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We prove that the normal curvatures of hyperspheres, the Rund curvature, and the Finsler curvature of circles in Hilbert geometry tend to 1 as the radii tend to infinity.

1. Introduction

A smooth connected manifold M^n is called a *Finsler* manifold [Bao et al. 2000] if there is a smooth positively homogeneous function $F : TM^n \rightarrow [0, \infty)$ on the coordinates in tangent spaces such that the symmetric bilinear form

$$g_y(u, v) = g_{ij}(x, y)u^i v^j : T_x M^n \times T_x M^n \rightarrow \mathbb{R}$$

is positively definite for each pair $(x, y) \in TM^n$, where $g_{ij}(x, y) = \frac{1}{2}[F^2(x, y)]_{y^i y^j}$.

Consider a bounded open convex domain U in \mathbb{R}^n with the Euclidean norm $\|\cdot\|$, and let ∂U be a C^3 hypersurface with positive normal curvatures. For a point $x \in U$ and a tangent vector $y \in T_x U = \mathbb{R}^n$, let x_- and x_+ be the intersection points of the rays $x + \mathbb{R}_- y$ and $x + \mathbb{R}_+ y$ with *absolute* ∂U . Then the Hilbert metric is defined as follows:

$$(1) \quad F(x, y) = \frac{1}{2}(\Theta(x, y) + \Theta(x, -y)),$$

where

$$\Theta(x, y) = \|y\| \frac{1}{\|x - x_+\|}, \quad \Theta(x, -y) = \|y\| \frac{1}{\|x - x_-\|}$$

are called the Funk metrics on U .

Hilbert geometries are the generalizations of Klein's model of the hyperbolic geometry. Hilbert geometries are also Finsler spaces of constant negative flag curvature -1 [Bao et al. 2000]. The Hilbert metric is invariant under projective transformations of \mathbb{R}^n leaving U bounded.

B. Colbois and P. Verovic [2002] proved that the Hilbert metric is asymptotically Riemannian at infinity. That means that in a given Hilbert geometry the unit sphere

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of the norm $F(x, \cdot)$ approaches the ellipsoid in C^0 topology as the point x tends to ∂U .

Unlike the Riemannian geometry, in the Finsler geometry there are several definitions of the curvature of a curve.

The normal curvature of a hypersurface in a Finsler space is defined as follows [Shen 2001]. Let $\varphi: N \rightarrow M^n$ be a hypersurface in a Finsler manifold M^n . A vector $\mathbf{n} \in T_{\varphi(x)}M^n$ is called a normal vector to N at the point $x \in N$ if $\mathbf{g}_n(y, \mathbf{n}) = 0$ for all $y \in T_x N$. The *normal curvature* \mathbf{k}_n at the point $x \in N$ in a direction $y \in T_x N$ is defined as

$$(2) \quad \mathbf{k}_n = \mathbf{g}_n(\nabla_{\dot{c}(s)}\dot{c}(s)|_{s=0}, \mathbf{n}),$$

where $\dot{c}(0) = y$, $c(s)$ is a geodesic in the induced connection on N , and \mathbf{n} is the chosen unit normal vector.

For a curve $c(s)$ parametrized by its arc length in M^n , it is possible to define two more curvatures.

The Finsler curvature of $c(s)$ [Finsler 1951; Rund 1959] is defined as

$$(3) \quad \mathbf{k}_F(c(s)) = \sqrt{\mathbf{g}_{\dot{c}(s)}(\nabla_{\dot{c}(s)}\dot{c}(s), \nabla_{\dot{c}(s)}\dot{c}(s))}.$$

The Rund curvature of $c(s)$ [Rund 1959] is defined as

$$(4) \quad \mathbf{k}_R(c(s)) = \sqrt{\mathbf{g}_{\nabla_{\dot{c}(s)}\dot{c}(s)}(\nabla_{\dot{c}(s)}\dot{c}(s), \nabla_{\dot{c}(s)}\dot{c}(s))}.$$

It is well-known that the normal curvatures of hyperspheres in the hyperbolic space \mathbb{H}^n are equal to $\coth(r)$ and tend to 1 as the radius r tends to infinity. We prove the same property for the Hilbert geometry.

Theorem 1.1. *The normal curvature, the Rund curvature, and the Finsler curvature of the circles centered at the same point in the 2-dimensional Hilbert geometry tend to 1 as their radii tend to infinity, uniformly at the point of the circle.*

Theorem 1.2. *The normal curvatures of the hyperspheres centered at the same point tend to 1 as their radii tend to infinity, uniformly at the point of the hypersphere and in the tangent vector at this point of the hypersphere.*

This can be interpreted as meaning that the Hilbert metric tends to the Riemannian metric of the hyperbolic space in C^2 -topology.

2. The choice of the coordinate system

Consider the Hilbert geometry based on a two-dimensional domain U in the Euclidean plane. Fix a point o in the domain U and a point $p \in \partial U$. Since ∂U is a convex curve, it admits the polar representation $\omega(\varphi)$ from the point o such that the point p corresponds to $\varphi = 0$.

Choose the coordinate system on the plane with the origin O at the point p ; let the axis x_2 be orthogonal to ∂U at p , x_1 be tangent to ∂U at p , and $U - \{p\}$ lie in the half-plane $x_2 > 0$.

