Pacific Journal of Mathematics

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Volume 254 No. 1 November 2011

PACIFIC JOURNAL OF MATHEMATICS
Vol. 254, No. 1, 2011

REGULARITY OF THE FIRST EIGENVALUE OF THE p-LAPLACIAN AND YAMABE INVARIANT ALONG GEOMETRIC FLOWS

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We first prove that the first eigenvalue of the p-Laplace operator and the Yamabe invariant are both locally Lipschitz along geometric flows under weak assumptions without assumptions on curvature. Secondly, the Yamabe invariant is found to be directionally differentiable along geometric flows. As an application, an open question about the Yamabe metric and Einstein metric is partially answered.

1. Introduction

Motivated by the Hamilton's Ricci flow, the method of geometric flow has been widely used to deal with geometric and topological properties of manifolds. We often encounter the derivative of geometric quantities when applying the method of geometric flow. Cao [2007; 2008] and Li [2007] consider the monotonicity of the first eigenvalue of $-\Delta + cR$ ($c \geq \frac{1}{4}$) based on their derivatives along Ricci flow. Ling [2007] proved a comparison theorem for the eigenvalue of the Laplace operator based on its derivative along Ricci flow. Unfortunately, there are many geometric quantities about which we don't know whether they are differentiable along the flow. Chang and Lu [2007] derive a formula for the derivative of the Yamabe constant along Ricci flow under a crucial technical assumption. Recently Wu, Wang and Zheng [Wu et al. 2010] considered the first eigenvalue of the p-Laplace operator, whose differentiability along Ricci flow is unknown.

For the first eigenvalue of a linear operator, we may assume that there is a C^1-family of smooth eigenvalues and eigenfunctions along geometric flow by eigenvalue perturbation theory. We have no uniform method to deal with the smoothness of the first eigenvalue of a nonlinear operator — even the continuity is unknown.

As the first eigenvalue can be seen as a minimum of a functional, we consider the regularity of geometric quantities of this type along geometric flow. Inspired

This work is partially supported by the NSFC10871069.
MSC2010: primary 58C40; secondary 53C44.
Keywords: Dini derivative, locally Lipschitz, first eigenvalue, p-Laplace operator, Yamabe invariant, geometric flow.

by the method used in [Hamilton 1986; Chow and Lu 2002] to prove the maximum principle for systems, we first study the relationship between the local Lipschitz property of continuous functions and their Dini derivatives.

Theorem 1.1. *Let $m(t)$ be a continuous function on an interval $\mathcal{I} \subset \mathbb{R}$. Suppose that for any $t \in \mathcal{I}$ there exists a C^1 function $M(t, s)$ of s defined on a neighborhood of t such that $M(t, t) = m(t)$ and $M(t, s) \geq m(s)$.*

(1) *If $(\partial M/\partial s)(t, t)$ is locally bounded, then $m(t)$ is locally Lipschitz.*

(2) *For any t in the interior of $\mathcal{I}$, if $m(t)$ is differentiable at t, then $m'(t) = (\partial M/\partial s)(t, t)$.*

Remark. By (2), if $m(t)$ is differentiable at an interior point t, then the derivative of $m(t)$ at this point is exactly $(\partial M/\partial s)(t, t)$, regardless of the choice of function $M(t, s)$.

Corollary 1.1.1. *In the same setting of Theorem 1.1, if $(\partial M/\partial s)(t, t)$ is locally bounded, then $m(t)$ is differentiable almost everywhere and $m'(t) = (\partial M/\partial s)(t, t)$ almost everywhere.*

Applying Theorem 1.1, we get the following results on the regularities of the first eigenvalues $\lambda_{1,p}$ of the p-Laplace operator and the Yamabe invariant along the general C^1 family of smooth geometric flows in this paper. We find that the first eigenvalue $\lambda_{1,p}$ of the p-Laplace operator is in general locally Lipschitz continuous. We also get local Lipschitz continuity of the Yamabe invariant and find its derivative with respect to t almost everywhere.

Theorem 1.2. *Let $g(x, t)$ be a C^1 family of smooth metrics on a n-dimensional compact Riemannian manifold M. Then the first eigenvalue $\lambda_{1,p}(g(t))$ of the p-Laplace operator is locally Lipschitz if $p \geq 2$ and M is closed or if $p > 1$ and M has nonempty boundary.*

Remark. In [Wu et al. 2010], a similar result on local Lipschitz continuity was obtained, but under some assumptions on curvature. Theorem 1.2 implies that local Lipschitz continuity should be available for more general smooth geometric flows without any curvature conditions.

Theorem 1.3. *Suppose M is an n-dimensional $(n \geq 3)$ closed connected Riemannian manifold, and $g(t)$, $t \in [0, T)$, is a C^1 family of smooth metrics on M. If $\bar{g}(t)$ is the Yamabe metric in the conformal class $[g(t)]$ for any $t \in [0, T)$, then the Yamabe invariant $\mathcal{Y}(g(t))$ is locally Lipschitz with respect to t, and*

$$\frac{d\mathcal{Y}(g(t))}{dt} \overset{\text{a.e.}}{=} -\int \frac{\bar{g}(t)}{g(t)} \left\langle \frac{\partial g}{\partial t}(t), \mathrm{Rc}^0(\bar{g}(t)) \right\rangle_{\bar{g}(t)} d\mu_{\bar{g}(t)} \operatorname{vol}(\bar{g}(t))^{-2/p}, \tag{1-1}$$

where a.e. stands for "almost everywhere", $p = \dfrac{2n}{n-2}$, and $\mathrm{Rc}^0(\bar{g}(t))$ is the trace-free part of Ricci curvature of $\bar{g}(t)$.

Let g_0 be a smooth metric on manifold M, $[g_0]$ be the conformal class of g_0, $\Lambda[g_0]$ be the collection of Yamabe metrics in $[g_0]$ and h be a smooth $(0,2)$-type symmetric tensor on M. Denote by $G_h(g_0,t)$ the collection of C^1 family of smooth metrics $g(t)$, $t\in[0,\varepsilon)$ with $g(0)=g_0$ and $(\partial g/\partial t)(0)=h$ for some $\varepsilon>0$, we define $\Lambda_h[g_0]$ by

$$\Lambda_h[g_0]:=\bigcup_{g(t)\in G_h(g_0,t)}\{\,\bar g_0\in\Lambda[g_0]:\bar g_0 \text{ is an accumulation point of } \Lambda[g(t)] \text{ as } t\to 0\,\},$$

where $\bar g_0$ generally exists by the compactness of $\Lambda[g_0]$ when $[g_0]\neq[g_{\mathrm{can}}]$, where g_{can} denotes the canonical metric on S^n [Anderson 2005]. (They prove that if $[g_i]\to[g_0]\neq[g_{\mathrm{can}}]$ smoothly, then every sequence of Yamabe metrics $(g^j)_i\in[g_i]$ has a subsequence converging smoothly to a Yamabe metric $[g^j]\in[g_0]$.) It is easy to see that if $\bar g_0\in\Lambda_h[g_0]$ then $c\bar g_0\in\Lambda_h[g_0]$ and $\bar g_0\in\Lambda_{ch}[g_0]$ for any $c>0$.

