we derive a relation between $Y_4(0,0,1)$ and $Y_5(0,1)$ used in Lemmas 4.1 and 4.2.

Lemma B.1. *In (2-9), the function H_0 admits the estimate*

$$\begin{aligned}H_0 = O(|U_2|^2+|U_3|^2) + O(|\nabla_{y_2,y_3}\rho|^2)\\ + O(|\nabla_{y_2,y_3}U_2|^2) + O(|\nabla_{y_2,y_3}U_3|^2) + O(|\nabla_{y_2,y_3}\xi|^2)\\ + O(1/X_0)(|U_2|+|U_3|+|\nabla_{y_2,y_3}\rho|+|\nabla_{y_2,y_3}U_2|+|\nabla_{y_2,y_3}U_3|+|\nabla_{y_2,y_3}\xi|).\end{aligned}$$

Proof. It follows from

$$\partial_{y_3}\Big(\frac{\Delta_2}{\Delta_1}(\xi(y_2,y_3),y_2,y_3)\Big) = \partial_{y_2}\Big(\frac{\Delta_3}{\Delta_1}(\xi(y_2,y_3),y_2,y_3)\Big)$$

that

$$\text{(B-1)}\qquad \partial_{y_3}\Delta_2 - \partial_{y_2}\Delta_3 = \frac{\Delta_2\partial_{y_3}\Delta_1 - \Delta_3\partial_{y_2}\Delta_1}{\Delta_1}.$$

Since

$$\begin{aligned}\partial_{y_3}\Delta_2 = \frac{y_1\tau\rho U_1}{1+(y_2^2+y_3^2)\tau^2}\Big(&(\partial_{y_3}\xi\partial_{y_1}+\partial_{y_3})U_2 + y_3^2\tau^2(\partial_{y_3}\xi\partial_{y_1}+\partial_{y_3})U_2\\ &- y_2y_3\tau^2(\partial_{y_3}\xi\partial_{y_1}+\partial_{y_3})U_3 + 2y_3\tau^2U_2 - y_2\tau^2U_3\Big)\\ &+ \frac{\partial_{y_3}\xi\tau\rho U_1 + \xi\tau(\partial_{y_3}\xi\partial_{y_1}+\partial_{y_3})(\rho U_1)}{1+(y_2^2+y_3^2)\tau^2}(U_2+y_3^2\tau^2U_2-y_2y_3\tau^2U_3)\\ &- \frac{2y_1y_3\tau^3\rho U_1}{(1+(y_2^2+y_3^2)\tau^2)^2}(U_2+y_3^2\tau^2U_2-y_2y_3\tau^2U_3),\end{aligned}$$

$$\begin{aligned}\partial_{y_2}\Delta_3 = \frac{y_1\tau\rho U_1}{1+(y_2^2+y_3^2)\tau^2}\Big(&(\partial_{y_2}\xi\partial_{y_1}+\partial_{y_2})U_3 + y_2^2\tau^2(\partial_{y_2}\xi\partial_{y_1}+\partial_{y_2})U_3\\ &- y_2y_3\tau^2(\partial_{y_2}\xi\partial_{y_1}+\partial_{y_2})U_2 + 2y_2\tau^2U_3 - y_3\tau^2U_2\Big)\\ &+ \frac{\partial_{y_2}\xi\tau\rho U_1 + \xi\tau(\partial_{y_2}\xi\partial_{y_1}+\partial_{y_2})(\rho U_1)}{1+(y_2^2+y_3^2)\tau^2}(U_3+y_2^2\tau^2U_3-y_2y_3\tau^2U_2)\\ &- \frac{2y_1y_2\tau^3\rho U_1}{(1+(y_2^2+y_3^2)\tau^2)^2}(U_3+y_2^2\tau^2U_3-y_2y_3\tau^2U_2),\end{aligned}$$