In this section we will construct a projective transformation P of the plane that sends U to \hat{U} and has the following properties:

- (1) $P(p) = p$.
- (2) The vector $u = (0, 1)$ is orthogonal to $\partial \hat{U}$ at the point p .
- (3) The tangent line to $\partial \hat{U}$ at the point p is parallel to the tangent line to $\partial \hat{U}$ at the point corresponding to $\varphi = \pi$.
- (4) $\partial \hat{U}$ is the graph of the function $x_2 = \hat{f}(x_1)$ such that $\hat{f}(0) = 0$, $\hat{f}'(0) = 0$, and $\hat{f}''(0) = \frac{1}{2}$ in the neighborhood of p .

We are going to give the explicit expression for this transformation and show that after this transformation the curvature of $\partial \hat{U}$ and the derivatives of f remain uniformly bounded.

The next lemma gives the upper bound on the angle between the radial and normal direction to the convex curve.

Lemma 2.1 [Borisenko 2002]. *Let γ be a closed embedded curve in the Euclidean plane whose curvature is greater than or equal to k . Let o be a point in the interior of the set bounded by γ , ω_0 the distance from o to γ , and φ the angle between the outer normal vector at the point $p \in \gamma$ and the vector op . Then*

$$(5) \quad \cos \angle(u_m, N(m)) \geq \omega_0 k.$$

Denote by k and K the minimum and maximum of the curvatures of ∂U . Also, $\omega_0 = \min_{\varphi} \omega(\varphi)$, $\omega_1 = \max_{\varphi} \omega(\varphi)$.

Let the length of the chord of U in the direction u equal H , the distance from o to the origin equal ω_u , $\omega_0 \leq \omega_u \leq \omega_1$, and the angle between u and x_2 equal α .

Step 1. Construct an affine transformation that makes the vector \overrightarrow{oO} parallel to x_2 . This transformation sends the points $(0, 0)$ and $(1, 0)$ to themselves, the point $(H \sin \alpha, H \cos \alpha) \in \partial U$ to the point $(0, H)$, and has the expression:

$$(6) \quad \tilde{x}_1 = x_1 - \tan \alpha x_2, \quad \tilde{x}_2 = \frac{x_2}{\cos \alpha}.$$

Denote the image of U as \tilde{U} . The point o now has the coordinates $(0, \omega_u)$. Denote by \tilde{k} the minimum of the curvature of $\partial \tilde{U}$ in the $(\tilde{x}_1, \tilde{x}_2)$ coordinate system, and by $\tilde{\omega}_0$ denote the distance from the point $(0, \omega_u)$ to $\partial \tilde{U}$. Note that the eigenvalues of the transformation (6) are equal to 1 and $1/\cos \alpha$. Hence

$$(7) \quad \omega_0 \leq \tilde{\omega}_0 \leq \frac{1}{\cos \alpha} \omega_0.$$

Lemma 2.1 then implies that the curvature of $\partial\tilde{U}$ remains bounded and separated from zero.

Step 2. Construct the transformation such that the tangent line

$$\tilde{x}_2 = -\tan \beta \tilde{x}_1 + H$$

to $\partial\tilde{U}$ at the point $(0, H)$ will be parallel to the axis \tilde{x}_1 , where β is the angle between \tilde{x}_2 and the normal vector to $\partial\tilde{U}$ at $(0, H)$. This transformation has the expression

$$(8) \quad \bar{x}_1 = \frac{H\tilde{x}_1}{H - \tan \beta \tilde{x}_1}, \quad \bar{x}_2 = \frac{H\tilde{x}_2}{H - \tan \beta \tilde{x}_1}.$$

Denote the image of \tilde{U} as \bar{U} .

We can estimate the angle $|\tan \beta|$. Using Lemma 2.1, we have

$$(9) \quad 0 \leq |\tan \beta| \leq \sqrt{\frac{1}{(\tilde{k}^2 \tilde{\omega}_0^2)} - 1}.$$

Estimate the curvature $\partial\bar{U}$. Let the curve $\partial\bar{U}$ be given in the parametric form $r(t) = (\tilde{x}_1(t), \tilde{x}_2(t))$. Then $\partial\bar{U}$ has the parametrization

$$\bar{r}(t) = \frac{Hr(t)}{H - \tan \beta \tilde{x}_1(t)}.$$

Differentiating leads to

$$\begin{aligned} \bar{r}'(t) &= \frac{Hr'(t)}{H - \tan \beta \tilde{x}_1(t)} + \frac{Hr(t) \tan \beta \tilde{x}'_1(t)}{(H - \tan \beta \tilde{x}_1(t))^2}, \\ \bar{r}''(t) &= \frac{2H \tan \beta r'(t) \tilde{x}'_1(t)}{(H - \tan \beta \tilde{x}_1(t))^2} + \frac{2Hr(t) \tan^2 \beta \tilde{x}'_1(t)^2}{(H - \tan \beta \tilde{x}_1(t))^3} \\ &\quad + \frac{Hr''(t)}{H - \tan \beta \tilde{x}_1(t)} + \frac{Hr(t) \tan \beta \tilde{x}''_1(t)}{(H - \tan \beta \tilde{x}_1(t))^2}. \end{aligned}$$

The strict convexity of $\partial\tilde{U}$ implies that $H - \tan \beta \tilde{x}_1(t) \geq \text{const} > 0$ for each t . This and the compactness argument leads to the maximum of the curvature of $\partial\bar{U}$ being bounded from above for some constant.

