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KENTARO NAGAO

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We study analogs of noncommutative Donaldson–Thomas invariants corresponding to the refined topological vertex for small crepant resolutions of toric Calabi–Yau 3-folds. We give three definitions of the invariants which are equivalent to each others and provide "wall-crossing" formulas for the invariants. In particular, we get normalized generating functions which are unchanged under wall-crossing.

Introduction

Donaldson–Thomas theory [Thomas 2000] is intersection theory on the moduli spaces of ideal sheaves on a smooth variety, which is conjecturally equivalent to Gromov–Witten theory [Maulik et al. 2006]. For a Calabi–Yau 3-fold, the virtual dimension of the moduli space is zero and hence Donaldson–Thomas invariants are said to be counting invariants of ideal sheaves. It is known that they coincide with the weighted Euler characteristics of the moduli spaces weighted by the Behrend functions [2009]. Recently, the Donaldson–Thomas invariants of Calabi–Yau 3-folds have been studied using categorical methods; see, for example, [Joyce 2008; 2007; Toda 2009; 2010; Kontsevich and Soibelman 2008; Joyce and Song 2010].

On the other hand, a smooth variety Y sometimes has a noncommutative associative algebra A such that the derived category of coherent sheaves on Y is equivalent to the derived category of A-modules. Derived McKay correspondence [Kapranov and Vasserot 2000; Bridgeland et al. 2001] and Van den Bergh's noncommutative crepant resolutions [2004] are typical examples. In such cases, B. Szendrői proposed to study counting invariants of A-modules (*noncommutative Donaldson–Thomas invariants*) and relations with the original Donaldson–Thomas invariants on Y [Szendrői 2008]. In [Nagao and Nakajima 2011; Nagao 2011a], we provided wall-crossing formulas which relate generating functions of the Donaldson–Thomas and noncommutative Donaldson–Thomas invariants for small crepant resolutions of toric Calabi–Yau 3-folds. (We say a resolution of a 3-fold is *small* if the dimension of each fiber is less than or equal to 1.)

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Keywords: Donaldson–Thomas theory, dimer model, topological vertex.

The aim of this paper is to propose new invariants generalizing noncommutative Donaldson–Thomas invariants and to provide "wall-crossing formulas" for small crepant resolutions of toric Calabi–Yau 3-folds. We have two directions of generalizations:

- "open" version:[1] corresponding to counting invariants of sheaves on Y with noncompact supports,[2]
- refined version: corresponding to refined topological vertex [Iqbal et al. 2009].[3]

Let $Y \to X$ be a projective small crepant resolution of an affine toric Calabi–Yau 3-fold. Recall that giving an affine toric Calabi–Yau 3-fold is equivalent to giving a convex lattice polygon. Existence of a small crepant resolution is equivalent to absence of interior lattice points in the polygon. It is easy to classify such polygons and X is one of the following:

- $X = X_{L^+,L^-} := \{\mathrm{xy} = \mathrm{z}^{L^+}\mathrm{w}^{L^-}\} \subset \mathbb{C}^4$ for $L^+ > 0$ and $L^- \geq 0$, or
- $X = X_{(\mathbb{Z}/2\mathbb{Z})^2} := \mathbb{C}^3/(\mathbb{Z}/2\mathbb{Z})^2$ where $(\mathbb{Z}/2\mathbb{Z})^2$ acts on $\mathbb{C}^3$ with weights $(1, 0)$, $(0, 1)$ and $(1, 1)$.

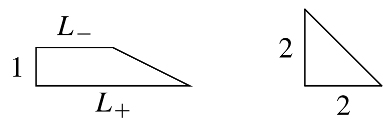


Figure 1. Polygons for X_{L^+,L^-} and $X_{(\mathbb{Z}/2\mathbb{Z})^2}$.

In this paper, we study the first case. We put $L := L^+ + L^-$. Note that $X_{1,1}$ is called the *conifold* and $X_{L,0}$ is isomorphic to $\mathbb{C} \times \mathbb{C}^2/(\mathbb{Z}/L\mathbb{Z})$.

Given a pair of Young diagrams $\nu = (\nu_+, \nu_-)$ and an L-tuple of Young diagrams

$$\lambda = \bigl(\lambda^{(1/2)}, \dots, \lambda^{(L-1/2)}\bigr),$$

the generating function of *refined open noncommutative Donaldson–Thomas invariants* (roncDT, in short)

$$\mathcal{Z}^{Y}_{\lambda,\nu}(\vec{q}) = \mathcal{Z}^{Y}_{\lambda,\nu}(q_+, q_-, q_1 \dots, q_{L-1}),$$

which is denoted by $\mathcal{Z}^{\mathrm{RTV}}_{\sigma,\lambda,\nu}$ in the body of this paper, is defined by counting the number of the following data:

- an $(L-1)$-tuple of Young diagrams $\vec{\nu} = (\nu^{(1)}, \dots, \nu^{(L-1)})$, and

[1] The word "open" stems from such terminologies as "open topological string theory". According to [Aganagic et al. 2005], open topological string partition function is given by summing up the generating functions of these invariants over Young diagrams.

[2] As far as we know, there is no definition of "open" invariants for general Calabi–Yau 3-folds.

[3] See [Behrend et al. 2009] for a geometric definition of refined invariants.

- an L-tuple of 3-dimensional Young diagrams $\vec{\Lambda} = (\Lambda^{(1/2)}, \dots, \Lambda^{(L-1/2)})$ such that $\Lambda^{(j)}$ is of type $(\lambda^{(j)}, \nu^{(j+1/2)}, {}^{\mathrm{t}}\nu^{(j-1/2)})$ or $(\lambda^{(j)}, {}^{\mathrm{t}}\nu^{(j-1/2)}, \nu^{(j+1/2)})$ (see Section 5.3 for details).

Such data parametrize torus fixed ideal sheaves on the small crepant resolution Y. In particular,

$$\mathcal{Z}^{Y}_{\varnothing,\varnothing}(\vec{q})\big|_{q_+=q_-}$$

coincides with the generating function of Euler characteristic versions of the Donaldson–Thomas invariants of Y.[4]

Let A be a noncommutative crepant resolution of X. Let $\mathbb{Z}_{\mathrm{h}}$ denote the set of half integers and let $\theta\colon \mathbb{Z}_{\mathrm{h}} \to \mathbb{Z}_{\mathrm{h}}$ be a bijection such that $\theta(h+L) = \theta(h)+L$ and such that

$$\theta(1/2)+\cdots+\theta(L-1/2) = 1/2+\cdots+(L-1/2).$$

We will define generating functions $\mathcal{Z}^{A}_{\lambda,\nu,\theta}(\vec{q})$, which are denoted by $\mathcal{Z}_{\sigma,\lambda,\nu,\theta}(\vec{q})$ in the body of this paper (see Section 3.4), satisfying these properties:

- $\mathcal{Z}^{A}_{\varnothing,\varnothing,\mathrm{id}}(\vec{q})\big|_{q_+=q_-=q_0^{1/2}}$ coincides with the generating function $\mathcal{Z}^{A}_{\mathrm{NCDT,eu}}$ of Euler characteristic versions[5] of noncommutative Donaldson–Thomas invariants for the noncommutative crepant resolution A; see [Mozgovoy and Reineke 2010] and the remark on page 184.
- "$\lim\limits_{\theta\to\infty}$" $\mathcal{Z}^{A}_{\lambda,\nu,\theta}(\vec{q}) = \mathcal{Z}^{Y}_{\lambda,\nu}(\vec{q})$; see Theorem 5.4.8. (The limit in this equation is, in fact, equivalent to a limit in the space of stability conditions for the category of finite-dimensional A-modules.)[6]

Moreover, for $i \in I := \mathbb{Z}/L\mathbb{Z}$ we can define the new bijection $\mu_i(\theta)\colon \mathbb{Z}_{\mathrm{h}} \to \mathbb{Z}_{\mathrm{h}}$ (see §1.2.1) and

- $\mathcal{Z}^{A}_{\lambda,\nu,\mu_i(\theta)}(\vec{q})/\mathcal{Z}^{A}_{\lambda,\nu,\theta}(\vec{q})$ is given explicitly (Theorem 4.2.2 and 4.4.2).

In [Nagao and Nakajima 2011; Nagao 2011a], we realized the $\mathcal{Z}^{A}_{\varnothing,\varnothing,\theta}(\vec{q})\big|_{q_+=q_-}$ as generating functions of virtual counting of certain moduli spaces and these moduli spaces are constructed using geometric invariant theory. In this story, θ determines a chamber in the space of stability parameters and the chamber corresponding to θ is adjacent to the chamber corresponding to $\mu_i(\theta)$ by a single wall. This is the reason we call Theorem 4.2.2 and 4.4.2 as wall-crossing formulas, even though our

[4] The Euler characteristic version of the Donaldson–Thomas invariant coincides with the Donaldson–Thomas invariant up to sign [Maulik et al. 2006].

[5] The Euler characteristic version of the noncommutative Donaldson–Thomas invariant coincides with the noncommutative Donaldson–Thomas invariant up to sign [Nagao 2011a; Mozgovoy and Reineke 2010].

[6] A moduli space of stable A-modules with the specific numerical data gives a crepant resolution of X [Ishii and Ueda 2008]. The direction in which we take limit in the space of stability conditions determines a stability parameter in the construction of a crepant resolution.

definition of the invariants and the proof of the formula are given in combinatorial ways. In fact, in the subsequent paper [Nagao 2011b] we provide an alternative geometric definition, in which θ determines a chamber in the space of Bridgeland's stability conditions for the category of finite-dimensional A-modules.

As consequences of the wall-crossing formula, we get

- Corollaries 4.5.2 and 5.5.2: $\mathcal{Z}^A_{\lambda,\nu,\theta}/\mathcal{Z}^A_{\lambda,\varnothing,\theta} = \mathcal{Z}^Y_{\lambda,\nu}/\mathcal{Z}^Y_{\lambda,\varnothing}$ for any θ, λ and ν.
- Corollaries 4.5.4 and 5.5.4: $\left(\mathcal{Z}^A_{\lambda,\nu,\theta}/\mathcal{Z}^A_{\varnothing,\varnothing,\theta}\right)\big|_{q_+=q_-} = \left(\mathcal{Z}^Y_{\lambda,\nu}/\mathcal{Z}^Y_{\varnothing,\varnothing}\right)\big|_{q_+=q_-}$ for any θ, λ and ν such that $c_\lambda[j]=0$ for any j (see §1.3.1 for notation).

By the results in [Nagao and Nakajima 2011; Nagao 2011a], these formulas should be interpreted as stability of the normalized generating functions under wall crossing. We can find such stability of normalized generating functions in other contexts such as flop invariance and DT-PT correspondence. Categorical interpretations of such normalized generating functions and their stability are expected.

Now, we summarize the prior study on noncommutative Donaldson–Thomas invariants:

- Szendrői's formula on the generating function of noncommutative Donaldson–Thomas invariants of the conifold was shown by B. Young [2009] in a purely combinatorial way. The main tool is an operation called *dimer shuffling*.
- J. Brian and Young [2010] generalized the Szendrői–Young formula for $X_{L,0}$ and $X_{(\mathbb{Z}/2\mathbb{Z})^2}$. The method is different from the one used in [Young 2009]: they use vertex operator method.
- In [Nagao and Nakajima 2011], we gave an interpretation of Szendrői–Young formula as a consequence of the wall-crossing formula. From our point of view, the argument there can be translated into combinatorial language by localization, yielding the argument in [Young 2009]. In particular, dimer shuffling is nothing but "mutation" in the categorical language.
- In [Nagao 2011a], we generalized the results in [Nagao and Nakajima 2011] for arbitrary small crepant resolutions of toric Calabi–Yau 3-folds.
- In [Joyce and Song 2010], the authors study noncommutative Donaldson–Thomas invariants of small crepant resolutions of toric Calabi–Yau 3-folds as examples of their theory of generalized Donaldson–Thomas invariants.
- T. Dimofte and S. Gukov [2010] provided a refined version of Szendrői–Young formula for the conifold.
- See [Jafferis and Moore 2008; Chuang and Jafferis 2009; Aganagic et al. 2011; Chuang and Pan 2010; Aganagic and Yamazaki 2010; Dimofte et al. 2011] for developments in physics.

In this paper, we define the roncDT invariants using a dimer model (Section 2), which is purely combinatorial.

In Section 3, we give an interpretation of the dimer model as a crystal melting model.[7] We construct an A-module $M^{\max}_{\sigma,\lambda,\nu,\theta}$ such that giving a dimer configuration is equivalent to giving a finite-dimensional torus invariant quotient module of $M^{\max}_{\sigma,\lambda,\nu,\theta}$. Hence the roncDT invariant coincides with the Euler characteristic of the moduli space of finite-dimensional quotient modules of $M^{\max}_{\sigma,\lambda,\nu,\theta}$; see [Nagao 2011b].[8]

In Section 4, we introduce the notion of dimer shuffling to prove the first main result of this paper: the wall-crossing formula (Theorems 4.2.2 and 4.4.2).

Finally we study the limit behavior of the dimer model in Section 5. The second main result is that the generating function given by the refined topological vertex for Y appears as the limit (Theorem 5.4.8).

While preparing the papers, the author was informed from J. Bryan that he and his collaborators C. Cadman and B. Young provided an explicit formula of $\mathcal{Z}^{A}_{\lambda,\nu,\mathrm{id}}|_{q_+=q_-}$ for $X_{L,0}$ and $X_{(\mathbb{Z}/2\mathbb{Z})^2}$ using vertex operator methods [Bryan et al. 2012; ≥ 2011]. In a subsequent paper [Nagao 2011b], we provide an explicit formula of $\mathcal{Z}^{A}_{\lambda,\nu,\theta}$ for X_{L_+,L_-} using vertex operator methods.

A physicist may refer to [Nagao and Yamazaki 2010], in which we explain the result of this paper in a physical context.

We conclude this introduction by definition some notation.

Indices. Let $\mathbb{Z}_{\mathrm{h}}$ denote the set of half integers and L be a positive integer. We set $I := \mathbb{Z}/L\mathbb{Z}$ and $I_{\mathrm{h}} := \mathbb{Z}_{\mathrm{h}}/L\mathbb{Z}$. The two natural projections $\mathbb{Z} \to I$ and $\mathbb{Z}_{\mathrm{h}} \to I_{\mathrm{h}}$ are denoted by the same symbol π. We sometimes identify I and I_{h} with $\{0, \dots, L-1\}$ and $\{1/2, \dots, L-1/2\}$ respectively.

The symbols n, h, i and j are used for elements in $\mathbb{Z}$, $\mathbb{Z}_{\mathrm{h}}$, I and I_{h} respectively.

For $n \in \mathbb{Z}$ and $h \in \mathbb{Z}_{\mathrm{h}}$, we define $c(n), c(h) \in \mathbb{Z}$ by

$$n = c(n) \cdot L + \pi(n), \quad h = c(h) \cdot L + \pi(h).$$

Young diagrams. A *Young diagram* ν is a map $\nu\colon \mathbb{Z} \to \mathbb{Z}$ such that $\nu(n) = |n|$ if $|n| \gg 0$ and $\nu(n) - \nu(n-1) = \pm 1$ for any $n \in \mathbb{Z}$. The map $\mathbb{Z}_{\mathrm{h}} \to \{\pm 1\}$ given by $j \mapsto \nu(j+1/2) - \nu(j-1/2)$ is also denoted by ν.

By an abuse of notation, we sometimes identify $+$ and $-$ with 1 and -1.

[7] From the geometric point of view, the crystal melting model is more natural. But in this paper we adapt the definition using the dimer model since it is more convenient when we prove some technical lemmas, which we also use in [Nagao 2011b].

[8] In the case when $\nu_+ = \nu_- = \varnothing$, the moduli spaces have symmetric obstruction theory and the invariant in this paper coincides with the weighted Euler characteristic up to sign.