Recently, Brendle [2008] with Marques [2009] gave counterexamples to the compactness for a full set of solutions to the Yamabe equation if the dimension of the manifold greater than 24. Later, Khuri, Marques and Schoen [Khuri et al. 2009] proved compactness if the dimension equal or less than 24.

In addition to Theorem 1.3, we have the following derivative calculation at $t=0$.

Theorem 1.4. *Let $g(t)$, $t\in[0,T)$, be a C^1 family of smooth metrics on a manifold M and g_{can} be the canonical metric on S^n. If $g(0)=g_0$ and $[g_0]\neq[g_{\mathrm{can}}]$, then*

$$\text{(1-2)}\quad \frac{d\mathcal{Y}(g(t))}{dt}\bigg|_{t=0}=\min_{\tilde g_0\in\Lambda[g_0]}\left\{-\int\frac{\tilde g_0}{g_0}\left\langle\frac{\partial g}{\partial t}(0),\mathrm{Rc}^0(\tilde g_0)\right\rangle_{\tilde g_0}d\mu_{\tilde g_0}\,\mathrm{vol}(\tilde g_0)^{-2/p}\right\}$$
$$=-\int\frac{\bar g_0}{g_0}\left\langle\frac{\partial g}{\partial t}(0),\mathrm{Rc}^0(\bar g_0)\right\rangle_{\bar g_0}d\mu_{\bar g_0}\,\mathrm{vol}(\bar g_0)^{-2/p},$$

where $p=2n/(n-2)$, $\bar g_0\in\Lambda_{(\partial g/\partial t)(0)}[g_0]$, $\mathrm{Rc}^0(\tilde g_0)$ is the trace-free part of Ricci curvature with respect to $\tilde g_0$ and $\mathrm{vol}(\tilde g_0)$ is the volume of M respect to $\tilde g_0$. In particular, $\mathcal{Y}$ is directionally differentiable at g_0.

Remark. This formula generalizes similar calculations in [Anderson 2005] where $\mathrm{tr}(\partial g/\partial t)(0)=0$, $\mathrm{vol}(g(t))=1$, and g_0 has constant scalar curvature. Meanwhile, when $\bar g_0\in\Lambda_{(\partial g/\partial t)(0)}[g_0]$, Equation (1-2) becomes more convenient to calculate, compared to the derivative calculation in [Anderson 2005] (in another form):

$$\text{(1-3)}\quad \min_{\tilde g_0}\left\{-\int\frac{\tilde g_0}{g_0}\left\langle\mathrm{Rc}^0(\tilde g_0),\frac{\partial g}{\partial t}(0)\right\rangle_{\tilde g_0}d\mu_{\tilde g_0}\right\},$$

where $\tilde g_0\in\Lambda_1[g_0]$ is taken over all accumulation points of $\Lambda_1[g(t)]$ as $t\to 0$ for $\Lambda_1[g_0]$ the set of unit volume Yamabe metrics in $[g_0]$. The derivative is difficult to calculate using this formula, but by Theorem 1.4 we can calculate this derivative if we know a Yamabe metric $\bar{g}_0$ in $\Lambda_{(\partial g/\partial t)(0)}[g_0]$. Moreover, the set to minimize in (1-3) has only one element by the last equality in (1-2).

In addition to the local Lipschitz property of the Yamabe invariant, we have:

Corollary 1.4.1. *With the same assumptions as in Theorem 1.3, the Yamabe invariant $\mathcal{Y}(g(t))$ is directionally differentiable at all t where $[g(t)] \neq [g_{\mathrm{can}}]$.*

In particular, in formula (1-2), if $(\partial g/\partial t)(0) = -2\,\mathrm{Rc}(g_0)$ and g_0 is a Yamabe metric in $\Lambda_{-\mathrm{Rc}(g_0)}[g_0]$, then $g_0 \in \Lambda_{-2\,\mathrm{Rc}(g_0)}[g_0]$ and $R(g_0)$ is constant, hence

$$\begin{aligned}\frac{d\mathcal{Y}(g(t))}{dt}\bigg|_{t=0} &= \int \langle \mathrm{Rc}(g_0), \mathrm{Rc}^0(g_0)\rangle_{g_0}\, d\mu_{g_0}\, \mathrm{vol}(g_0)^{-2/p} \\ &= \int \left|\mathrm{Rc}^0(g_0)\right|^2 d\mu_{g_0}\, \mathrm{vol}(g_0)^{-2/p} \geq 0.\end{aligned}$$

Ricci flow evolves sphere to sphere, so we have the following conclusion along the Ricci flow.

Corollary 1.4.2. *Let M^n be a closed and connected manifold with $n \geq 3$ and $g(t), t \in [0, T)$, be a solution of Ricci flow $\partial g/\partial t = -2\,\mathrm{Rc}$ on M with $g(0) = g_0$. If $g_0 \in \Lambda_{-\mathrm{Rc}}[g_0]$, then $d\mathcal{Y}(g(t))/dt|_{t=0} \geq 0$ and $d\mathcal{Y}(g(t))/dt|_{t=0} = 0$ if only if g_0 is a Einstein metric.*

Remark. There is a similar result in [Chang and Lu 2007] under the assumption that there exists a C^1 family of $\phi(t) > 0$ such that $\phi(t)^{4/(n-2)}g(t)$ is a Yamabe metric and $\phi(0)$ is constant. From the definition of $\Lambda_{-\mathrm{Rc}}[g_0]$ we can see that our assumption is weaker.

Let $\mathscr{C}$ denote the set of unit volume constant scalar curvature metrics on a connect closed manifold M; it is well-known (see [Besse 1987]) that generically $\mathscr{C}$ is an infinite-dimensional manifold. Let $s : \mathscr{C} \mapsto \mathbb{R}$ be the scalar curvature function. It has long been an open problem whether a Yamabe metric which is a local maximizer of s is necessarily an Einstein metric [Besse 1987]. Some progress on this question was made in [Bessieres et al. 2003] in dimension 3 and in [Anderson 2005] in any dimension. Let $\mathcal{M}$ be the collection of all smooth metrics on M and $\mathcal{Y} : \mathcal{M} \mapsto \mathbb{R}$ be the Yamabe invariant function. By the definition of the Yamabe invariant, $s(g) \geq \mathcal{Y}(g)$ for any $g \in \mathscr{C}$, hence if a Yamabe metric is a local maximizer of s, it must be a local maximizer of $\mathcal{Y}$. Now, we consider whether a Yamabe metric that is a local maximizer of the Yamabe invariant is necessarily an Einstein metric. Following from Corollary 1.4.2, the next result gives a partial answer.

Corollary 1.4.3. *Let M^n be a closed and connected manifold with $n \geq 3$ and suppose a Yamabe metric g is a local maximum of the Yamabe invariant functional $\mathcal{Y}(\cdot)$. If $g \in \Lambda_{-\mathrm{Rc}}[g]$, then g is Einstein.*

In Section 2, we give a basic introduction to Dini derivatives and the proof of Theorem 1.1. In Section 3, we prove the Lipschitz property of the first eigenvalue of the p-Laplace operator along geometric flows. In Section 4, we show that the Yamabe invariant is locally Lipschitz and directionally differentiable along geometric flows.