If the curve $\partial\tilde{U}$ is the graph $\tilde{x}_2 = f(\tilde{x}_1)$ and $f(0) = f'(0) = 0$, then its curvature at the point $(0, 0)$ after the transformation (8) will not change. Indeed,

$$\begin{aligned} &\tilde{x}'_1(t)^2 + \tilde{x}'_2(t)^2 \\ &= \left(\frac{Ht \tan \beta}{(H - t \tan \beta)^2} + \frac{H}{H - t \tan \beta} \right)^2 + \left(\frac{H \tan \beta f(t)}{(H - t \tan \beta)^2} + \frac{Hf'(t)}{H - t \tan \beta} \right)^2, \end{aligned}$$

$$\begin{aligned} & \tilde{x}'_1(t)\tilde{x}''_2(t) - \tilde{x}''_1(t)\tilde{x}'_2(t) \\ &= -\left(\frac{2Ht \tan^2 \beta}{(H-t \tan \beta)^3} + \frac{2H \tan \beta}{(H-t \tan \beta)^2}\right)\left(\frac{H \tan \beta f(t)}{(H-t \tan \beta)^2} + \frac{Hf'(t)}{H-t \tan \beta}\right) \\ &+ \left(\frac{Ht \tan \beta}{(H-t \tan \beta)^2} + \frac{H}{H-t \tan \beta}\right)\left(\frac{2H \tan^2 \beta f(t)}{(H-t \tan \beta)^3} + \frac{2H \tan \beta f'(t)}{(H-t \tan \beta)^2} + \frac{Hf''(t)}{H-t \tan \beta}\right). \end{aligned}$$

We obtain the claim after substituting the equalities $f(0) = f'(0) = 0$. So the curvature of $\partial\bar{U}$ at the origin is still separated from zero.

Step 3. Construct a transformation such that the distance from $(0, \omega_u)$ to the origin is equal to 1 and the curvature of $\partial\bar{U}$ at the origin is equal to $\frac{1}{2}$. This transformation has the expression:

$$(10) \quad \hat{x}_1 = \frac{\bar{x}_1}{\omega_u}, \quad \hat{x}_2 = \frac{\bar{x}_2}{2\omega_u^2 \bar{k}(0)}.$$

Denote the image of \bar{U} as \hat{U} . It is obvious that the curvature of $\partial\hat{U}$ remains bounded.

The announced transformation P is the composition of the transformations (6), (8), and (10), and the following proposition holds:

Proposition 2.2. *There exists a constant C_0 depending on U such that the curvature of $P(\partial U)$ is bounded from above by C_0 .*

Let ∂U be the graph of the function $x_2 = f(x_1)$ in the initial coordinate system. After the transformation P , $P(\partial U)$ can be considered the graph of the function $x_2 = \hat{f}(x_1)$ such that $\hat{f}(0) = 0$, $\hat{f}'(0) = 0$, and $\hat{f}''(0) = \frac{1}{2}$ in the neighborhood of p .

Finally, estimate the third derivative $\hat{f}'''(0)$. Evidently, under the affine transformations (6) and (10) the third derivative remains bounded. We only need to control $f'''(0)$ at Step 2.

So let the curve $\partial\tilde{U}$ be the graph $\tilde{x}_2 = \tilde{f}(\tilde{x}_1)$ and after the transformation (8) we obtain the graph \bar{f} . The rules for differentiation lead to

$$(11) \quad \bar{f}'''(0) = \tilde{f}'''(0) - \frac{\tan \beta \tilde{k}(0)}{H}.$$

As ∂U is the compact curve, we obtain:

Proposition 2.3. *There exist constants C_1, C_2 depending on U , such that*

$$C_1 \leq \hat{f}'''(0) \leq C_2.$$

Analogously we can estimate all higher derivatives.

The Hilbert metrics for the domains U and \hat{U} are isometric. Therefore, without loss of generality, we will consider the Hilbert metric for the domain \hat{U} and will denote \hat{U} by U .

3. Series expansions for the metric tensor of the Hilbert metric

From the decomposition of the Hilbert metric through the Funk metrics (1), we conclude

$$\begin{aligned}
 g_{ij}(x, y) &= F(x, y)F_{y^i y^j}(x, y) + F_{y^i}(x, y)F_{y^j}(x, y) \\
 &= \frac{1}{2}F(x, y)(\Theta_{y^i y^j}(x, y) + \Theta_{y^i y^j}(x, -y)) \\
 &\quad + \frac{1}{4}(\Theta_{y^i}(x, y) - \Theta_{y^i}(x, -y))(\Theta_{y^j}(x, y) - \Theta_{y^j}(x, -y)).
 \end{aligned}$$