A Young diagram can be represented by a nonincreasing sequence of positive integers. We fix the notation as in Figure 2.

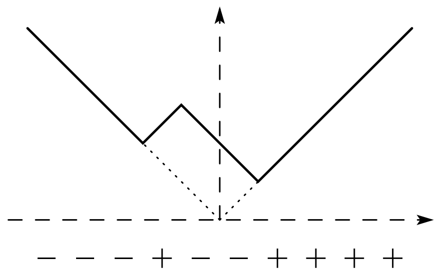


Figure 2. $\nu = (1, 1)$, ${}^{\mathrm{t}}\nu = (2)$.

Formal variables. Let q_+, q_- and $q_0, \dots, q_{L-1}$ be formal variables. We use q_+, q_- and $q_1, \dots, q_{L-1}$ for generating functions of refined invariants. Substituting $q_+ = q_- = (q_0)^{1/2}$, we get generating functions of nonrefined invariants.

Let $P := \mathbb{Z} \cdot I$ be the lattice with the basis $\{\alpha_i \mid i \in I\}$. For an element $\alpha = \sum \alpha^i \cdot \alpha_i \in P \quad (\alpha^i \in \mathbb{Z})$, we put $q^\alpha := \prod (q_i)^{\alpha^i}$.

For $\alpha, \alpha' \in P$, we say $\alpha < \alpha'$ or $q^\alpha < q^{\alpha'}$ if $\alpha' - \alpha \in P^+ := \mathbb{Z}_{\geq 0} \cdot I$.

1. Preliminaries

1.1. *Affine root system.*

1.1.1. For $h, h' \in \mathbb{Z}_{\mathrm{h}}$, we define $\alpha_{[h,h']} \in P$ by

$$\alpha_{[h,h']} := \sum_{n=h+1/2}^{h'-1/2} \alpha_{\pi(n)}$$

if $h < h'$, $\alpha_{[h,h']} = 1$ if $h = h'$ and $\alpha_{[h,h']} = -\alpha_{[h',h]}$ if $h > h'$. We set

$$\Lambda := \{\alpha_{[h,h']} \in P \mid h \neq h'\},$$
$$\Lambda^{\mathrm{re},+} := \{\alpha_{[h,h']} \in \Lambda \mid h < h',\ h \not\equiv h' \ (\mathrm{mod}\, L)\}.$$

An element in Λ (resp. $\Lambda^{\mathrm{re},+}$) is called a *root* (resp. *positive real root*) of the affine root system of type A_{L-1}.

1.1.2. The element $\delta := \alpha_0 + \cdots \alpha_{L-1} \in P$ is called the *minimal imaginary root*. We set

$$\Lambda^{\mathrm{fin},+} := \left\{\alpha_{[j,j']} \in \Lambda \mid 1/2 \leq j < j' \leq L - 1/2\right\}$$

and

$$\Lambda^{\mathrm{re},+}_{+} := \left\{\alpha_{[j,j']} + N\delta \mid \alpha_{[j,j']} \in \Lambda^{\mathrm{fin},+},\ N \geq 0\right\}. \tag{1-1}$$

Example 1.1.3. In the case of $L=4$, we have

$$\Lambda^{\mathrm{fin},+} := \big\{\alpha_1,\, \alpha_2,\, \alpha_3,\, \alpha_2+\alpha_3,\, \alpha_1+\alpha_2,\, \alpha_1+\alpha_2+\alpha_3\big\}.$$

1.1.4. For a root $\alpha \in \Lambda$, we take h and h' such that $\alpha = \alpha_{[h,h']}$ and set

$$j_-(\alpha) := \pi(h), \text{ and } j_+(\alpha) := \pi(h').$$

We also put

$$B^\alpha := \big\{(h,h') \in \big(\mathbb{Z}_{\mathrm{h}}\big)^2 \,\big|\, \alpha_{[h,h']} = \alpha\big\}.$$

1.1.5. Let Θ denote the set of bijections $\theta\colon \mathbb{Z}_{\mathrm{h}} \to \mathbb{Z}_{\mathrm{h}}$ such that

- $\theta(h+L) = \theta(h)+L$ for any $h \in \mathbb{Z}_{\mathrm{h}}$, and
- $\displaystyle\sum_{h=1/2}^{L-1/2} \theta(h) = \sum_{h=1/2}^{L-1/2} h.$

Example 1.1.6. In the case of $L=4$, the correspondence

$$\tfrac{1}{2} \mapsto -\tfrac{1}{2}, \quad \tfrac{3}{2} \mapsto \tfrac{3}{2}, \quad \tfrac{5}{2} \mapsto \tfrac{5}{2}, \quad \tfrac{7}{2} \mapsto \tfrac{9}{2}$$

gives an elements in Θ. Let $\mu_0(\mathrm{id})$ denote this map (see §1.2.1 for notation).

1.1.7. For $\theta \in \Theta$ and $i \in I$, we define $\alpha(\theta,i) := \alpha_{[\theta(n-1/2),\theta(n+1/2)]}$ $(n \in \pi^{-1}(i))$.

Example 1.1.8.

$$\begin{aligned}
\alpha(\mathrm{id},0) &= \alpha_0, & \alpha(\mu_0(\mathrm{id}),0) &= -\alpha_0,\\
\alpha(\mathrm{id},1) &= \alpha_1, & \alpha(\mu_0(\mathrm{id}),1) &= \alpha_0+\alpha_1,\\
\alpha(\mathrm{id},2) &= \alpha_2, & \alpha(\mu_0(\mathrm{id}),2) &= \alpha_2,\\
\alpha(\mathrm{id},3) &= \alpha_3, & \alpha(\mu_0(\mathrm{id}),3) &= \alpha_0+\alpha_3.
\end{aligned}$$

1.1.9. If $\alpha = \alpha_{[h,h']}$ is a positive real root, we write $\theta(\alpha) > 0$ if $\theta^{-1}(h) > \theta^{-1}(h')$, and we write $\theta(\alpha) < 0$ if $\theta^{-1}(h) < \theta^{-1}(h')$. We set

$$\Lambda_\theta^{\mathrm{re},+} := \big\{\alpha \in \Lambda^{\mathrm{re},+} \,\big|\, \theta(\alpha) > 0\big\}. \tag{1-2}$$

Example 1.1.10. We have $\Lambda_{\mathrm{id}}^{\mathrm{re},+} = \varnothing$ and $\Lambda_{\mu_0(\mathrm{id})}^{\mathrm{re},+} = \{\alpha_0\}$.

Remark. As we mentioned in the introduction, we studied moduli spaces of representations of a noncommutative crepant resolution of X_{L^+,L^-} in [Nagao 2011a]. In this theory, the space of stability conditions can be canonically identified with $P^* \otimes \mathbb{R}$ and the walls are classified as follows:

- the walls $W_\alpha := (\mathbb{R}\cdot\alpha)^\perp \subset P^*\otimes\mathbb{R}$ $\quad(\alpha \in \Lambda^{\mathrm{re},+})$, and
- the wall $W_\delta := (\mathbb{R}\cdot\delta)^\perp$, which separates the Donaldson–Thomas and Pandharipande–Thomas chambers.

The maps $\theta\colon \mathbb{Z}_{\mathrm{h}} \to \mathbb{Z}_{\mathrm{h}}$ as above parametrize the chambers on one side of the wall W_δ. The notation $\theta(\alpha) \gtrless 0$ respects this parametrization.

1.2. ***Wall-crossing.***

1.2.1. For $i \in I$, let $\mu_i : \mathbb{Z}_{\mathrm{h}} \to \mathbb{Z}_{\mathrm{h}}$ be the map given by

$$\mu_i(h) = \begin{cases} h-1 & \text{if } \pi(h-1/2) = i, \\ h+1 & \text{if } \pi(h+1/2) = i, \\ h & \text{otherwise.} \end{cases}$$

For $\theta \in \Theta$, we put $\mu_i(\theta) := \theta \circ \mu_i$.

Remark. The chambers corresponding to θ and $\mu_i(\theta)$ are separated by the wall $W_{\alpha(\theta,i)}$, which is the reason for the title of this subsection. From the viewpoint of the affine root system, wall crossing corresponds to simple reflection; from the viewpoint of noncommutative crepant resolutions, it corresponds to mutation; and from the viewpoint of dimer models, to dimer shuffling.

1.2.2. Let $\boldsymbol{i} = (i_1, i_2, \dots) \in I^{\mathbb{Z}_{>0}}$ be a sequence of elements in I. For $b > 0$, we define

$$\theta_{\boldsymbol{i},b} := \mu_{i_{b-1}}(\cdots(\mu_{i_1}(\mathrm{id}))\cdots) \in \Theta, \quad \alpha_{\boldsymbol{i},b} := \alpha(\theta_{\boldsymbol{i},b}, i_b).$$

We say $\boldsymbol{i} \in I^{\mathbb{Z}_{>0}}$ is a *minimal expression* if $\theta_{\boldsymbol{i},b}(\alpha_{\boldsymbol{i},b}) < 0$ for any $b > 0$. For a minimal expression $\boldsymbol{i}$, we have

$$\Lambda^{\mathrm{re},+}_{\theta_{\boldsymbol{i},b}} = \{\alpha_{\boldsymbol{i},1}, \dots, \alpha_{\boldsymbol{i},b-1}\}.$$

1.3. ***Core and quotient of a Young diagram.***

1.3.1. Let $\sigma : I_{\mathrm{h}} \to \{\pm\}$ and $\lambda : \mathbb{Z}_{\mathrm{h}} \to \{\pm\}$ be maps such that $\lambda(h) = \pm\sigma(\pi(h))$ if $\pm h \gg 0$. We define integers $c_\lambda[j]$ and Young diagrams $\lambda^{[j]}$ for $j \in I_{\mathrm{h}}$ by

$$\lambda(h) = \lambda^{[\pi(h)]}\big(\sigma(j(h)) \cdot (c(h) - c_\lambda[\pi(h)] + 1/2)\big).$$

Remark. In the case $\sigma \equiv +$ and $\sum c_\lambda[j] = 0$, the sequence $\big(c_\lambda[j]\big)$ of integers and the sequence $\big(\lambda^{[j]}\big)$ of Young diagrams are called the L-core and the L-quotient of the Young diagram λ.

1.3.2. We put

$$B^{\alpha,\pm}_{\sigma,\lambda} := \{(h,h') \in B^\alpha \mid -\lambda(h)\sigma(h) = \lambda(h')\sigma(h') = \pm\}. \tag{1-3}$$

Lemma 1.3.3.

$$\big|B^{\alpha,+}_{\sigma,\lambda}\big| - \big|B^{\alpha,-}_{\sigma,\lambda}\big| = \alpha^0 + c_\lambda[j_-(\alpha)] - c_\lambda[j_+(\alpha)].$$

Proof. We write simply $j_\pm$ for $j_\pm(\alpha)$. Note that we have

$$B^\alpha = \big\{\big(cL + j_-, (c+\alpha^0)L + j_+\big) \bigm| c \in \mathbb{Z}\big\}.$$

For an integer N, we set

$$B^\alpha_N := \big\{\big(cL + j_-, (c+\alpha^0)L + j_+\big) \bigm| c \in [-N, N-1]\big\}.$$

Take a sufficiently large integer N. Then

$$B^{\alpha,+}_{\sigma,\lambda},\, B^{\alpha,-}_{\sigma,\lambda} \subset B^{\alpha}_{N}$$

and so

$$\begin{aligned}
&\bigl|B^{\alpha,+}_{\sigma,\lambda}\bigr| - \bigl|B^{\alpha,-}_{\sigma,\lambda}\bigr| \\
&\quad = -\sharp\bigl\{(h,h') \in B^{\alpha}_{N} \bigm| \lambda(h)\sigma(h) = +\bigr\} + \sharp\bigl\{(h,h') \in B^{\alpha}_{N} \bigm| \lambda(h')\sigma(h') = +\bigr\} \\
&\quad = -\sharp\bigl\{c \in [-N, N-1] \bigm| \lambda^{[j_-]}\bigl(\sigma(j_-)\cdot(c - c_\lambda[j_-] + 1/2)\bigr) = \sigma(j_-)\bigr\} \\
&\qquad\quad + \sharp\bigl\{c \in [-N, N-1] \bigm| \lambda^{[j_+]}\bigl(\sigma(j_+)\cdot(c + \alpha^0 - c_\lambda[j_+] + 1/2)\bigr) = \sigma(j_+)\bigr\} \\
&\quad = -(N - c_\lambda[j_-] - 1/2) + (N + \alpha^0 - c_\lambda[j_+] - 1/2) \\
&\quad = \alpha^0 + c_\lambda[j_-] - c_\lambda[j_+]. \qquad \square
\end{aligned}$$

For σ, λ, θ and i, we put

$$B^{i,\pm}_{\sigma,\lambda,\theta} := \bigl\{n \in \pi^{-1}(i) \mid (\theta(n-1/2), \theta(n+1/2)) \in B^{\alpha(\theta,i),\pm}_{\sigma,\lambda}\bigr\}. \tag{1-4}$$

2. Dimer model

2.1. *Dimer configurations.*

2.1.1. We fix the following data:

- a map $\sigma\colon I_{\mathrm{h}} \to \{\pm\}$,
- a map $\lambda\colon \mathbb{Z}_{\mathrm{h}} \to \{\pm\}$ such that $\lambda(h) = \pm\sigma(\pi(h))$ for $\pm h \gg 0$,
- a pair of Young diagrams $\nu = (\nu_+, \nu_-)$,
- a bijection $\theta\colon \mathbb{Z}_{\mathrm{h}} \to \mathbb{Z}_{\mathrm{h}}$ in Θ.

We put $\tilde\sigma := \sigma \circ \pi \circ \theta$, $\tilde\lambda := \lambda \circ \theta$ and $L_\pm := |\sigma^{-1}(\pm)|$.

2.1.2. We consider the following graph in the (x, y)-plane. First, we set

$$H(\sigma,\theta) := \bigl\{n \in \mathbb{Z} \mid \tilde\sigma(n-1/2) = \tilde\sigma(n+1/2)\bigr\}, \quad I_H(\sigma,\theta) := \pi(H(\sigma,\theta)), \tag{2-1}$$

$$S(\sigma,\theta) := \bigl\{n \in \mathbb{Z} \mid \tilde\sigma(n-1/2) \neq \tilde\sigma(n+1/2)\bigr\}, \quad I_S(\sigma,\theta) := \pi(S(\sigma,\theta)) \tag{2-2}$$

and for $n \in H(\sigma,\theta)$ we put $\tilde\sigma(n) := \tilde\sigma(n \pm 1/2)$.

The set of the vertices is given by

$$\begin{aligned}
\mathcal{V} := &\{(n,m) \mid n \in S(\sigma,\theta),\ n-m\colon \text{odd}\} \\
&\sqcup \{(n-1/2, m) \mid n \in H(\sigma,\theta),\ n-m\colon \text{odd}\} \\
&\sqcup \{(n+1/2, m) \mid n \in H(\sigma,\theta),\ n-m\colon \text{odd}\},
\end{aligned}$$

which are denoted by $v(n,m)$, $v_{\mathrm{l}}(n-1/2, m)$ and $v_{\mathrm{r}}(n+1/2, m)$ respectively.

The set of the edges is given by

$$\mathcal{E} := \{e_{\mathrm{h}}(n,m) \mid n \in H(\sigma,\theta),\ n-m : \mathrm{odd}\} \sqcup \{e_{\mathrm{s}}(h,k) \mid h,k \in \mathbb{Z}_{\mathrm{h}}\},$$

where

- $e_{\mathrm{h}}(n,m)$ connects $v_{\mathrm{l}}(n-1/2,m)$ and $v_{\mathrm{r}}(n+1/2,m)$,
- $e_{\mathrm{s}}(h,k)$ connects $v(h-1/2,k+1/2)$ or $v_{\mathrm{r}}(h,k+1/2)$ and $v(h+1/2,k-1/2)$ or $v_{\mathrm{l}}(h,k-1/2)$ if $h-k$ is even, and
- $e_{\mathrm{s}}(h,k)$ connects $v(h-1/2,k-1/2)$ or $v_{\mathrm{r}}(h,k-1/2)$ and $v(h+1/2,k+1/2)$ or $v_{\mathrm{l}}(h,k+1/2)$ if $h-k$ is odd.