2. Dini derivatives and the proof of Theorem 1.1

In this section, we first recall the definitions of Dini derivatives and semicontinuity. Then we give some propositions about Dini derivatives. Lastly, we prove Theorem 1.1.

Hamilton [1986] studied properties of Lipschitz functions by means of their Dini derivatives, and from this derived the maximum principle for systems on closed manifolds. Chow [2002] proved similar results in weaker settings. Dini derivatives provide a powerful way to deal with nonregular functions.

These definitions of Dini derivatives and semicontinuity also appear in [Chow et al. 2008].

Definition 2.1 (Dini derivatives). Let $f(t)$ be a function on (a, b). The upper Dini derivative is the lim sup of forward difference quotients:

$$\frac{d^+ f}{dt}(t) := \limsup_{h \to 0^+} \frac{f(t+h) - f(t)}{h}.$$

The lower Dini derivative is the lim inf of forward difference quotients:

$$\frac{d^- f}{dt}(t) := \liminf_{h \to 0^+} \frac{f(t+h) - f(t)}{h}.$$

The upper converse Dini derivative is the lim sup of backward difference quotients:

$$\frac{d_+ f}{dt}(t) := \limsup_{h \to 0^+} \frac{f(t) - f(t-h)}{h}.$$

The lower converse Dini derivative is the lim inf of backward difference quotients:

$$\frac{d_- f}{dt}(t) := \liminf_{h \to 0^+} \frac{f(t) - f(t-h)}{h}.$$

If the function f is also defined at a, we can define its upper Dini derivative and lower Dini derivative at a; and if the function f is also defined at b, we can define its upper converse Dini derivative and lower converse Dini derivative at b.

Since we don't make any assumption on the function $f(t)$, it is possible that any one of the Dini derivatives of $f(t)$ above may take the value $+\infty$ or $-\infty$.

Definition 2.2 (semicontinuity). Let $f(t)$ be a function on an interval. We say f is right upper semicontinuous if $\limsup_{h \to 0^+} f(t+h) \le f(t)$; we say f is right lower semicontinuous if $\liminf_{h\to 0^+} f(t+h) \ge f(t)$; we say f is left upper semicontinuous if $\limsup_{h\to 0^+} f(t-h) \le f(t)$; we say f is left lower semicontinuous if $\liminf_{h\to 0^+} f(t-h) \ge f(t)$.

Lemma 2.3. *If $f(t):(a,b)\to\mathbb{R}$ is left lower semicontinuous with $(d^+f/dt)(t)\le 0$, then $f(t)$ is decreasing.*

Proof. Given $\varepsilon > 0$, define $f_\varepsilon(t) := f(t) - \varepsilon t$. We shall show that $f_\varepsilon(t) \le f_\varepsilon(s)$ for any $a < s \le t < b$. The lemma then follows from taking $\varepsilon \to 0$.

Since $(d^+f/dt)(s) \le 0$, we have $(d^+f_\varepsilon/dt)(s) \le -\varepsilon$, then there exists a number $\delta(s,\varepsilon) > 0$ such that $(f_\varepsilon(s+h) - f_\varepsilon(s))/h \le -\varepsilon/2 < 0$ for all $h \in (0, \delta(s,\varepsilon))$, hence $f_\varepsilon(t) \le f_\varepsilon(s)$ on $h \in [s, s+\delta(s,\varepsilon))$. Define $\tau(\varepsilon, s) \in [s,b]$ by

$$\tau := \sup\left\{\tau' \in [s,b] : f_\varepsilon(t) \le f_\varepsilon(s) \text{ for all } t \in [s,\tau')\right\}.$$

then $\tau \ge s+\delta(s,\varepsilon) > s$. One can check that, in fact, $f_\varepsilon(t) \le f_\varepsilon(s)$ for all $t \in [s,\tau)$. We now prove $\tau = b$ to complete the proof. If for some s and $\varepsilon > 0$, we have $\tau < b$, then there exists a sequence of times $\{\tau_i\} \nearrow \tau$, such that $f_\varepsilon(s) \ge f_\varepsilon(\tau_i - 1/2^i)$ when i is large enough. Hence

$$f_\varepsilon(s) \ge \liminf_{i\to\infty} f_\varepsilon(\tau_i - 1/2^i) \ge \liminf_{h\to 0^+} f_\varepsilon(\tau - h) \ge f_\varepsilon(\tau)$$

follows from the left lower semicontinuity of $f_\varepsilon(t)$. Applying the above procedure again by replacing s with τ gives $f_\varepsilon(t) \le f_\varepsilon(\tau) \le f_\varepsilon(s)$ when $t \in [\tau, \tau+\delta(\tau,\varepsilon))$, hence $f_\varepsilon(t) \le f_\varepsilon(s)$ when $t \in [s, \tau+\delta(\tau,\varepsilon))$. This is a contradiction since the definition of τ implies $\delta(\tau,\varepsilon) \le 0$. □

Note. A similar conclusion can be found in [Chow et al. 2008]. There, the domain of f is $[0,T)$, hence f must be both left lower semicontinuous and right upper semicontinuous. Here we choose the domain of f to be (a,b), so we can weaken the assumptions on f.

Proposition 2.4. (a) *If $f(t) : (a,b) \to \mathbb{R}$ is left lower semicontinuous, then $d^+f/dt \le 0$ if and only if $f(t)$ is decreasing.*

(b) *If $f(t) : (a,b) \to \mathbb{R}$ is right upper semicontinuous, then $d_+f/dt \le 0$ if and only if $f(t)$ is decreasing.*

(c) *If $f(t) : (a,b) \mapsto \mathbb{R}$ is left upper semicontinuous, then $d^-f/dt \ge 0$ if and only if $f(t)$ is increasing.*

(d) *If $f(t) : (a,b) \mapsto \mathbb{R}$ is right lower semicontinuous, then $d_-f/dt \ge 0$ if and only if $f(t)$ is increasing.*

Proof. (a) If $f(t)$ is decreasing then $d^+f/dt \le 0$. The other direction follows from Lemma 2.3.

(b)–(d) follows from applying part (a) to the functions $-f(-t)$, $-f(t)$, and $f(-t)$, respectively. $\square$

From this propositions, we see that a semicontinuous function is monotonic if certain types of its Dini derivatives have a definite sign. A further analysis shows that monotonicity can be a nice bridge between Dini derivatives of different type.