Okada’s lemma [Shen 2001] for Funk metrics gives the expression of the derivatives of $\Theta(x, y)$ with respect to the coordinates on tangent spaces through the derivatives with respect to the coordinates on U :

$$\Theta(x, y)_{x^k} = \Theta(x, y)\Theta(x, y)_{y^k}.$$

Using this lemma, we can write:

$$\begin{aligned}
 (12) \quad g_{ij}(x, y) &= \frac{1}{2}F(x, y) \frac{\Theta_{x^i x^j}(x, y)\Theta(x, y) - 2\Theta_{x^i}(x, y)\Theta_{x^j}(x, y)}{\Theta(x, y)^3} \\
 &\quad + \frac{1}{2}F(x, y) \frac{\Theta_{x^i x^j}(x, -y)\Theta(x, -y) - 2\Theta_{x^i}(x, -y)\Theta_{x^j}(x, -y)}{\Theta(x, -y)^3} \\
 &\quad + \frac{1}{4} \left(\frac{\Theta_{x^i}(x, y)}{\Theta(x, y)} - \frac{\Theta_{x^i}(x, -y)}{\Theta(x, -y)} \right) \left(\frac{\Theta_{x^j}(x, y)}{\Theta(x, y)} - \frac{\Theta_{x^j}(x, -y)}{\Theta(x, -y)} \right).
 \end{aligned}$$

For convenience we will use lower indices x_i for coordinates. Let $F(x_1, x_2, y_1, y_2)$ be a two-dimensional Hilbert metric and $\Theta(x_1, x_2, y_1, y_2)$ the corresponding Funk metric. Assume that the point (x_1, x_2) is sufficiently close to ∂U . Then we can express ∂U as the graph $x_2 = f(x_1)$ such that $f(0) = 0$, $f'(0) = 0$, and $f''(0) = \frac{1}{2}$. Consider a point (x_1, x_2) above the graph $x_2 = f(x_1)$. Denote by $r(x_1, x_2, y_1, y_2)$ the distance between the point (x_1, x_2) and the intersection point of the line passing through (x_1, x_2) in the direction (y_1, y_2) with the curve $x_2 = f(x_1)$. Then

$$(13) \quad \Theta(x_1, x_2, y_1, y_2) = \sqrt{y_1^2 + y_2^2} r(x_1, x_2, y_1, y_2)^{-1}.$$

Now we obtain the derivatives of $r(x_1, x_2, y_1, y_2)$ on x_1, x_2 . The parameter $t(x_1, x_2, y_1, y_2)$ corresponding to the intersection points of the curve $x_2 = f(x_1)$ with the line

$$x_1(t) = x_1 + ty_1, \quad x_2(t) = x_2 + ty_2$$

satisfies the functional equation

$$(14) \quad x_2 + ty_2 = f(x_1 + t(x_1, x_2, y_1, y_2)y_1).$$

Differentiate (14) on x_1, x_2 :

$$(15) \quad t_{x_1}y_2 = f'(x_1 + ty_1)(1 + t_{x_1}y_1), \quad 1 + t_{x_2}y_2 = f'(x_1 + ty_1)t_{x_2}y_1.$$

We obtain the explicit expressions for t_{x_1}, t_{x_2} :

$$(16) \quad t_{x_1} = \frac{f'(x_1 + ty_1)}{y_2 - y_1 f'(x_1 + ty_1)}, \quad t_{x_2} = \frac{1}{y_1 f'(x_1 + ty_1) - y_2}.$$

Differentiating (15) leads to

$$(17) \quad \begin{aligned} y_2 t_{x_1 x_1} &= f''(x_1 + ty_1)(1 + y_1 t_{x_1})^2 + f'(x_1 + ty_1)y_1 t_{x_1 x_1}, \\ y_2 t_{x_1 x_2} &= f''(x_1 + ty_1)(1 + y_1 t_{x_1})y_2 t_{x_2} + f'(x_1 + ty_1)y_1 t_{x_1 x_2}, \\ y_2 t_{x_2 x_2} &= f''(x_1 + ty_1)(y_1 t_{x_2})^2 + f'(x_1 + ty_1)y_1 t_{x_2 x_2}. \end{aligned}$$

We obtain the expressions for the second derivatives of t :

$$(18) \quad \begin{aligned} t_{x_1 x_1} &= \frac{f''(x_1 + ty_1)(1 + y_1 t_{x_1})^2}{y_2 - y_1 f'(x_1 + ty_1)}, \\ t_{x_1 x_2} &= \frac{f''(x_1 + ty_1)(1 + y_1 t_{x_1})y_1 t_{x_2}}{y_2 - y_1 f'(x_1 + ty_1)}, \\ t_{x_2 x_2} &= \frac{f''(x_1 + ty_1)(y_1 t_{x_2})^2}{y_2 - y_1 f'(x_1 + ty_1)}. \end{aligned}$$

We need the derivatives of $r(x_1, x_2, y_1, y_2)$. By definition,

$$r(x_1, x_2, y_1, y_2) = \sqrt{(y_1 t)^2 + (y_2 t)^2} = \sqrt{y_1^2 + y_2^2} t(x_1, x_2, y_1, y_2).$$

Hence $r_{x_k} = t_{x_k}$ and $r_{x_k x_l} = t_{x_k x_l}$.

Now it is possible to calculate the derivatives of the Funk metric. Formula (13) implies

$$(19) \quad \Theta_{x_k} = -\sqrt{y_1^2 + y_2^2} \frac{r_{x_k}}{r^2}.$$

After differentiating (19), we obtain

$$(20) \quad \Theta_{x_k x_l} = -\sqrt{y_1^2 + y_2^2} \frac{r_{x_k x_l} r^2 - 2r r_{x_l} r_{x_k}}{r^4} = \sqrt{y_1^2 + y_2^2} (2\Theta^3 r_{x_k} r_{x_l} - \Theta^2 r_{x_k x_l}).$$

Finally, from (12) it is possible to obtain the coefficients of the metric tensor. We will need the values of $g_{ij}(x_1, x_2, y_1, y_2)$ at the points $(x_1, x_2) = (0, x_2)$.

