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EXTENSION OF AN ANALYTIC DISC AND DOMAINS IN $\mathbb{C}^2$ WITH NONCOMPACT AUTOMORPHISM GROUP

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Let Ω be a smoothly bounded domain in $\mathbb{C}^2$ such that the Bergman representative map near the boundary continues to be diffeomorphic up to the boundary. If such a domain admits a holomorphic automorphism group orbit accumulating at a boundary point of finite D'Angelo type $2m$, we show that the domain Ω is biholomorphic to the Thullen domain

$$\{(z, w) \in \mathbb{C}^2 : |z|^{2m} + |w|^2 < 1\}.$$

This result refines the well-known theorem of E. Bedford and S. Pinchuk.

1. Introduction

Denote by $\mathrm{Aut}(\Omega)$ the set of biholomorphic self-maps of a domain (that is, an open connected set) Ω in the n-dimensional complex Euclidean space $\mathbb{C}^n$. By [Cartan 1932], $\mathrm{Aut}(\Omega)$ is a (real) Lie group with respect to the law of composition and the topology of uniform convergence on compact subsets. One of the traditional important questions is:

Which bounded domains admit a noncompact automorphism group?

There are several well-known results concerning this question; see, for example, [Wong 1977; Bedford and Pinchuk 1988; Kim 1992]. This paper also pertains to this line of research. Recall the following theorem:

Theorem 1.1 [Bedford and Pinchuk 1988]. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^2$ with a real analytic boundary. If Ω has a noncompact automorphism group, then Ω is biholomorphic to the Thullen domain*

$$E_{2m} := \{(z, w) \in \mathbb{C}^2 : |z|^{2m} + |w|^2 < 1\}$$

for some positive integer m.

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The main thrust of this article is to try to *localize* this theorem. Theorem 1.1 and its generalizations and refinements (in [Bedford and Pinchuk 1991], for example) rely upon global assumptions (partly local but not local) that the boundary is globally real analytic (or, at least, of finite D'Angelo type). Such assumptions were needed in order use the orbit accumulation point not of the original noncompact automorphism orbit, but of a 1-parameter subgroup produced by the initial scaling method; the finite D'Angelo type of that orbit accumulation boundary point is that exponent $2m$ in Bedford and Pinchuk's theorem. Keeping this in mind, we state our main result here:

Theorem 1.2. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^2$ with smooth (C^∞) boundary satisfying Condition BR* (see Definition 3.3)*. Suppose there is a point $p_0 \in \partial\Omega$ of finite D'Angelo type $2m$, a point $q \in \Omega$, and a sequence $\{\varphi_j\} \subset \mathrm{Aut}(\Omega)$ such that*

$$\lim_{j\to\infty} \varphi_j(q) = p_0 \in \partial\Omega.$$

Then

$$\Omega \cong E_{2m} := \{(z, w) \in \mathbb{C}^2 : |z|^{2m} + |w|^2 < 1\}.$$

The key step of the proof is showing the smooth extension of a certain holomorphic disc in the given domain. Since Fefferman's celebrated work [1974], analysis on the Bergman kernel function has been regarded as one of the most powerful tools in understanding the smooth extension of holomorphic mappings. In the equidimensional case, Bell and Ligocka [1980] introduced the so-called Conditions A and B on the Bergman kernel function, which guarantee the smooth extension of biholomorphic mappings. In contrast with the equidimensional case, Conditions A and B seem insufficient to prove the smooth extension of holomorphic discs in a bounded domain in $\mathbb{C}^2$. This is the reason why we define a new criterion for the smooth extension, which we call Condition BR.

According to [Ligocka 1980], Condition B holds if the Bergman representative maps, introduced by S. Bergman, form holomorphic coordinates near the boundary. Inspired by Ligocka's observation, we say that a domain with smooth boundary satisfies Condition BR if for every boundary point p, there is an interior point q at which the Bergman representative map gives rise to a smooth coordinate system in a relative open neighborhood of q that includes the boundary point p (see also Definition 3.3).

Outline of paper. In Section 2 we briefly explain Berteloot's argument on the Pinchuk scaling method without proof. The smooth extension of holomorphic disc under Condition BR is proved in Section 3 (see Proposition 3.7). The main theorem is proved in the last section.

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2. Berteloot's two-dimensional analysis on Pinchuk's scaling

Scaling. Let Ω be a domain in $\mathbb{C}^2$ and let p_0 belong to $\partial\Omega$. Assume that $\partial\Omega$ is of class C^∞, pseudoconvex and of finite type in a neighborhood of p_0. Let $2m$ be the type of $\partial\Omega$ at p_0 in the sense of [D'Angelo 1982]. We may assume that $p_0 = (0, 0)$ and that $\operatorname{Re}(\partial/\partial w)$ is the outward normal vector to $\partial\Omega$ at p_0.