We put

$$\mathcal{F} := \{(n,m) \in \mathbb{Z}^2 \mid n+m : \mathrm{even}\}, \quad \mathcal{F}_i := \{(n,m) \in \mathcal{F} \mid n \in \pi^{-1}(i)\} \tag{2-3}$$

for $i \in I$. Note that $\mathcal{E}$ divides the plain into disjoint hexagons and quadrilaterals. The hexagons are parametrized by the set

$$\mathcal{F}_{\mathrm{H}} := \{(n,m) \in \mathcal{F} \mid n \in H(\sigma,\theta)\}$$

and the quadrilaterals are parametrized by the set

$$\mathcal{F}_{\mathrm{S}} := \{(n,m) \in \mathcal{F} \mid n \in S(\sigma,\theta)\}.$$

For $(n,m) \in \mathcal{F}$, let $f(n,m)$ denote the corresponding hexagon or quadrilateral.

Example 2.1.3. In Figure 3, we show the graph associated with $L=3$, σ given by

$$\sigma(1/2) = +, \quad \sigma(3/2) = -, \quad \sigma(5/2) = -,$$

and $\theta = \mathrm{id}$ ($L_+ = 1$, $L_- = 2$).

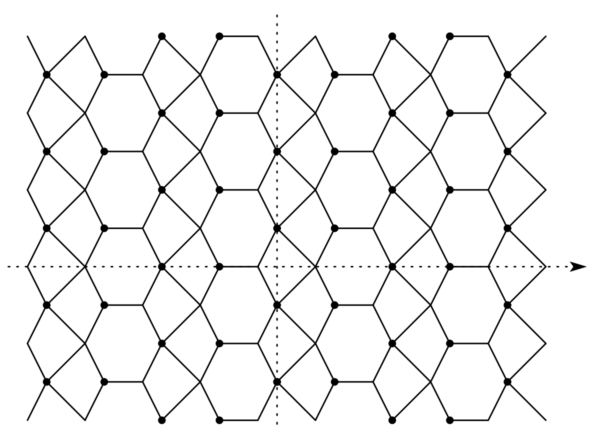


Figure 3. Graph and $\mathcal{V}_+$ for Example 2.1.3.

2.1.4. We set

$$\begin{aligned}\mathcal{V}_{\pm} := \{v(n,m) \mid \tilde{\sigma}(n+1/2) = \pm\} \\ \sqcup \{v_{\mathrm{l}}(n-1/2,m) \mid \tilde{\sigma}(n) = \mp\} \sqcup \{v_{\mathrm{r}}(n+1/2,m) \mid \tilde{\sigma}(n) = \pm\}.\end{aligned}$$

Note that $\mathcal{V} = \mathcal{V}_+ \sqcup \mathcal{V}_-$ and each element in $\mathcal{E}$ connects an element in $\mathcal{V}_+$ and an element in $\mathcal{V}_-$ (see Figure 3 for example).

A *perfect matching* is a subset of $\mathcal{E}$ giving a bijection between $\mathcal{V}_+$ and $\mathcal{V}_-$.

2.1.5. We define the map $F_{\sigma,\lambda,\theta}\colon \mathbb{Z} \to \mathbb{Z}$ by $F_{\sigma,\lambda,\theta}(0) = 0$ and

$$F_{\sigma,\lambda,\theta}(n) = F_{\sigma,\lambda,\theta}(n-1) - \tilde{\lambda}(n-1/2). \tag{2-4}$$

For $k \in \mathbb{Z}_{\mathrm{h}}$, we set

$$\begin{aligned}\mathcal{P}^{k,\pm}_{\sigma,\lambda,\theta} := \bigl\{e_{\mathrm{h}}\bigl(n,\, F_{\sigma,\lambda,\theta}(n) + 2k\bigr) \bigm| n \in \mathbb{Z},\ \tilde{\sigma}(n) = \mp\bigr\} \\ \sqcup \bigl\{e_{\mathrm{s}}\bigl(h,\, \tfrac{1}{2}(F_{\sigma,\lambda,\theta}(h-1/2) + F_{\sigma,\lambda,\theta}(h+1/2)) + 2k\bigr) \bigm| h \in \mathbb{Z}_{\mathrm{h}},\ \tilde{\sigma}(h) = \pm\bigr\}.\end{aligned}$$

For a Young diagram η, define the perfect matching

$$\mathcal{P}^{\eta}_{\sigma,\lambda,\theta} := \bigsqcup_{k \in \mathbb{Z}_{\mathrm{h}}} \mathcal{P}^{k,\eta(k)}_{\sigma,\lambda,\theta}.$$

Example 2.1.6. In Figure 4, we show the perfect matching associated with σ as in Example 2.1.3, $\theta = \mathrm{id}$, $\eta = \varnothing$, and λ given by

$$\lambda(h) = \begin{cases} + & \text{if } h = -5/2, \\ - & \text{if } h = 1/2, \\ \mathrm{sgn}(h)\sigma(h) & \text{otherwise.} \end{cases}$$

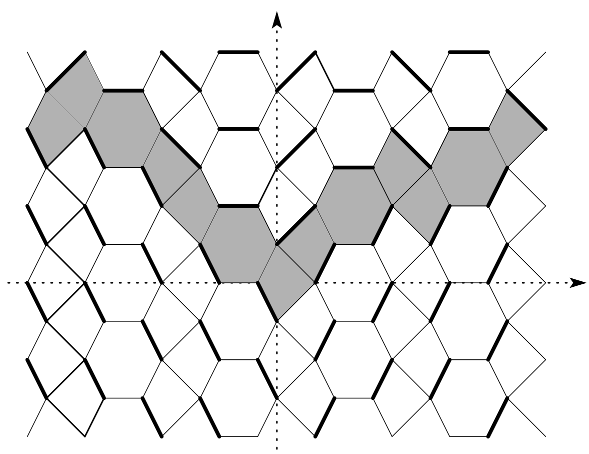


Figure 4. Example 2.1.6: $\{f(n,\, F_{\sigma,\lambda,\mathrm{id}}(n)) \mid n \in \mathbb{Z}\}$ and $\mathcal{P}^{\varnothing}_{\sigma,\lambda,\mathrm{id}}$.

2.1.7. Define the perfect matching

$$\mathcal{P}^{\pm}_{\sigma,\lambda,\theta} := \left\{e_{\mathrm{h}}(n,m) \mid \tilde{\sigma}(n) = \mp\right\} \sqcup \left\{e_{\mathrm{s}}(h,k) \mid \tilde{\sigma}(h) = \pm,\ h \cdot \tilde{\lambda}(h) - k : \text{even}\right\}.$$

Definition 2.1.8. A perfect matching D is said to be *a dimer configuration of type* $(\sigma, \lambda, \nu, \theta)$ if D coincides with $\mathcal{P}^{\nu_{\pm}}_{\sigma,\lambda,\theta}$ in the area $\{\pm x > m\}$ and $\mathcal{P}^{\pm}_{\sigma,\lambda,\theta}$ in the area $\{\pm y > m\}$ for $m \gg 0$.

Remark. A dimer configuration of type $(\sigma, \vec{\varnothing}, \vec{\varnothing}, \mathrm{id})$ is "a perfect matching congruent to the canonical perfect matching" in the terminology of [Mozgovoy and Reineke 2010].

2.1.9. For $f \in \mathcal{F}$, let $\partial f \subset \mathcal{E}$ denote the set of edges surrounding the face f. By moving f around clockwise, we can determine an orientation for each element in ∂f. Let $\partial^{\pm} f \subset \partial f$ denote the subset of edges starting from elements in $\mathcal{V}_{\pm}$.

For an edge $e \in \mathcal{E}$, let $f^{\pm}(e)$ denote the unique face such that $e \in \partial^{\pm} f^{\pm}(e)$.

2.2. *Weights.*

2.2.1. For $h \in \mathbb{Z}_{\mathrm{h}}$, we define the monomials $w_{\sigma,\lambda}(h)$ by the conditions

$$w_{\sigma,\lambda}(h) = \begin{cases} \left(Q_{\sigma(h)}\right)^{c(h)-c_\lambda[j(h)]} q^{(j(h))}_{\sigma(h)} & \text{if } h \gg 0, \\ \left(Q_{-\sigma(h)}\right)^{c(h)-c_\lambda[j(h)]} q^{(j(h))}_{-\sigma(h)} & \text{if } h \ll 0, \end{cases}$$

and

$$w_{\sigma,\lambda}(h)/w_{\sigma,\lambda}(h-L) = q_{\lambda(h)} \cdot q_{\lambda(h-L)} \cdot q_1 \cdot \cdots \cdot q_{L-1},$$

where

$$Q_{\pm} := (q_{\pm})^2 \cdot q_1 \cdot \cdots \cdot q_{L-1},\quad q^{(j)}_{\pm} := q_{\pm} \cdot q_1 \cdot \cdots \cdot q_{j-1/2}.$$

Note that for $h \neq h'$ we have

$$w_\lambda(h')/w_\lambda(h)\big|_{q_+=q_-=(q_0)^{1/2}} = q^{\alpha_{[h,h']}}. \tag{2-5}$$

Example 2.2.2. Figure 5 shows the weight $w_{\sigma,\lambda}$ for σ and λ as in Example 2.1.6.

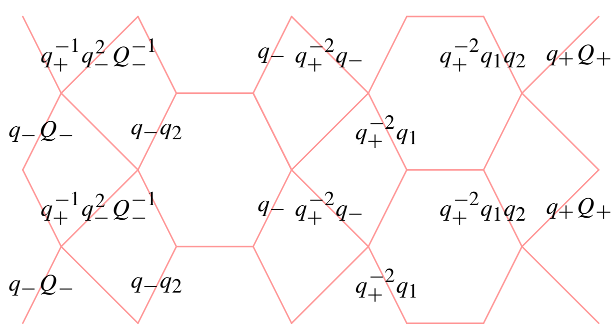


Figure 5. The weight $w_{\sigma,\lambda}$.

2.2.3. To an edge $e \in \mathscr{E}$ we associate the weight $w_{\sigma,\lambda,\theta}(e)$ by

$$w_{\sigma,\lambda,\theta}(e_{\mathrm{s}}(h,k)) := \begin{cases} w_{\sigma,\lambda}(\theta(h))^{\tilde{\sigma}(h)\cdot\tilde{\lambda}(h)} & \text{if } h\cdot\tilde{\lambda}(h)-k \text{ is odd,} \\ 1 & \text{if } h\cdot\tilde{\lambda}(h)-k \text{ is even,} \end{cases} \tag{2-6}$$

$$w_{\lambda,\sigma,\theta}(e_{\mathrm{h}}(n,m)) := 1. \tag{2-7}$$

2.2.4. Fix σ and λ. Then the set $\bigsqcup_{\alpha\in\Lambda^{\mathrm{re},+}} B^{\alpha,-}_{\sigma,\lambda}$ is finite. We define

$$F^{\alpha}_{\sigma,\lambda} := \prod_{(h,h')\in B^{\alpha,-}_{\sigma,\lambda}} \frac{w_{\sigma,\lambda}(h')}{w_{\sigma,\lambda}(h)}, \qquad F^{\theta}_{\sigma,\lambda} := \prod_{\substack{\alpha\in\Lambda^{\mathrm{re},+};\ \theta(\alpha)<0,\\ \sigma(j^-(\alpha))\neq\sigma(j^+(\alpha)).}} F^{\alpha}_{\sigma,\lambda}. \tag{2-8}$$

2.2.5. Note that for a dimer configuration D of type $(\sigma,\lambda,\nu,\theta)$ we have only a finite number of $e \in D$ such that $w_{\sigma,\lambda,\theta}(e) \neq 1$.

Definition 2.2.6. For a dimer configuration D of type $(\sigma,\lambda,\nu,\theta)$, we define the weight $w_{\sigma,\lambda,\theta}(D)$ by

$$w_{\sigma,\lambda,\theta}(D) := F^{\theta}_{\sigma,\lambda}\cdot\prod_{e\in D} w_{\sigma,\lambda,\theta}(e). \tag{2-9}$$

(See (2-6)–(2-8) for notation.)

Remark. We will define the generating function $\mathscr{Z}_{\sigma,\lambda,\nu,\theta}$ by the sum of weighs of all dimer configurations of type $(\sigma,\lambda,\nu,\theta)$.[9]

2.2.7. For a finite subset $\mathscr{E}' \subset \mathscr{E}$, we put

$$w_{\sigma,\lambda,\theta}(\mathscr{E}') := \prod_{e\in\mathscr{E}'} w_{\sigma,\lambda,\theta}(e)$$

and for a face $f \in \mathscr{F}$ we put

$$w_{\sigma,\lambda,\theta}(f) := \frac{w_{\sigma,\lambda,\theta}(\partial^- f)}{w_{\sigma,\lambda,\theta}(\partial^+ f)}. \tag{2-10}$$

For an integer n we set

$$w_{\sigma,\lambda,\theta}(n) := \frac{w_{\sigma,\lambda}(\theta(n+1/2))}{w_{\sigma,\lambda}(\theta(n-1/2))};$$

then

$$w_{\sigma,\lambda,\theta}(f(n,m)) = w_{\sigma,\lambda,\theta}(n)$$

for any $(n,m)\in\mathscr{F}$. By (2-5), we have

$$w_{\sigma,\lambda,\theta}(n)\big|_{q_+=q_-=(q_0)^{1/2}} = q^{\alpha(\theta,i)}.$$

[9] We will leave the definition of the generating function until Section 3.4 since we will use Proposition 3.3.9 to prove that the number of dimer configurations of type $(\sigma,\lambda,\nu,\theta)$ is finite.

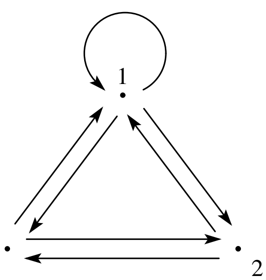
3. The viewpoint of noncommutative crepant resolutions

3.1. ***Noncommutative crepant resolutions.*** Let Γ be a lattice in the (x, y)-plane generated by $(L, 0)$ and $(0, 2)$. The graph given in §2.1.2 is invariant under the action of Γ and so gives a graph on the torus $\mathbb{R}^2/\Gamma$. This gives a quiver with a potential $A = (Q_{\sigma,\theta}, w_{\sigma,\theta})$ as in [Nagao 2011a]. The vertices of $Q_{\sigma,\theta}$ are parametrized by I and the arrows are given by

$$\Bigl(\bigsqcup_{j\in I_{\mathrm{h}}} h_j^+\Bigr) \sqcup \Bigl(\bigsqcup_{j\in I_{\mathrm{h}}} h_j^-\Bigr) \sqcup \Bigl(\bigsqcup_{i\in I_H(\sigma,\theta)} r_i\Bigr)$$

(see (2-1) for notation). Here h_j^+ (resp. h_j^-) is an edge from $j - 1/2$ to $j + 1/2$ (resp. from $j + 1/2$ to $j - 1/2$) and r_i is an edge from i to itself. See [Nagao 2011a, §1.2] for the definition of the potential $w_{\sigma,\theta}$.

Example 3.1.1. Here is the quiver $Q_{\sigma,\mathrm{id}}$ for σ as in Example 2.1.6:



Remarks.