$$\begin{aligned}\partial_{y_2}\Delta_1 = \rho\Big(&2(1+y_3^2\tau^2)U_2(\partial_{y_2}\xi\partial_{y_1}+\partial_{y_2})U_2 - 2y_2y_3\tau^2 U_3(\partial_{y_2}\xi\partial_{y_1}+\partial_{y_2})U_2 \\ &- 2y_2y_3\tau^2 U_2(\partial_{y_2}\xi\partial_{y_1}+\partial_{y_2})U_3 + 2(1+y_2^2\tau^2)U_3(\partial_{y_2}\xi\partial_{y_1}+\partial_{y_2})U_3\Big) \\ &+ (1+(y_2^2+y_3^2)\tau^2)[(\partial_{y_2}\xi\partial_{y_1}+\partial_{y_2})P] + 2y_2\tau^2[P] \\ &+ (\partial_{y_2}\xi\partial_{y_1}+\partial_{y_2})\rho((1+y_3^2\tau^2)U_2^2 - 2y_2y_3\tau^2U_2U_3 + (1+y_2^2\tau^2)U_3^2) \\ &+ \rho(2y_2\tau^2U_3^2 - 2y_3\tau^2U_2U_3),\end{aligned}$$

$$\begin{aligned}\partial_{y_3}\Delta_1 = \rho\Big(&2(1+y_3^2\tau^2)U_2(\partial_{y_3}\xi\partial_{y_1}+\partial_{y_3})U_2 - 2y_2y_3\tau^2 U_3(\partial_{y_3}\xi\partial_{y_1}+\partial_{y_3})U_2 \\ &- 2y_2y_3\tau^2 U_2(\partial_{y_3}\xi\partial_{y_1}+\partial_{y_3})U_3 + 2(1+y_2^2\tau^2)U_3(\partial_{y_3}\xi\partial_{y_1}+\partial_{y_3})U_3\Big) \\ &+ (1+(y_2^2+y_3^2)\tau^2)[(\partial_{y_3}\xi\partial_{y_1}+\partial_{y_3})P] + 2y_3\tau^2[P] \\ &+ (\partial_{y_3}\xi\partial_{y_1}+\partial_{y_3})\rho((1+y_3^2\tau^2)U_2^2 - 2y_2y_3\tau^2U_2U_3 + (1+y_2^2\tau^2)U_3^2) \\ &+ \rho(2y_3\tau^2U_2^2 - 2y_2\tau^2U_2U_3),\end{aligned}$$

substituting these expressions into (B-1) yields

$$\begin{aligned}&(\partial_{y_3}\xi\partial_{y_1}+\partial_{y_3})U_2 - (\partial_{y_2}\xi\partial_{y_1}+\partial_{y_2})U_3 \\ &\qquad = H_0(y_2, y_3, \rho, U_2, U_3, \xi, \nabla_{y_2,y_3}\rho, \nabla_{y_2,y_3}U_2, \nabla_{y_2,y_3}U_3, \nabla_{y_2,y_3}\xi),\end{aligned}$$

where

$$\begin{aligned}H_0 = {}&O(|U_2|^2+|U_3|^2) + O(|\nabla_{y_2,y_3}\rho|^2) + O(|\nabla_{y_2,y_3}U_2|^2) + O(|\nabla_{y_2,y_3}U_3|^2) + O(|\nabla_{y_2,y_3}\xi|^2) \\ &+ O(1/X_0)\big(|U_2|+|U_3|+|\nabla_{y_2,y_3}\rho|+|\nabla_{y_2,y_3}U_2|+|\nabla_{y_2,y_3}U_3|+|\nabla_{y_2,y_3}\xi|\big).\end{aligned}$$

This completes the proof of Lemma B.1. □

Lemma B.2. *Under the assumptions of Theorem 2.1, we have*

$$\begin{cases}\partial_{z_2}Y_2 + \partial_{z_3}Y_3 = F_3 & \text{in } E_+, \\ \partial_{z_3}Y_2 - \partial_{z_2}Y_3 = F_4 & \text{in } E_+, \\ z_2Y_2 + z_3Y_3 = 0 & \text{on } z_2^2+z_3^2 = 1,\end{cases}$$