Claim 2.5. *Let $\mathcal{I} \subset \mathbb{R}$ be an interval and $\mathring{\mathcal{I}}$ be its interior.*

(a) *If $f(t) : \mathcal{I} \mapsto \mathbb{R}$ is right upper semicontinuous and left lower semicontinuous,*

$$\frac{d^+f}{dt} \le 0 \text{ in } \mathring{\mathcal{I}} \iff f(t) \text{ is decreasing on } \mathcal{I} \iff \frac{d_+f}{dt} \le 0 \text{ in } \mathring{\mathcal{I}}.$$

(b) *If $f(t) : \mathcal{I} \mapsto \mathbb{R}$ is right lower semicontinuous and left upper semicontinuous,*

$$\frac{d^-f}{dt} \ge 0 \text{ in } \mathring{\mathcal{I}} \iff f(t) \text{ is increasing on } \mathcal{I} \iff \frac{d_-f}{dt} \ge 0 \text{ in } \mathring{\mathcal{I}}.$$

Proof. We prove the first part; the second is similar. If $f(t) : \mathcal{I} \mapsto \mathbb{R}$ is right upper semicontinuous and left lower semicontinuous, then $f(t)$ is decreasing on $\mathcal{I}$ if and only if $f(t)$ is decreasing on $\mathring{\mathcal{I}}$. The conclusion then follows from parts (a) and (b) of Proposition 2.4. $\square$

Theorem 2.6. *If $f : (a, b) \mapsto \mathbb{R}$ is a continuous function with d^+f/dt or d_+f/dt locally bounded from above and d^-f/dt or d_-f/dt locally bounded from below, then f is locally Lipschitz.*

Proof. Given any $s \in (a, b)$, let $U(s)$ be a compact and connected neighborhood of s in (a, b). Then on $U(s)$, without loss of generality, we can assume $d^+f/dt \le A$ or $d_+f/dt \le A$ and $d^-f/dt \ge -A$ or $d_-f/dt \ge -A$, where $A > 0$ is a constant. Hence $d^+(f - At)/dt \le 0$ (or $d_+(f - At)/dt \le 0$) by parts (a) and (b) of Proposition 2.4, and $d_-(f + At)/dt \ge 0$ (or $d^-(f + At)/dt \ge 0$) by parts (c) and (d). Then $f - At$ is decreasing and $f + At$ is increasing on $U(s)$ by Claim 2.5. Thus $|f(t_2) - f(t_1)| \le A\,|t_2 - t_1|$ for any $t_1, t_2 \in U(s)$, so f is locally Lipschitz. $\square$

Proof of Theorem 1.1. Since $M(t, t) = m(t)$ and $M(t, s) \ge m(s)$ in a neighborhood of t, we have

$$\frac{d^+m}{dt}(t) = \limsup_{h\to 0^+} \frac{m(t+h) - m(t)}{h} \le \limsup_{h\to 0^+} \frac{M(t, t+h) - M(t, t)}{h} = \frac{\partial M}{\partial s}(t, t), \tag{2-1}$$

$$\frac{d_-m}{dt}(t) = \liminf_{h\to 0^+} \frac{m(t) - m(t-h)}{h} \ge \liminf_{h\to 0^+} \frac{M(t, t) - M(t, t-h)}{h} = \frac{\partial M}{\partial s}(t, t). \tag{2-2}$$

Since $(\partial M/\partial s)(t,t)$ is locally bounded, $(d^+m/dt)(t)$ is bounded from above and $(d_-m/dt)(t)$ is bounded from below. Then by Theorem 2.6, the function $m(t)$ is locally Lipschitz in the interior of $\mathcal{I}$.

Let a be the left endpoint of $\mathcal{I}$, b be the right endpoint of $\mathcal{I}$. If $a \in \mathcal{I}$, let $c = \min\{a+1, (a+b)/2\}$. Since $(\partial M/\partial s)(t,t)$ is locally bounded on $\mathcal{I}$, we can assume that $|(\partial M/\partial s)(t,t)| \le A$ (A is a constant) on $[a,c]$. Then $d^+(m(t) - At)/dt \le 0$ on $[a,c)$ and $d_-(m(t) + At)/dt \ge 0$ on $(a,c]$ by (2-1) and (2-2). Hence by part (a) of Claim 2.5, the function $m(t) - At$, is decreasing on $[a,c]$, and by part (b), the function $m(t) + At$ is increasing on $[a,c]$. Then $|m(t_1) - m(t_2)| \le A\,|t_1 - t_2|$ for any $t_1, t_2 \in [a,c]$, so $m(t)$ is locally Lipschitz at $t = a$. Similarly, if $b \in \mathcal{I}$, then $m(t)$ is locally Lipschitz at $t = b$. In conclusion, $m(t)$ is locally Lipschitz on $\mathcal{I}$.

For any t in the interior of $\mathcal{I}$, if $m(t)$ is differentiable at this point, then by (2-1) we have $m'(t) = (d^+m/dt)(t) \le (\partial M/\partial s)(t,t)$, and by (2-2) we have $m'(t) = (d_-m/dt)(t) \ge (\partial M/\partial s)(t,t)$. Hence $m'(t) = (\partial M/\partial s)(t,t)$. $\square$

3. First eigenvalue of the p-Laplacian

In this section we consider the local Lipschitz property of the p-Laplace operator along general geometric flows. Let (M, g) be a compact connected Riemannian manifold. Define

$$G(f,g) := \frac{\int_M |\nabla f|_g^p \, d\mu_g}{\int_M |f|^p \, d\mu_g},$$

where $\nabla f = df$ is a covariant vector. Recalling the definition of the first eigenvalue $\lambda_{1,p}(g)$ of the p-Laplace operator, it is known that if $\partial M \neq \varnothing$ then

$$\lambda_{1,p}(g) := \inf\bigl\{G(f,g) : f \in W_0^{1,p}(M),\ \ f \neq 0\bigr\}$$

and if M is closed then

$$\lambda_{1,p}(g) := \inf\bigl\{G(f,g) : f \in W^{1,p}(M),\ \int_M |f|^{p-2} f \, d\mu_g = 0,\ \ f \neq 0\bigr\}.$$

The minimum (a positive number) is achieved by a $C^{1,\alpha}$ $(0 < \alpha < 1)$ eigenfunction f (see [Serrin 1964; Tolksdorf 1984]). This eigenfunction f satisfies the Euler–Lagrange equation

$$\Delta_p f = -\lambda_{1,p}(g)|f|^{p-2} f,$$

where Δ_p $(p > 1)$ is the p-Laplace operator with respect to g given by

$$\Delta_p f = \operatorname{div}_g(|\nabla f|_g^{p-2}\, \nabla f).$$

The following theorem implies that $\lambda(g(t))$ is continuous with respect to t along general geometric flows.

Theorem 3.1 [Wu et al. 2010]. *If g_1 and g_2 are two metrics on M which satisfy $(1+\varepsilon)^{-1}g_1 \le g_2 \le (1+\varepsilon)g_1$, then for any $p > 1$, we have*

$$(1+\varepsilon)^{-(n+p/2)} \le \frac{\lambda_{1,p}(g_1)}{\lambda_{1,p}(g_2)} \le (1+\varepsilon)^{(n+p/2)}.$$

Let $f \in C^{1,\alpha}(M)$ be nonconstant and $g(x,t)$, $t \in [0,T)$, be a C^1 family of smooth metrics on M. Define a function of $c \in (-\infty,\infty)$ and $t \in [0,T)$:

$$P(c,t) := \int_M |f+c|^{p-2}(f+c)\, d\mu_{g(t)}, \quad p \ge 2.$$

The function $P(c,t)$ is C^1 with respect to c and t, since

$$\frac{\partial P}{\partial c} = (p-1)\int_M |f+c|^{p-2}\, d\mu_{g(t)} > 0.$$

Then by the implicit function theorem, given any c_0 and t_0 there exists a C^1 function $c(t)$ defined on a neighborhood of t_0 such that $P(c(t),t) = P(c_0,t_0)$.

