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# REGULARITY OF THE FIRST EIGENVALUE OF THE $p$ -LAPLACIAN AND YAMABE INVARIANT ALONG GEOMETRIC FLOWS

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**We first prove that the first eigenvalue of the  $p$ -Laplace operator and the Yamabe invariant are both locally Lipschitz along geometric flows under weak assumptions without assumptions on curvature. Secondly, the Yamabe invariant is found to be directionally differentiable along geometric flows. As an application, an open question about the Yamabe metric and Einstein metric is partially answered.**

## 1. Introduction

Motivated by the Hamilton's Ricci flow, the method of geometric flow has been widely used to deal with geometric and topological properties of manifolds. We often encounter the derivative of geometric quantities when applying the method of geometric flow. Cao [2007; 2008] and Li [2007] consider the monotonicity of the first eigenvalue of  $-\Delta + cR$  ( $c \geq \frac{1}{4}$ ) based on their derivatives along Ricci flow. Ling [2007] proved a comparison theorem for the eigenvalue of the Laplace operator based on its derivative along Ricci flow. Unfortunately, there are many geometric quantities about which we don't know whether they are differentiable along the flow. Chang and Lu [2007] derive a formula for the derivative of the Yamabe constant along Ricci flow under a crucial technical assumption. Recently Wu, Wang and Zheng [Wu et al. 2010] considered the first eigenvalue of the  $p$ -Laplace operator, whose differentiability along Ricci flow is unknown.

For the first eigenvalue of a linear operator, we may assume that there is a  $C^1$ -family of smooth eigenvalues and eigenfunctions along geometric flow by eigenvalue perturbation theory. We have no uniform method to deal with the smoothness of the first eigenvalue of a nonlinear operator — even the continuity is unknown.

As the first eigenvalue can be seen as a minimum of a functional, we consider the regularity of geometric quantities of this type along geometric flow. Inspired

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by the method used in [Hamilton 1986; Chow and Lu 2002] to prove the maximum principle for systems, we first study the relationship between the local Lipschitz property of continuous functions and their Dini derivatives.

**Theorem 1.1.** *Let  $m(t)$  be a continuous function on an interval  $\mathcal{I} \subset \mathbb{R}$ . Suppose that for any  $t \in \mathcal{I}$  there exists a  $C^1$  function  $M(t, s)$  of  $s$  defined on a neighborhood of  $t$  such that  $M(t, t) = m(t)$  and  $M(t, s) \geq m(s)$ .*

- (1) *If  $(\partial M/\partial s)(t, t)$  is locally bounded, then  $m(t)$  is locally Lipschitz.*
- (2) *For any  $t$  in the interior of  $\mathcal{I}$ , if  $m(t)$  is differentiable at  $t$ , then  $m'(t) = (\partial M/\partial s)(t, t)$ .*

**Remark.** By (2), if  $m(t)$  is differentiable at an interior point  $t$ , then the derivative of  $m(t)$  at this point is exactly  $(\partial M/\partial s)(t, t)$ , regardless of the choice of function  $M(t, s)$ .

**Corollary 1.1.1.** *In the same setting of Theorem 1.1, if  $(\partial M/\partial s)(t, t)$  is locally bounded, then  $m(t)$  is differentiable almost everywhere and  $m'(t) = (\partial M/\partial s)(t, t)$  almost everywhere.*

Applying Theorem 1.1, we get the following results on the regularities of the first eigenvalues  $\lambda_{1,p}$  of the  $p$ -Laplace operator and the Yamabe invariant along the general  $C^1$  family of smooth geometric flows in this paper. We find that the first eigenvalue  $\lambda_{1,p}$  of the  $p$ -Laplace operator is in general locally Lipschitz continuous. We also get local Lipschitz continuity of the Yamabe invariant and find its derivative with respect to  $t$  almost everywhere.

**Theorem 1.2.** *Let  $g(x, t)$  be a  $C^1$  family of smooth metrics on a  $n$ -dimensional compact Riemannian manifold  $M$ . Then the first eigenvalue  $\lambda_{1,p}(g(t))$  of the  $p$ -Laplace operator is locally Lipschitz if  $p \geq 2$  and  $M$  is closed or if  $p > 1$  and  $M$  has nonempty boundary.*

**Remark.** In [Wu et al. 2010], a similar result on local Lipschitz continuity was obtained, but under some assumptions on curvature. Theorem 1.2 implies that local Lipschitz continuity should be available for more general smooth geometric flows without any curvature conditions.

**Theorem 1.3.** *Suppose  $M$  is an  $n$ -dimensional ( $n \geq 3$ ) closed connected Riemannian manifold, and  $g(t), t \in [0, T)$ , is a  $C^1$  family of smooth metrics on  $M$ . If  $\bar{g}(t)$  is the Yamabe metric in the conformal class  $[g(t)]$  for any  $t \in [0, T)$ , then the Yamabe invariant  $\mathfrak{Y}(g(t))$  is locally Lipschitz with respect to  $t$ , and*

$$(1-1) \quad \frac{d\mathfrak{Y}(g(t))}{dt} \stackrel{\text{a.e.}}{=} - \int \frac{\bar{g}(t)}{g(t)} \left\langle \frac{\partial g}{\partial t}(t), \text{Rc}^0(\bar{g}(t)) \right\rangle_{\bar{g}(t)} d\mu_{\bar{g}(t)} \text{vol}(\bar{g}(t))^{-2/p},$$

where a.e. stands for “almost everywhere”,  $p = \frac{2n}{n-2}$ , and  $\text{Rc}^0(\bar{g}(t))$  is the trace-free part of Ricci curvature of  $\bar{g}(t)$ .

Let  $g_0$  be a smooth metric on manifold  $M$ ,  $[g_0]$  be the conformal class of  $g_0$ ,  $\Lambda[g_0]$  be the collection of Yamabe metrics in  $[g_0]$  and  $h$  be a smooth  $(0, 2)$ -type symmetric tensor on  $M$ . Denote by  $G_h(g_0, t)$  the collection of  $C^1$  family of smooth metrics  $g(t)$ ,  $t \in [0, \varepsilon)$  with  $g(0) = g_0$  and  $(\partial g / \partial t)(0) = h$  for some  $\varepsilon > 0$ , we define  $\Lambda_h[g_0]$  by

$$\Lambda_h[g_0] := \bigcup_{g(t) \in G_h(g_0, t)} \{ \bar{g}_0 \in \Lambda[g_0] : \bar{g}_0 \text{ is an accumulation point of } \Lambda[g(t)] \text{ as } t \rightarrow 0 \},$$

where  $\bar{g}_0$  generally exists by the compactness of  $\Lambda[g_0]$  when  $[g_0] \neq [g_{\text{can}}]$ , where  $g_{\text{can}}$  denotes the canonical metric on  $S^n$  [Anderson 2005]. (They prove that if  $[g_i] \rightarrow [g_0] \neq [g_{\text{can}}]$  smoothly, then every sequence of Yamabe metrics  $(g^j)_i \in [g_i]$  has a subsequence converging smoothly to a Yamabe metric  $[g^j] \in [g_0]$ .) It is easy to see that if  $\bar{g}_0 \in \Lambda_h[g_0]$  then  $c\bar{g}_0 \in \Lambda_h[g_0]$  and  $\bar{g}_0 \in \Lambda_{ch}[g_0]$  for any  $c > 0$ .