Let $\{q_j\}$ be a sequence of points in Ω which converges to $(0, 0)$. For every j large enough, there exists a unique boundary point $p_j \in \partial\Omega$ which satisfies

$$q_j + (0, \varepsilon_j) = p_j, \quad \text{for some } \varepsilon_j > 0.$$

According to [Catlin 1989], if we let $2m$ be the D'Angelo type of $\partial\Omega$ at the origin, there exists a homogeneous subharmonic polynomial $H(z, \bar{z})$ of degree $2m$ with no harmonic terms such that, for a certain open neighborhood $\mathcal{U}$ of $(0, 0)$,

$$(z, w) \in \Omega \cap \mathcal{U} \iff \operatorname{Re} w + H(z, \bar{z}) + R(z, \operatorname{Im} w) < 0,$$

with $R(z, \operatorname{Im} w) := o(|z|^{2m} + \operatorname{Im} w)$.

Consider the sequence of maps $A_j : \mathbb{C}^2 \to \mathbb{C}^2$ defined by

$$A_j(z, w) = (z - a_j, w - b_j + c_j(z - a_j)),$$

where $p_j = (a_j, b_j)$ and $c_j \in \mathbb{C}$ is chosen so that the complex tangent line of $\partial A_j(\Omega)$ at $(0, 0)$ is $\{(z, w) \in \mathbb{C}^2 : w = (0, 0)\}$. Then we have $A_j(p_j) = (0, 0)$, $A_j(q_j) = (0, -\varepsilon_j)$, and

$$(z, w) \in A_j(\Omega \cap \mathcal{U}) \iff \operatorname{Re} w + \sum_{k=2}^{2m} P_{k,j}(z, \bar{z}) + R_j(z, \bar{z}, \operatorname{Im} w) < 0,$$

where the $P_{k,j}(z, \bar{z})$ are homogeneous polynomials of degree k with no harmonic terms, and

$$R_j(z, \bar{z}, \operatorname{Im} w) = o(|z|^{2m+1} + |\operatorname{Im} w|), \qquad \lim_{j \to \infty} R_j(z, \operatorname{Im} w) = R(z, \operatorname{Im} w).$$

Since the set of polynomials of degree not exceeding k is a finite dimensional vector space, we simply give an inner product. Then choose $\delta_j > 0$ so that

$$\left\| \varepsilon_j^{-1} \sum_{k=2}^{2m} P_{k,j}(\delta_j z) \right\| = 1.$$

Since $\lim_{j\to\infty} P_{k,j} = 0$ for $k < 2m$ and $P_{2m,j}$ converges to some homogeneous subharmonic polynomial of degree $2m$ with no harmonic terms, it follows that $\delta_j^{2m} \le C\varepsilon_j$ for some constant C.

Then consider the dilation map $\Lambda_j : \mathbb{C}^2 \to \mathbb{C}^2$ defined by

$$\Lambda_j(z, w) = \left(\frac{z}{\delta_j}, \frac{w}{\varepsilon_j}\right).$$

Denote by $T_j : \Omega \cap \mathcal{U} \to \mathbb{C}^2$ the transformation defined by $T_j := \Lambda_j \circ A_j \circ \varphi_j$, for each j. This T_j is called the sequence of *scaling maps*. Note that

$$(z, w) \in T_j(\Omega \cap \mathcal{U}) \iff \operatorname{Re} w + \frac{1}{\varepsilon_j} \sum_{k=2}^{2m} P_{k,j}(\delta_j z, \delta_j \bar{z}) + \frac{1}{\varepsilon_j} R_j(\delta_j z, \delta_j \bar{z}, \varepsilon_j \operatorname{Im} w) < 0.$$

Note that the sequence of polynomials $\left\{\varepsilon_j^{-1} \sum_{k=2}^{2m} P_{k,j}(\delta_j z)\right\}$ is bounded in norm. Thus it contains a convergent subsequence, converging to some polynomial $H(z, \bar{z})$ of degree at most $2m$. Since the remainder term of the defining function tends to zero as $j \to \infty$, we see that the sequence of domains $T_j(\Omega \cap \mathcal{U})$ converges to a domain $M_H := \{(z, w) \in \mathbb{C}^2 : \operatorname{Re} w + H(z, \bar{z}) < 0\}$ with $\|H\| = 1$. According to [Berteloot 1994], the scaling sequence forms a normal family of holomorphic mappings and the polynomial $H(z, \bar{z})$ turns out to be a homogeneous polynomial. Moreover:

Theorem 2.1 [Berteloot 1994]. *Let Ω be a domain in $\mathbb{C}^2$, and let p_0 belong to $\partial\Omega$. Assume that there exists a sequence $\{\varphi_j\}$ in* $\operatorname{Aut}(\Omega)$ *and a point $q \in \Omega$ such that* $\lim_{j\to\infty} \varphi_j(q) = p_0$. *If $\partial\Omega$ is a pseudoconvex and finite D'Angelo type near p_0, then Ω is biholomorphically equivalent to the model domain*

$$M_H := \{(z, w) \in \mathbb{C}^2 : \operatorname{Re} w + H(z, \bar{z}) < 0\},$$

where $H(z, \bar{z})$ is a homogeneous subharmonic polynomial that does not contain any harmonic terms.