with

$$\begin{aligned}F_3 = {}&O(1/X_0)\cdot(Y_1, X_0^{-1}Y_2, X_0^{-1}Y_3, Y_4, Y_5) \\ &+ O(\varepsilon)\cdot(\partial_{z_1}Y_2, \partial_{z_1}Y_3, \partial_{z_2}Y_4, \partial_{z_2}Y_5, \partial_{z_3}Y_4, \partial_{z_3}Y_5) \\ &+ O(1)\cdot(\partial_{z_1}Y_1, \partial_{z_1}Y_4),\end{aligned}$$

$$\begin{aligned}F_4 = {}&O(\varepsilon)\cdot(l^1, l^2) + O(1)\cdot(\partial_{z_3}Y_2, \partial_{z_2}Y_3)(0, \beta_1(z), \beta_2(z)) \\ &+ O(1/X_0)\cdot(\varepsilon Y_1, X_0^{-2}Y_2, X_0^{-2}Y_3, \varepsilon Y_4, X_0^{-2}Y_5) \\ &+ O(\varepsilon)\cdot(\partial_{z_1}Y_1, \partial_{z_1}Y_4, \partial_{z_2}Y_1, \partial_{z_3}Y_1) \\ &+ O(1/X_0^2)\cdot(\partial_{z_1}Y_2, \partial_{z_1}Y_3, \partial_{z_2}Y_2, X_0\partial_{z_2}Y_3, \partial_{z_2}Y_4, X_0^{-1}\partial_{z_2}Y_5, \\ &\qquad X_0\partial_{z_3}Y_2, \partial_{z_3}Y_3, \partial_{z_3}Y_4, \partial_{z_3}Y_5),\end{aligned}$$

where l^i and β_i for $i = 1, 2$ are defined as in Lemma 3.5.

Proof. By the first and the second equations in (2-11) we obtain

$$\text{(B-2)}\quad \begin{aligned} &c^2(\rho)\big((1+z_2^2\tau^2)(1+z_3^2\tau^2)-z_2^2z_3^2\tau^4\big)D_2\rho \\ &\quad=(1+z_3^2\tau^2)(\rho U_1D_1U_2+\rho U_2D_2U_2+\rho U_3D_3U_2) \\ &\qquad-z_2z_3\tau^2(\rho U_1D_1U_3+\rho U_2D_2U_3+\rho U_3D_3U_3) \\ &\qquad+\rho D_0((1+z_3^2\tau^2)U_2-z_2z_3\tau^2U_3)(U_1-z_2\tau U_2-z_2\tau U_3) \end{aligned}$$

$$\text{(B-3)}\quad \begin{aligned} &c^2(\rho)\big((1+z_2^2\tau^2)(1+z_3^2\tau^2)-z_2^2z_3^2\tau^4\big)D_3\rho \\ &\quad=(1+z_2^2\tau^2)(\rho U_1D_1U_3+\rho U_2D_2U_3+\rho U_3D_3U_3) \\ &\qquad-z_2z_3\tau^2(\rho U_1D_1U_2+\rho U_2D_2U_2+\rho U_3D_3U_2) \\ &\qquad+\rho D_0((1+z_2^2\tau^2)U_3-z_2z_3\tau^2U_2)(U_1-z_2\tau U_2-z_2\tau U_3). \end{aligned}$$

Applying ∂_{y_3} to (B-2) and ∂_{y_2} to (B-3), and then subtracting them results in

$$\text{(B-4)}\quad \begin{aligned} &(\rho U_1D_1+\rho U_2D_2+\rho U_3D_3)\big(\partial_{z_3}U_2-\partial_{z_2}U_3+O(\varepsilon)\partial_{z_1}U_2+O(\varepsilon)\partial_{z_1}U_3\big) \\ &\quad+(\rho U_1D_1+\rho U_2D_2+\rho U_3D_3)\big(z_2z_3\tau^2\partial_{z_2}U_2-z_2^2\tau^2\partial_{z_2}U_3 \\ &\qquad\qquad+z_3^2\tau^2\partial_{z_3}U_2-z_2z_3\tau^2\partial_{z_3}U_3\big) \\ &\qquad\qquad=H_4(z,U,\rho,\nabla U,\nabla\rho), \end{aligned}$$

where

$$\begin{aligned} H_4(z,\rho,U,\nabla\rho,\nabla U)&=O(|U_2|^2+|U_3|^2)+O(|\nabla U|^2)+O(|\nabla\rho|^2) \\ &\quad+O(1/X_0+\varepsilon)(|U_2|+|U_3|+|\nabla\rho|+|\nabla U|). \end{aligned}$$