Recently, Brendle [2008] with Marques [2009] gave counterexamples to the compactness for a full set of solutions to the Yamabe equation if the dimension of the manifold greater than 24. Later, Khuri, Marques and Schoen [Khuri et al. 2009] proved compactness if the dimension equal or less than 24.

In addition to Theorem 1.3, we have the following derivative calculation at  $t = 0$ .

**Theorem 1.4.** *Let  $g(t)$ ,  $t \in [0, T)$ , be a  $C^1$  family of smooth metrics on a manifold  $M$  and  $g_{\text{can}}$  be the canonical metric on  $S^n$ . If  $g(0) = g_0$  and  $[g_0] \neq [g_{\text{can}}]$ , then*

$$(1-2) \quad \left. \frac{d^{\mathfrak{O}}(g(t))}{dt} \right|_{t=0} = \min_{\tilde{g}_0 \in \Lambda[g_0]} \left\{ - \int \frac{\tilde{g}_0}{g_0} \left\langle \frac{\partial g}{\partial t}(0), \text{Rc}^0(\tilde{g}_0) \right\rangle_{\tilde{g}_0} d\mu_{\tilde{g}_0} \text{vol}(\tilde{g}_0)^{-2/p} \right\} \\ = - \int \frac{\bar{g}_0}{g_0} \left\langle \frac{\partial g}{\partial t}(0), \text{Rc}^0(\bar{g}_0) \right\rangle_{\bar{g}_0} d\mu_{\bar{g}_0} \text{vol}(\bar{g}_0)^{-2/p},$$

where  $p = 2n/(n - 2)$ ,  $\bar{g}_0 \in \Lambda_{(\partial g / \partial t)(0)}[g_0]$ ,  $\text{Rc}^0(\bar{g}_0)$  is the trace-free part of Ricci curvature with respect to  $\bar{g}_0$  and  $\text{vol}(\bar{g}_0)$  is the volume of  $M$  respect to  $\bar{g}_0$ . In particular,  $\mathfrak{O}$  is directionally differentiable at  $g_0$ .

**Remark.** This formula generalizes similar calculations in [Anderson 2005] where  $\text{tr}(\partial g / \partial t)(0) = 0$ ,  $\text{vol}(g(t)) = 1$ , and  $g_0$  has constant scalar curvature. Meanwhile, when  $\bar{g}_0 \in \Lambda_{(\partial g / \partial t)(0)}[g_0]$ , Equation (1-2) becomes more convenient to calculate, compared to the derivative calculation in [Anderson 2005] (in another form):

$$(1-3) \quad \min_{\tilde{g}_0} \left\{ - \int \frac{\tilde{g}_0}{g_0} \left\langle \text{Rc}^0(\tilde{g}_0), \frac{\partial g}{\partial t}(0) \right\rangle_{\tilde{g}_0} d\mu_{\tilde{g}_0} \right\},$$

where  $\tilde{g}_0 \in \Lambda_1[g_0]$  is taken over all accumulation points of  $\Lambda_1[g(t)]$  as  $t \rightarrow 0$  for  $\Lambda_1[g_0]$  the set of unit volume Yamabe metrics in  $[g_0]$ . The derivative is difficult to calculate using this formula, but by Theorem 1.4 we can calculate this derivative

if we know a Yamabe metric  $\bar{g}_0$  in  $\Lambda_{(\partial g/\partial t)(0)}[g_0]$ . Moreover, the set to minimize in (1-3) has only one element by the last equality in (1-2).

In addition to the local Lipschitz property of the Yamabe invariant, we have:

**Corollary 1.4.1.** *With the same assumptions as in Theorem 1.3, the Yamabe invariant  $\mathfrak{Y}(g(t))$  is directionally differentiable at all  $t$  where  $[g(t)] \neq [g_{\text{can}}]$ .*

In particular, in formula (1-2), if  $(\partial g/\partial t)(0) = -2 \text{Rc}(g_0)$  and  $g_0$  is a Yamabe metric in  $\Lambda_{-\text{Rc}(g_0)}[g_0]$ , then  $g_0 \in \Lambda_{-2 \text{Rc}(g_0)}[g_0]$  and  $R(g_0)$  is constant, hence

$$\begin{aligned} \left. \frac{d\mathfrak{Y}(g(t))}{dt} \right|_{t=0} &= \int \langle \text{Rc}(g_0), \text{Rc}^0(g_0) \rangle_{g_0} d\mu_{g_0} \text{vol}(g_0)^{-2/p} \\ &= \int |\text{Rc}^0(g_0)|^2 d\mu_{g_0} \text{vol}(g_0)^{-2/p} \geq 0. \end{aligned}$$

Ricci flow evolves sphere to sphere, so we have the following conclusion along the Ricci flow.

**Corollary 1.4.2.** *Let  $M^n$  be a closed and connected manifold with  $n \geq 3$  and  $g(t), t \in [0, T)$ , be a solution of Ricci flow  $\partial g/\partial t = -2 \text{Rc}$  on  $M$  with  $g(0) = g_0$ . If  $g_0 \in \Lambda_{-\text{Rc}}[g_0]$ , then  $d\mathfrak{Y}(g(t))/dt|_{t=0} \geq 0$  and  $d\mathfrak{Y}(g(t))/dt|_{t=0} = 0$  if only if  $g_0$  is a Einstein metric.*

**Remark.** There is a similar result in [Chang and Lu 2007] under the assumption that there exists a  $C^1$  family of  $\phi(t) > 0$  such that  $\phi(t)^{4/(n-2)}g(t)$  is a Yamabe metric and  $\phi(0)$  is constant. From the definition of  $\Lambda_{-\text{Rc}}[g_0]$  we can see that our assumption is weaker.

Let  $\mathcal{C}$  denote the set of unit volume constant scalar curvature metrics on a connect closed manifold  $M$ ; it is well-known (see [Besse 1987]) that generically  $\mathcal{C}$  is an infinite-dimensional manifold. Let  $s : \mathcal{C} \mapsto \mathbb{R}$  be the scalar curvature function. It has long been an open problem whether a Yamabe metric which is a local maximizer of  $s$  is necessarily an Einstein metric [Besse 1987]. Some progress on this question was made in [Bessieres et al. 2003] in dimension 3 and in [Anderson 2005] in any dimension. Let  $\mathcal{M}$  be the collection of all smooth metrics on  $M$  and  $\mathfrak{Y} : \mathcal{M} \mapsto \mathbb{R}$  be the Yamabe invariant function. By the definition of the Yamabe invariant,  $s(g) \geq \mathfrak{Y}(g)$  for any  $g \in \mathcal{C}$ , hence if a Yamabe metric is a local maximizer of  $s$ , it must be a local maximizer of  $\mathfrak{Y}$ . Now, we consider whether a Yamabe metric that is a local maximizer of the Yamabe invariant is necessarily an Einstein metric. Following from Corollary 1.4.2, the next result gives a partial answer.