From this point on, we denote the biholomorphism by $\Psi : \Omega \to M_H$. We have $\Psi(q) = (0, -1) \in \mathbb{C}^2$, since $T_j(q) = (0, -1)$ for every j.

Embedded totally geodesic disc. Consider the set $\{(0, w) \in \mathbb{C}^2\} \cap M_H \subset M_H$ which is just the left half plane in the complex plane $\{0\} \times \mathbb{C}$. Let $\widetilde{D}$ be the left half plane and D the open unit disc in $\mathbb{C}$. Hence we consider the biholomorphism $\mu : D \to \widetilde{D}$ defined by

$$\mu(\zeta) = \frac{\zeta + 1}{\zeta - 1},$$

and denote the injection map by $\iota : \widetilde{D} \hookrightarrow M_H$, that is, $\iota(\zeta) = (0, \zeta)$.

There are two families of automorphisms of M_H that preserve $\widetilde{D}$:

$$\tau_s : (z, w) \mapsto (z, w + is) \qquad \text{with } s \in \mathbb{R},$$
$$\eta_t : (z, w) \mapsto (t^{1/(2m)} z, tw) \quad \text{with } t > 0.$$

Since D and Ω are bounded domains, they admit Bergman metrics. We denote them by β_D and β_Ω, respectively, and for unbounded domains M_H and $\widetilde{D}$, their Bergman metrics can be defined through pull-backs. Since the mappings Ψ and μ are biholomorphisms, we define the Bergman metric on M_H by $\beta_{M_H} := (\Psi^{-1})^* \beta_\Omega$ and the Bergman metric on $\widetilde{D}$ by $\beta_{\widetilde{D}} := (\mu^{-1})^* \beta_D$. We also have:

Proposition 2.2. *The inclusion ι is an isometric embedding up to a positive constant multiple, that is, $\iota^* \beta_{M_H} = \lambda \beta_{\widetilde{D}}$ for some constant $\lambda > 0$.*

Proof. Denote by $\Gamma_{\widetilde{D}}$ the set of automorphisms of M_H that preserve $\widetilde{D}$. Then by the observation above, the action $(\gamma, x) \mapsto \gamma(x) \colon \Gamma_{\widetilde{D}} \times \widetilde{D} \to \widetilde{D}$ is transitive. Furthermore, this action is isometric with respect to the restricted Bergman metric $\beta_{M_H}|_{\widetilde{D}}$, so $\beta_{M_H}|_{\widetilde{D}}$ has constant (negative) curvature. Also, $\beta_{\widetilde{D}}$ is a positive constant multiple of the Poincaré metric. Thus the assertion follows. $\square$

3. Extension of totally geodesic disc

In this section, we discuss the extension problem up to the boundary of the isometric embedding $g := \Psi^{-1} \circ \iota \circ \mu \colon D \to \Omega$ of the unit disc D into Ω.

This g is an injective proper holomorphic mapping. Since ι is an isometric embedding, g is also an isometric embedding (up to a constant multiple). Namely, $g^* \beta_\Omega = \lambda \cdot \beta_D$, for the same constant $\lambda > 0$ as above. Set $\widehat{D} := g(D)$, the image of D by g.

The Bergman representative map. For a bounded domain Ω in $\mathbb{C}^n$, let K_Ω denote the Bergman kernel function. Following S. Bergman's original exposition, we recite the definition of his "representative domain". Since this is actually a mapping, we call it the *Bergman representative map*. The definition we use in this article is as follows:

Definition 3.1. The *Bergman representative map $b_{\Omega,p}$* is defined by

$$b_{\Omega,p}(z) = \big(b^1_{\Omega,p}(z), \ldots, b^n_{\Omega,p}(z)\big),$$

where

$$b^k_{\Omega,p}(z) = \frac{\partial}{\partial \overline{w}_k}\bigg|_{w=p} \log \frac{K_\Omega(z, w)}{K_\Omega(w, w)}.$$

This "mapping", if well-defined, maps Ω into $\mathbb{C}^n$. This map, for each $p \in \Omega$, is known to be a local biholomorphism of a neighborhood of p onto its image that is an open neighborhood of the origin in $\mathbb{C}^n$. In this regard, we shall frequently call this map the Bergman coordinate system throughout the rest of this article.