Finally, due to the first equation in (2-4) and (B-4), a direct computation implies

$$\begin{cases} \partial_{z_2}Y_2+\partial_{z_3}Y_3=F_3 & \text{in } E_+, \\ \partial_{z_3}Y_2-\partial_{z_2}Y_3=F_4 & \text{in } E_+, \\ z_2Y_2+z_3Y_3=0 & \text{on } z_2^2+z_3^2=1, \end{cases}$$

and F_i for $i=3,4$ has the same properties as stated in Lemma B.2. □

Lemma B.3. *Assume that the problem*

$$\text{(B-5)}\quad \begin{cases} \partial_2u_1+\partial_3u_2=f_1(x_1,x_2,x_3) & \textit{in } \Omega=\{(x_1,x_2,x_3):[0,1]\times B_1(0)\}, \\ \partial_3u_1-\partial_2u_2=f_2(x_1,x_2,x_3) & \textit{in } \Omega=\{(x_1,x_2,x_3):[0,1]\times B_1(0)\}, \\ \partial_1u_1=f_3(x_1,x_2,x_3) & \textit{in } \Omega=\{(x_1,x_2,x_3):[0,1]\times B_1(0)\}, \\ \partial_1u_2=f_4(x_1,x_2,x_3) & \textit{in } \Omega=\{(x_1,x_2,x_3):[0,1]\times B_1(0)\}, \\ x_2u_1+x_3u_2=0 & \textit{on } \Gamma=\{(x_1,x_2,x_3):[0,1]\times \partial B_1(0)\} \end{cases}$$

has a $C^{2,\alpha}(\bar{\Omega})$ *solution* (u_1, u_2), *where* $f_i \in C^{1,\alpha}(\bar{\Omega})$ *for* $i = 1, 2, 3, 4$. *Then*

$$\sum_{i=2}^{3}\sum_{j=1}^{2} \|\partial_{x_i} u_j\|_{C^{1,\alpha}(\bar{\Omega})} \le C\left(\|\nabla f_1\|_{C^{\alpha}(\bar{\Omega})} + \sum_{i=2}^{4} \|f_i\|_{C^{1,\alpha}(\bar{\Omega})}\right). \tag{B-6}$$

Proof. Set $\Sigma_1 = \{(0, x_2, x_3) : x_2^2 + x_3^2 \le 1\}$ and $\Sigma_2 = \{(1, x_2, x_3) : x_2^2 + x_3^2 \le 1\}$.

First, we assert

$$\sum_{i=1}^{2} \|u_j\|_{C^{2,\alpha}(\Sigma_i)} + \|u_j\|_{C^{2,\alpha}(\Gamma)} \le C \sum_{k=1}^{4} \|f_k\|_{C^{1,\alpha}(\bar{\Omega})}, \quad j = 1, 2. \tag{B-7}$$

Indeed, it follows from (B-5) that on Σ_i for $i = 1, 2$,

$$\begin{cases} \partial_2 u_1 + \partial_3 u_2 = f_1(i-1, x_2, x_3) & \text{in } B_1(0), \\ \partial_3 u_1 - \partial_2 u_2 = f_2(i-1, x_2, x_3) & \text{in } B_1(0), \\ x_2 u_1 + x_2 u_2 = 0 & \text{on } \Sigma_i. \end{cases}$$

Thus, by the solution of the index -2 Hilbert problem in [Bers 1950; 1951; Vekua 1952],

$$\|u_1\|_{C^{2,\alpha}(\Sigma_i)} + \|u_2\|_{C^{2,\alpha}(\Sigma_i)} \le C(\|f_1\|_{C^{1,\alpha}(\Sigma_i)} + \|f_2\|_{C^{1,\alpha}(\Sigma_i)}), \quad i = 1, 2. \tag{B-8}$$

For notational convenience, set $w_1 = x_2 u_1 + x_3 u_2$ and $w_2 = x_3 u_2 - x_2 u_1$.