**Corollary 1.4.3.** *Let  $M^n$  be a closed and connected manifold with  $n \geq 3$  and suppose a Yamabe metric  $g$  is a local maximum of the Yamabe invariant functional  $\mathfrak{Y}(\cdot)$ . If  $g \in \Lambda_{-\text{Rc}}[g]$ , then  $g$  is Einstein.*

In Section 2, we give a basic introduction to Dini derivatives and the proof of Theorem 1.1. In Section 3, we prove the Lipschitz property of the first eigenvalue of the  $p$ -Laplace operator along geometric flows. In Section 4, we show that the Yamabe invariant is locally Lipschitz and directionally differentiable along geometric flows.

## 2. Dini derivatives and the proof of Theorem 1.1

In this section, we first recall the definitions of Dini derivatives and semicontinuity. Then we give some propositions about Dini derivatives. Lastly, we prove Theorem 1.1.

Hamilton [1986] studied properties of Lipschitz functions by means of their Dini derivatives, and from this derived the maximum principle for systems on closed manifolds. Chow [2002] proved similar results in weaker settings. Dini derivatives provide a powerful way to deal with nonregular functions.

These definitions of Dini derivatives and semicontinuity also appear in [Chow et al. 2008].

**Definition 2.1** (Dini derivatives). Let  $f(t)$  be a function on  $(a, b)$ . The upper Dini derivative is the lim sup of forward difference quotients:

$$\frac{d^+f}{dt}(t) := \limsup_{h \rightarrow 0^+} \frac{f(t+h) - f(t)}{h}.$$

The lower Dini derivative is the lim inf of forward difference quotients:

$$\frac{d^-f}{dt}(t) := \liminf_{h \rightarrow 0^+} \frac{f(t+h) - f(t)}{h}.$$

The upper converse Dini derivative is the lim sup of backward difference quotients:

$$\frac{d_+f}{dt}(t) := \limsup_{h \rightarrow 0^+} \frac{f(t) - f(t-h)}{h}.$$

The lower converse Dini derivative is the lim inf of backward difference quotients:

$$\frac{d_-f}{dt}(t) := \liminf_{h \rightarrow 0^+} \frac{f(t) - f(t-h)}{h}.$$

If the function  $f$  is also defined at  $a$ , we can define its upper Dini derivative and lower Dini derivative at  $a$ ; and if the function  $f$  is also defined at  $b$ , we can define its upper converse Dini derivative and lower converse Dini derivative at  $b$ .

Since we don't make any assumption on the function  $f(t)$ , it is possible that any one of the Dini derivatives of  $f(t)$  above may take the value  $+\infty$  or  $-\infty$ .

**Definition 2.2** (semicontinuity). Let  $f(t)$  be a function on an interval. We say  $f$  is right upper semicontinuous if  $\limsup_{h \rightarrow 0^+} f(t+h) \leq f(t)$ ; we say  $f$  is right

lower semicontinuous if  $\liminf_{h \rightarrow 0^+} f(t+h) \geq f(t)$ ; we say  $f$  is left upper semicontinuous if  $\limsup_{h \rightarrow 0^+} f(t-h) \leq f(t)$ ; we say  $f$  is left lower semicontinuous if  $\liminf_{h \rightarrow 0^+} f(t-h) \geq f(t)$ .

**Lemma 2.3.** *If  $f(t) : (a, b) \rightarrow \mathbb{R}$  is left lower semicontinuous with  $(d^+f/dt)(t) \leq 0$ , then  $f(t)$  is decreasing.*

*Proof.* Given  $\varepsilon > 0$ , define  $f_\varepsilon(t) := f(t) - \varepsilon t$ . We shall show that  $f_\varepsilon(t) \leq f_\varepsilon(s)$  for any  $a < s \leq t < b$ . The lemma then follows from taking  $\varepsilon \rightarrow 0$ .

Since  $(d^+f/dt)(s) \leq 0$ , we have  $(d^+f_\varepsilon/dt)(s) \leq -\varepsilon$ , then there exists a number  $\delta(s, \varepsilon) > 0$  such that  $(f_\varepsilon(s+h) - f_\varepsilon(s))/h \leq -\varepsilon/2 < 0$  for all  $h \in (0, \delta(s, \varepsilon))$ , hence  $f_\varepsilon(t) \leq f_\varepsilon(s)$  on  $h \in [s, s + \delta(s, \varepsilon))$ . Define  $\tau(\varepsilon, s) \in [s, b]$  by

$$\tau := \sup \{ \tau' \in [s, b] : f_\varepsilon(t) \leq f_\varepsilon(s) \text{ for all } t \in [s, \tau'] \}.$$

then  $\tau \geq s + \delta(s, \varepsilon) > s$ . One can check that, in fact,  $f_\varepsilon(t) \leq f_\varepsilon(s)$  for all  $t \in [s, \tau)$ . We now prove  $\tau = b$  to complete the proof. If for some  $s$  and  $\varepsilon > 0$ , we have  $\tau < b$ , then there exists a sequence of times  $\{\tau_i\} \nearrow \tau$ , such that  $f_\varepsilon(s) \geq f_\varepsilon(\tau_i - 1/2^i)$  when  $i$  is large enough. Hence

$$f_\varepsilon(s) \geq \liminf_{i \rightarrow \infty} f_\varepsilon(\tau_i - 1/2^i) \geq \liminf_{h \rightarrow 0^+} f_\varepsilon(\tau - h) \geq f_\varepsilon(\tau)$$

follows from the left lower semicontinuity of  $f_\varepsilon(t)$ . Applying the above procedure again by replacing  $s$  with  $\tau$  gives  $f_\varepsilon(t) \leq f_\varepsilon(\tau) \leq f_\varepsilon(s)$  when  $t \in [\tau, \tau + \delta(\tau, \varepsilon))$ , hence  $f_\varepsilon(t) \leq f_\varepsilon(s)$  when  $t \in [s, \tau + \delta(\tau, \varepsilon))$ . This is a contradiction since the definition of  $\tau$  implies  $\delta(\tau, \varepsilon) \leq 0$ . □

**Note.** A similar conclusion can be found in [Chow et al. 2008]. There, the domain of  $f$  is  $[0, T)$ , hence  $f$  must be both left lower semicontinuous and right upper semicontinuous. Here we choose the domain of  $f$  to be  $(a, b)$ , so we can weaken the assumptions on  $f$ .