3.1. Expansions for $g_{ij}(\mathbf{0}, x_2, \mathbf{1}, \mathbf{0})$. Note that the strict convexity of ∂U implies that $f'(t(x_1, x_2)) \neq 0$ for $t(x_1, x_2) \neq 0$. Then from (16) we deduce

$$(21) \quad t_{x_1}(0, x_2, 1, 0) = -1,$$

$$(22) \quad t_{x_2}(0, x_2, 1, 0) = \frac{1}{f'(t(0, x_2, 1, 0))},$$

and from (18)

$$(23) \quad t_{x_1 x_1}(0, x_2, 1, 0) = t_{x_1 x_2}(0, x_2, 1, 0) = 0,$$

$$(24) \quad t_{x_2 x_2}(0, x_2, 1, 0) = -\frac{f''(t(0, x_2, 1, 0))}{f'(t(0, x_2, 1, 0))^3}.$$

Expanding the functional equation (14) in a power series with respect to t as $x_2 \rightarrow 0$, we find the expansions of $t(0, x_2, 1, 0)$.

$$(25) \quad x_2 = \frac{1}{4}t^2 + \frac{1}{6}f'''(0)t^3 + O(t^4).$$

We will find t in expanded form

$$(26) \quad t = A + B\sqrt{x_2} + Cx_2 + Dx_2^{3/2} + O(x_2^2)$$

After substituting (26) into (25) and transposing all members in the left side, we obtain the following system of equations:

$$(27) \quad 3A^2 + 2A^3 f'''(0) + (6AB + 6A^2 f'''(0)B)\sqrt{x_2} \\ + (-12 + 3B^2 + 6A f'''(0)B^2 + 6AC + 6A^2 f'''(0)C)x_2 + (2f'''(0)B^2 \\ + 6BC + 12A f'''(0)BC + 6AD + 6A^2 f'''(0)D)x_2^{3/2} + O(x_2^2) = 0.$$

Choose the coefficients A , B , C , and D so that the left side of (27) is $O(x_2^2)$. Equating the coefficients under the powers of x_2 to zero we obtain two expansions for t which correspond to the directions $(1, 0)$ and $(-1, 0)$.

$$(28) \quad t(0, x_2, \pm 1, 0) = \pm 2\sqrt{x_2} - \frac{4}{3}f'''(0)x_2 + O(x_2^2).$$

In our case $r = t$, so we get

$$(29) \quad r(0, x_2, 1, 0) = 2\sqrt{x_2} - \frac{4}{3}f'''(0)x_2 + O(x_2^2).$$

Later on, all power series will be considered in the neighborhood of 0. The series expansion for the metric F is

$$\begin{aligned}
 F(0, x_2, 1, 0) &= \frac{1}{2} \left(\frac{1}{r(0, x_2, 1, 0)} + \frac{1}{r(0, x_2, -1, 0)} \right) \\
 &= \frac{1}{2} \left(\frac{1}{2\sqrt{x_2} - \frac{4}{3}f'''(0)x_2 + O(x_2^2)} + \frac{1}{2\sqrt{x_2} + \frac{4}{3}f'''(0)x_2 + O(x_2^2)} \right) \\
 &= \frac{9}{\sqrt{x_2}(18 - 8f'''(0)^2x_2) + O(x_2^{3/2})}. \\
 (30) \quad F(0, x_2, 1, 0) &= \frac{1}{2\sqrt{x_2}} + \frac{2f'''(0)^2}{9}\sqrt{x_2} + O(x_2^{3/2}).
 \end{aligned}$$

We will also need the difference:

$$\begin{aligned}
 (31) \quad \Theta(0, x_2, 1, 0) - \Theta(0, x_2, -1, 0) &= \frac{1}{r(0, x_2, 1, 0)} - \frac{1}{r(0, x_2, -1, 0)} \\
 &= \frac{-6f'''(0) + O(x_2)}{4f'''(0)^2x_2 - 9 + O(x_2^{3/2})} \\
 &= \frac{2}{3}f'''(0) + O(x_2).
 \end{aligned}$$

From (21), using $r_{x_k} = t_{x_k}$, we get

$$(32) \quad r_{x_1}(0, x_2, 1, 0) = -1.$$

Expand the denominator of (22) with respect to t :

$$\begin{aligned}
 r_{x_2}(0, x_2, 1, 0) &= \frac{1}{f'(t(0, x_2, 1, 0))} \\
 &= \frac{1}{f''(0)t(0, x_2, 1, 0) + \frac{1}{2}f'''(0)t(0, x_2, 1, 0)^2 + O(t(0, x_2, 1, 0)^3)}.
 \end{aligned}$$

Using the fact that $f'(0) = 0$ and $f''(0) = \frac{1}{2}$ and substituting the value of t from (28), we obtain

$$r_{x_2}(0, x_2, 1, 0) = \frac{1}{\frac{1}{2}(2\sqrt{x_2} - \frac{4}{3}f'''(0)x_2) + \frac{1}{2}f'''(0)(2\sqrt{x_2} - \frac{4}{3}f'''(0)x_2)^2 + O(x_2^2)}.$$