In this and the next sections, if f is a real function on M, we simply write $\sup f$ instead of $\sup_{x\in M} f(x)$. Let $g(t)$ be a family of Riemannian metrics on manifold. If $\alpha(t)$ is a family of $(0,2)$-type tensors, we denote by $\operatorname{tr}\alpha(t) = g^{ij}(t)\alpha_{ij}(t)$ its trace with respect to $g(t)$ and by

$$|\alpha(t)|_{g(s)} = \sqrt{g^{ij}(s)g^{kl}(s)\alpha_{ik}(t)\alpha_{jl}(t)}$$

its norm with respect to $g(s)$; if $\beta(t)$ is also a family of $(0,2)$-type tensors, we denote by

$$\langle \alpha(t), \beta(t)\rangle_{g(s)} = \sqrt{g^{ij}(s)g^{kl}(s)\alpha_{ik}(t)\beta_{jl}(t)}$$

the inner product derived from the metric $g(s)$. Moreover, we use $|\alpha(t)|$ instead of $|\alpha(t)|_{g(t)}$ and $\langle \alpha(t), \beta(t)\rangle$ instead of $\langle \alpha(t), \beta(t)\rangle_{g(t)}$ for simplicity.

Proof of Theorem 1.2. For any t_0, let $f(t_0)$ be a minimizer of $G(\,\cdot\,, g(t_0))$. If M is closed and $p \ge 2$, then $f(t_0)$ is a nonconstant C^1 function on M with

$$\int_M |f(t_0)|^{p-2} f(t_0)\, d\mu_{g(t_0)} = 0.$$

Hence there is a continuous differentiable function $c(t_0,s)$ of s defined in a neighborhood of t_0 such that $c(t_0,t_0) = 0$ and

$$\int_M |f(t_0)+c(t_0,s)|^{p-2}(f(t_0)+c(t_0,s))\, d\mu_{g(s)} = 0.$$

Otherwise if $\partial M \ne \varnothing$ and $p > 1$, we can just take $c(t_0,s) \equiv 0$.

Let $N(t,s)=G(f(t)+c(t,s),\,g(s))$. Then $N(t,t)=\lambda_{1,p}(g(t))$ and $N(t,s)\geq\lambda_{1,p}(g(s))$. A simple calculation gives

$$\begin{aligned}
&\frac{\partial N}{\partial s}(t,t)\\
&=\frac{\partial}{\partial s}\Big|_{s=t}\frac{\int_M|\nabla(f(t)+c(t,s))|^p_{g(s)}\;d\mu_{g(s)}}{\int_M|f(t)+c(t,s)|^p\;d\mu_{g(s)}}\\
&=\Bigg(\int_M\bigg(|\nabla f(t)|^p_{g(t)}\frac{\operatorname{tr}(\partial g/\partial s)(t)}{2}-\frac{p}{2}\,|\nabla f(t)|^{p-2}_{g(t)}\,\frac{\partial g}{\partial s}(t)(\nabla f(t),\nabla f(t))\bigg)\,d\mu_{g(t)}\\
&\qquad-\lambda_{1,p}(g(t))\int_M\bigg(|f(t)|^p\;\frac{\operatorname{tr}(\partial g/\partial s)(t)}{2}+p\,|f(t)|^{p-2}\;f(t)\frac{\partial c}{\partial s}(t,t)\bigg)\,d\mu_{g(t)}\Bigg)\\
&\qquad\qquad\times\bigg(\int_M|f(t)|^p\;d\mu_{g(t)}\bigg)^{-1}.
\end{aligned}$$

To simplify this formula, we use that

$$\int_M p\,|f(t)|^{p-2}\;f(t)\frac{\partial c}{\partial s}(t,t)\,d\mu_{g(t)}=0.$$

When M is closed, this follows from $\int_M p\,|f(t)|^{p-2}\;f(t)\,d\mu_{g(t)}=0$, and when $\partial M\neq\varnothing$, from $c(t,s)\equiv 0$. Hence we get

$$\begin{aligned}
\frac{\partial N}{\partial s}(t,t)=\Bigg(\int_M\bigg(&|\nabla f(t)|^p_{g(t)}\;\frac{\operatorname{tr}(\partial g/\partial s)(t)}{2}\\
&-\frac{p}{2}\,|\nabla f(t)|^{p-2}_{g(t)}\;\frac{\partial g}{\partial s}(t)(\nabla f(t),\,\nabla f(t))\bigg)\,d\mu_{g(t)}\\
&-\lambda_{1,p}(g(t))\int_M|f(t)|^p\;\frac{\operatorname{tr}(\partial g/\partial s)(t)}{2}\;d\mu_{g(t)}\Bigg)\\
&\times\bigg(\int_M|f(t)|^p\;d\mu_{g(t)}\bigg)^{-1}.
\end{aligned}\tag{3-1}$$

Now apply the Cauchy–Schwarz formula

$$\left|\frac{\partial g}{\partial s}(t)(\nabla f(t),\,\nabla f(t))\right|\leq\left|\frac{\partial g}{\partial s}(t)\right|_{g(t)}\,|\nabla f(t)|^2$$

and the fact that

$$\left|\operatorname{tr}\frac{\partial g}{\partial s}(t)\right|=\left|\left\langle g(t),\,\frac{\partial g}{\partial s}(t)\right\rangle\right|\leq|g(t)|\left|\frac{\partial g}{\partial s}(t)\right|_{g(t)}=\sqrt{n}\left|\frac{\partial g}{\partial s}(t)\right|_{g(t)}$$

to obtain

$$\left|\frac{\partial N}{\partial s}(t,t)\right|\leq\Big(\sqrt{n}+\frac{p}{2}\Big)\,\lambda_{1,p}(g(t))\,\sup\left|\frac{\partial g}{\partial s}(t)\right|_{g(t)}.$$

By the compactness of M, $\lambda_{1,p}(g(t))$ and $\sup|(\partial g/\partial s)(t)|_{g(t)}$ are both continuous. Then $|(\partial N/\partial s)(t,t)|$ is locally bounded, so Theorem 1.1 implies Theorem 1.2. $\square$

4. The Yamabe invariant

In this section, we consider the local Lipschitz property of the Yamabe invariant along general geometric flows and use the constants

$$p=\frac{2n}{n-2},\quad a=\frac{4(n-1)}{n-2},\quad b=\frac{4}{n-2}.$$

With no specification, M is a n-dimensional ($n\geq 3$) connected closed smooth Riemannian manifold, g is a smooth metric on it. Denote by R its scalar curvature, by Rc its Ricci curvature, and by $\mathrm{Rc}^0=\mathrm{Rc}-\frac{1}{n}Rg$ its trace-free Ricci curvature. The conformal class $[g]$ of metric g is defined by

$$[g]:=\{\phi^b g:\phi\in C^\infty(M),\ \phi>0\},$$

and the homogeneous total scalar curvature $\mathbf{S}(g)$ is defined by

$$\mathbf{S}(g):=\int_M R\,d\mu_g\Big/\int_M d\mu_g,$$

where $d\mu_g$ is the volume form with respect to metric g. Then the Yamabe invariant is defined by