**Proposition 2.4.** (a) *If  $f(t) : (a, b) \rightarrow \mathbb{R}$  is left lower semicontinuous, then  $d^+f/dt \leq 0$  if and only if  $f(t)$  is decreasing.*

(b) *If  $f(t) : (a, b) \rightarrow \mathbb{R}$  is right upper semicontinuous, then  $d_+f/dt \leq 0$  if and only if  $f(t)$  is decreasing.*

(c) *If  $f(t) : (a, b) \mapsto \mathbb{R}$  is left upper semicontinuous, then  $d^-f/dt \geq 0$  if and only if  $f(t)$  is increasing.*

(d) *If  $f(t) : (a, b) \mapsto \mathbb{R}$  is right lower semicontinuous, then  $d_-f/dt \geq 0$  if and only if  $f(t)$  is increasing.*

*Proof.* (a) If  $f(t)$  is decreasing then  $d^+f/dt \leq 0$ . The other direction follows from Lemma 2.3.

(b)–(d) follows from applying part (a) to the functions  $-f(-t)$ ,  $-f(t)$ , and  $f(-t)$ , respectively.  $\square$

From this propositions, we see that a semicontinuous function is monotonic if certain types of its Dini derivatives have a definite sign. A further analysis shows that monotonicity can be a nice bridge between Dini derivatives of different type.

**Claim 2.5.** *Let  $\mathcal{I} \subset \mathbb{R}$  be an interval and  $\mathring{\mathcal{I}}$  be its interior.*

(a) *If  $f(t) : \mathcal{I} \mapsto \mathbb{R}$  is right upper semicontinuous and left lower semicontinuous,*

$$\frac{d^+f}{dt} \leq 0 \text{ in } \mathring{\mathcal{I}} \iff f(t) \text{ is decreasing on } \mathcal{I} \iff \frac{d_+f}{dt} \leq 0 \text{ in } \mathring{\mathcal{I}}.$$

(b) *If  $f(t) : \mathcal{I} \mapsto \mathbb{R}$  is right lower semicontinuous and left upper semicontinuous,*

$$\frac{d^-f}{dt} \geq 0 \text{ in } \mathring{\mathcal{I}} \iff f(t) \text{ is increasing on } \mathcal{I} \iff \frac{d_-f}{dt} \geq 0 \text{ in } \mathring{\mathcal{I}}.$$

*Proof.* We prove the first part; the second is similar. If  $f(t) : \mathcal{I} \mapsto \mathbb{R}$  is right upper semicontinuous and left lower semicontinuous, then  $f(t)$  is decreasing on  $\mathcal{I}$  if and only if  $f(t)$  is decreasing on  $\mathring{\mathcal{I}}$ . The conclusion then follows from parts (a) and (b) of Proposition 2.4.  $\square$

**Theorem 2.6.** *If  $f : (a, b) \mapsto \mathbb{R}$  is a continuous function with  $d^+f/dt$  or  $d_+f/dt$  locally bounded from above and  $d^-f/dt$  or  $d_-f/dt$  locally bounded from below, then  $f$  is locally Lipschitz.*

*Proof.* Given any  $s \in (a, b)$ , let  $U(s)$  be a compact and connected neighborhood of  $s$  in  $(a, b)$ . Then on  $U(s)$ , without loss of generality, we can assume  $d^+f/dt \leq A$  or  $d_+f/dt \leq A$  and  $d^-f/dt \geq -A$  or  $d_-f/dt \geq -A$ , where  $A > 0$  is a constant. Hence  $d^+(f - At)/dt \leq 0$  (or  $d_+(f - At)/dt \leq 0$ ) by parts (a) and (b) of Proposition 2.4, and  $d_-(f + At)/dt \geq 0$  (or  $d^-(f + At)/dt \geq 0$ ) by parts (c) and (d). Then  $f - At$  is decreasing and  $f + At$  is increasing on  $U(s)$  by Claim 2.5. Thus  $|f(t_2) - f(t_1)| \leq A|t_2 - t_1|$  for any  $t_1, t_2 \in U(s)$ , so  $f$  is locally Lipschitz.  $\square$

*Proof of Theorem 1.1.* Since  $M(t, t) = m(t)$  and  $M(t, s) \geq m(s)$  in a neighborhood of  $t$ , we have

$$\begin{aligned} (2-1) \quad \frac{d^+m}{dt}(t) &= \limsup_{h \rightarrow 0^+} \frac{m(t+h) - m(t)}{h} \leq \limsup_{h \rightarrow 0^+} \frac{M(t, t+h) - M(t, t)}{h} \\ &= \frac{\partial M}{\partial s}(t, t), \end{aligned}$$

$$\begin{aligned} (2-2) \quad \frac{d_-m}{dt}(t) &= \liminf_{h \rightarrow 0^+} \frac{m(t) - m(t-h)}{h} \geq \liminf_{h \rightarrow 0^+} \frac{M(t, t) - M(t, t-h)}{h} \\ &= \frac{\partial M}{\partial s}(t, t). \end{aligned}$$



Since  $(\partial M/\partial s)(t, t)$  is locally bounded,  $(d^+m/dt)(t)$  is bounded from above and  $(d_-m/dt)(t)$  is bounded from below. Then by Theorem 2.6, the function  $m(t)$  is locally Lipschitz in the interior of  $\mathcal{I}$ .

Let  $a$  be the left endpoint of  $\mathcal{I}$ ,  $b$  be the right endpoint of  $\mathcal{I}$ . If  $a \in \mathcal{I}$ , let  $c = \min\{a+1, (a+b)/2\}$ . Since  $(\partial M/\partial s)(t, t)$  is locally bounded on  $\mathcal{I}$ , we can assume that  $|(\partial M/\partial s)(t, t)| \leq A$  ( $A$  is a constant) on  $[a, c]$ . Then  $d^+(m(t) - At)/dt \leq 0$  on  $[a, c)$  and  $d_-(m(t) + At)/dt \geq 0$  on  $(a, c]$  by (2-1) and (2-2). Hence by part (a) of Claim 2.5, the function  $m(t) - At$ , is decreasing on  $[a, c]$ , and by part (b), the function  $m(t) + At$  is increasing on  $[a, c]$ . Then  $|m(t_1) - m(t_2)| \leq A|t_1 - t_2|$  for any  $t_1, t_2 \in [a, c]$ , so  $m(t)$  is locally Lipschitz at  $t = a$ . Similarly, if  $b \in \mathcal{I}$ , then  $m(t)$  is locally Lipschitz at  $t = b$ . In conclusion,  $m(t)$  is locally Lipschitz on  $\mathcal{I}$ .