- The center of A is isomorphic to $R := \mathbb{C}[\mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{w}]/(\mathrm{xy} = \mathrm{z}^{L_+}\mathrm{w}^{L_-})$. In [Nagao 2011a, Theorem 1.14 and 1.19], we showed that A is a noncommutative crepant resolution of $X = \operatorname{Spec} R$.
- The affine 3-fold X is toric. In fact,

$$T = \operatorname{Spec} \tilde{R} := \operatorname{Spec} \mathbb{C}[\mathrm{x}^\pm, \mathrm{y}^\pm, \mathrm{z}^\pm, \mathrm{w}^\pm]/(\mathrm{xy} = \mathrm{z}^{L_+}\mathrm{w}^{L_-}) \subset X$$

  is a 3-dimensional torus.

3.2. ***Dimer model and noncommutative crepant resolution.***

3.2.1. We will construct an A-module $M(D)$ for a dimer configuration D. Let $V_i = V_i(D)$ $(i \in I)$ be vector space with the basis

$$\bigl\{b[D; x, y, z] \mid (x, y) \in \mathcal{F}_i,\ z \in \mathbb{Z}_{\geq 0}\bigr\}$$

(see (2-3) for notation). We define the map $h_j^\pm\colon V_{j\mp 1/2} \to V_{j\pm 1/2}$ by setting

$$h_j^\pm(b[D; x, y, z]) = \begin{cases} b[D; x \pm 1, y - \tilde{\sigma}(j), z] & \text{if } e_{\mathrm{s}}\bigl(x \pm \frac{1}{2}, y - \frac{1}{2}\tilde{\sigma}(j)\bigr) \notin D, \\ b[D; x \pm 1, y - \tilde{\sigma}(j), z+1] & \text{if } e_{\mathrm{s}}\bigl(x \pm \frac{1}{2}, y - \frac{1}{2}\tilde{\sigma}(j)\bigr) \in D, \end{cases}$$

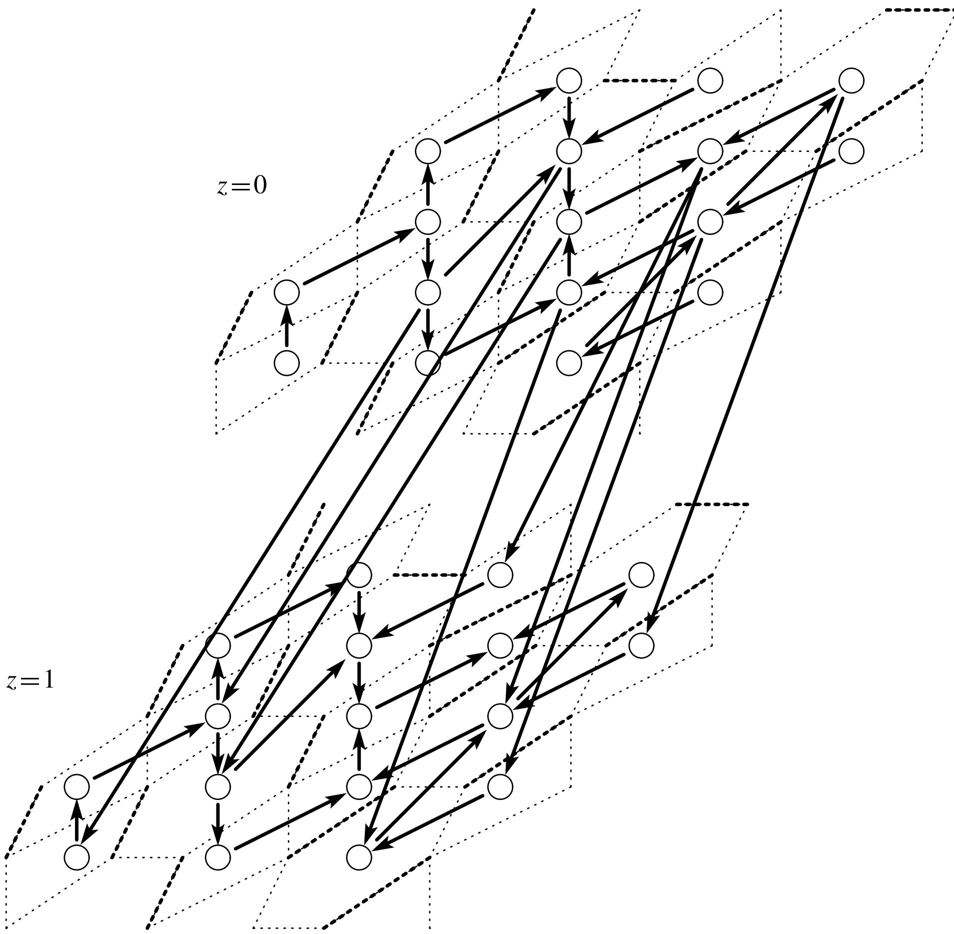


Figure 6. An example of $M(D)$.

and $r_i : V_i \to V_i$ by

$$r_i(b[D; x, y, z]) = \begin{cases} b[D; x, y+\tilde{\sigma}(j), z] & \text{if } e_{\mathrm{h}}(x, y+\tilde{\sigma}(j)/2) \notin D, \\ b[D; x, y+\tilde{\sigma}(j), z+1] & \text{if } e_{\mathrm{h}}(x, y+\tilde{\sigma}(j)/2) \in D. \end{cases}$$

3.2.2. Let $\mathscr{C} \subset \mathscr{E}$ be a subset which gives a closed zigzag curve without self-intersection. By moving along the zigzag curve clockwisely, we can determine an orientation for each element in $\mathscr{C}$. Let $\mathscr{C}^{\pm} \subset \mathscr{C}$ denote the subset of edges starting from elements in $\mathscr{V}_{\pm}$.

Let D be a dimer configuration of type $(\sigma, \lambda, \nu, \theta)$. A subset $\mathscr{C}$ as above is said to be a *positive cycle* with respect to D if $\mathscr{C} \cap D = \mathscr{C}^{+}$, and it is said to be a *negative cycle* with respect to D if $\mathscr{C}^{-}$.

3.2.3. Given a dimer configuration D and a positive cycle $\mathscr{C}$ with respect to D, let $D_{\mathscr{C}}$ be the dimer configuration given by

$$D_{\mathscr{C}} = (D \backslash \mathscr{C}^+) \cup \mathscr{C}^-.$$

Then we can check the following lemma:

Lemma 3.2.4. *The surjection $M(D) \to M(D_{\mathscr{C}})$ given by*

$$b[D; x, y, z] \mapsto \begin{cases} 0 & \text{if } (x, y) \in \mathscr{C}^\circ \text{ and } z = 0, \\ b[D_{\mathscr{C}}; x, y, z-1] & \text{if } (x, y) \in \mathscr{C}^\circ \text{ and } z \geq 1, \\ b[D_{\mathscr{C}}; x, y, z] & \text{if } (x, y) \notin \mathscr{C}^\circ, \end{cases}$$

is a homomorphism of A-modules, where $\mathscr{C}^\circ$ is the interior of the closed zigzag curve. Moreover,

$$w_{\sigma,\lambda,\theta}(D_{\mathscr{C}}) = w_{\sigma,\lambda,\theta}(D) \cdot \prod_{f \in \mathscr{C}^\circ} w_{\sigma,\lambda,\theta}(f).$$

3.3. *Crystal melting interpretation.* In this subsection, we show that a dimer configuration of type $(\sigma, \lambda, \nu, \theta)$ corresponds to a (torus invariant) quotient A-module of the A-module $M^{\max} = M^{\max}_{\sigma,\lambda,\nu,\theta}$. In the physicists' terminology, studying such quotient modules is called *the crystal melting model* (see [Ooguri and Yamazaki 2009]) and $M^{\max}$ is called the *grand state* of the model.

3.3.1. We define a Young diagram $G_{\sigma,\lambda,\theta} : \mathbb{Z} \to \mathbb{Z}$ by the following conditions:

- $G_{\sigma,\lambda,\theta}(n) = |n|$ if $|n| \gg 0$, and
- $G_{\sigma,\lambda,\theta}(n) = G_{\sigma,\lambda,\theta}(n-1) + \tilde{\sigma}(n-1/2)\tilde{\lambda}(n-1/2)$ for any n.

We define a map $G_{\sigma,\lambda,\theta} : \mathscr{F} \to \mathbb{Z}$ by

$$G_{\sigma,\lambda,\theta}(n, m) := G(n)_{\sigma,\lambda,\theta} + 2 \cdot |m - F_{\sigma,\lambda,\theta}(n)|, \tag{3-1}$$

where $F_{\sigma,\lambda,\theta}(n)$ is given in (2-4).

Example 3.3.2. In the case of Example 2.1.6, we have

$$(G_{\sigma,\lambda,\mathrm{id}}(n))_{n \in \mathbb{Z}} = (\ldots, 6, 5, 4, 3, 4, 3, 2, 1, 2, 3, 4, 5, 6, \ldots)$$

and $G_{\sigma,\lambda,\mathrm{id}}(n, m)$ is given in Figure 7.

3.3.3. We define two maps $F^{\pm}_{\sigma,\lambda,\theta} : \mathbb{Z} \to \mathbb{Z}$ by the following conditions:

- $F^{\pm}_{\sigma,\lambda,\theta}(n) = F_{\sigma,\lambda,\theta}(n)$ if $\pm n \gg 0$.
- $F^{\pm}_{\sigma,\lambda,\theta}(n) = F^{\pm}_{\sigma,\lambda,\theta}(n-1) \mp \tilde{\sigma}(n-1/2)$ for any n.

Then we define two maps $G^{\nu_\pm,\pm}_{\sigma,\lambda,\theta} : \mathscr{F} \to \mathbb{Z}$ by

$$G^{\nu_\pm,\pm}_{\sigma,\lambda,\theta}(n, m) := \nu_\pm(m - F^{\pm}_{\sigma,\lambda,\theta}(n)) \pm n. \tag{3-2}$$

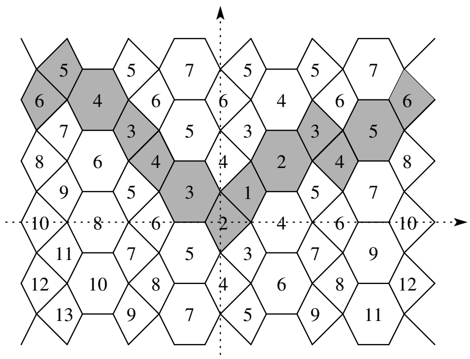


Figure 7. $G_{\sigma,\lambda,\mathrm{id}}(n, m)$.

Example 3.3.4. Figure 8 shows $G^{\varnothing,+}_{\sigma,\lambda,\mathrm{id}}$ and $G^{\square,-}_{\sigma,\lambda,\mathrm{id}}$ for σ and λ as in Example 2.1.6.

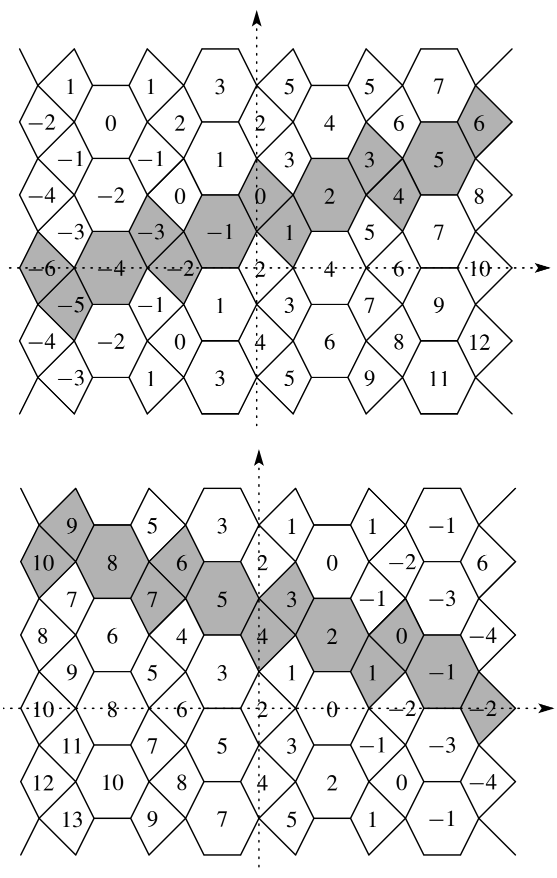


Figure 8. $G^{\varnothing,+}_{\sigma,\lambda,\theta}$ (top) and $G^{\square,-}_{\sigma,\lambda,\theta}$ (bottom).

3.3.5. We define a map $G^{\nu}_{\sigma,\lambda,\theta}\colon \mathscr{F} \to \mathbb{Z}$ by

$$G^{\nu}_{\sigma,\lambda,\theta}(n,m) := \max\bigl(G_{\sigma,\lambda,\theta}(n,m),\, G^{\nu_+,+}_{\sigma,\lambda,\theta}(n,m),\, G^{\mu_-,-}_{\sigma,\lambda,\theta}(n,m)\bigr).$$

We can verify that

$$G^{\nu}_{\sigma,\lambda,\theta}(f^+(e)) = G^{\nu}_{\sigma,\lambda,\theta}(f^-(e)) + 1 \text{ or } G^{\nu}_{\sigma,\lambda,\theta}(f^-(e)) - 3.$$

for an edge $e \in \mathscr{E}$ (see §2.1.9 for notation). We define a perfect matching $D^{\max} = D^{\max}_{\sigma,\lambda,\nu,\theta}$ by

$$e \in D^{\max} \iff G^{\nu}_{\sigma,\lambda,\theta}(f^+(e)) = G^{\nu}_{\sigma,\lambda,\theta}(f^-(e)) - 3.$$

Let $M^{\max} = M^{\max}_{\sigma,\lambda,\nu,\theta} := M(D^{\max}_{\sigma,\lambda,\nu,\theta})$ denote the corresponding A-module.

Example 3.3.6. In Figure 9, we show $G^{\vec{\varnothing}}_{\sigma,\lambda,\mathrm{id}}$ and $D^{\max}_{\sigma,\lambda,\vec{\varnothing},\mathrm{id}}$ for σ and λ as in Example 2.1.6.

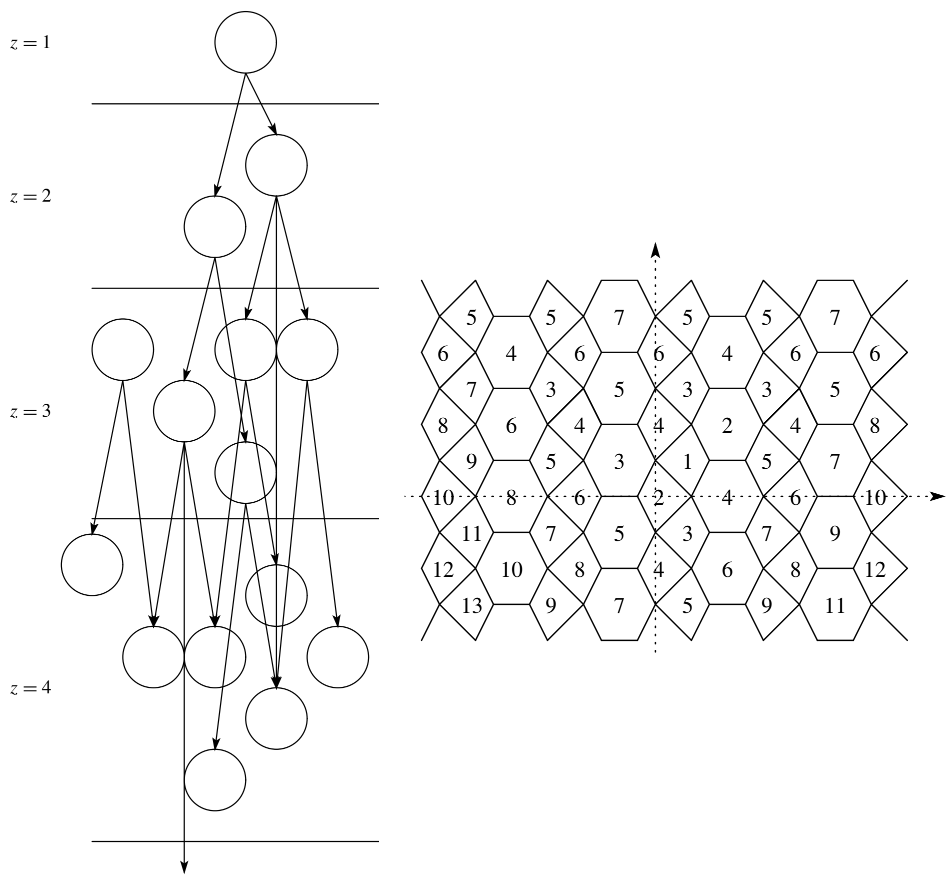


Figure 9. $G^{\vec{\varnothing}}_{\sigma,\lambda,\mathrm{id}}$ and $D^{\max}_{\sigma,\lambda,\vec{\varnothing},\mathrm{id}}$.