$r_{x_2}(0, x_2, -1, 0)$ is analogous. Finally,

$$(33) \quad r_{x_2}(0, x_2, 1, 0) = \frac{1}{\sqrt{x_2}} - \frac{4f'''(0)}{3} + \frac{40f'''(0)^2}{9}\sqrt{x_2} + O(x_2).$$

The second derivative has the form $r_{x_k x_l} = t_{x_k x_l}$. From (23) we obtain

$$(34) \quad r_{x_1 x_1}(0, x_2, 1, 0) = r_{x_1 x_2}(0, x_2, 1, 0) = 0.$$

And (24) implies

$$r_{x_2 x_2}(0, x_2, 1, 0) = -\frac{f''(t(0, x_2, 1, 0))}{f'(t(0, x_2, 1, 0))^3}.$$

We now expand the numerator and denominator in a series with respect to t and use $f'(0) = 0$, $f''(0) = \frac{1}{2}$, and (28):

$$\begin{aligned} r_{x_2x_2}(0, x_2, 1, 0) &= -\frac{\frac{1}{2} + f'''(0)t(0, x_2, 1, 0) + \frac{1}{2}f^{(4)}(0)t(0, x_2, 1, 0)^2 + O(t^3)}{(f''(0)t(0, x_2, 1, 0) + \frac{1}{2}f'''(0)t(0, x_2, 1, 0)^2 + O(t(0, x_2, 1, 0)^3))^3} \\ &= \frac{-\frac{1}{2} - 2f'''(0)\sqrt{x_2} + (\frac{4}{3}f'''(0)^2 - 4f^{(4)}(0))x_2 + \frac{16}{3}f'''(0)f^{(4)}(0)x_2^{3/2} + O(x_2^2)}{x_2^{3/2} + 4f'''(0)x_2^2 + O(x_2^{5/2})} \end{aligned}$$

Thus

$$(35) \quad r_{x_2x_2}(0, x_2, 1, 0) = -\frac{1}{2x_2^{3/2}} - \frac{2f^{(4)}(0)}{\sqrt{x_2}} + O(1).$$

From (19), (29), (32) we find that

$$\Theta_{x_1}(0, x_2, 1, 0) = \frac{1}{(2\sqrt{x_2} - \frac{4}{3}f'''(0)x_2 + O(x_2^2))^2}.$$

Analogously, acting for the vector $(-1, 0)$, we get

$$(36) \quad \Theta_{x_1}(0, x_2, \pm 1, 0) = \pm \frac{1}{4x_2} + \frac{f'''(0)}{3\sqrt{x_2}} + O(1).$$

From (29) and (33) we deduce

$$\Theta_{x_2}(0, x_2, 1, 0) = -\frac{1/\sqrt{x_2} - 4f'''(0)/3 + (40f'''(0)/9)\sqrt{x_2} + O(x_2)}{(2\sqrt{x_2} - \frac{4}{3}f'''(0)x_2 + O(x_2^2))^2},$$

and finally,

$$(37) \quad \Theta_{x_2}(0, x_2, \pm 1, 0) = -\frac{1}{4x_2^{3/2}} - \frac{f'''(0)^2}{\sqrt{x_2}} + O(1).$$

Using the formulae (20), (29), (33), and (35), we obtain the expression for the second derivatives of the Funk metric:

$$(38) \quad \Theta_{x_2x_2}(0, x_2, \pm 1, 0) = \frac{3}{8x_2^{5/2}} + \frac{13f'''(0)^2 + 3f^{(4)}(0)}{6x_2^{3/2}} + O\left(\frac{1}{x_2}\right).$$

Finally we can estimate the metric coefficients. From (13), (29), and (36) we get

$$(39) \quad \frac{\Theta_{x_1}(0, x_2, 1, 0)}{\Theta(0, x_2, 1, 0)} - \frac{\Theta_{x_1}(0, x_2, -1, 0)}{\Theta(0, x_2, -1, 0)} = \frac{1}{\sqrt{x_2}} + \frac{4f'''(0)^2}{9}\sqrt{x_2} + O(x_2^{3/2}).$$

It follows from (13), (29), and (37) that

$$(40) \quad \frac{\Theta_{x_2}(0, x_2, 1, 0)}{\Theta(0, x_2, 1, 0)} - \frac{\Theta_{x_2}(0, x_2, -1, 0)}{\Theta(0, x_2, -1, 0)} = \frac{2f'''(0)}{3\sqrt{x_2}} + O(1).$$

Note that

$$(41) \quad \Theta_{x_1x_1}(0, x_2, \pm 1, 0)\Theta(0, x_2, \pm 1, 0) - 2\Theta_{x_1}(0, x_2, \pm 1, 0)\Theta_{x_1}(0, x_2, \pm 1, 0) \\ = (2\Theta^3 r_{x_1} r_{x_1} - \Theta^2 r_{x_1x_1})\Theta - 2\Theta^2 r_{x_1} \Theta^2 r_{x_1} = 0,$$

since $r_{x_1x_1} = 0$, and analogously

$$(42) \quad \Theta_{x_1x_2}(0, x_2, \pm 1, 0)\Theta(0, x_2, \pm 1, 0) - 2\Theta_{x_1}(0, x_2, \pm 1, 0)\Theta_{x_2}(0, x_2, \pm 1, 0) \\ = 0.$$