For any  $t$  in the interior of  $\mathcal{I}$ , if  $m(t)$  is differentiable at this point, then by (2-1) we have  $m'(t) = (d^+m/dt)(t) \leq (\partial M/\partial s)(t, t)$ , and by (2-2) we have  $m'(t) = (d_-m/dt)(t) \geq (\partial M/\partial s)(t, t)$ . Hence  $m'(t) = (\partial M/\partial s)(t, t)$ . □

### 3. First eigenvalue of the $p$ -Laplacian

In this section we consider the local Lipschitz property of the  $p$ -Laplace operator along general geometric flows. Let  $(M, g)$  be a compact connected Riemannian manifold. Define

$$G(f, g) := \frac{\int_M |\nabla f|_g^p d\mu_g}{\int_M |f|^p d\mu_g},$$

where  $\nabla f = df$  is a covariant vector. Recalling the definition of the first eigenvalue  $\lambda_{1,p}(g)$  of the  $p$ -Laplace operator, it is known that if  $\partial M \neq \emptyset$  then

$$\lambda_{1,p}(g) := \inf\{G(f, g) : f \in W_0^{1,p}(M), f \neq 0\}$$

and if  $M$  is closed then

$$\lambda_{1,p}(g) := \inf\{G(f, g) : f \in W^{1,p}(M), \int_M |f|^{p-2} f d\mu_g = 0, f \neq 0\}.$$

The minimum (a positive number) is achieved by a  $C^{1,\alpha}$  ( $0 < \alpha < 1$ ) eigenfunction  $f$  (see [Serrin 1964; Tolksdorf 1984]). This eigenfunction  $f$  satisfies the Euler–Lagrange equation

$$\Delta_p f = -\lambda_{1,p}(g)|f|^{p-2} f,$$

where  $\Delta_p$  ( $p > 1$ ) is the  $p$ -Laplace operator with respect to  $g$  given by

$$\Delta_p f = \operatorname{div}_g(|\nabla f|_g^{p-2} \nabla f).$$

The following theorem implies that  $\lambda(g(t))$  is continuous with respect to  $t$  along general geometric flows.

**Theorem 3.1** [Wu et al. 2010]. *If  $g_1$  and  $g_2$  are two metrics on  $M$  which satisfy  $(1 + \varepsilon)^{-1}g_1 \leq g_2 \leq (1 + \varepsilon)g_1$ , then for any  $p > 1$ , we have*

$$(1 + \varepsilon)^{-(n+p/2)} \leq \frac{\lambda_{1,p}(g_1)}{\lambda_{1,p}(g_2)} \leq (1 + \varepsilon)^{(n+p/2)}.$$

Let  $f \in C^{1,\alpha}(M)$  be nonconstant and  $g(x, t)$ ,  $t \in [0, T]$ , be a  $C^1$  family of smooth metrics on  $M$ . Define a function of  $c \in (-\infty, \infty)$  and  $t \in [0, T]$ :

$$P(c, t) := \int_M |f + c|^{p-2} (f + c) d\mu_{g(t)}, \quad p \geq 2.$$

The function  $P(c, t)$  is  $C^1$  with respect to  $c$  and  $t$ , since

$$\frac{\partial P}{\partial c} = (p - 1) \int_M |f + c|^{p-2} d\mu_{g(t)} > 0.$$

Then by the implicit function theorem, given any  $c_0$  and  $t_0$  there exists a  $C^1$  function  $c(t)$  defined on a neighborhood of  $t_0$  such that  $P(c(t), t) = P(c_0, t_0)$ .

In this and the next sections, if  $f$  is a real function on  $M$ , we simply write  $\sup f$  instead of  $\sup_{x \in M} f(x)$ . Let  $g(t)$  be a family of Riemannian metrics on manifold. If  $\alpha(t)$  is a family of  $(0, 2)$ -type tensors, we denote by  $\text{tr} \alpha(t) = g^{ij}(t)\alpha_{ij}(t)$  its trace with respect to  $g(t)$  and by

$$|\alpha(t)|_{g(s)} = \sqrt{g^{ij}(s)g^{kl}(s)\alpha_{ik}(t)\alpha_{jl}(t)}$$

its norm with respect to  $g(s)$ ; if  $\beta(t)$  is also a family of  $(0, 2)$ -type tensors, we denote by

$$\langle \alpha(t), \beta(t) \rangle_{g(s)} = \sqrt{g^{ij}(s)g^{kl}(s)\alpha_{ik}(t)\beta_{jl}(t)}$$

the inner product derived from the metric  $g(s)$ . Moreover, we use  $|\alpha(t)|$  instead of  $|\alpha(t)|_{g(t)}$  and  $\langle \alpha(t), \beta(t) \rangle$  instead of  $\langle \alpha(t), \beta(t) \rangle_{g(t)}$  for simplicity.

*Proof of Theorem 1.2.* For any  $t_0$ , let  $f(t_0)$  be a minimizer of  $G(\cdot, g(t_0))$ . If  $M$  is closed and  $p \geq 2$ , then  $f(t_0)$  is a nonconstant  $C^1$  function on  $M$  with

$$\int_M |f(t_0)|^{p-2} f(t_0) d\mu_{g(t_0)} = 0.$$

Hence there is a continuous differentiable function  $c(t_0, s)$  of  $s$  defined in a neighborhood of  $t_0$  such that  $c(t_0, t_0) = 0$  and

$$\int_M |f(t_0) + c(t_0, s)|^{p-2} (f(t_0) + c(t_0, s)) d\mu_{g(s)} = 0.$$

Otherwise if  $\partial M \neq \emptyset$  and  $p > 1$ , we can just take  $c(t_0, s) \equiv 0$ .

Let  $N(t, s) = G(f(t) + c(t, s), g(s))$ . Then  $N(t, t) = \lambda_{1,p}(g(t))$  and  $N(t, s) \geq \lambda_{1,p}(g(s))$ . A simple calculation gives

$$\begin{aligned} & \frac{\partial N}{\partial s}(t, t) \\ &= \frac{\partial}{\partial s} \Big|_{s=t} \frac{\int_M |\nabla(f(t) + c(t, s))|_{g(s)}^p d\mu_{g(s)}}{\int_M |f(t) + c(t, s)|^p d\mu_{g(s)}} \\ &= \left( \int_M \left( |\nabla f(t)|_{g(t)}^p \frac{\text{tr}(\partial g / \partial s)(t)}{2} - \frac{p}{2} |\nabla f(t)|_{g(t)}^{p-2} \frac{\partial g}{\partial s}(t) (\nabla f(t), \nabla f(t)) \right) d\mu_{g(t)} \right. \\ & \quad \left. - \lambda_{1,p}(g(t)) \int_M \left( |f(t)|^p \frac{\text{tr}(\partial g / \partial s)(t)}{2} + p |f(t)|^{p-2} f(t) \frac{\partial c}{\partial s}(t, t) \right) d\mu_{g(t)} \right) \\ & \quad \times \left( \int_M |f(t)|^p d\mu_{g(t)} \right)^{-1}. \end{aligned}$$

To simplify this formula, we use that

$$\int_M p |f(t)|^{p-2} f(t) \frac{\partial c}{\partial s}(t, t) d\mu_{g(t)} = 0.$$