Remark. The graph of the map $m \mapsto G^{\nu}_{\sigma,\lambda,\theta}(n,m)$ determines a Young diagram. This is what we denote by $\mathcal{V}_{\min}(n)$ in [Nagao 2011b, §3.1].

Lemma 3.3.7. *There is no positive cycle with respect to $D^{\max}$.*

Proof. Assume that we have a positive cycle $\mathscr{C}$. For an edge $e \in \partial\mathscr{C}$, let $f_{\text{in}}(e)$ (resp. $f_{\text{out}}(e)$) be the unique face such that $e \in \partial f_{\text{in}}(e)$ and $f_{\text{in}}(e) \in \mathscr{C}^{\circ}$ (resp. $e \in \partial f_{\text{out}}(e)$ and $f_{\text{out}}(e) \notin \mathscr{C}^{\circ}$). Then we have

$$G^{\nu}_{\sigma,\lambda,\theta}(f_{\text{in}}(e)) > G^{\nu}_{\sigma,\lambda,\theta}(f_{\text{out}}(e)). \tag{3-3}$$

Take a face $(n,m) \in \mathscr{C}^{\circ}$. If $G^{\nu}_{\sigma,\lambda,\theta}(n,m) = G^{\nu,\pm}_{\sigma,\lambda,\theta}(n,m)$, then

$$(n \pm n',\, F^{\pm}_{\sigma,\lambda,\theta}(n \pm n') - F^{\pm}_{\sigma,\lambda,\theta}(n) + m) \in \mathscr{C}^{\circ}$$

for any $n' \geq 0$ by (3-2) and (3-3), and this is a contradiction. On the other hands, if $G^{\mu}_{\sigma,\lambda,\theta}(n,m) = G_{\sigma,\lambda,\theta}(n,m)$ and $\pm m \mp F_{\sigma,\lambda,\theta}(n) \geq 0$, then $(n, m \pm m') \in \mathscr{C}^{\circ}$ for any $m' \geq 0$ by (3-1) and (3-3), and this is also a contradiction. Hence the claim follows. □

3.3.8. For a map $H : \mathscr{F} \to \mathbb{Z}_{\geq 0}$, let $V_i^H \subset V_i(D^{\max})$ ($i \in I$) be the subspace spanned by the elements

$$\bigl\{ b[D^{\max}; x, y, z] \mid (x,y) \in \mathscr{F}_i,\ z \geq H(x,y) \bigr\}.$$

The following proposition gives a one-to-one correspondence between dimer configurations of type $(\sigma,\lambda,\nu,\theta)$ and finite-dimensional quotient modules of $M^{\max}_{\sigma,\lambda,\nu,\theta}$.

Proposition 3.3.9. *Given a monomial $\boldsymbol{q}$, we have a natural bijection between*

- *the set of dimer configurations of type $(\sigma,\lambda,\nu,\theta)$ with weight $\boldsymbol{q}$, and*
- *the set of maps $H : \mathscr{F} \to \mathbb{Z}_{\geq 0}$ satisfying the following conditions*:
  - *$H(f) = 0$ except for only a finite number of $f \in \mathscr{F}$,*
  - *$(V_i^H)_{i \in I}$ is stable under the action of A, and*
  - *$w_{\sigma,\lambda,\theta}(D^{\max}) \cdot \prod_f w_{\sigma,\lambda,\theta}(f)^{H(f)} = \boldsymbol{q}$.*

Proof. Let D be a dimer configuration of type $(\sigma,\lambda,\nu,\theta)$. By Lemma 3.3.7, $(D \cup D^{\max}) \backslash (D \cap D^{\max})$ is a disjoint union $\sqcup \mathscr{C}_\gamma$ of a finite number of positive cycles. We define a map $H_D : \mathscr{F} \to \mathbb{Z}_{\geq 0}$ by

$$H_D(f) := \sharp\{\mathscr{C}_\gamma \mid f \in \mathscr{C}^{\circ}_\gamma\}.$$

Then we can verify the claim using Lemma 3.2.4. □

Remark. The graph of the map $m \mapsto G^{\nu}_{\sigma,\lambda,\theta}(n,m) + 2H(n,m)$ determines a Young diagram. This is what we denote by $\mathcal{V}(n)$ in [Nagao 2011b, §3.1].

3.4. ***Generating function.*** From the description given by Proposition 3.3.9, we can verify that, fixing a monomial $\boldsymbol{q}$, we have only a finite number of dimer configurations of type $(\sigma, \lambda, \nu, \theta)$ with weight $\boldsymbol{q}$.

Definition 3.4.1. We define the generating function by

$$\mathcal{Z}_{\sigma,\lambda,\nu,\theta} = \mathcal{Z}_{\sigma,\lambda,\nu,\theta}(\vec{q}) := \sum_{D} w_{\sigma,\lambda,\theta}(D),$$

where the sum is taken over all dimer configurations of type $(\sigma, \lambda, \nu, \theta)$. In particular, we put

$$\mathcal{Z}^{\mathrm{NCDT}}_{\sigma,\lambda,\nu} := \mathcal{Z}_{\sigma,\lambda,\nu,\mathrm{id}_{\mathbb{Z}_\mathrm{h}}}.$$

Remark. Note that $\mathcal{Z}^{\mathrm{NCDT}}_{\sigma,\lambda,\nu} \cdot w_{\sigma,\lambda,\theta}\big(D^{\max}_{\sigma,\lambda,\nu.\mathrm{id}}\big)^{-1}$ is a formal power series in q_+, q_- and $q_1, \dots, q_{L-1}$.

4. Dimer shuffling and wall-crossing formula

4.1. ***Dimer shuffling at a hexagon.*** In this and next subsections, we study the relation between dimer configurations of type $(\sigma, \lambda, \nu, \theta)$ and of type $(\sigma, \lambda, \nu, \mu_i(\theta))$ for $i \in I_H(\sigma, \theta)$.

4.1.1. For $(n, m) \in \mathcal{F}$ and $M \in \mathbb{Z}_{>0} \sqcup \{\infty\}$. we put

$$f(n, m; \pm, M) := \bigcup_{m'=0}^{M-1} f(n, m \pm m')$$

We define $\partial f(n, m; \pm, M)$ and $\partial^{\pm} f(n, m; \pm, M)$ in the same way as in §2.1.9 and §3.2.2.

4.1.2. For a dimer configuration D and $n \in B^{i,\pm}_{\sigma,\lambda,\theta}$, let $m(D, n)$ denote the unique integer such that

$$\partial f(n, m(D, n); \sigma(i), \infty) \cap D = \partial^{\pm} f(n, m(D, n); \sigma(i), \infty).$$

4.1.3. For a dimer configuration D and $i \in I$, we consider the following conditions:

(4-1) $\partial f \cap D \neq \partial^- f$ for any $f \in \mathcal{F}_i$,

(4-2) $\partial f \cap D \neq \partial^+ f$ if $f \in \mathcal{F}_i \backslash \{f(n, m(D, n)) \mid n \in B^{i,\pm}_{\sigma,\lambda,\theta}\}$,

(4-3) $\partial f(n, m(D, n) - 2\sigma(i)) \cap D \neq \partial^- f(n, m(D, n) - 2\sigma(i))$ for $n \in B^{i,\pm}_{\sigma,\lambda,\theta}$.

4.1.4. For a dimer configuration D° of type $(\sigma, \lambda, \nu, \theta)$ satisfying the condition (4-1), we set

$$E_i(D^\circ) := \big\{(n, m) \in \mathcal{F}_i \mid \partial f(n, m) \cap D^\circ = \partial^+ f(n, m)\big\},$$

and define the map $M^i_{D^\circ}\colon E_i(D^\circ)\to\mathbb{Z}_{>0}\sqcup\{\infty\}$ by

$$M^i_{D^\circ}(n,m):=\max\{M\mid \partial f(n,m;\sigma(i),M)\cap D^\circ=\partial^+ f(n,m;\sigma(i),M)\}.$$

Note that

$$\bigl(M^i_{D^\circ}\bigr)^{-1}(\infty)=\bigl\{(n,m_n)\mid n\in B^{i,+}_{\sigma,\lambda,\theta}\bigr\}.$$

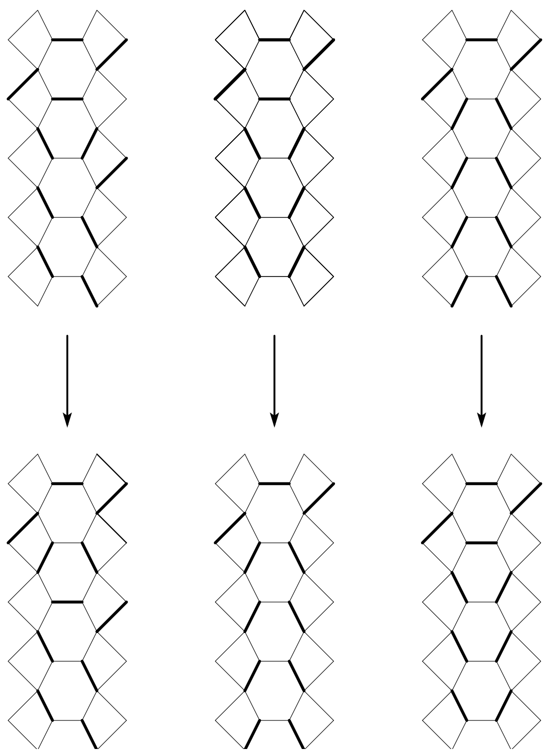
We put $E^{\mathrm{fin}}_i(D^\circ):=E_i(D^\circ)\backslash\bigl(M^i_{D^\circ}\bigr)^{-1}(\infty)$.

Definition 4.1.5. For a dimer configuration D° of type $(\sigma,\lambda,\nu,\theta)$ satisfying the condition (4-1), let $\mu_i(D^\circ)$ be the a dimer configuration of type $(\sigma,\lambda,\nu,\mu_i(\theta))$ given by

$$\Biggl(D^\circ\Bigl\backslash\Biggl(\bigcup_{(n,m)\in E_i(D^\circ)}\partial^+ f\bigl(n,m;\sigma(i),M^i_{D^\circ}(n,m)\bigr)\cup\bigcup_{n\in B^{i,-}_{\sigma,\lambda,\theta}}\partial^- f(n,m;\sigma(i),\infty)\Biggr)\Biggr)$$
$$\sqcup\Biggl(\bigcup_{(n,m)\in E_i(D^\circ)}\partial^- f\bigl(n,m;\sigma(i),M^i_{D^\circ}(n,m)\bigr)\cup\bigcup_{n\in B^{i,-}_{\sigma,\lambda,\theta}}\partial^+ f(n,m;\sigma(i),\infty)\Biggr).$$

Note that $\mu_i(D^\circ)$ satisfies the condition (4-2) and (4-3).

Example 4.1.6. Here are some examples of dimer shuffling at hexagons.



Lemma 4.1.7. $$w_{\sigma,\lambda,\mu_i(\theta)}(\mu_i(D^\circ)) = w_{\sigma,\lambda,\theta}(D^\circ).$$

Proof. For $n \in \pi^{-1}(i)$ and $m \in \mathbb{Z}$ such that $n+m$ is odd, we put

$$D^\circ(n,m) := \{e_{\mathrm{s}}(n+\varepsilon_1, m+\varepsilon_2)\ (\varepsilon_1, \varepsilon_2 = \pm 1/2)\} \cap D^\circ.$$

Assume that

$$(n,m-1),\ (n,m+1) \notin \bigcup_{(n,m)\in E_i(D^\circ)} f\big(n,m;\sigma(i), M^i_{D^\circ}(n,m)\big). \tag{4-4}$$

Then $D^\circ(n,m)$ is one of the following:

$$\varnothing, \quad \{e_{\mathrm{s}}(n\pm 1/2, m\pm 1/2)\}, \quad \{e_{\mathrm{s}}(n\pm 1/2, m\mp 1/2)\}.$$

In particular, we have

$$w_{\sigma,\lambda,\theta}(D^\circ(n,m)) = w_{\sigma,\lambda,\mu_i(\theta)}(D^\circ(n,m)).$$

Hence

$$w_{\sigma,\lambda,\theta}\big(D^\circ \cap \mu_i(D^\circ)\big) = w_{\sigma,\lambda,\mu_i(\theta)}\big(D^\circ \cap \mu_i(D^\circ)\big).$$

The claim follows from this and the fact that

$$w_{\sigma,\lambda,\theta}\big(\partial^\pm f(n,m,M)\big) = w_{\sigma,\lambda,\mu_i(\theta)}\big(\partial^\mp f(n,m,M)\big)$$

for $n \in \pi^{-1}(i)$. □

4.2. *Wall-crossing formula at a hexagon.*

Lemma 4.2.1.

$$\mathfrak{Z}_{\sigma,\lambda,\nu,\theta} = \sum_{D^\circ} w_{\sigma,\lambda,\theta}(D^\circ) \prod_{n\in B^{i,+}_{\sigma,\lambda,\theta}} \frac{1}{1+w_{\sigma,\lambda,\theta}(n)} \prod_{(n,m)\in E_i^{\mathrm{fin}}(D^\circ)} \frac{1+w_{\sigma,\lambda,\theta}(n)^{M^i_{D^\circ}(n,m)+1}}{1+w_{\sigma,\lambda,\theta}(n)},$$

where the sum is taken over all dimer configurations D° of type $(\sigma,\lambda,\nu,\theta)$ satisfying the condition (4-1).