$$\mathcal{Y}(g):=\inf_{\bar g\in[g]}\mathbf{S}(\bar g). \tag{4-1}$$

The minimizer metric is called a Yamabe metric. For the conformal transformation of the scalar curvature $R(g)$ and the trace-free Ricci curvature $\mathrm{Rc}^0(g)$, we have (see [Besse 1987])

$$\phi^{p-1}R(\phi^b g)=R(g)\phi-a\,\Delta\phi, \tag{4-2}$$

$$\mathrm{Rc}^0(\phi^2 g)=\mathrm{Rc}^0(g)+(n-2)\phi(\nabla\nabla\phi^{-1})^0, \tag{4-3}$$

where Δ is the Laplace–Beltrami operator with respect to the metric g, $\alpha^0=\alpha-\frac{1}{n}\operatorname{tr}(\alpha)\,\alpha$ is the trace-free part of $(0,2)$-type tensor α. If we define

$$E(\phi,g):=\int(a\,|\nabla\phi|_g^2+R(g)\phi^2)\,d\mu_g,$$

$$Q(\phi,g):=\frac{E(\phi,g)}{\left(\int\phi^p\,d\mu_g\right)^{2/p}}=E(\phi,g)\,\|\phi\|_{p,g}^{-2},$$

where $\nabla\phi=d\phi$ is a covariant vector and $\|\phi\|_{p,g}=\left(\int\phi^p\,d\mu_g\right)^{1/p}$ is the L^p norm with respect to metric g. Then the Yamabe invariant $\mathcal{Y}(g)$ can also be defined by

$$\mathcal{Y}(g):=\inf\{Q(\phi,g):\phi\in C^\infty(M),\phi>0\}. \tag{4-4}$$

The existence of a minimizer u follows from the solution of the Yamabe problem (see [Lee and Parker 1987] for the history). Hence $u^b g$ is a Yamabe metric, moreover the minimizer u satisfies the Euler–Lagrange function

$$R(g)u - a\,\Delta u = \alpha\, u^{p-1}, \tag{4-5}$$

where

$$\alpha = E(u,g)\,\|u\|_{p,g}^{-p} = \mathcal{Y}(g)\,\|u\|_{p,g}^{2-p}\,.$$

Denote by g_{can} the canonical metric on S^n, and consider the set $\Lambda[g]$ of all smooth Yamabe metrics in a given conformal class $[g]$. By the solution to the Yamabe problem, the sets $\Lambda[g]$ as g varies are also compact in the following sense (see [Anderson 2005]): if $g_i \to g$ smoothly and $[g] \neq [g_{\mathrm{can}}]$, then any sequence of Yamabe metrics $\bar{g}_i \in \Lambda[g_i]$ has a subsequence converging smoothly to a Yamabe metric $\bar{g} \in \Lambda[g]$.

The Yamabe constant $\mathcal{Y}(g)$ is continuous with respect to g under the C^2-topology of the space of metrics on M (see [Besse 1987, Proposition 4.31]).

Proof of Theorem 1.3. Since each $\bar{g}(t) \in [g(t)]$ is a Yamabe metric, we can assume $\bar{g}(t) = \phi^b(t)g(t)$. Then $0 < \phi(t) \in C^1(M)$ and $\phi(t)$ minimizes $Q(\,\cdot\,, g(t))$. Defining $N(t,s) := Q(\phi(t), g(s))$, then $\mathcal{Y}(g(t)) = N(t,t)$ and $\mathcal{Y}(g(s)) \le N(t,s)$. We compute

$$\begin{aligned}
&\frac{\partial N}{\partial s}(t,t) \\
&= \frac{\partial}{\partial s}\Big|_{s=t}\left(\frac{\int (a\,|\nabla\phi(t)|^2_{g(s)} + R(s)\phi(t)^2)\,d\mu_{g(s)}}{\left(\int \phi(t)^p\,d\mu_{g(s)}\right)^{2/p}}\right) \\
&= \int\left(-a\frac{\partial g}{\partial s}(t)(\nabla\phi(t),\nabla\phi(t)) + \frac{\partial R}{\partial s}(t)\phi(t)^2\right)d\mu_{g(t)}\left(\int \phi(t)^p\,d\mu_{g(t)}\right)^{-2/p} \\
&\quad + \int \frac{1}{2}\left(a\,|\nabla\phi(t)|^2_{g(s)} + R(t)\phi(t)^2\right)\operatorname{tr}\frac{\partial g}{\partial s}(t)d\mu_{g(t)}\left(\int \phi(t)^p\,d\mu_{g(t)}\right)^{-2/p} \\
&\quad - \frac{1}{p}\mathcal{Y}(g(t))\int \phi(t)^p\operatorname{tr}\frac{\partial g}{\partial s}(t)d\mu_{g(t)}\left(\int \phi(t)^p\,d\mu_{g(t)}\right)^{-1},
\end{aligned} \tag{4-6}$$

so that

$$\begin{aligned}
&\left|\frac{\partial N}{\partial s}(t,t)\right| \\
&\le \left(1+\frac{\sqrt{n}}{2}\right)\sup\left|\frac{\partial g}{\partial s}(t)\right|\cdot\frac{\int a\,|\nabla\phi(t)|^2_{g(t)}\,d\mu_{g(t)}}{\left(\int\phi(t)^p\,d\mu_{g(t)}\right)^{2/p}} + \frac{\sqrt{n}\,|\mathcal{Y}(g(t))|}{p}\sup\left|\frac{\partial g}{\partial s}(t)\right| \\
&\quad + \left(\sup\left|\frac{\partial R}{\partial s}(t)\right| + \frac{\sqrt{n}}{2}\sup|R(g(t))|\cdot\sup\left|\frac{\partial g}{\partial s}(t)\right|\right)\frac{\int\phi(t)^2\,d\mu_{g(t)}}{\left(\int\phi(t)^p\,d\mu_{g(t)}\right)^{2/p}}.
\end{aligned} \tag{4-7}$$

Next, we process the two integral terms in the above formula. Since $p>2$, applying Hölder's inequality gives

$$\frac{\int \phi^2\,d\mu_g}{\left(\int \phi^p\,d\mu_g\right)^{2/p}} \le \mathrm{vol}(g)^{1-2/p}. \tag{4-8}$$

By the definition of $Q(\phi, g)$, we have

$$\begin{aligned}\frac{\int a\,|\nabla\phi|_g^2\,d\mu_g}{\left(\int \phi^p\,d\mu_g\right)^{2/p}} &= Q(\phi, g) - \frac{\int R\,\phi^2\,d\mu_g}{\left(\int \phi^p\,d\mu_g\right)^{2/p}} \\ &\le Q(\phi, g) + \sup|R(g)|\,\frac{\int \phi^2\,d\mu_g}{\left(\int \phi^p\,d\mu_g\right)^{2/p}} \\ &\le Q(\phi, g) + \sup|R(g)|\,\mathrm{vol}(g)^{1-2/p}.\end{aligned} \tag{4-9}$$

Substituting (4-8) and (4-9) into (4-7), we come to

$$\begin{aligned}\left|\frac{\partial N}{\partial s}(t,t)\right| \le \left(\left(1+\sqrt{n}\right)\sup\left|\frac{\partial g}{\partial s}(t)\right|\cdot\sup|R(g(t))| + \sup\left|\frac{\partial R}{\partial s}(t)\right|\right)\mathrm{vol}(g(t))^{1-2/p} \\ + \left(1+\frac{\sqrt{n}}{2}+\frac{\sqrt{n}}{p}\right)\sup\left|\frac{\partial g}{\partial s}(t)\right|\cdot|\mathcal{Y}(g(t))|.\end{aligned}$$

Since $\sup|\partial g/\partial s(t)|$, $\sup|\partial R/\partial s(t)|$, $\sup|R(g(t))|$, $\mathrm{vol}(g(t))^{1-2/p}$, and $|\mathcal{Y}(g(t))|$ are all continuous on the closed manifold M, we conclude that $(\partial N/\partial s)(t,t)$ is locally bounded, hence $\mathcal{Y}(g(t))$ is locally Lipschitz by Theorem 1.1.