Then from (13), (29), (37), and (38) we get

$$(43) \quad \frac{\Theta_{x_2x_2}(0, x_2, \pm 1, 0)\Theta(0, x_2, \pm 1, 0) - 2\Theta_{x_2}(0, x_2, \pm 1, 0)\Theta_{x_2}(0, x_2, \pm 1, 0)}{\Theta(0, x_2, \pm 1, 0)^3} \\ = \frac{1}{2x_2^{3/2}} + \frac{2f^{(4)}(0)}{\sqrt{x_2}} + O(1).$$

Finally, using (30), (12), (39), (40), (41), (42), and (43), we obtain the series expansions of the metric tensor of the Hilbert metric.

$$(44) \quad g_{11}(0, x_2, 1, 0) = \frac{1}{4x_2} + O(1), \\ g_{12}(0, x_2, 1, 0) = \frac{f'''(0)}{6x_2} + O(1), \\ g_{22}(0, x_2, 1, 0) = \frac{1}{4x_2^2} + \frac{2f'''(0)^2 + 9f^{(4)}(0)}{18x_2} + O(1).$$

3.2. Expansions for $g_{ij}(0, x_2, 0, 1)$. The formulae in (16) imply that, at $(0, x_2)$,

$$t_{x_1}(0, x_2, 0, \pm 1) = 0, \\ t_{x_2}(0, x_2, 0, \pm 1) = -1, \\ t_{x_1x_2}(0, x_2, 0, \pm 1) = t_{x_2x_2}(0, x_2, 0, \pm 1) = 0.$$

Note that the functions $t(0, x_2, 0, \pm 1)$ have the representations

$$t(0, x_2, 0, -1) = -x_2, \quad t(0, x_2, 0, 1) = H - x_2.$$

Here H denotes the length of the chord of ∂U in the direction $(0, 1)$. Then

$$\Theta(0, x_2, 0, -1) = \frac{1}{x_2}, \quad \Theta(0, x_2, 0, 1) = \frac{1}{H - x_2}.$$

Consequently,

$$(45) \quad F(0, x_2, 0, 1) = \frac{1}{2} \left(\frac{1}{H-x_2} + \frac{1}{x_2} \right) = \frac{1}{2x_2} + O(1).$$

We can estimate the derivatives of the Funk metrics $\Theta(0, x_2, 0, \pm 1)$. It follows from (19) and (20) that

$$(46) \quad \Theta_{x_2}(0, x_2, 0, -1) = \frac{1}{x_2^2}, \quad \Theta_{x_2}(0, x_2, 0, 1) = -\frac{1}{(H-x_2)^2},$$

$$(47) \quad \Theta_{x_2x_2}(0, x_2, 0, -1) = \frac{2}{x_2^3}, \quad \Theta_{x_2x_2}(0, x_2, 0, 1) = \frac{2}{(H-x_2)^3}.$$

Using (12), (46), and (47), we get the expansions:

$$(48) \quad \begin{aligned} g_{12}(0, x_2, 0, 1) &= 0, \\ g_{22}(0, x_2, 0, 1) &= \frac{1}{4} \left(\frac{1}{H-x_2} + \frac{1}{x_2} \right)^2 = \frac{1}{4x_2^2} + O\left(\frac{1}{x_2}\right). \end{aligned}$$

We will also need the values $F(0, x_2, l, \frac{1}{2})$.

We have

$$\begin{aligned} t(0, x_2, -l, -\frac{1}{2}) &= -2x_2 + 2l^2x_2^2 + O(x_2^3), \\ t(0, x_2, l, \frac{1}{2}) &= L + O(x_2). \end{aligned}$$

Then

$$\begin{aligned} F(0, x_2, l, \frac{1}{2}) &= \frac{\sqrt{\frac{1}{4} + l^2}}{2\sqrt{\frac{1}{4}t(0, x_2, l, \frac{1}{2})^2 + (lt(0, x_2, l, \frac{1}{2}))^2}} \\ &\quad + \frac{\sqrt{\frac{1}{4} + l^2}}{2\sqrt{\frac{1}{4}t(0, x_2, -l, -\frac{1}{2})^2 + (lt(0, x_2, -l, -\frac{1}{2}))^2}} \\ &= \frac{\sqrt{\frac{1}{4} + l^2}}{2\sqrt{\frac{1}{4} + l^2}} \left(\frac{1}{t(0, x_2, l, \frac{1}{2})} - \frac{1}{t(0, x_2, -l, -\frac{1}{2})} \right). \end{aligned}$$

Finally,

$$(49) \quad F(0, x_2, l, \frac{1}{2}) = \frac{1}{4x_2} + \frac{1}{2L} + O(x_2).$$

4. Proof of the theorems

The Chern–Rund covariant derivative along the curve $c(t)$ in the Finsler space equipped with the Hilbert metric F is given by the formula [Shen 2001]

$$(50) \quad \nabla_{c'(t)}c'(t) = \{c''(t)^i + (\Theta(c(t), c'(t)) - \Theta(c(t), -c'(t)))c'(t)^i\} \frac{\partial}{\partial x^i}.$$

For calculating the normal curvature (2), the Finsler curvature (3), and the Rund curvature (4), we need the covariant derivative $\nabla_{\dot{c}(s)}\dot{c}(s)$ of the curve $c(s)$ parametrized by its arc length.