Proof. For a map $s\colon E_i(D^\circ) \to \mathbb{Z}_{\geq 0}$ such that $s(n,m) \leq M^i_{D^\circ}(n,m)$, we define the dimer configuration

$$D^\circ_s := \left(D^\circ \Big\backslash \bigcup_{(n,m)\in E_i(D^\circ)} \partial^+ f(n,m;\sigma(i), s(n,m))\right) \sqcup \bigcup_{(n,m)\in E_i(D^\circ)} \partial^- f(n,m;\sigma(i),s(n,m)).$$

Then

$$w_{\sigma,\lambda,\theta}(D^\circ_s) = w_{\sigma,\lambda,\theta}(D^\circ) \prod_{(n,m)\in E_i(D^\circ)} w_{\sigma,\lambda,\theta}(n)^{s(n,m)}.$$

Note that any dimer configuration D is uniquely realized as $D^\circ(s)$ by some D° and s. Hence we have

$$\begin{aligned}\mathfrak{X}_{\sigma,\lambda,\nu,\theta} &= \sum_{D^\circ} w_{\sigma,\lambda,\theta}(D^\circ)\cdot\Bigg(\sum_s \prod_{(n,m)\in E_i(D^\circ)} w_{\sigma,\lambda,\theta}(n)^{s(n,m)}\Bigg)\\ &= \sum_{D^\circ} w_{\sigma,\lambda,\nu,\theta}(D^\circ)\prod_{n\in B^{i,+}_{\sigma,\lambda,\theta}}\frac{1}{1-w_{\sigma,\lambda,\theta}(n)}\\ &\qquad\qquad\times\prod_{(n,m)\in E_i^{\mathrm{fin}}(D^\circ)}\frac{1+w_{\sigma,\lambda,\theta}(n)^{M^i_{D^\circ}(n,m)+1}}{1+w_{\sigma,\lambda,\theta}(n)}.\end{aligned}$$ □

Theorem 4.2.2.

$$\mathfrak{X}_{\sigma,\lambda,\nu,\mu_i(\theta)} = \mathfrak{X}_{\sigma,\lambda,\nu,\theta}\prod_{n\in B^{i,+}_{\sigma,\lambda,\theta}}\big(1-w_{\sigma,\lambda,\theta}(n)\big)\prod_{n\in B^{i,-}_{\sigma,\lambda,\theta}}\frac{1}{1-w_{\sigma,\lambda,\theta}(n)}.$$

Proof. As Lemma 4.2.1, we get

$$\begin{aligned}\mathfrak{X}_{\sigma,\lambda,\nu,\mu_i(\theta)} &= \sum_{D^\bullet} w_{\sigma,\lambda,\mu_i(\theta)}(D^\bullet)\prod_{n\in B^{i,+}_{\sigma,\lambda,\mu_i(\theta)}}\frac{1}{1-w_{\sigma,\lambda,\mu_i(\theta)}(n)^{-1}}\\ &\qquad\qquad\times\prod_{(n,m)\in\check{E}_i(D^\bullet)}\frac{1+w_{\sigma,\lambda,\mu_i(\theta)}(n)^{-\check{M}^i_{D^\bullet}(n,m)-1}}{1+w_{\sigma,\lambda,\mu_i(\theta)}(n)^{-1}},\end{aligned}$$

where the sum is taken over all dimer configurations $D^\bullet$ of type $(\sigma,\lambda,\nu,\mu_i(\theta))$ satisfying (4-2), (4-3), and

$$\begin{aligned}\check{E}_i(D^\bullet) &:= \big\{(n,m)\in\mathcal{F}_i \;\big|\; \partial f(n,m)\cap D^\bullet = \partial^- f(n,m)\big\},\\ \check{M}^i_{D^\bullet}(n,m) &:= \max\big\{M \;\big|\; \partial f(n,m;\sigma(i),M)\cap D^\bullet = \partial^- f(n,m;-\sigma(i),M)\big\}.\end{aligned}$$

Note that μ_i gives a one-to-one correspondence between dimer configurations of type $(\sigma,\lambda,\nu,\theta)$ satisfying (4-1) and those of type $(\sigma,\lambda,\nu,\mu_i(\theta))$ satisfying (4-2) and (4-3). Hence the claim follows from

- $B^{i,\pm}_{\sigma,\lambda,\mu_i(\theta)} = B^{i,\mp}_{\sigma,\lambda,\theta}$,
- $w_{\sigma,\lambda,\mu_i(\theta)}(n) = w_{\sigma,\lambda,\theta}(n)^{-1}$ for $n\in\pi^{-1}(i)$,
- $(n,m)\mapsto(n, m+\sigma(i)\cdot(M^i_{D^\circ}(n,m)-1))$ gives a bijection between $E_i^{\mathrm{fin}}(D^\circ)$ and $\check{E}_i(\mu_i(D^\circ))$ which respects $M^i_{D^\circ}$ and $\check{M}^i_{\mu_i(D^\circ)}$,

and Lemma 4.2.1. □

4.3. ***Dimer shuffling at a quadrilateral.*** In this subsection, we study the relation between dimer configurations of type $(\sigma,\lambda,\nu,\theta)$ and of type $(\sigma,\lambda,\nu,\mu_i(\theta))$ for $i\in I_S(\sigma,\theta)$.

4.3.1. For a dimer configuration D° of type $(\sigma, \lambda, \nu, \theta)$ satisfying the condition (4-1) and $n \in \pi^{-1}(i)$, we define

$$E_n^1(D^\circ) := \{(n, m) \in \mathcal{F} \mid \partial f(n, m) \cap D^\circ = \partial^+ f(n, m)\},$$
$$E_n^2(D^\circ) := \{(n, m) \in \mathcal{F} \mid \partial f(n, m) \cap D^\circ = \varnothing\}.$$

Lemma 4.3.2. $$|E_n^1(D^\circ)| - |E_n^2(D^\circ)| = \begin{cases} \mp 1 & \textit{if } n \in B^{i,\pm}_{\sigma,\lambda,\theta}, \\ 0 & \textit{otherwise}. \end{cases}$$

(See (1-4) for notation.)

Proof. For $n, m \in \mathbb{Z}$ such that $n + m$ is odd, we define $\varepsilon_{D^\circ}(n, m)$ by

$$\varepsilon_{D^\circ}(n, m) := \begin{cases} + & \text{if } e_{\mathrm{s}}(n+1/2, m+1/2), e_{\mathrm{s}}(n-1/2, m+1/2) \notin D, \\ - & \text{if } e_{\mathrm{s}}(n+1/2, m-1/2), e_{\mathrm{s}}(n-1/2, m-1/2) \notin D. \end{cases}$$

Then for $(n, m) \in \mathcal{F}$, we have

$$(n, m) \in E_n^1(D^\circ) \iff \varepsilon_{D^\circ}(n, m \pm 1) = \pm,$$
$$(n, m) \in E_n^2(D^\circ) \iff \varepsilon_{D^\circ}(n, m \pm 1) = \mp,$$

and $\varepsilon_{D^\circ}(n, m) = \mp\tilde{\lambda}(n \pm 1/2)$ if $\tilde{\sigma}(n \pm 1/2) \cdot m \gg 0$. The claim follows. $\square$

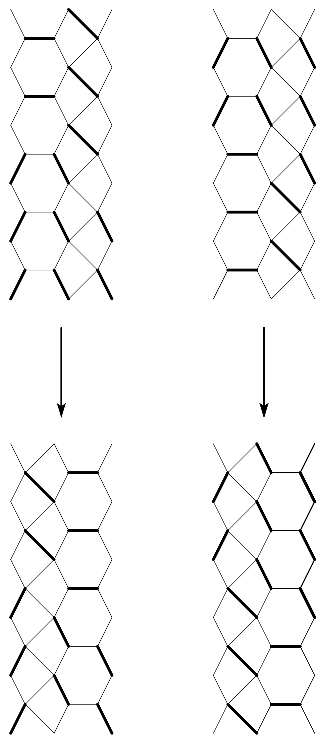
4.3.3. For a dimer configuration D° of type $(\sigma, \lambda, \nu, \theta)$ satisfying the condition (4-1), we define a dimer configuration $\mu_i(D^\circ)$ of type $(\sigma, \lambda, \nu, \mu_i(\theta))$ as follows:

- If $\pi(h) \neq i \pm 1/2$, we have
$$e_{\mathrm{s}}(h, k) \in D^\circ \iff e_{\mathrm{s}}(h, k) \in \mu_i(D^\circ),$$
- If $n \in I_H(\sigma, \theta)$ and $\pi(n) \neq i \pm 1$, we have
$$e_{\mathrm{h}}(n, m) \in D^\circ \iff e_{\mathrm{h}}(n, m) \in \mu_i(D^\circ),$$
- For $(n, m) \in \mathcal{F}_i$ we have
$$D^\circ(f(n, m)) = \varnothing \iff \mu_i(D^\circ)(f(n, m)) = \partial^-_{\sigma,\mu_i(\theta)}(f(n, m)),$$
$$D^\circ(f(n, m)) = \partial^+_{\sigma,\theta}(f(n, m)) \iff \mu_i(D^\circ)(f(n, m)) = \varnothing,$$
(Here we use notation such as $\partial^\pm_{\sigma,\theta}(f(n, m))$ in order to emphasize that the notions like $\partial^\pm(f(n, m))$ given in §2.1.9 depend on σ and θ.)
- If $D^\circ(f(n, m)) \neq \varnothing, \partial^+_{\sigma,\theta}(f(n, m))$ for $(n, m) \in \mathcal{F}_i$, we have
$$e_{\mathrm{s}}(n + \varepsilon_1, m + \varepsilon_2) \in D^\circ \iff e_{\mathrm{s}}(n - \varepsilon_1, m - \varepsilon_2) \in \mu_i(D^\circ) \quad (\varepsilon_1, \varepsilon_2 = \pm 1/2),$$
- If $\sigma(i \pm 3/2) \neq \sigma(i \pm 1/2)$, we have
$$e_{\mathrm{s}}(n \pm 1/2, m - 1), e_{\mathrm{s}}(n \pm 1/2, m + 1) \notin D^\circ \iff e_{\mathrm{h}}(n \pm 1, m) \in \mu_i(D^\circ).$$

Note that $\mu_i(D^\circ)$ satisfies the condition

$$D(f) \neq \partial^+ f \text{ for any } f \in \mathcal{F}_i. \tag{4-5}$$

Example 4.3.4. Here are some examples of dimer shuffling at squares.



Lemma 4.3.5. $$w_{\sigma,\lambda,\mu_i(\theta)}(\mu_i(D^\circ)) = w_{\sigma,\lambda,\theta}(D^\circ).$$

Proof. We have $w_{\sigma,\lambda,\theta}(\partial^+_{\sigma,\theta} f) = w_{\sigma,\lambda,\mu_i(\theta)}(\partial^-_{\sigma,\mu_i(\theta)} f)$ for $f \in \mathcal{F}_i$, and

$$w_{\sigma,\lambda,\theta}(\partial^+_{\sigma,\theta} f) = \begin{cases} 1 & \text{if } n \in B^{i,+}_{\sigma,\lambda,\theta}, \\ w_{\sigma,\lambda,\theta}(n)^{-1} & \text{if } n \in B^{i,-}_{\sigma,\lambda,\theta}. \end{cases}$$

Thus, the claim follows from Lemma 4.3.2 and (2-9). □

4.4. *Wall-crossing formula at a quadrilateral.*

Lemma 4.4.1. $\mathcal{Z}_{\sigma,\lambda,\nu,\theta} = \sum_{D^\circ} w_{\sigma,\lambda,\theta}(D^\circ) \cdot \prod_{n\in\pi^{-1}(i)} \big(1 + w_{\sigma,\lambda,\theta}(n)\big)^{|E^1_n(D^\circ)|}$.

Proof. We set

$$E^1_i(D^\circ) := \bigcup_{n\in\pi^{-1}(i)} E^1_n(D^\circ), \quad E^2_i(D^\circ) := \bigcup_{n\in\pi^{-1}(i)} E^2_n(D^\circ).$$

Given a subset $S \subset E_i^1(D^\circ)$, we get a dimer configuration D_S° of type $(\sigma, \lambda, \nu, \theta)$ such that

$$D_S^\circ := \big(D \backslash \bigcup \partial^+ f\big) \cup \bigcup \partial^+ f,$$

and we have

$$w_{\sigma,\lambda,\theta}(D_S^\circ) = w_{\sigma,\lambda,\theta}(D^\circ) \prod_{(n,m)\in S} w_{\sigma,\lambda,\theta}(n).$$

Note that any dimer configuration D is uniquely realized as D_S° by some D° and S. Hence we have

$$\begin{aligned}
\mathfrak{L}_{\sigma,\lambda,\nu,\theta} &= \sum_{D^\circ} w_{\sigma,\lambda,\theta}(D^\circ)\Big(\sum_S \prod_{(n,m)\in S} w_{\sigma,\lambda,\theta}(n)\Big) \\
&= \sum_{D^\circ} w_{\sigma,\lambda,\theta}(D^\circ) \prod_{(n,m)\in E_i^1(D^\circ)} \big(1 + w_{\sigma,\lambda,\theta}(n)\big) \\
&= \sum_{D^\circ} w_{\sigma,\lambda,\theta}(D^\circ) \prod_{n\in\pi^{-1}(i)} \big(1 + w_{\sigma,\lambda,\theta}(n)\big)^{|E_n^1(D^\circ)|}. \qquad \square
\end{aligned}$$

Theorem 4.4.2.

$$\mathfrak{L}_{\sigma,\lambda,\nu,\mu_i(\theta)} = \mathfrak{L}_{\sigma,\lambda,\nu,\theta} \prod_{n\in B^{i,+}_{\sigma,\lambda,\theta}} \big(1 + w_{\sigma,\lambda,\theta}(n)\big)^{-1} \prod_{n\in B^{i,-}_{\sigma,\lambda,\theta}} \big(1 + w_{\sigma,\lambda,\theta}(n)\big).$$

Proof. Let $D^\bullet$ be a dimer configuration of type $(\sigma, \lambda, \nu, \mu_i(\theta))$ satisfying (4-5). We put

$$\tilde{E}_n^1(D^\bullet) := \big\{(n, m) \in \mathcal{F} \,\big|\, \partial_{\sigma,\mu_i(\theta)} f(n, m) \cap D^\bullet = \partial^-_{\sigma,\mu_i(\theta)} f(n, m)\big\}.$$

Then, as Lemma 4.4.1, we get

$$\mathfrak{L}_{\sigma,\lambda,\nu,\mu_i(\theta)} = \sum_{D^\bullet} w_{\sigma,\lambda,\mu_i(\theta)}(D^\bullet) \prod_{n\in\pi^{-1}(i)} \big(1 + w_{\sigma,\lambda,\mu_i(\theta)}(n)^{-1}\big)^{|\tilde{E}_n^1(D^\bullet)|},$$

where the sum is taken over all dimer configurations $D^\bullet$ of type $(\sigma, \lambda, \nu, \mu_i(\theta))$ satisfying the condition (4-5). Note that μ_i gives a one-to-one correspondence of dimer configurations of type $(\sigma, \lambda, \nu, \theta)$ satisfying the condition (4-1) and ones of type $(\sigma, \lambda, \nu, \mu_i(\theta))$ satisfying the condition (4-5). Hence the claim follows from the equalities $\tilde{E}_n^1(\mu_i(D^\circ)) = E_n^2(D^\circ)$ and $w_{\sigma,\lambda,\mu_i(\theta)}(n) = w_{\sigma,\lambda,\theta}(n)^{-1}$, both valid for $n \in \pi^{-1}(i)$, together with Lemma 4.4.1. $\square$

4.5. ***Conclusion.*** For σ and $\alpha \in \Lambda^{\mathrm{re},+}$, we put

$$\sigma(\alpha) := \sigma(j^-(\alpha)) \cdot \sigma(j^+(\alpha)). \tag{4-6}$$

Combining Theorem 4.2.2 and 4.4.2, we get:

Theorem 4.5.1. *$\mathcal{Z}_{\sigma,\lambda,\nu,\theta}$ has the value*

$$\mathcal{Z}^{\mathrm{NCDT}}_{\sigma,\lambda,\nu}\prod_{\alpha\in\Lambda_\theta^{\mathrm{re},+}}\Bigg(\prod_{(h,h')\in B^{\alpha,+}_{\sigma,\lambda}}\Big(1-\sigma(\alpha)\frac{w_\lambda(h')}{w_\lambda(h)}\Big)^{\sigma(\alpha)}\prod_{(h,h')\in B^{\alpha,-}_{\sigma,\lambda}}\Big(1-\sigma(\alpha)\frac{w_\lambda(h')}{w_\lambda(h)}\Big)^{-\sigma(\alpha)}\Bigg).$$

(See (1-2) and (1-3) for notation.)