When  $M$  is closed, this follows from  $\int_M p |f(t)|^{p-2} f(t) d\mu_{g(t)} = 0$ , and when  $\partial M \neq \emptyset$ , from  $c(t, s) \equiv 0$ . Hence we get

$$\begin{aligned} (3-1) \quad \frac{\partial N}{\partial s}(t, t) &= \left( \int_M \left( |\nabla f(t)|_{g(t)}^p \frac{\text{tr}(\partial g / \partial s)(t)}{2} \right. \right. \\ & \quad \left. \left. - \frac{p}{2} |\nabla f(t)|_{g(t)}^{p-2} \frac{\partial g}{\partial s}(t) (\nabla f(t), \nabla f(t)) \right) d\mu_{g(t)} \right. \\ & \quad \left. - \lambda_{1,p}(g(t)) \int_M |f(t)|^p \frac{\text{tr}(\partial g / \partial s)(t)}{2} d\mu_{g(t)} \right) \\ & \quad \times \left( \int_M |f(t)|^p d\mu_{g(t)} \right)^{-1}. \end{aligned}$$

Now apply the Cauchy–Schwarz formula

$$\left| \frac{\partial g}{\partial s}(t) (\nabla f(t), \nabla f(t)) \right| \leq \left| \frac{\partial g}{\partial s}(t) \right|_{g(t)} |\nabla f(t)|^2$$

and the fact that

$$\left| \text{tr} \frac{\partial g}{\partial s}(t) \right| = \left| \left\langle g(t), \frac{\partial g}{\partial s}(t) \right\rangle \right| \leq |g(t)| \left| \frac{\partial g}{\partial s}(t) \right|_{g(t)} = \sqrt{n} \left| \frac{\partial g}{\partial s}(t) \right|_{g(t)}$$

to obtain

$$\left| \frac{\partial N}{\partial s}(t, t) \right| \leq \left( \sqrt{n} + \frac{p}{2} \right) \lambda_{1,p}(g(t)) \sup \left| \frac{\partial g}{\partial s}(t) \right|_{g(t)}.$$

By the compactness of  $M$ ,  $\lambda_{1,p}(g(t))$  and  $\sup |(\partial g/\partial s)(t)|_{g(t)}$  are both continuous. Then  $|(\partial N/\partial s)(t, t)|$  is locally bounded, so Theorem 1.1 implies Theorem 1.2.  $\square$

#### 4. The Yamabe invariant

In this section, we consider the local Lipschitz property of the Yamabe invariant along general geometric flows and use the constants

$$p = \frac{2n}{n-2}, \quad a = \frac{4(n-1)}{n-2}, \quad b = \frac{4}{n-2}.$$

With no specification,  $M$  is a  $n$ -dimensional ( $n \geq 3$ ) connected closed smooth Riemannian manifold,  $g$  is a smooth metric on it. Denote by  $R$  its scalar curvature, by  $Rc$  its Ricci curvature, and by  $Rc^0 = Rc - \frac{1}{n}Rg$  its trace-free Ricci curvature. The conformal class  $[g]$  of metric  $g$  is defined by

$$[g] := \{\phi^b g : \phi \in C^\infty(M), \phi > 0\},$$

and the homogeneous total scalar curvature  $S(g)$  is defined by

$$S(g) := \int_M R d\mu_g / \int_M d\mu_g,$$

where  $d\mu_g$  is the volume form with respect to metric  $g$ . Then the Yamabe invariant is defined by

$$(4-1) \quad \mathfrak{Y}(g) := \inf_{\bar{g} \in [g]} S(\bar{g}).$$

The minimizer metric is called a Yamabe metric. For the conformal transformation of the scalar curvature  $R(g)$  and the trace-free Ricci curvature  $Rc^0(g)$ , we have (see [Besse 1987])

$$(4-2) \quad \phi^{p-1} R(\phi^b g) = R(g)\phi - a \Delta\phi,$$

$$(4-3) \quad Rc^0(\phi^2 g) = Rc^0(g) + (n-2)\phi(\nabla\nabla\phi^{-1})^0,$$

where  $\Delta$  is the Laplace–Beltrami operator with respect to the metric  $g$ ,  $\alpha^0 = \alpha - \frac{1}{n} \text{tr}(\alpha) \alpha$  is the trace-free part of  $(0, 2)$ -type tensor  $\alpha$ . If we define

$$E(\phi, g) := \int (a |\nabla\phi|_g^2 + R(g)\phi^2) d\mu_g,$$

$$Q(\phi, g) := \frac{E(\phi, g)}{(\int \phi^p d\mu_g)^{2/p}} = E(\phi, g) \|\phi\|_{p,g}^{-2},$$

where  $\nabla\phi = d\phi$  is a covariant vector and  $\|\phi\|_{p,g} = (\int \phi^p d\mu_g)^{1/p}$  is the  $L^p$  norm with respect to metric  $g$ . Then the Yamabe invariant  $\mathfrak{Y}(g)$  can also be defined by

$$(4-4) \quad \mathfrak{Y}(g) := \inf\{Q(\phi, g) : \phi \in C^\infty(M), \phi > 0\}.$$

The existence of a minimizer  $u$  follows from the solution of the Yamabe problem (see [Lee and Parker 1987] for the history). Hence  $u^b g$  is a Yamabe metric, moreover the minimizer  $u$  satisfies the Euler–Lagrange function

$$(4-5) \quad R(g)u - a \Delta u = \alpha u^{p-1},$$

where

$$\alpha = E(u, g) \|u\|_{p,g}^{-p} = \mathfrak{Y}(g) \|u\|_{p,g}^{2-p}.$$

Denote by  $g_{\text{can}}$  the canonical metric on  $S^n$ , and consider the set  $\Lambda[g]$  of all smooth Yamabe metrics in a given conformal class  $[g]$ . By the solution to the Yamabe problem, the sets  $\Lambda[g]$  as  $g$  varies are also compact in the following sense (see [Anderson 2005]): if  $g_i \rightarrow g$  smoothly and  $[g] \neq [g_{\text{can}}]$ , then any sequence of Yamabe metrics  $\bar{g}_i \in \Lambda[g_i]$  has a subsequence converging smoothly to a Yamabe metric  $\bar{g} \in \Lambda[g]$ .

The Yamabe constant  $\mathfrak{Y}(g)$  is continuous with respect to  $g$  under the  $C^2$ -topology of the space of metrics on  $M$  (see [Besse 1987, Proposition 4.31]).