We should remark that our definition above is not the canonical Bergman representative "domain". However, the canonical Bergman representative "domain" differs from ours by a composition of an invertible complex-linear map.

The following proposition demonstrates the role of Bergman's representative map in our context.

Proposition 3.2. *Let Ω_1 and Ω_2 be bounded domains in $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively, and let $b_{\Omega_1,p}$ and $b_{\Omega_2,q}$ be the Bergman coordinate systems at p in Ω_1 and at q in Ω_2, respectively. If $f:\Omega_1\to\Omega_2$ is a Bergman isometry (not necessarily onto) with $f(p)=q$, there exists a linear map $A:\mathbb{C}^n\to\mathbb{C}^m$ such that $b_{\Omega_1,p}=A\circ b_{\Omega_2,q}\circ f$.*

Proof. Let $(z_1,\dots,z_m)$ and $(w_1,\dots,w_m)$ represent the standard complex Euclidean coordinate expressions for points in $\mathbb{C}^m$ and

$$(Z_1,\dots,Z_n)\quad\text{and}\quad(W_1,\dots,W_n)$$

for points in $\mathbb{C}^n$. We write K_1 for the Bergman kernel K_{Ω_1} and K_2 for K_{Ω_2}.

Since $f^*\beta_{\Omega_2}=\beta_{\Omega_1}$,

$$\frac{\partial^2\log K_1(z,z)}{\partial z_a\partial\bar z_b}\bigg|_{z=x}=\sum_{j,k=1}^{n}\left(\frac{\partial^2\log K_2(Z,Z)}{\partial Z_j\partial\bar Z_k}\bigg|_{Z=f(x)}\right)\cdot\frac{\partial f_j}{\partial z_a}\bigg|_x\overline{\frac{\partial f_k}{\partial z_b}\bigg|_x}.$$

For each $x,y\in\Omega_1$, set $K(x,y):=K_2(f(x),f(y))$. Then,

$$\begin{aligned}\frac{\partial^2\log K(z,z)}{\partial z_a\partial\bar z_b}\bigg|_{z=x}&=\frac{\partial^2\log K_2(f(z),f(z))}{\partial z_a\partial\bar z_b}\bigg|_{z=x}\\&=\sum_{j,k=1}^{n}\left(\frac{\partial^2\log K_2(Z,Z)}{\partial Z_j\partial\bar Z_k}\bigg|_{Z=f(x)}\right)\cdot\frac{\partial f_j}{\partial z_a}\bigg|_x\overline{\frac{\partial f_k}{\partial z_b}\bigg|_x}.\end{aligned}$$

Hence, for each $a,b=1,\dots,m$,

$$\frac{\partial^2}{\partial z_a\partial\bar z_b}\bigg|_{z=x}\{\log K_1(z,z)-\log K(z,z)\}=0,\ \text{ for every }x\in\Omega_1,$$

or equivalently,

$$\log K_1(z,w)-\log K(z,w)=\varphi(z)+\overline{\varphi(w)}$$

for some holomorphic function $\varphi:\Omega_1\to\mathbb{C}$.

Consequently we obtain

$$\frac{\partial}{\partial \overline{w}_a}\bigg|_{w=p}\left\{\log\frac{K_1(z,w)}{K_1(w,w)}-\log\frac{K(z,w)}{K(w,w)}\right\}$$

$$=\frac{\partial}{\partial \overline{w}_a}\bigg|_{w=p}\Big((\log K_1(z,w)-\log K(z,w))-(\log K_1(w,w)-\log K(w,w))\Big)$$

$$=\frac{\partial}{\partial \overline{w}_a}\bigg|_{w=p}\Big(\varphi(z)+\overline{\varphi(w)}-(\varphi(w)+\overline{\varphi(w)})\Big)=\frac{\partial}{\partial \overline{w}_a}\bigg|_{w=p}(\varphi(z)-\varphi(w))=0$$

for every $z, p \in \Omega_1$. In short,

$$\frac{\partial}{\partial \overline{w}_a}\bigg|_{w=p}\log\frac{K_1(z,w)}{K_1(w,w)}=\frac{\partial}{\partial \overline{w}_a}\bigg|_{w=p}\log\frac{K(z,w)}{K(w,w)}.$$