Since the second term in this expression does not depend on ν, we have:

Corollary 4.5.2.
$$\frac{\mathcal{Z}_{\sigma,\lambda,\nu,\theta}}{\mathcal{Z}_{\sigma,\lambda,\vec{\varnothing},\theta}}=\frac{\mathcal{Z}^{\mathrm{NCDT}}_{\sigma,\lambda,\nu}}{\mathcal{Z}^{\mathrm{NCDT}}_{\sigma,\lambda,\vec{\varnothing}}}.$$

Lemma 1.3.3 and Theorem 4.5.1 yield:

Theorem 4.5.3. (See (1-2) for notation.)

$$\begin{aligned}&\mathcal{Z}_{\sigma,\lambda,\nu,\theta}\Big|_{q_+=q_-=(q_0)^{1/2}}\\&\qquad=\mathcal{Z}^{\mathrm{NCDT}}_{\sigma,\lambda,\nu}\Big|_{q_+=q_-=(q_0)^{1/2}}\prod_{\alpha\in\Lambda_\theta^{\mathrm{re},+}}\big(1-\sigma(\alpha)\cdot q^{\alpha}\big)^{\sigma(\alpha)[\alpha^0+c_\lambda(j_-(\alpha))-c_\lambda(j_+(\alpha))]}.\end{aligned}$$

Since the second term on the right depends only on the $c_\lambda[j]$ and not on λ and ν, we have:

Corollary 4.5.4. *If $c_\lambda[j]=0$ for any j, we have*

$$\frac{\mathcal{Z}_{\sigma,\lambda,\nu,\theta}}{\mathcal{Z}_{\sigma,\vec{\varnothing},\vec{\varnothing},\theta}\Big|_{q_+=q_-}}=\frac{\mathcal{Z}^{\mathrm{NCDT}}_{\sigma,\lambda,\nu}}{\mathcal{Z}^{\mathrm{NCDT}}_{\sigma,\vec{\varnothing},\vec{\varnothing}}\Big|_{q_+=q_-}}.$$

5. Refined topological vertex via dimer model

5.1. *Refined topological vertex for $\mathbb{C}^3$.*

5.1.1. A Young diagram can be regarded as a subset of $(\mathbb{Z}_{\geq 0})^2$. For a Young diagram λ, let

$$\begin{aligned}\Lambda^{\mathrm{x}}(\lambda)&=\big\{(x,y,z)\in(\mathbb{Z}_{\geq 0})^3\mid(y,z)\in\lambda\big\},\\ \Lambda^{\mathrm{y}}(\lambda)&=\big\{(x,y,z)\in(\mathbb{Z}_{\geq 0})^3\mid(z,x)\in\lambda\big\},\\ \Lambda^{\mathrm{z}}(\lambda)&=\big\{(x,y,z)\in(\mathbb{Z}_{\geq 0})^3\mid(x,y)\in\lambda\big\}.\end{aligned}$$

5.1.2. Given a triple $(\lambda_x,\lambda_y,\lambda_z)$ of Young diagrams, define

$$\Lambda^{\min}:=\Lambda^{\mathrm{x}}(\lambda_x)\cup\Lambda^{\mathrm{y}}(\lambda_y)\cup\Lambda^{\mathrm{z}}(\lambda_z)\subset(\mathbb{Z}_{\geq 0})^3.$$

5.1.3. A subset Λ of $(\mathbb{Z}_{\geq 0})^3$ is said to be a 3-*dimensional Young diagram of type* $(\lambda_x, \lambda_y, \lambda_z)$ if the following conditions are satisfied:

- If $(x, y, z) \notin \Lambda$, then $(x+1, y, z),\ (x, y+1, z),\ (x, y, z+1) \notin \Lambda$.
- $\Lambda \supset \Lambda^{\min}$.
- $|\Lambda \backslash \Lambda^{\min}| < \infty$.

5.1.4. For a Young diagram λ, we define a monomial $w_\lambda(m)$ for each $m \in \mathbb{Z}$ by

$$w_\lambda(m) = q_{\lambda(m-1/2)} \cdot q_{\lambda(m+1/2)} \cdot q_1 \cdot \cdots \cdot q_{L-1}. \tag{5-1}$$

For a finite subset S of $(\mathbb{Z}_{\geq 0})^3$ we define the weight $w(S)$ by

$$w(S) := \prod_{(x,y,z) \in S} w_{\lambda_x}(y-z).$$

For a positive integer N, we set $C_N := [0, N]^3$. Given a 3-dimensional Young diagram Λ of type $(\lambda_x, \lambda_y, \lambda_z)$, we take a sufficiently large N such that $\Lambda \backslash \Lambda^{\min} \subset C_N$ and define the weight $w(\Lambda)$ of Λ by

$$w(\Lambda) := \frac{w(\Lambda \cap C_N)}{w(\Lambda^{\mathrm{x}}(\lambda_x) \cap C_N) w(\Lambda^{\mathrm{y}}(\lambda_y) \cap C_N)\, w(\Lambda^{\mathrm{z}}(\lambda_z) \cap C_N)}.$$

Note that this is well-defined.

Remarks. • In the definition of $w(\Lambda)$, the three axes do not play the same role. The x-axis is called the *preferred axis* for the refined topological vertex.

- If we replace the definition (5-1) with

  $$(q_{\lambda(m-1/2)})^2 \cdot q_1 \cdot \cdots \cdot q_{L-1},$$

  then the weight coincides with the one in [Iqbal et al. 2009]. Our weight coincides with the one in [Dimofte and Gukov 2010].

  We define the generating function

  $$G_{\lambda_x,\lambda_y,\lambda_z}(\vec{q}\,) := \sum w(\Lambda),$$

where the sum is taken over all 3-dimensional Young diagrams of type $(\lambda_x, \lambda_y, \lambda_z)$.

5.2. *Dimer model for $L = 1$.* In the case $L = 1$, the graph in §2.1.2 gives a hexagon lattice. As we have only two choices of σ, we put $\sigma(1/2) = +$. We take id as θ. We omit σ and id from the notation in this subsection. Note that λ is a single 2-dimensional Young diagram.

It is well-known that giving a dimer configuration of type (λ, ν) is equivalent to giving a 3-dimensional Young diagram of type $(\lambda, \nu_+, {}^{\mathrm{t}}\nu_-)$. Let $D(\Lambda)$ be the dimer configuration corresponding to a 3-dimensional Young diagram Λ.

For a Young diagram $\eta = (\eta_{(1)}, \eta_{(2)}, \dots)$ and a monomial p, we put

$$w(\eta; p, Q) := \prod \big(pQ^{i-1}\big)^{\eta_{(i)}}.$$

Then we can verify the following:

$$w_\lambda(D(\Lambda)) = w(\nu_-; q_+, Q_+)\, w(\nu_+; q_-, Q_-)\, w(\Lambda). \tag{5-2}$$

Example 5.2.1. As we show in Figure 10, we have

$$\begin{aligned} w_\varnothing(\Lambda^{\min}_{\varnothing,(1),\varnothing}) &= w((1); q_-, Q_-) &&= q_-,\\ w_\varnothing(\Lambda^{\min}_{\varnothing,(2),\varnothing}) &= w((2); q_-, Q_-) &&= q_-^2,\\ w_\varnothing(\Lambda^{\min}_{\varnothing,(1,2),\varnothing}) &= w((2,1); q_-, Q_-) &&= q_-^3 Q_-. \end{aligned}$$

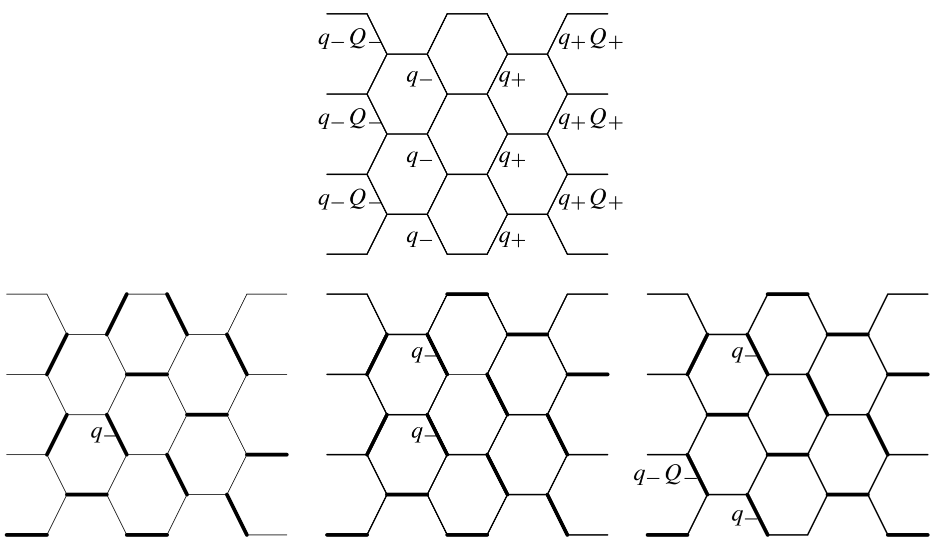


Figure 10. $D(\Lambda^{\min}_{\varnothing,(1),\varnothing})$, $D(\Lambda^{\min}_{\varnothing,(2),\varnothing})$ and $D(\Lambda^{\min}_{\varnothing,(1,2),\varnothing})$.

In particular, we have

$$\mathscr{Z}_{\lambda,\nu} = w(\nu_-; q_+, Q_+) \cdot w(\nu_+; q_-, Q_-) \cdot G_{\lambda,\nu_+,{}^{\mathrm{t}}\nu_-},$$

where $\mathscr{Z}_{\lambda,\nu}$ is the generating function given in Definition 3.4.1.

5.3. ***Refined topological vertex for a small resolution.*** We will define generating functions $\mathscr{Z}^{\mathrm{RTV}}_{\sigma,\lambda,\nu}(\vec{q}\,)$. First, we consider the following data: let $\vec{\nu} = (\nu^{(1)}, \dots, \nu^{(L-1)})$ be an $(L-1)$-tuple of Young diagrams and $\vec{\Lambda} = (\Lambda^{(1/2)}, \dots, \Lambda^{(L-1/2)})$ be an L-tuple of 3-dimensional Young diagrams such that $\Lambda^{(j)}$ is

- of type $(\lambda^{(j)}, \nu^{(j+1/2)}, {}^{\mathrm{t}}\nu^{(j-1/2)})$ if $\sigma(j) = +$,
- of type $(\lambda^{(j)}, {}^{\mathrm{t}}\nu^{(j-1/2)}, \nu^{(j+1/2)})$ if $\sigma(j) = -$,

where we put $\nu^{(0)} := \nu_-$ and $\nu^{(L)} := \nu_+$. We say that the data $(\vec{\Lambda}, \vec{\nu})$ is of type (σ, λ, ν). We define the weight $w(\vec{\Lambda}, \vec{\nu})$ of the data $(\vec{\Lambda}, \vec{\nu})$ by

$$w_\sigma(\vec{\Lambda}, \vec{\nu}) := w(\nu_+; q_-, Q_-) \cdot w(\nu_-; q_+, Q_+) \Bigg(\prod_{j=1/2}^{L-1/2} w(\Lambda^{(j)}) \Bigg) \Bigg(\prod_{i=1}^{L-1} w^i_\sigma(\mu^{(i)}) \Bigg),$$

where $w^i_\sigma(\mu^{(i)})$ is given by

$$\text{(5-3)} \quad w^i_\sigma(\mu^{(i)}) := \prod_{(\alpha,\beta)\in\mu^i} \begin{cases} q_i \cdot Q^{2\alpha+1} & \text{if } \sigma(i-\frac{1}{2}) = \sigma(i+\frac{1}{2}) = +, \\ q_i \cdot Q^{2\beta+1} & \text{if } \sigma(i-\frac{1}{2}) = \sigma(i+\frac{1}{2}) = -, \\ q_i \cdot Q \cdot Q_+^\alpha \cdot Q_-^\beta & \text{if } \sigma(i-\frac{1}{2}) = +,\ \ \sigma(i+\frac{1}{2}) = -, \\ q_i \cdot Q \cdot Q_-^\alpha \cdot Q_+^\beta & \text{if } \sigma(i-\frac{1}{2}) = -,\ \ \sigma(i+\frac{1}{2}) = +. \end{cases}$$

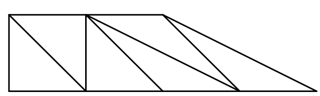
We consider the generating function

$$\mathcal{Z}^{\mathrm{RTV}}_{\sigma,\lambda,\mu}(\vec{q}\,) := \sum w_\sigma(\vec{\Lambda}, \vec{\nu})$$

where the sum is taken over all the data as above.

Remark. This is the generating function of the refined topological vertex associated to Y_σ, where $Y_\sigma \to X$ is the crepant resolution constructed from σ (see [Nagao 2011a, §1.1] for the construction of Y_σ). Here is the polygon corresponding to Y_σ, for σ given by

$$(\sigma(1/2), \ldots, \sigma(11/2)) = (+, -, +, +, -, +):$$



5.4. *Limit behavior of the dimer model.*

5.4.1. Let $\boldsymbol{i} \in I^{\mathbb{Z}_{>0}}$ be a minimal expression such that for any $N \in \mathbb{Z}_{\geq 0}$ we have $b(N) \in \mathbb{Z}_{>0}$ such that $\alpha_{\boldsymbol{i},b} > N\delta$ for any $b > b(N)$.

Lemma 5.4.2. *Given σ, λ and a monomial $\boldsymbol{q}$, there exists an integer B_1 such that the following condition holds: for any $b \geq B_1$,*

- *any dimer configuration of type $(\sigma, \lambda, \nu, \theta_{\boldsymbol{i},b})$ with weight $\boldsymbol{q}$ satisfies* (4-1),
- *any dimer configuration of type $(\sigma, \lambda, \nu, \theta_{\boldsymbol{i},b+1})$ with weight $\boldsymbol{q}$ satisfies* (4-2), *and*
- *μ_{i_b} gives a one-to-one correspondence between dimer configurations of type $(\sigma, \lambda, \nu, \theta_{\boldsymbol{i},b})$ with weight $(\sigma, \lambda, \nu, \theta_{\boldsymbol{i},b+1})$ with weight $\boldsymbol{q}$.*

Proof. Take N_2 such that

$$q^{N_2\delta} > \boldsymbol{q} \cdot w_{\sigma,\lambda,\theta}\big(D^{\max}_{\sigma,\lambda,\nu.\mathrm{id}}\big)^{-1}.$$

By Theorem 4.5.3 and the remark just before Section 4,

$$\mathcal{Z}_{\sigma,\lambda,\nu,\theta} \cdot w_{\sigma,\lambda,\theta}\big(D^{\max}_{\sigma,\lambda,\nu.\mathrm{id}}\big)^{-1}\big|_{q_+=q_-=(q_0)^{1/2}}$$

is a polynomial in $q_0, \dots, q_{L-1}$. Thus, there does not exist any dimer configuration with weight $\boldsymbol{q} - \alpha(\boldsymbol{i}, b)$ for any $b > b(N_2) =: B_1$, where $b(N_2)$ is taken as in §5.4.1.

Assume that we have a dimer configuration type $(\sigma, \lambda, \nu, \theta_{\boldsymbol{i},b})$ with weight $\boldsymbol{q}$ and $f \in \mathcal{F}$ such that $D(f) = \partial^-(f)$. Then we get a dimer configuration $D \cup \partial^+(f)\backslash\partial^-(f)$ with weight $\boldsymbol{q} - \alpha(\theta, i)$, which is a contradiction. We can check the second claim similarly and the third claim immediately follows from the first and second ones. □

5.4.3. Given σ, λ, we can take an integer N_2 such that

- $\tilde{\sigma}(h) = \pm\tilde{\lambda}(h)$ for any $h \in \mathbb{Z}_{\mathrm{h}}$ such that $\pm h > N_2 L$,
- $e^{\mathrm{s}}(h, k) \notin D^{\max}_{\sigma,\lambda,\theta_{\boldsymbol{i},B_1}}$ for any h and k such that $h < N_2 L$ and $h \cdot \tilde{\sigma}(h) - k$ is even, and
- $e^{\mathrm{s}}(h, k) \notin D^{\max}_{\sigma,\lambda,\theta_{\boldsymbol{i},B_1}}$ for any h and k such that $h > N_2 L$ and $h \cdot \tilde{\sigma}(h) - k$ is odd.