Equation (B-5) implies that w_1 and w_2 satisfy the following the second-order elliptic equations, respectively:

$$\begin{cases} (\partial_1^2 + \partial_2^2 + \partial_3^2) w_1 = \partial_1(x_2 f_2 + x_3 f_4) \\ \qquad\qquad + \partial_2(x_2 f_1 - x_3 f_2) + \partial_3(x_2 f_2 + x_3 f_3) & \text{in } \Omega, \\ w_1 = 0 & \text{on } \Gamma, \end{cases} \tag{B-9}$$

$$\begin{cases} (\partial_1^2 + \partial_2^2 + \partial_3^2) w_2 = \partial_1(x_3 f_3 - x_2 f_4) \\ \qquad\qquad + \partial_2(x_2 f_2 + x_3 f_1) - \partial_3(x_2 f_1 - x_3 f_2) & \text{in } \Omega, \\ (x_2\partial_2 + x_3\partial_3) w_2 = f_2 & \text{on } \Gamma. \end{cases} \tag{B-10}$$

For the problem (B-9), it follows from [Gilbarg and Trudinger 1983, Theorem 3.7, Theorem 6.6] that

$$\|w_1\|_{C^{2,\alpha}(\bar{\Omega})} \le C\left(\sum_{i=1}^{2} \|w_1\|_{C^{2,\alpha}(\Sigma_i)} + \sum_{j=1}^{4} \|f_j\|_{C^{1,\alpha}(\bar{\Omega})}\right). \tag{B-11}$$

For (B-10), the compatibility conditions at corners and $C^{2,\alpha}$ estimates of solutions to second-order elliptic equations with mixed boundary conditions in [Xin et al.

2009, Lemma A] imply that

(B-12)
$$\|w_2\|_{C^{2,\alpha}(\bar\Omega)} \le C\Big(\sum_{i=1}^{2}\|w_2\|_{C^{2,\alpha}(\Sigma_i)} + \sum_{j=1}^{4}\|f_j\|_{C^{1,\alpha}(\bar\Omega)}\Big).$$

Transforming w_1 and w_2 back to u_1 and u_2 via

$$u_1 = \frac{x_2w_1 + x_3w_2}{x_2^2+x_3^2} \quad\text{and}\quad u_2 = \frac{x_3w_1 - x_2w_2}{x_2^2+x_3^2}$$

gives

(B-13)
$$\begin{aligned}\|u_1\|_{C^{2,\alpha}(\Gamma)} + \|u_2\|_{C^{2,\alpha}(\Gamma)} &\le C(\|w_1\|_{C^{2,\alpha}(\bar\Omega)} + \|w_2\|_{C^{2,\alpha}(\bar\Omega)})\\ &\le C\Big(\sum_{i=1}^{2}(\|w_1\|_{C^{2,\alpha}(\Sigma_i)} + \|w_2\|_{C^{2,\alpha}(\Sigma_i)}) + \sum_{j=1}^{4}\|f_j\|_{C^{1,\alpha}(\bar\Omega)}\Big).\end{aligned}$$

This, together with (B-5), yields (B-7).

Next, we derive the second-order elliptic equations on u_1 and u_2. By (B-5),

$$\begin{cases}(\partial_1^2+\partial_2^2+\partial_3^2)u_1 = \partial_1 f_3 + \partial_2 f_1 + \partial_3 f_2 & \text{in } \Omega,\\ (\partial_1^2+\partial_2^2+\partial_3^2)u_2 = \partial_1 f_4 - \partial_2 f_2 + \partial_3 f_1 & \text{in } \Omega.\end{cases}$$