*Proof of Theorem 1.3.* Since each  $\bar{g}(t) \in [g(t)]$  is a Yamabe metric, we can assume  $\bar{g}(t) = \phi^b(t)g(t)$ . Then  $0 < \phi(t) \in C^1(M)$  and  $\phi(t)$  minimizes  $Q(\cdot, g(t))$ . Defining  $N(t, s) := Q(\phi(t), g(s))$ , then  $\mathfrak{Y}(g(t)) = N(t, t)$  and  $\mathfrak{Y}(g(s)) \leq N(t, s)$ . We compute

$$(4-6) \quad \begin{aligned} & \frac{\partial N}{\partial s}(t, t) \\ &= \frac{\partial}{\partial s} \Big|_{s=t} \left( \frac{\int (a |\nabla \phi(t)|_{g(s)}^2 + R(s)\phi(t)^2) d\mu_{g(s)}}{(\int \phi(t)^p d\mu_{g(s)})^{2/p}} \right) \\ &= \int \left( -a \frac{\partial g}{\partial s}(t) (\nabla \phi(t), \nabla \phi(t)) + \frac{\partial R}{\partial s}(t) \phi(t)^2 \right) d\mu_{g(t)} \left( \int \phi(t)^p d\mu_{g(t)} \right)^{-2/p} \\ & \quad + \int \frac{1}{2} (a |\nabla \phi(t)|_{g(s)}^2 + R(t)\phi(t)^2) \text{tr} \frac{\partial g}{\partial s}(t) d\mu_{g(t)} \left( \int \phi(t)^p d\mu_{g(t)} \right)^{-2/p} \\ & \quad - \frac{1}{p} \mathfrak{Y}(g(t)) \int \phi(t)^p \text{tr} \frac{\partial g}{\partial s}(t) d\mu_{g(t)} \left( \int \phi(t)^p d\mu_{g(t)} \right)^{-1}, \end{aligned}$$

so that

$$(4-7) \quad \begin{aligned} & \left| \frac{\partial N}{\partial s}(t, t) \right| \\ & \leq \left( 1 + \frac{\sqrt{n}}{2} \right) \sup \left| \frac{\partial g}{\partial s}(t) \right| \cdot \frac{\int a |\nabla \phi(t)|_{g(t)}^2 d\mu_{g(t)}}{(\int \phi(t)^p d\mu_{g(t)})^{2/p}} + \frac{\sqrt{n} |\mathfrak{Y}(g(t))|}{p} \sup \left| \frac{\partial g}{\partial s}(t) \right| \\ & \quad + \left( \sup \left| \frac{\partial R}{\partial s}(t) \right| + \frac{\sqrt{n}}{2} \sup |R(g(t))| \cdot \sup \left| \frac{\partial g}{\partial s}(t) \right| \right) \frac{\int \phi(t)^2 d\mu_{g(t)}}{(\int \phi(t)^p d\mu_{g(t)})^{2/p}}. \end{aligned}$$

Next, we process the two integral terms in the above formula. Since  $p > 2$ , applying Hölder's inequality gives

$$(4-8) \quad \frac{\int \phi^2 d\mu_g}{\left(\int \phi^p d\mu_g\right)^{2/p}} \leq \text{vol}(g)^{1-2/p}.$$

By the definition of  $Q(\phi, g)$ , we have

$$(4-9) \quad \begin{aligned} \frac{\int a |\nabla \phi|_g^2 d\mu_g}{\left(\int \phi^p d\mu_g\right)^{2/p}} &= Q(\phi, g) - \frac{\int R \phi^2 d\mu_g}{\left(\int \phi^p d\mu_g\right)^{2/p}} \\ &\leq Q(\phi, g) + \sup |R(g)| \frac{\int \phi^2 d\mu_g}{\left(\int \phi^p d\mu_g\right)^{2/p}} \\ &\leq Q(\phi, g) + \sup |R(g)| \text{vol}(g)^{1-2/p}. \end{aligned}$$

Substituting (4-8) and (4-9) into (4-7), we come to

$$\begin{aligned} \left| \frac{\partial N}{\partial s}(t, t) \right| &\leq \left( (1 + \sqrt{n}) \sup \left| \frac{\partial g}{\partial s}(t) \right| \cdot \sup |R(g(t))| + \sup \left| \frac{\partial R}{\partial s}(t) \right| \right) \text{vol}(g(t))^{1-2/p} \\ &\quad + \left( 1 + \frac{\sqrt{n}}{2} + \frac{\sqrt{n}}{p} \right) \sup \left| \frac{\partial g}{\partial s}(t) \right| \cdot |\mathfrak{Y}(g(t))|. \end{aligned}$$

Since  $\sup |\partial g / \partial s(t)|$ ,  $\sup |\partial R / \partial s(t)|$ ,  $\sup |R(g(t))|$ ,  $\text{vol}(g(t))^{1-2/p}$ , and  $|\mathfrak{Y}(g(t))|$  are all continuous on the closed manifold  $M$ , we conclude that  $(\partial N / \partial s)(t, t)$  is locally bounded, hence  $\mathfrak{Y}(g(t))$  is locally Lipschitz by Theorem 1.1.

Next, we simplify the formula (4-6). By (4-5) we have

$$R\phi(t) - a \Delta \phi(t) = \mathfrak{Y}(g(t)) \|\phi(t)\|_{p,g(t)}^{2-p} \phi(t)^{p-1}.$$

Multiplying both sides by  $\phi(t) \text{tr}(\partial g / \partial s)(t)$  and integrating by parts gives

$$(4-10) \quad \begin{aligned} \mathfrak{Y}(g(t)) \|\phi(t)\|_{p,g(t)}^{2-p} \int \phi(t)^p \text{tr} \frac{\partial g}{\partial s}(t) d\mu_{g(t)} \\ = \int \left( R\phi(t)^2 + a |\nabla \phi(t)|^2 - \frac{a}{2} \Delta(\phi(t)^2) \right) \text{tr} \frac{\partial g}{\partial s}(t) d\mu_{g(t)}. \end{aligned}$$

Substituting (4-10) into (4-6), we get

$$(4-11) \quad \begin{aligned} \frac{\partial N}{\partial s}(t, t) &= \left( \int \frac{a}{2p} \Delta(\phi(t)^2) \text{tr} \frac{\partial g}{\partial s}(t) + \phi(t)^2 \left( \frac{\partial R}{\partial s}(t) + \frac{1}{n} R \text{tr} \frac{\partial g}{\partial s}(t) \right) \right) d\mu_{g(t)} \\ &\quad - \int a \left\langle \left( \frac{\partial g}{\partial s}(t) \right)^0, \nabla \phi(t) \otimes \nabla \phi(t) \right\rangle d\mu_{g(t)} \|\phi(t)\|_{p,g(t)}^{-2}. \end{aligned}$$

The evolution function of scalar curvature  $R$  is

$$\frac{\partial R}{\partial s} = \operatorname{div} \left( \operatorname{div} \frac{\partial g}{\partial s} \right) - \Delta \operatorname{tr} \frac{\partial g}{\partial s} - \left\langle \operatorname{Rc}, \frac{\partial g}{\partial s} \right\rangle.$$

Substituting this into (4-11) gives

$$\begin{aligned} \frac{\partial N}{\partial s}(t, t) &= \int \left\langle \frac{\partial g}{\partial s}(t), (\nabla \nabla \phi(t))^2 - \phi(t)^2 \operatorname{Rc} - a \nabla \phi(t) \otimes \nabla \phi(t) \right\rangle^0 d\mu_{g(t)} \|\phi(t)\|_{p, g(t)}^{-2} \\ &= - \int \left\langle \frac{\partial g}{\partial s}(t), \phi(t)^2 (\operatorname{Rc} + (n-2)\phi(t)^{b/2} \nabla \nabla \phi(t)^{-b/2}) \right\rangle^0 d\mu_{g(t)} \|\phi(t)\|_{p, g(t)}^{-2}. \end{aligned}$$