Next, we simplify the formula (4-6). By (4-5) we have

$$R\phi(t) - a\,\Delta\phi(t) = \mathcal{Y}(g(t))\|\phi(t)\|_{p,g(t)}^{2-p}\phi(t)^{p-1}.$$

Multiplying both sides by $\phi(t)\,\mathrm{tr}(\partial g/\partial s)(t)$ and integrating by parts gives

$$\begin{aligned}&\mathcal{Y}(g(t))\|\phi(t)\|_{p,g(t)}^{2-p}\int \phi(t)^p\,\mathrm{tr}\,\frac{\partial g}{\partial s}(t)\,d\mu_{g(t)} \\ &\qquad = \int\left(R\phi(t)^2 + a\,|\nabla\phi(t)|^2 - \frac{a}{2}\,\Delta\left(\phi(t)^2\right)\right)\mathrm{tr}\,\frac{\partial g}{\partial s}(t)\,d\mu_{g(t)}.\end{aligned} \tag{4-10}$$

Substituting (4-10) into (4-6), we get

$$\begin{aligned}&\frac{\partial N}{\partial s}(t,t) \\ &= \left(\int \frac{a}{2p}\Delta(\phi(t)^2)\,\mathrm{tr}\,\frac{\partial g}{\partial s}(t) + \phi(t)^2\left(\frac{\partial R}{\partial s}(t) + \frac{1}{n}R\,\mathrm{tr}\,\frac{\partial g}{\partial s}(t)\right)d\mu_{g(t)} \right.\\ &\qquad \left. - \int a\left\langle\left(\frac{\partial g}{\partial s}(t)\right)^0, \nabla\phi(t)\otimes\nabla\phi(t)\right\rangle d\mu_{g(t)}\right)\|\phi(t)\|_{p,g(t)}^{-2}.\end{aligned} \tag{4-11}$$

The evolution function of scalar curvature R is

$$\frac{\partial R}{\partial s} = \operatorname{div}\left(\operatorname{div}\frac{\partial g}{\partial s}\right) - \Delta \operatorname{tr}\frac{\partial g}{\partial s} - \left\langle \mathrm{Rc}, \frac{\partial g}{\partial s}\right\rangle.$$

Substituting this into (4-11) gives

$$\begin{aligned}\frac{\partial N}{\partial s}&(t,t)\\ &= \int \left\langle \frac{\partial g}{\partial s}(t), \left(\nabla\nabla\phi(t)^2 - \phi(t)^2\,\mathrm{Rc} - a\nabla\phi(t)\otimes\nabla\phi(t)\right)^0\right\rangle d\mu_{g(t)}\|\phi(t)\|_{p,g(t)}^{-2}\\ &= -\int \left\langle \frac{\partial g}{\partial s}(t), \phi(t)^2\left(\mathrm{Rc} + (n-2)\phi(t)^{b/2}\nabla\nabla\phi(t)^{-b/2}\right)^0\right\rangle d\mu_{g(t)}\|\phi(t)\|_{p,g(t)}^{-2}.\end{aligned}$$

By the conformal transformation of trace-free Ricci curvature (4-3),

$$\frac{\partial N}{\partial s}(t,t) = -\int \phi^{-b}(t)\left\langle \frac{\partial g}{\partial s}(t), \mathrm{Rc}^0(\phi^b(t)g(t))\right\rangle d\mu_{\phi^b(t)g(t)}\operatorname{vol}(\phi^b(t)g(t))^{-2/p}.$$

Since $\phi^b(t) = \bar{g}(t)/g(t)$, we get

$$\frac{\partial N}{\partial s}(t,t) = -\int \frac{\bar{g}(t)}{g(t)}\left\langle \frac{\partial g}{\partial s}(t), \mathrm{Rc}^0(\bar{g}(t))\right\rangle_{\bar{g}(t)} d\mu_{\bar{g}(t)}\operatorname{vol}(\bar{g}(t))^{-2/p}. \tag{4-12}$$

Then the theorem follows from Corollary 1.1.1. □

Proof of Theorem 1.4. Let $\phi(t)$ be any minimizer of $Q(\,\cdot\,, g(t))$. Then

$$\tilde{g}(t) = \phi^b(t)g(t)$$

is the Yamabe metric in the conformal class $[g(t)]$. Define

$$\mathcal{N}(\tilde{g}(s), g(t)) := Q(\phi(s), g(t)).$$

Then

$$\mathcal{Y}(g(t)) = \mathcal{N}(\tilde{g}(t), g(t)), \qquad \mathcal{Y}(g(t)) \le \mathcal{N}(\tilde{g}(s), g(t)).$$

Hence, when $t > 0$,

$$\begin{aligned}\frac{\mathcal{N}(\tilde{g}(t), g(t)) - \mathcal{N}(\tilde{g}(t), g(0))}{t} &\le \frac{\mathcal{Y}(g(t)) - \mathcal{Y}(g(0))}{t}\\ &\le \frac{\mathcal{N}(\tilde{g}(0), g(t)) - \mathcal{N}(\tilde{g}(0), g(0))}{t}.\end{aligned} \tag{4-13}$$

By (4-12) and the definitions of $\mathcal{N}$ and N, we get

$$\frac{\partial \mathcal{N}}{\partial t}(\tilde{g}(t), g(t)) = -\int \frac{\tilde{g}(t)}{g(t)}\left\langle \frac{\partial g}{\partial t}(t), \mathrm{Rc}^0(\tilde{g}(t))\right\rangle_{\tilde{g}(t)} d\mu_{\tilde{g}(t)}\operatorname{vol}(\tilde{g}(t))^{-2/p}. \tag{4-14}$$

It is easy to see that $\mathcal{N}(\tilde{g}(s), g(t))$ and $(\partial\mathcal{N}/\partial t)(\tilde{g}(s), g(t))$ as functionals of $\tilde{g}(s)$ and $g(t)$ are continuous with the C^2-topology on the space of metrics. Applying the mean value theorem to the variable t in the function $\mathcal{N}(\tilde{g}(s), g(t))$, there exists a number $0 < \beta(s,t) < t$ such that

$$\mathcal{N}(\tilde{g}(s), g(t)) - \mathcal{N}(\tilde{g}(s), g(0)) = t\,\frac{\partial\mathcal{N}}{\partial t}(\tilde{g}(s), g(\beta(s,t))).$$