Thus,

(B-14)
$$\|u_1\|_{C^{2,\alpha}(\bar\Omega)} + \|u_2\|_{C^{2,\alpha}(\bar\Omega)} \le C\Big(\sum_{i=1}^{2}\|u_j\|_{C^{2,\alpha}(\Sigma_i)} + \|u_j\|_{C^{2,\alpha}(\Gamma)} + \sum_{i=1}^{4}\|f_i\|_{C^{1,\alpha}(\bar\Omega)}\Big).$$

Substituting (B-7) into (B-14) yields

(B-15)
$$\|u_1\|_{C^{2,\alpha}(\bar\Omega)} + \|u_2\|_{C^{2,\alpha}(\bar\Omega)} \le C\sum_{i=1}^{4}\|f_i\|_{C^{1,\alpha}(\bar\Omega)}.$$

For each $x_1 \in [0, 1]$, from the first and the fifth equations in (B-5) it follows that

$$\int_{B_1(0)} f_1(x_1, x_2, x_3)\,dx_2\,dx_3 = \int_{\partial B_1(0)} (x_2u_1 + x_3u_2)\,dl = 0,$$

so by $f_1 \in C^{1,\alpha}(\Omega)$ and the integral mean value theorem, there exists some point $(x_2^*(x_1), x_3^*(x_1)) \in B_1(0)$ such that

$$f_1(x_1, x_2^*(x_1), x_3^*(x_1)) = 0.$$

This implies

$$\|f_1\|_{C^{1,\alpha}(\bar{\Omega})} \le C\|\nabla f_1\|_{C^\alpha(\bar{\Omega})}. \tag{B-16}$$

Substituting (B-16) into (B-15) yields (B-6). □

Lemma B.4. *Under the assumptions of Lemma 4.2, at the point* $(0, 0, 1)$,

$$Y_4 = a_0 Y_5 + O(1/X_0^2)Y_5, \quad Y_5 = O(X_0)Y_4,$$

where $a_0 < 0$ *and* $a_0 = O(1/X_0)$.

Proof. In the coordinate (y_1, y_2, y_3), the background solution $(\rho_0^\pm(y_1), U_0^\pm(y_1))$ satisfies (see Appendix A),

$$\begin{cases} \dfrac{d\rho_0^\pm(y_1)}{dy_1} = \dfrac{2(M_0^\pm(y_1))^2\rho_0^\pm(y_1)}{y_1(1-(M_0^\pm(y_1))^2)}, \\ \dfrac{dU_0^\pm(y_1)}{dy_1} = -\dfrac{2U_0^\pm(y_1)}{y_1(1-(M_0^\pm(y_1))^2)}, \\ \dfrac{dM_0^\pm(y_1)}{dy_1} = -\dfrac{M_0^\pm(y_1)(2+(\gamma-1)M_0^\pm(y_1))}{y_1(1-(M_0^\pm(y_1))^2)}, \end{cases} \tag{B-17}$$

where

$$M_0^\pm(y_1) = \frac{U_0^\pm(y_1)}{c(\rho_0^\pm(y_1))}$$

denote the Mach numbers of supersonic coming flow and subsonic flow, respectively.

By (B-17) and (2-16)–(2-17),

$$\begin{cases} M_0^-(y_1) = M_0^-(X_0) + O(1/X_0), \\ \rho_0^-(y_1) = \rho_0^-(X_0) + O(1/X_0), \\ U_{1,0}^-(y_1) = U_0^-(X_0) + O(1/X_0). \end{cases} \tag{B-18}$$