Altogether,

$$\begin{aligned} b^a_{\Omega_1,p}(z) &= \frac{\partial}{\partial \overline{w}_a}\bigg|_{w=p}\log\frac{K_1(z,w)}{K_1(w,w)}=\frac{\partial}{\partial \overline{w}_a}\bigg|_{w=p}\log\frac{K(z,w)}{K(w,w)} \\ &= \frac{\partial}{\partial \overline{w}_a}\bigg|_{w=p}\log\frac{K_2(f(z),f(w))}{K_2(f(w),f(w))} \\ &= \sum_{k=1}^{n}\left(\frac{\partial}{\partial \overline{W}_k}\bigg|_{W=f(p)}\log\frac{K_2(f(z),W)}{K_2(W,W)}\right)\cdot\overline{\frac{\partial f_k}{\partial z_a}\bigg|_p} \\ &= \sum_{k=1}^{n}\left(\overline{\frac{\partial f_k}{\partial z_a}\bigg|_p}\right)\cdot b^k_{\Omega_2,f(p)}(f(z)). \end{aligned}$$

So it suffices to set

$$\overline{A}:=\begin{pmatrix}\frac{\partial f_1}{\partial z_1}(p) & \cdots & \frac{\partial f_n}{\partial z_1}(p)\\ \vdots & \ddots & \vdots\\ \frac{\partial f_1}{\partial z_m}(p) & \cdots & \frac{\partial f_n}{\partial z_m}(p)\end{pmatrix},$$

so that $b_{\Omega_1,p}=A\circ b_{\Omega_2,q}\circ f$. □

Now we present Condition BR precisely.

Definition 3.3. A domain $\Omega \Subset \mathbb{C}^n$ is said to satisfy Condition BR if, for any $q \in \partial\Omega$, there exists an open neighborhood $\mathcal{U}$ of q such that the Bergman representative map $b_{\Omega,p}$ centered at p is a C^∞- coordinate system on $\mathcal{U}\cap\overline{\Omega}$ for some $p \in \mathcal{U}\cap\Omega$.

Remark 3.4. Greene and Krantz [1982, Lemma 5.7] proved that every bounded domain with smooth strongly pseudoconvex boundary satisfies Condition BR by estimating the derivatives of the Bergman kernel function near the boundary. However, for a general bounded domain, it seems nontrivial to characterize Condition

BR in terms of other boundary (geometric) invariants. For instance, it is unknown whether bounded domains with real analytic boundary satisfy Condition BR.

Despite nontriviality for the characterization of the condition, the statement of the main theorem still makes sense, since the domain E_{2m} admits *global* Bergman representative coordinates: Let $\Omega^\alpha := \{z = (z_1, z_2) \in \mathbb{C}^2 : |z_1|^{2/\alpha} + |z_2|^2 < 1\}$ for a positive real number α. The explicit formula of the Bergman kernel of Ω^α is given by

$$K_{\Omega^\alpha}(z, w) = \frac{1}{\pi^2}(1 - z_2\overline{w}_2)^{\alpha-2}\frac{(\alpha+1)(1 - z_2\overline{w}_2)^\alpha + (\alpha-1)z_1\overline{w}_1}{\big((1 - z_2\overline{w}_2)^\alpha - z_1\overline{w}_1\big)^3}.$$

By a straightforward computation,

$$b_{\Omega^\alpha,0}(z) = \left(\frac{4\alpha+2}{\alpha+1}\,z_1,\ (\alpha+2)z_2\right),$$

and so

$$\det\left(\frac{\partial}{\partial z_j}b_{\Omega^\alpha,0}(z)\right) = \frac{2(2\alpha+1)(\alpha+2)}{\alpha+1} \neq 0.$$

In particular, the Bergman representative map of $E_{2m} = \Omega^{1/m}$ at the origin gives rise to a global coordinate system of the domain.

Remark 3.5. E. Ligocka [1980] showed that any bounded domain with smooth boundary satisfying Condition BR should satisfy Condition B, which says that Bell–Ligocka coordinates continue to be diffeomorphic up to the boundary. It may be reasonable to expect the converse to be true, but that has yet to be clarified as far as the author is aware.

We continue our proof of the extension of g to the boundary in the next section.

Proof of extension of g. Since $g^*\beta_\Omega = \lambda \cdot \beta_D$, Proposition 3.2 implies:

Corollary 3.6. *For $\zeta \in D$ and $g(\zeta) = \hat{\zeta} \in \Omega$,*

$$(\dagger) \qquad \lambda \cdot b_{D,\zeta}(z) = \overline{g_1'(\zeta)} \cdot b^1_{\Omega,\hat{\zeta}}(g(z)) + \overline{g_2'(\zeta)} \cdot b^2_{\Omega,\hat{\zeta}}(g(z)).$$