Take a monomial $\boldsymbol{q}$. Since we have only a finite number of dimer configuration of type $(\sigma, \lambda, \nu, \theta_{\boldsymbol{i},B_1})$ with weight $\boldsymbol{q}$ and each dimer configuration has only finite difference with $D^{\max}_{\sigma,\lambda,\nu,\theta_{\boldsymbol{i},B_1}}$, we can take an integer N_4 such that

- $\tilde{\sigma}(h) = \pm\tilde{\lambda}(h)$ for any $h \in \mathbb{Z}_{\mathrm{h}}$ such that $\pm h > LN_4$,
- $e^{\mathrm{s}}(h, k) \notin D$ for any h and k such that $h < LN_4$ and $h \cdot \tilde{\sigma}(h) - k$ is even, and
- $e^{\mathrm{s}}(h, k) \notin D$ for any h and k such that $h > LN_4$ and $h \cdot \tilde{\sigma}(h) - k$ is odd.

Lemma 5.4.4. *Let D be a dimer configuration of type $(\sigma, \lambda, \nu, \theta)$ satisfying the condition* (4-1). *Take $h \in \pi^{-1}(i + 1/2)$ such that $\tilde{\sigma}(h) = \tilde{\lambda}(h)$ and assume that $e_{\mathrm{s}}(h, k) \notin D$ for any $k \in \mathbb{Z}_{\mathrm{h}}$ such that $h\tilde{\sigma}(h) - k$ is odd. Then $e_{\mathrm{s}}(h - 1, k - \tilde{\sigma}(h))$ is not in $\mu_i(D)$.*

Similarly, take $h \in \pi^{-1}(i + 1/2)$ such that $\tilde{\sigma}(h) = -\tilde{\lambda}(h)$ and assume that $e_{\mathrm{s}}(h, k) \notin D$ for any $k \in \mathbb{Z}_{\mathrm{h}}$ such that $h\tilde{\sigma}(h) - k$ is even. Then $e_{\mathrm{s}}(h + 1, k + \tilde{\sigma}(h))$ is not in $\mu_i(D)$.

Proof. In the case $i \in I_S$, for any $h, k \in \mathbb{Z}_{\mathrm{h}}$ such that $\tilde{\sigma}(h) = \tilde{\lambda}(h)$ and $h\tilde{\sigma}(h) - k$ is odd, we can verify

$$e_{\mathrm{s}}(h, k) \notin D \Longrightarrow e_{\mathrm{s}}(h - 1, k - \tilde{\sigma}(h)) \notin \mu_i(D)$$

from the definition of $\mu_i(D)$ in §4.3.3.

In the case $i \in I_S$, assume we have $k \in \mathbb{Z}_{\mathrm{h}}$ such that $h\tilde{\sigma}(h) - k$ is odd and $e_{\mathrm{s}}(h-1, k-\tilde{\sigma}(h)) \in \mu_i(D)$. From Definition 4.1.5, we have $e_{\mathrm{s}}(h-1, k-\tilde{\sigma}(h)) \in D$. Since $e_{\mathrm{s}}(h, k-2\tilde{\sigma}(h)) \notin D$, we have $e_{\mathrm{s}}(h, k-\tilde{\sigma}(h)) \in D$. Then, since $\tilde{\sigma}(h) = \tilde{\lambda}(h)$, there exists m such that $\sigma(i)(m-k) > 0$ and $\partial f(h-1/2, m) \cap D = \partial^- f(h-1/2, m)$, which is a contradiction. □

5.4.5. Given σ, λ and a monomial $\boldsymbol{q}$, take B_1 and N_4 as in Lemma 5.4.2 and §5.4.3. By the definition of N_4 and Lemma 5.4.4, we have the following lemma:

Lemma 5.4.6. *For any $b \geq B_1$ and any dimer configuration of type $(\sigma, \lambda, \nu, \theta_{\boldsymbol{i},b})$ with weight $\boldsymbol{q}$, we have*

- *$e_{\mathrm{s}}(h, k) \notin D$ for any h and k such that $h < \theta_{\boldsymbol{i},b}^{-1}(\pi(h)) - 2LN_4$ and $h \cdot \tilde{\sigma}(h) - k$ is even, and*
- *$e_{\mathrm{s}}(h, k) \notin D$ for any h and k such that $h < \theta_{\boldsymbol{i},b}^{-1}(\pi(h)) + 2LN_4$ and $h \cdot \tilde{\sigma}(h) - k$ is odd.*

5.4.7. We assume that

$$\theta_{\boldsymbol{i},b}^{-1}(1/2) < \theta_{\boldsymbol{i},b}^{-1}(3/2) < \cdots < \theta_{\boldsymbol{i},b}^{-1}(L - 1/2)$$

for any $b > 0$.

Given σ, λ and a monomial $\boldsymbol{q}$, take B_5 such that $B_5 > b(2N_4)$ and $B_5 > B_1$. The following theorem is the main result of this section:

Theorem 5.4.8. *For any $b > B_5$, we have a bijection between*

- *the set of dimer configurations of type $(\sigma, \lambda, \nu, \theta_{\boldsymbol{i},b})$ with weights $\boldsymbol{q}$, and*
- *the set of data $(\vec{\Lambda}, \vec{\nu})$ as in Section 5.3 of type (σ, λ, ν) with weights $\boldsymbol{q}$.*

Proof. First, we divide the (x, y)-plane into the following $2L + 1$ areas:

$$C_j := \{\theta^{-1}(j) - 2LN_4 < x < \theta^{-1}(j) + 2LN_4\} \quad (j \in I_{\mathrm{h}}),$$
$$C_0 := \{x < \theta^{-1}(1/2) - 2LN_4\},$$
$$C_i := \{\theta^{-1}(i - 1/2) + 2LN_4 < x < \theta^{-1}(i + 1/2) - 2LN_4\} \quad (1 \leq i \leq L - 1),$$
$$C_L := \{\theta^{-1}(L - 1/2) + 2LN_4 < x\}.$$

By Lemma 5.4.6, in the area C_j we have

- $e^{\mathrm{s}}[h, k] \notin D$ for any h and k such that $\pi(h) > j$ and $h \cdot \tilde{\sigma}(h) - k$ is even;
- $e^{\mathrm{s}}[h, k] \notin D$ for any h and k such that $\pi(h) < j$ and $h \cdot \tilde{\sigma}(h) - k$ is odd.

Removing these edges, we get a new graph. A face of the new graph is a union of L-tuple of elements in $\mathcal{F}$. If we regard such a union as a hexagon, the dimer configuration D gives a dimer configuration for the hexagon lattice — in other words, a three-dimensional diagram. Let $\Lambda^{(j)}$ denote this three-dimensional diagram. (See Example 5.4.9.)

Similarly, in the area C_j we have

- $e^{\mathrm{s}}[h,k] \notin D$ for any h and k such that $\pi(h) > i$ and $h \cdot \tilde{\sigma}(h) - k$ is even;
- $e^{\mathrm{s}}[h,k] \notin D$ for any h and k such that $\pi(h) < i$ and $h \cdot \tilde{\sigma}(h) - k$ is odd.

Removing these edges, we get a new graph, which is an infinite disjoint union of zigzag paths. For each zigzag path, we have two choices of perfect matching and so the dimer configuration D gives a Young diagram $\nu^{(i)}$. We can verify that the datum $(\vec{\Lambda}, \vec{\nu})$ satisfies the conditions in Section 5.3. Note that the reverse construction also works.

We have to check the correspondence above respects the weights. Note that all edges of in the area C_i have weights $= 1$. By (5-2), the contribution of the part in the area C_j is given by

$$w\big(\nu^{(j-1/2)}; q_+^{(s_i(j))}, Q_+\big)\, w\big(\nu^{(j+1/2)}; (q_+^{(s_i(j))})^{-1} Q, Q_-\big)\, w(\Lambda^{(j)}) \text{if } \sigma(j) = +,$$
$$w\big({}^{\mathrm{t}}\nu^{(j-1/2)}; q_+^{(s_i(j))}, Q_+\big)\, w\big({}^{\mathrm{t}}\nu^{(j+1/2)}; (q_+^{(s_i(j))})^{-1} Q, Q_-\big)\, w(\Lambda^{(j)}) \text{if } \sigma(j) = -.$$

Combining these contributions, we get the claim. □

Example 5.4.9. We take σ as in Example 2.1.3 and $\lambda = \varnothing$. Assume that $\theta(1/2) = N + 1/2$ and $\theta(5/2) = -N + 5/2$ for $N \gg 0$. In Figure 11, we show the weight (after putting $q_+ = q_- = q_0^{1/2}$) of edges in the area $C_{1/2}$. We can idenfity the graph in the area $C_{1/2}$ with a hexagon lattice as shown in Figure 12.

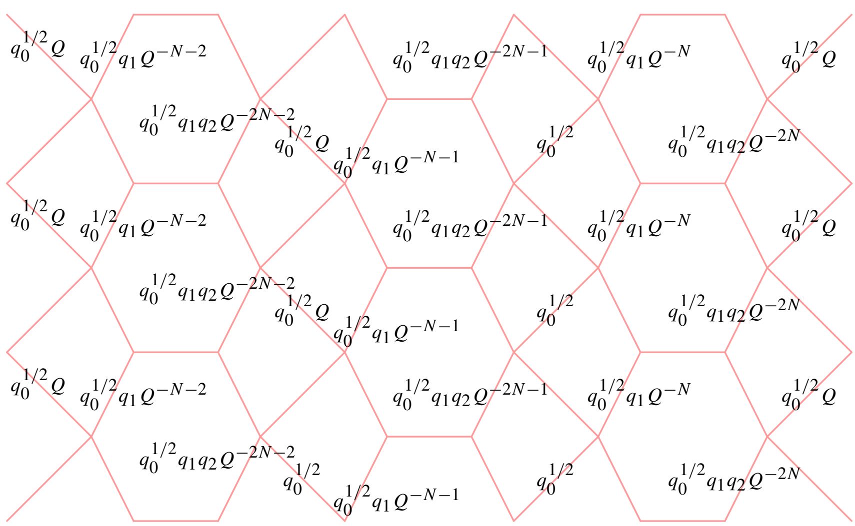


Figure 11. The graph in the area $C_{1/2}$.

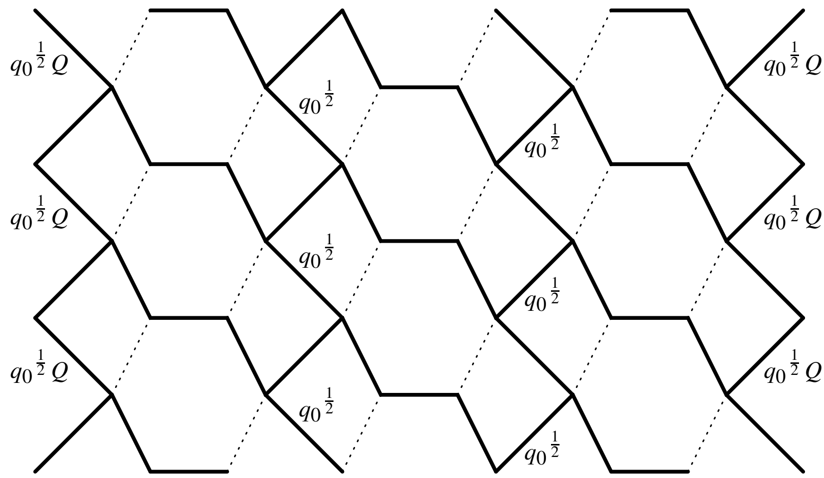


Figure 12. Identification with a hexagon lattice.

Remark. In general, we have the permutation $s_i \in \mathfrak{S}_{I_\mathrm{h}}$ of the set I_h satisfying the following condition: for sufficiently large b we have

$$\theta_{i,b}^{-1}(s_i(1/2)) < \theta_{i,b}^{-1}(s_i(3/2)) < \cdots < \theta_{i,b}^{-1}(s_i(L-1/2)).$$

The permutation s_i determines the dirENCTION in which we take limit in the space of stability conditions. It is the refine topological vertex associated to $Y_{\sigma\circ s_i}$ what we get in the limit.

5.5. *Conclusion.* Note that

$$\bigcup_{b=1}^{\infty} \Lambda_{\theta_{i,b}}^{\mathrm{re},+} = \Lambda_{+}^{\mathrm{re},+}.$$

Combining Theorem 4.5.1 and Theorem 5.4.8, we have:

Theorem 5.5.1. $\mathcal{Z}_{\sigma,\lambda,\nu}^{\mathrm{RTV}}$ *has the value*

$$\mathcal{Z}_{\sigma,\lambda,\nu}^{\mathrm{NCDT}} \prod_{\alpha\in\Lambda_{+}^{\mathrm{re},+}} \Biggl(\prod_{(h,h')\in B_{\sigma,\lambda}^{\alpha,+}} \Bigl(1-\sigma(\alpha)\frac{w_\lambda(h')}{w_\lambda(h)}\Bigr)^{\sigma(\alpha)} \prod_{(h,h')\in B_{\sigma,\lambda}^{\alpha,-}} \Bigl(1-\sigma(\alpha)\frac{w_\lambda(h')}{w_\lambda(h)}\Bigr)^{-\sigma(\alpha)} \Biggr).$$

(See (1-1), (1-3) and (4-6) for notation.)

Since the second term in this expression does not depend on ν, we have:

Corollary 5.5.2.
$$\frac{\mathcal{Z}_{\sigma,\lambda,\nu}^{\mathrm{RTV}}}{\mathcal{Z}_{\sigma,\lambda,\vec{\varnothing}}^{\mathrm{RTV}}} = \frac{\mathcal{Z}_{\sigma,\lambda,\nu}^{\mathrm{NCDT}}}{\mathcal{Z}_{\sigma,\lambda,\vec{\varnothing}}^{\mathrm{NCDT}}}.$$

Combining Theorem 4.5.3 and Theorem 5.4.8, we have:

Theorem 5.5.3.

$$\mathcal{Z}^{\mathrm{RTV}}_{\sigma,\lambda,\nu}\Big|_{q_+=q_-=(q_0)^{1/2}} = \mathcal{Z}^{\mathrm{NCDT}}_{\sigma,\lambda,\nu}\Big|_{q_+=q_-=(q_0)^{1/2}} \prod_{\alpha\in\Lambda_+^{\mathrm{re},+}} \big(1-\sigma(\alpha)\cdot q^{\alpha}\big)^{\sigma(\alpha)\cdot[\alpha^0+c_\lambda(j_-(\alpha))-c_\lambda(j_+(\alpha))]}.$$

(See (1-1), (1-3) and (4-6) for notation.)

Since the second term in the right-hand side depend only on $c_\lambda[j]$'s but not on λ and ν, we have the following:

Corollary 5.5.4. *If $c_\lambda[j]=0$ for any j, we have*

$$\frac{\mathcal{Z}^{\mathrm{RTV}}_{\sigma,\lambda,\nu}}{\mathcal{Z}^{\mathrm{RTV}}_{\sigma,\vec{\varnothing},\vec{\varnothing}}\Big|_{q_+=q_-}} = \frac{\mathcal{Z}^{\mathrm{NCDT}}_{\sigma,\lambda,\nu}}{\mathcal{Z}^{\mathrm{NCDT}}_{\sigma,\vec{\varnothing},\vec{\varnothing}}\Big|_{q_+=q_-}}.$$

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KENTARO NAGAO
GRADUATE SCHOOL OF MATHEMATICS
NAGOYA UNIVERSITY
FUROCHO, CHIKUSAKU
NAGOYA, 464-8602
JAPAN

kentaron@math.nagoya-u.ac.jp

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