In addition, it follows from (2-5) that at the point $z = (0, 0, 1)$,

$$\begin{cases} \rho Y_1 + V_1 Y_4 \\ \quad = \big(\rho_0^-(\xi_1(0,1))\overline{U}_0^-(\xi_1(0,1),0,1) - \rho_0^-(\xi_2(0,1))\overline{U}_0^-(\xi_2(0,1),0,1)\big) \\ \qquad + O(\varepsilon^2)Y_1 + O(\varepsilon)Y_2 + O(\varepsilon)Y_3 + O(\varepsilon^2)Y_4 + O(\varepsilon^2)Y_5, \\ \rho(U_1+V_1)Y_1 + V_1^2 Y_4 + (1+\tau^2)c^2(\tilde{\rho})Y_4 \\ \quad = \big((\rho_0^-(\overline{U}_0^-)^2)(\xi_1(0,1),0,1) - (\rho_0^-(\overline{U}_0^-)^2)(\xi_2(0,1),0,1)\big) \\ \qquad + (1+\tau^2)\big(P_0^-(\xi_1(0,1)) - P_0^-(\xi_2(0,1))\big) + O(\varepsilon^2)Y_1 + O(\varepsilon)Y_2 + O(\varepsilon)Y_3 \\ \qquad + O(\varepsilon^2)Y_4 + O(\varepsilon^2)Y_5. \end{cases}$$

Using this and a direct computation gives

$$
\begin{aligned}
\text{(B-19)}\quad & \big((1+\tau^2)c^2(\tilde\rho)-U_1(\xi_1(0,1),0,1)V_1(\xi_2(0,1),0,1)\big)Y_4\\
&=(1+\tau^2)\big(P_0^-(\xi_1(0,1))-P_0^-(\xi_2(0,1))\big)\\
&\quad+(\rho_0^-(U_0^-)^2)(\xi_1(0,1),0,1)-(\rho_0^-(U_0^-)^2)(\xi_2(0,1),0,1)\\
&\quad-\Big(\big((\rho_0^-U_0^-)(\xi_1(0,1),0,1)-(\rho_0^-U_0^-)(\xi_2(0,1),0,1)\big)\\
&\qquad\qquad\times\big(U_1(\xi_1(0,1),0,1)+V_1(\xi_2(0,1),0,1)\big)\Big)\\
&\quad+O(\varepsilon^2)Y_1+O(\varepsilon)Y_2+O(\varepsilon)Y_3+O(\varepsilon^2)Y_4+O(\varepsilon^2)Y_5.
\end{aligned}
$$

Since

$$
\begin{cases}
\dfrac{d(\rho_0^-(r)U_0^-(r))}{dr}=-\dfrac{2\rho_0^-(r)U_0^-(r)}{r},\\[2mm]
\dfrac{d\big(\rho_0^-(r)(U_0^-(r))^2+P_0^-(r)\big)}{dr}=-\dfrac{2\rho_0^-(r)(U_0^-(r))^2}{r},
\end{cases}
$$

we have

$$
\begin{aligned}
\text{(B-20)}\quad & (1+\tau^2)\big(P_0^-(\xi_1(0,1))-P_0^-(\xi_2(0,1))\big)\\
&\quad+(\rho_0^-(\overline{U}_0^-)^2)(\xi_1(0,1),0,1)-(\rho_0^-(\overline{U}_0^-)^2)(\xi_2(0,1),0,1)\\
&\quad-\Big(\big((\rho_0^-\overline{U}_0^-)(\xi_1(0,1),0,1)-(\rho_0^-\overline{U}_0^-)(\xi_2(0,1),0,1)\big)\\
&\qquad\qquad\times\big(U_1(\xi_1(0,1),0,1)+V_1(\xi_2(0,1),0,1)\big)\Big)\\
&=-\frac{2\rho_0^-(\tilde\xi)(\overline{U}_0^-(\tilde\xi))^2}{\tilde\xi}Y_5(0,1)\\
&\quad+\frac{2\rho_0^-(\tilde\xi)\overline{U}_0^-(\tilde\xi)}{\tilde\xi}\big(U_1(\xi_1(0,1),0,1)+V_1(\xi_2(0,1),0,1)\big)Y_5(0,1)\\
\text{(B-21)}\quad & \big((1+\tau^2)c^2(\tilde\rho)-U_1V_1\big)Y_4\\
&=-\frac{2(\rho_0^-U_0^-)(\tilde\xi)}{\tilde\xi}\Big(U_0^-(\tilde\xi)-\big(U_1(\xi_1(0,1))+V_1(\xi_2(0,1))\big)\Big)Y_5\\
&\quad+O(\varepsilon^2)Y_1+O(\varepsilon)Y_2+O(\varepsilon)Y_3+O(\varepsilon^2)Y_4+O(\varepsilon^2)Y_5,
\end{aligned}
$$