Now consider the reflection map $r : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $r(z, w) = (-z, w)$, which is an automorphism of M_H. The fixed point set in M_H of r is exactly equal to $\widetilde{D}$, that is, $\{p \in M_H : r(p) = p\} = \widetilde{D}$. If we set $\hat{r} := \Psi^{-1} \circ r \circ \Psi : \Omega \to \Omega$, then $\hat{r}$ is an automorphism of Ω and the fixed point set of $\hat{r}$ is equal to $\widehat{D}(= g(D))$. If we choose a particular point $\hat{\zeta} = g(\zeta)$, then $\hat{r}$ is a linear reflection with respect to the $b_{\Omega,\hat{\zeta}}$-coordinates. The definition of $\hat{r}$ implies that it has two eigenvalues, $+1$ and -1. Moreover, $\widehat{D}$ is a subset of a 1-dimensional linear subspace of $\mathbb{C}^2$. Thus there exists a linear isomorphism $L : \mathbb{C}^2 \to \mathbb{C}^2$ with the matrix representation $L = (L_{jk})_{j,k=1,2}$ such that

$$(\ddagger) \qquad L_{21} \cdot b^1_{\Omega,\hat{\zeta}}(g(z)) + L_{22} \cdot b^2_{\Omega,\hat{\zeta}}(g(z)) = 0.$$

Note that $\lambda \cdot b_{D,\zeta}(z)$ is never zero near the boundary of D. Thus $(\overline{g_1'(\zeta)}, \overline{g_2'(\zeta)})$ and (L_{21}, L_{22}) are linearly independent. Thus we may apply Cramer's rule to (†) and (‡) to deduce

$$b^1_{\Omega,\hat{\xi}}(g(z)) = \frac{\lambda \cdot L_{22}}{L_{22} \cdot \overline{g_1'(\zeta)} - L_{21} \cdot \overline{g_2'(\zeta)}} b_{D,\zeta}(z),$$
$$b^2_{\Omega,\hat{\xi}}(g(z)) = -\frac{\lambda \cdot L_{21}}{L_{22} \cdot \overline{g_1'(\zeta)} - L_{21} \cdot \overline{g_2'(\zeta)}} b_{D,\zeta}(z).$$

We may emphasize that $b_{\Omega,\hat{\xi}}\big(g(z)\big)$ is equal to $b_{D,\zeta}(z)$ multiplied by the constant vector

$$\frac{\lambda}{L_{22} \cdot \overline{g_1'(\zeta)} - L_{21} \cdot \overline{g_2'(\zeta)}}(L_{22}, -L_{21})$$

in $\mathbb{C}^2$. This now yields what we wanted: the map $G := b_{\Omega,\hat{\xi}} \circ g \circ b^{-1}_{D,\zeta}$ is linear and hence smooth everywhere. Consequently the map $g = b^{-1}_{\Omega,\hat{\xi}} \circ G \circ b_{D,\zeta}$ extends smoothly up to the boundary of D. In summary, we have

Proposition 3.7. *Let D be a unit disc in $\mathbb{C}$, and define $g : D \to \Omega$ by*

$$g(\zeta) := \Psi^{-1}\Big(0, \frac{\zeta - 1}{\zeta + 1}\Big).$$

Then the map g can extends smoothly (C^∞) up to the boundary.

Remark 3.8. This proposition does not follow directly from the general extension theorems in several complex variables; notice that the dimensions of the domains involved are not equal. It may be worth noting the existence of an example by Globevnik and Stout [1986, Example III.5]. For the unit ball $\mathbb{B}^2$ in $\mathbb{C}^2$, there exists a proper holomorphic embedding $f : D \to \mathbb{B}^2$ such that the Hausdorff dimension of the boundary of $f(D)$ (precisely speaking the set of radial boundary limit values of f) is strictly larger than 1. In particular, f cannot even extend continuously to the boundary.

4. Application to the Bedford–Pinchuk theorem

We now present the proof of the main result of this article, restating it here for convenience:

Theorem 1.2. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^2$ with smooth (C^∞) boundary satisfying Condition BR* (see Definition 3.3). *Suppose there is a point $p_0 \in \partial\Omega$ of finite D'Angelo type $2m$, a point $q \in \Omega$, and a sequence $\{\varphi_j\} \subset \mathrm{Aut}(\Omega)$ such that*

$$\lim_{j\to\infty} \varphi_j(q) = p_0 \in \partial\Omega.$$

Then

$$\Omega \cong E_{2m} := \{(z, w) \in \mathbb{C}^2 : |z|^{2m} + |w|^2 < 1\}.$$

Start with the biholomorphism $\Psi : \Omega \to M_H$ in Theorem 2.1, with $\Psi(q) = (0, -1)$, and recall the automorphisms τ_s and η_t of M_H defined as follows:

$$\tau_s(z, w) := (z, w + is) \quad \text{for } s \in \mathbb{R}$$
$$\eta_t(z, w) := (t^{1/2m} z, tw) \quad \text{for } t > 0.$$

Define the automorphism h_t of Ω by $h_t := \Psi^{-1} \circ \eta_t \circ \Psi$. Since η_t preserves $\widetilde{D}$, there exists $\ell_t \in \mathrm{Aut}(D)$ such that $g \circ \ell_t = h_t \circ g$. (Note here that every automorphism of the unit disc D extends holomorphically across the boundary of D.)