By the conformal transformation of trace-free Ricci curvature (4-3),

$$\frac{\partial N}{\partial s}(t, t) = - \int \phi^{-b}(t) \left\langle \frac{\partial g}{\partial s}(t), \operatorname{Rc}^0(\phi^b(t)g(t)) \right\rangle d\mu_{\phi^b(t)g(t)} \operatorname{vol}(\phi^b(t)g(t))^{-2/p}.$$

Since  $\phi^b(t) = \bar{g}(t)/g(t)$ , we get

$$(4-12) \quad \frac{\partial N}{\partial s}(t, t) = - \int \frac{\bar{g}(t)}{g(t)} \left\langle \frac{\partial g}{\partial s}(t), \operatorname{Rc}^0(\bar{g}(t)) \right\rangle_{\bar{g}(t)} d\mu_{\bar{g}(t)} \operatorname{vol}(\bar{g}(t))^{-2/p}.$$

Then the theorem follows from Corollary 1.1.1. □

*Proof of Theorem 1.4.* Let  $\phi(t)$  be any minimizer of  $Q(\cdot, g(t))$ . Then

$$\tilde{g}(t) = \phi^b(t)g(t)$$

is the Yamabe metric in the conformal class  $[g(t)]$ . Define

$$\mathcal{N}(\tilde{g}(s), g(t)) := Q(\phi(s), g(t)).$$

Then

$$\mathfrak{Y}(g(t)) = \mathcal{N}(\tilde{g}(t), g(t)), \quad \mathfrak{Y}(g(t)) \leq \mathcal{N}(\tilde{g}(s), g(t)).$$

Hence, when  $t > 0$ ,

$$(4-13) \quad \begin{aligned} \frac{\mathcal{N}(\tilde{g}(t), g(t)) - \mathcal{N}(\tilde{g}(t), g(0))}{t} &\leq \frac{\mathfrak{Y}(g(t)) - \mathfrak{Y}(g(0))}{t} \\ &\leq \frac{\mathcal{N}(\tilde{g}(0), g(t)) - \mathcal{N}(\tilde{g}(0), g(0))}{t}. \end{aligned}$$

By (4-12) and the definitions of  $\mathcal{N}$  and  $N$ , we get

$$(4-14) \quad \frac{\partial \mathcal{N}}{\partial t}(\tilde{g}(t), g(t)) = - \int \frac{\tilde{g}(t)}{g(t)} \left\langle \frac{\partial g}{\partial t}(t), \operatorname{Rc}^0(\tilde{g}(t)) \right\rangle_{\tilde{g}(t)} d\mu_{\tilde{g}(t)} \operatorname{vol}(\tilde{g}(t))^{-2/p}.$$

It is easy to see that  $\mathcal{N}(\tilde{g}(s), g(t))$  and  $(\partial\mathcal{N}/\partial t)(\tilde{g}(s), g(t))$  as functionals of  $\tilde{g}(s)$  and  $g(t)$  are continuous with the  $C^2$ -topology on the space of metrics. Applying the mean value theorem to the variable  $t$  in the function  $\mathcal{N}(\tilde{g}(s), g(t))$ , there exists a number  $0 < \beta(s, t) < t$  such that

$$\mathcal{N}(\tilde{g}(s), g(t)) - \mathcal{N}(\tilde{g}(s), g(0)) = t \frac{\partial\mathcal{N}}{\partial t}(\tilde{g}(s), g(\beta(s, t))).$$

Substituting into (4-13), we come to

$$(4-15) \quad \frac{\partial\mathcal{N}}{\partial t}(\tilde{g}(t), g(\beta(t, t))) \leq \frac{\mathfrak{Y}(g(t)) - \mathfrak{Y}(g(0))}{t} \leq \frac{\partial\mathcal{N}}{\partial t}(\tilde{g}(0), g(\beta(0, t))).$$

Letting  $t \rightarrow 0$ , then  $\beta(0, t) \rightarrow 0$  and  $\beta(t, t) \rightarrow 0$  follows from  $0 < \beta(s, t) < t$ . Hence

$$(4-16) \quad \limsup_{t \rightarrow 0} \frac{\mathfrak{Y}(g(t)) - \mathfrak{Y}(g(0))}{t} \leq \frac{\partial\mathcal{N}}{\partial t}(\tilde{g}(0), g(0)) \quad \text{for all } \tilde{g}(0) \in \Lambda[g(0)].$$

Pick  $t_i > 0, t_i \rightarrow 0$  such that

$$(4-17) \quad \liminf_{t \rightarrow 0} \frac{\mathfrak{Y}(g(t)) - \mathfrak{Y}(g(0))}{t} = \lim_{i \rightarrow \infty} \frac{\mathfrak{Y}(g(t_i)) - \mathfrak{Y}(g(0))}{t_i}.$$

Using the compactness of  $\Lambda[g_0]$ , there exists a subsequence of  $t_i$  (denoted again by  $t_i$  for simplicity) and a Yamabe metric  $\bar{g}_0 \in \Lambda[g_0]$  such that

$$\lim_{i \rightarrow \infty} \tilde{g}(t_i) = \bar{g}_0.$$

Then by the first inequality in (4-15),

$$(4-18) \quad \begin{aligned} \lim_{i \rightarrow \infty} \frac{\mathfrak{Y}(g(t_i)) - \mathfrak{Y}(g(0))}{t_i} &\geq \lim_{i \rightarrow \infty} \frac{\partial\mathcal{N}}{\partial t}(\tilde{g}(t_i), g(\beta(t_i, t_i))) \\ &= \frac{\partial\mathcal{N}}{\partial t}(\bar{g}_0, g(0)). \end{aligned}$$

Hence by (4-16) and (4-17),  $\mathfrak{Y}(g(t))$  is differentiable at  $t = 0$  and

$$(4-19) \quad \lim_{t \rightarrow 0} \frac{\mathfrak{Y}(g(t)) - \mathfrak{Y}(g(0))}{t} = \frac{\partial\mathcal{N}}{\partial t}(\bar{g}_0, g(0)).$$

This implies the first equality in (1-2) by (4-16) and (4-14). We now know that the  $t_i$  chosen after (4-17) can be any sequence of  $t_i > 0, t_i \rightarrow 0$ . Then the  $\bar{g}_0$  in (4-19) can be any accumulation point of  $\bar{g}(t)$  as  $t \rightarrow 0$  in  $\Lambda[g_0]$ , hence any accumulation point of  $\Lambda[g(t)]$  as  $t \rightarrow 0$  in  $\Lambda[g_0]$ . The second equality in (1-2) follows from applying this to other metrics  $g(t) \in G_{(\partial g/\partial t)(0)}(g_0, t)$ .  $\square$



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