Substituting into (4-13), we come to

$$\frac{\partial\mathcal{N}}{\partial t}(\tilde{g}(t), g(\beta(t,t))) \le \frac{\mathcal{Y}(g(t)) - \mathcal{Y}(g(0))}{t} \le \frac{\partial\mathcal{N}}{\partial t}(\tilde{g}(0), g(\beta(0,t))). \tag{4-15}$$

Letting $t \to 0$, then $\beta(0,t) \to 0$ and $\beta(t,t) \to 0$ follows from $0 < \beta(s,t) < t$. Hence

$$\limsup_{t\to 0} \frac{\mathcal{Y}(g(t)) - \mathcal{Y}(g(0))}{t} \le \frac{\partial\mathcal{N}}{\partial t}(\tilde{g}(0), g(0)) \quad \text{for all } \tilde{g}(0) \in \Lambda[g(0)]. \tag{4-16}$$

Pick $t_i > 0$, $t_i \to 0$ such that

$$\liminf_{t\to 0} \frac{\mathcal{Y}(g(t)) - \mathcal{Y}(g(0))}{t} = \lim_{i\to\infty} \frac{\mathcal{Y}(g(t_i)) - \mathcal{Y}(g(0))}{t_i}. \tag{4-17}$$

Using the compactness of $\Lambda[g_0]$, there exists a subsequence of t_i (denoted again by t_i for simplicity) and a Yamabe metric $\bar{g}_0 \in \Lambda[g_0]$ such that

$$\lim_{i\to\infty} \tilde{g}(t_i) = \bar{g}_0.$$

Then by the first inequality in (4-15),

$$\begin{aligned} \lim_{i\to\infty} \frac{\mathcal{Y}(g(t_i)) - \mathcal{Y}(g(0))}{t_i} &\ge \lim_{i\to\infty} \frac{\partial\mathcal{N}}{\partial t}(\tilde{g}(t_i), g(\beta(t_i, t_i))) \\ &= \frac{\partial\mathcal{N}}{\partial t}(\bar{g}_0, g(0)). \end{aligned} \tag{4-18}$$

Hence by (4-16) and (4-17), $\mathcal{Y}(g(t))$ is differentiable at $t = 0$ and

$$\lim_{t\to 0} \frac{\mathcal{Y}(g(t)) - \mathcal{Y}(g(0))}{t} = \frac{\partial\mathcal{N}}{\partial t}(\bar{g}_0, g(0)). \tag{4-19}$$

This implies the first equality in (1-2) by (4-16) and (4-14). We now know that the t_i chosen after (4-17) can be any sequence of $t_i > 0$, $t_i \to 0$. Then the $\bar{g}_0$ in (4-19) can be any accumulation point of $\bar{g}(t)$ as $t \to 0$ in $\Lambda[g_0]$, hence any accumulation point of $\Lambda[g(t)]$ as $t \to 0$ in $\Lambda[g_0]$. The second equality in (1-2) follows from applying this to other metrics $g(t) \in G_{(\partial g/\partial t)(0)}(g_0, t)$. $\square$

Acknowledgement

We thank everyone who helped us with useful discussions and ideas. We also thank the referee for comments and suggestions that helped improve this paper.

References

[Anderson 2005] M. T. Anderson, "On uniqueness and differentiability in the space of Yamabe metrics", *Commun. Contemp. Math.* **7**:3 (2005), 299–310. MR 2006d:53031 Zbl 1082.58013

[Besse 1987] A. L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik (3) **10**, Springer, Berlin, 1987. MR 88f:53087 Zbl 0613.53001

[Bessieres et al. 2003] L. Bessieres, J. Lafontaine, and L. Rozoy, "Scalar curvature and black holes", preprint, 2003, Available at www-fourier.ujf-grenoble.fr/~rozoy/sca.pdf.

[Brendle 2008] S. Brendle, "Blow-up phenomena for the Yamabe equation", *J. Amer. Math. Soc.* **21**:4 (2008), 951–979. MR 2009m:53084 Zbl 1206.53041

[Brendle and Marques 2009] S. Brendle and F. C. Marques, "Blow-up phenomena for the Yamabe equation II", *J. Differential Geom.* **81**:2 (2009), 225–250. MR 2010k:53050 Zbl 0589.90031

[Cao 2007] X. Cao, "Eigenvalues of $(-\Delta + \frac{R}{2})$ on manifolds with nonnegative curvature operator", *Math. Ann.* **337**:2 (2007), 435–441. MR 2007g:53071 Zbl 1105.53051

[Cao 2008] X. Cao, "First eigenvalues of geometric operators under the Ricci flow", *Proc. Amer. Math. Soc.* **136**:11 (2008), 4075–4078. MR 2009f:53098 Zbl 1166.58007

[Chang and Lu 2007] S.-C. Chang and P. Lu, "Evolution of Yamabe constant under Ricci flow", *Ann. Global Anal. Geom.* **31**:2 (2007), 147–153. MR 2008e:53121 Zbl 1112.53048

[Chow and Lu 2002] B. Chow and P. Lu, "The time-dependent maximum principle for systems of parabolic equations subject to an avoidance set", preprint, 2002. arXiv math/0211209

[Chow et al. 2008] B. Chow, S.-C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo, and L. Ni, *The Ricci flow: techniques and applications, Part II: Analytic aspects*, Mathematical Surveys and Monographs **144**, American Mathematical Society, Providence, RI, 2008. MR 2008j:53114 Zbl 1157.53035

[Hamilton 1986] R. S. Hamilton, "Four-manifolds with positive curvature operator", *J. Differential Geom.* **24**:2 (1986), 153–179. MR 87m:53055 Zbl 0628.53042

[Khuri et al. 2009] M. A. Khuri, F. C. Marques, and R. M. Schoen, "A compactness theorem for the Yamabe problem", *J. Differential Geom.* **81**:1 (2009), 143–196. MR 2010e:53065 Zbl 1162.53029

[Lee and Parker 1987] J. M. Lee and T. H. Parker, "The Yamabe problem", *Bull. Amer. Math. Soc.* (*N.S.*) **17**:1 (1987), 37–91. MR 88f:53001 Zbl 0633.53062

[Li 2007] J.-F. Li, "Eigenvalues and energy functionals with monotonicity formulae under Ricci flow", *Math. Ann.* **338**:4 (2007), 927–946. MR 2008c:53068 Zbl 1127.53059

[Ling 2007] J. Ling, "A comparison theorem and a sharp bound via the Ricci flow", preprint, 2007. arXiv math.DG/0710.2574

[Serrin 1964] J. Serrin, "Local behavior of solutions of quasi-linear equations", *Acta Math.* **111** (1964), 247–302. MR 30 #337 Zbl 0128.09101

[Tolksdorf 1984] P. Tolksdorf, "Regularity for a more general class of quasilinear elliptic equations", *J. Differential Equations* **51**:1 (1984), 126–150. MR 85g:35047

[Wu et al. 2010] J.-Y. Wu, E.-M. Wang, and Y. Zheng, "First eigenvalue of the p-Laplace operator along the Ricci flow", *Ann. Global Anal. Geom.* **38**:1 (2010), 27–55. MR 2011i:53103 Zbl 1201.53077

Received September 20, 2010. Revised May 16, 2011.

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

PACIFIC JOURNAL OF MATHEMATICS

Volume 254 No. 1 November 2011