Lemma 4.2. *There exists a unique boundary point $\tilde{p}$ of Ω such that*

$$\lim_{t \to 0} h_t(q) = \tilde{p}.$$

Proof. Since $q = g(0)$, $h_t(q) = h_t(g(0))$. So

$$\lim_{t \to 0} h_t(q) = \lim_{t \to 0} h_t \circ g(0) = \lim_{t \to 0} g \circ \ell_t(0) = g\big(\lim_{t \to 0} \ell_t(0)\big) = g(1),$$

since $g : D \to \Omega$ extends to the boundary. Thus it suffices to let $g(1) = \tilde{p}$. Notice that $\tilde{p} \in \partial\Omega$ since g is proper. □

Note that h_t, for any $0 < t < 1$, fixes the boundary point $\tilde{p}$, and also that

$$h_t \in \mathrm{Aut}(\Omega) \cap \mathrm{Diff}(\overline{\Omega})$$

due to Condition BR. Notice that $dh_t|_{\tilde{p}}$ has two eigenvalues, t and $t^{1/2m}$. Hence Lemma 4.2 implies that h_t is a contracting automorphism at $\tilde{p}$. At this step, note that whether $\partial\Omega$ is of D'Angelo finite type at $\tilde{p}$ is unclear. So we apply the following result:

Theorem 4.3 [Kim and Yoccoz 2011]. *Suppose that Ω is a bounded domain in $\mathbb{C}^n$ with a smooth boundary. If there exists $h \in \mathrm{Aut}(\Omega) \cap \mathrm{Diff}(\overline{\Omega})$ that is contracting at a boundary point $\tilde{p}$, then $\partial\Omega$ at $\tilde{p}$ is of finite type in the sense of D'Angelo. Moreover, the boundary $\partial\Omega$ is defined by a weighted homogeneous polynomial determined completely by the resonance set of the contraction h.*

Therefore our $\tilde{p}$ is of finite type in the sense of D'Angelo and Ω is biholomorphic to the domain M_P defined by $M_P := \{(z, w) \in \mathbb{C}^2 : \mathrm{Re}\, w + P(z, \bar{z}) < 0\}$, where $P(z, \bar{z})$ is a weighted homogeneous polynomial. But since z is a single variable, our P is in fact homogeneous. According to Oeljeklaus [1993], $\deg P = \deg H = 2m$. Therefore the domain Ω is biholomorphic to the domain which is defined by the homogeneous polynomial of degree $2m$.

It remains to show that the homogeneous polynomial P actually is equal to $|z|^{2m}$. For this purpose we shall follow the original method of Bedford and Pinchuk by constructing a parabolic automorphisms fixing $\tilde{p}$.

Define the automorphism k_s of Ω by $k_s := \Psi^{-1} \circ \tau_s \circ \Psi$. As before, there exists an automorphism m_s of D such that $g \circ m_s = k_s \circ g$.

Lemma 4.4. $\lim_{s\to\pm\infty} k_s(q)$ *is a single boundary point of* Ω. *Moreover, this limit point is the same as* $\tilde{p}$.

Proof. Since $q = g(0)$, $k_s(q) = k_s(g(0))$. So,

$$\lim_{s\to\pm\infty} k_s(q) = \lim_{s\to\pm\infty} k_s \circ g(0) = \lim_{s\to\pm\infty} g \circ m_s(0) = g\big(\lim_{s\to\pm\infty} m_s(0)\big) = g(1).$$

Hence the assertion follows. □

Notice again that $k_s \in \mathrm{Aut}(\Omega) \cap \mathrm{Diff}(\overline{\Omega})$ by Condition BR. Moreover, k_s preserves $\partial\Omega$ and fixes $\tilde{p}$. Hence Lemma 4.4 implies that k_s is parabolic with the limit point at $\tilde{p}$. Altogether, $\tilde{p}$ is the point fixed by the contraction h_t and the parabolic automorphisms k_s.

This allows to use the analysis of [Bedford and Pinchuk 1988] so that we may conclude that $H(z, \bar{z}) = c|z|^{2m}$. Therefore Ω is biholomorphic to the Thullen domain $E_{2m} := \{(z, w) \in \mathbb{C}^2 : \mathrm{Re}\, w + |z|^{2m} < 0\}$.