where $\tilde\rho$ and $\tilde\xi$ are the values derived by the mean value theorem on the functions $P(\rho)-P(q)$ and $G(\xi_1(0,1))-G(\xi_2(0,1))$ with

$$
\begin{aligned}
G(y_1)&=(1+\tau^2)P_0^-(y_1)+(\rho_0^-(U_0^-)^2)(y_1,0,1)\\
&\qquad-(\rho_0^-U_0^-)(y_1,0,1)\big(U_1(\xi_1(0,1),0,1)+V_1(\xi_2(0,1),0,1)\big),
\end{aligned}
$$

respectively.

Substituting (B-19)–(B-20) into (B-18) yields

$$\text{(B-22)}\quad \big((1+\tau^2)c^2(\tilde\rho)-U_1V_1\big)Y_4 = -\frac{2(\rho_0^-U_0^-)(\tilde\xi)}{\tilde\xi}\bigg(U_0^-(\tilde\xi)-\bigg(\frac{(\rho_0^-U_0^-)(\xi_1(0,1))}{\rho(\xi_1(0,1))} + \frac{(\rho_0^-U_0^-)(\xi_2(0,1))}{q(\xi_2(0,1))}\bigg)\bigg)Y_5 + O(\varepsilon^2)Y_1+O(\varepsilon)Y_2+O(\varepsilon)Y_3+O(\varepsilon)Y_4+O(\varepsilon^2)Y_5.$$

Due to the assumptions in Theorem 2.1, we have

$$\rho(\xi_1(0,1)) = \widehat{\rho_0}^+(r_0)+O(\varepsilon),$$
$$q(\xi_2(0,1)) = \widehat{\rho_0}^+(r_0)+O(\varepsilon),$$
$$\rho_0^-(\xi_i(0,1)) = \rho_0^-(r_0)+O(\varepsilon),\quad i=1,2.$$

Then for $\rho_0^+(r_0) > 2\rho_0^-(r_0)$ and small ε,

$$\text{(B-23)}\quad \begin{cases}\rho(\xi_1(0,1)) > 2\rho_0^-(\xi_1(0,1)),\\ q(\xi_2(0,1)) > 2\rho_0^-(\xi_2(0,1)).\end{cases}$$

Moreover,

$$U_0^-(\tilde\xi) = U_0^-(\xi_1(0,1)) + O(1/X_0)(\xi_1(0,1)-\tilde\xi),$$
$$U_0^-(\tilde\xi) = U_0^-(\xi_2(0,1)) + O(1/X_0)(\xi_2(0,1)-\tilde\xi),$$
$$\tilde\rho = \rho(\xi_1(0,1)) + O(1)Y_4,$$
$$V_1 = U_1 + O(1)Y_1.$$

So (B-22) becomes

$$\text{(B-24)}\quad \big((1+\tau^2)c^2(\rho(\xi_1(0,1))-U_1^2\big)Y_4 = -\frac{(\rho_0^-U_0^-)(\tilde\xi)}{\tilde\xi}\bigg(U_0^-(\xi_1(0,1))+U_0^-(\xi_2(0,1)) - \bigg(\frac{2(\rho_0^-U_0^-)(\xi_1(0,1))}{\rho(\xi_1(0,1))}+\frac{2(\rho_0^-U_0^-)(\xi_2(0,1))}{q(\xi_2(0,1))}\bigg)\bigg)Y_5 + O(\varepsilon^2)Y_1+O(\varepsilon)Y_2+O(\varepsilon)Y_3+O(\varepsilon)Y_4+O(1/X_0^2)Y_5.$$

By (4-4), (4-6) and (B-23)–(B-24), we obtain that at the point (0, 0, 1)

$$Y_4 = a_0Y_5 + O(1/X_0^2)Y_5 \quad\text{and}\quad Y_5 = O(X_0)Y_4,$$

where $a_0 < 0$ and $a_0 = O(1/X_0)$, which completes the proof of Lemma B.4. □

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