Remark 4.5. In Bedford and Pinchuk's result (Theorem 1.1), the exponent $2m$ for the Thullen domain in its conclusion is not clearly specified, since it comes from the type of the boundary point that arises as the limit point of the parabolic orbit produced in the proof. With the assumption of noncompactness of the automorphism group, Pinchuk's scaling produces a parabolic orbit. But the location of the limit point of this parabolic orbit is arbitrary. That is why the global finiteness of the D'Angelo type of the boundary (which follows in particular by the real analyticity) was assumed in the first place. In our case, on the other hand, we prove that the limit point of the parabolic orbit is also the limit point of a contraction — which follows by the extension theorem of the special totally geodesic disc (Proposition 3.7) — and hence the limit point has to be of D'Angelo finite type by the Kim–Yoccoz result (Theorem 4.3). Then we could further show, combining these results with that of a theorem of Oeljeklaus, that the exponent must actually be the D'Angelo type of the original boundary orbit accumulation point, as stated.

References

[Bedford and Pinchuk 1988] E. Bedford and S. I. Pinchuk, "Domains in $\mathbb{C}^2$ with noncompact groups of holomorphic automorphisms", *Mat. Sb.* (*N.S.*) **135**:2 (1988), 147–157. In Russian; translated in *Math. USSR Sbornik* **63**:1 (1989), 141–151. MR 937803 (89d:32054)

[Bedford and Pinchuk 1991] E. Bedford and S. Pinchuk, "Domains in $\mathbb{C}^{n+1}$ with noncompact automorphism group", *J. Geom. Anal.* **1**:3 (1991), 165–191. MR 92f:32024 Zbl 0733.32014

[Bell and Ligocka 1980] S. Bell and E. Ligocka, "A simplification and extension of Fefferman's theorem on biholomorphic mappings", *Invent. Math.* **57**:3 (1980), 283–289. MR 81i:32017 Zbl 0411.32010

[Berteloot 1994] F. Berteloot, "Characterization of models in $\mathbb{C}^2$ by their automorphism groups", *Internat. J. Math.* **5**:5 (1994), 619–634. MR 95i:32040 Zbl 0817.32010

[Cartan 1932] H. Cartan, "Sur les fonctions de plusieurs variables complexes. L'itération des transformations intérieures d'un domaine borné", *Math. Z.* **35**:1 (1932), 760–773. MR 1545327 Zbl 0004.40602

[Catlin 1989] D. W. Catlin, "Estimates of invariant metrics on pseudoconvex domains of dimension two", *Math. Z.* **200**:3 (1989), 429–466. MR 90e:32029 Zbl 0661.32030

[D'Angelo 1982] J. P. D'Angelo, "Real hypersurfaces, orders of contact, and applications", *Ann. of Math.* (2) **115**:3 (1982), 615–637. MR 84a:32027 Zbl 0488.32008

[Fefferman 1974] C. Fefferman, "The Bergman kernel and biholomorphic mappings of pseudoconvex domains", *Invent. Math.* **26** (1974), 1–65. MR 50 #2562 Zbl 0289.32012

[Globevnik and Stout 1986] J. Globevnik and E. L. Stout, "The ends of varieties", *Amer. J. Math.* **108**:6 (1986), 1355–1410. MR 88m:32008 Zbl 0678.32005

[Greene and Krantz 1982] R. E. Greene and S. G. Krantz, "The automorphism groups of strongly pseudoconvex domains", *Math. Ann.* **261**:4 (1982), 425–446. MR 84c:32032 Zbl 0531.32016

[Kim 1992] K.-T. Kim, "Domains in $\mathbb{C}^n$ with a piecewise Levi flat boundary which possess a noncompact automorphism group", *Math. Ann.* **292**:4 (1992), 575–586. MR 93h:32024 Zbl 0735.32021

[Kim and Yoccoz 2011] K.-T. Kim and J.-C. Yoccoz, "CR manifolds admitting a CR contraction", *J. Geom. Anal.* **21**:2 (2011), 476–493. MR 2772081 Zbl 1219.32017

[Ligocka 1980] E. Ligocka, "Some remarks on extension of biholomorphic mappings", pp. 350–363 in *Analytic functions (Proc. Seventh Conf.)* (Kozubnik, 1979), edited by J. Lawrynowicz, Lecture Notes in Math. **798**, Springer, Berlin, 1980. MR 82b:32037 Zbl 0458.32008

[Oeljeklaus 1993] K. Oeljeklaus, "On the automorphism group of certain hyperbolic domains in $\mathbb{C}^2$", pp. 193–216 in *Colloque d'analyse complexe et géométrie* (Marseille, 1992), Astérisque **217**, 1993. MR 94i:32052 Zbl 0792.32019

[Wong 1977] B. Wong, "Characterization of the unit ball in $\mathbb{C}^n$ by its automorphism group", *Invent. Math.* **41**:3 (1977), 253–257. MR 58 #11521 Zbl 0385.32016

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