For a given curve $c(t)$, we will denote by the dot the derivative with respect to the arc length s , and by the prime the derivative with respect to t . Then let $t = t(s)$ be the reparametrization. We get

$$\dot{c}(s) = c'(t)t'_s.$$

Using that s in the length parameter, we get

$$1 = F(c(t), c'(t))t'_s.$$

Hence

$$\dot{c}(s) = \frac{c'(t)}{F(c(t), c'(t))}.$$

The next step is to calculate $\nabla_{\dot{c}(s)}\dot{c}(s)$.

$$\begin{aligned} \nabla_{\dot{c}(s)}\dot{c}(s) &= \nabla_{c'(t)/F(c(t), c'(t))} \frac{c'(t)}{F(c(t), c'(t))} \\ &= \frac{1}{F(c(t), c'(t))} \left(\nabla_{c'(t)} \left(\frac{1}{F(c(t), c'(t))} \right) c'(t) + \frac{1}{F(c(t), c'(t))} \nabla_{c'(t)} c'(t) \right). \end{aligned}$$

According to [Bao et al. 2000],

$$\nabla_{c'(t)} \left(\frac{1}{F(c(t), c'(t))} \right) = - \frac{\mathbf{g}_{c'(t)}(\nabla_{c'(t)} c'(t), c'(t))}{F(c(t), c'(t))^3}.$$

Then the derivative $\nabla_{\dot{c}(s)}\dot{c}(s)$ has the form

$$\nabla_{\dot{c}(s)}\dot{c}(s) = \frac{1}{F(c(t), c'(t))^2} \left(\nabla_{c'(t)} c'(t) - \frac{\mathbf{g}_{c'(t)}(\nabla_{c'(t)} c'(t), c'(t))}{F(c(t), c'(t))^2} c'(t) \right).$$

Finally, using (50), we get the formula:

$$(51) \quad \nabla_{\dot{c}(s)}\dot{c}(s) = \frac{c''(t) + c'(t) \left(\Theta(c(t), c'(t)) - \Theta(c(t), -c'(t)) - \frac{\mathbf{g}_{c'(t)}(\nabla_{c'(t)} c'(t), c'(t))}{F(c(t), c'(t))^2} \right)}{F(c(t), c'(t))^2}.$$

As in Section 2 fix a point o in the domain U and a point $p \in \partial U$. The curve ∂U admits the polar representation $\omega(\varphi)$ from the point o such that the point p corresponds to $\varphi = 0$. According to Section 2, we assume that U satisfies the conditions (1)–(4).

Then one can get that $\omega'(0) = 0$, $\omega(0) = 1$, $\omega''(0) = \frac{1}{2}$, $\omega'(\pi) = 0$. Set

$$C = \frac{1 + \omega(\pi)}{\omega(\pi)}.$$

In [Borisenko and Olin 2008] the polar function $\rho_r(u)$ of the hypersphere of radius r was obtained:

$$(52) \quad \rho_r(u) = \frac{\omega(-u)\omega(u)(e^{2r} - 1)}{\omega(u) + \omega(-u)e^{2r}}.$$

As $r \rightarrow \infty$,

$$(53) \quad \omega(u) - \rho_r(u) = \omega(u) \left(\frac{\omega(u)}{\omega(-u)} + 1 \right) e^{-2r} + o(e^{-2r}).$$

From (52) we get that the circle of radius r admits the parametrization

$$c(\varphi) = \left(\frac{\omega(\pi - \varphi)\omega(\varphi)(e^{2r} - 1)}{\omega(\varphi) + \omega(\pi - \varphi)e^{2r}} \sin \varphi, \frac{\omega(\pi - \varphi)\omega(\varphi)(e^{2r} - 1)}{\omega(\varphi) + \omega(\pi - \varphi)e^{2r}} \cos \varphi \right),$$

where $\omega(\varphi)$ is the polar function of ∂U .

Then

$$(54) \quad c'(0) = \frac{\omega(\pi)(e^{2r} - 1)}{1 + \omega(\pi)e^{2r}}(1, 0) = (1 - Ce^{-2r} + O(e^{-3r}), 0), \quad r \rightarrow \infty.$$

The second derivative:

$$(55) \quad c''(0) = \frac{(e^{2r}\omega(\pi)^2(\omega''(0) - 1) - \omega(\pi) + \omega'(\pi))(e^{2r} - 1)}{(1 + e^{2r}\omega(\pi))^2}(0, 1),$$

$$(55) \quad c''(0) = \left(0, -\frac{1}{2} + O(e^{-2r}) \right), \quad r \rightarrow \infty.$$

From (53) we get that at the point of the circle the second coordinate is

$$(56) \quad x_2 = \omega(0) - \frac{\omega(\pi)\omega(0)(e^{2r} - 1)}{\omega(0) + \omega(\pi)e^{2r}} = Ce^{-2r} + O(e^{-3r}).$$

Estimate the derivative $\nabla_{\dot{c}(0)}\dot{c}(0)$ using the formulae (51), (31), and (56):

$$\Theta(c(0), c'(0)) - \Theta(c(0), -c'(0)) = \Theta(0, Ce^{-2r} + O(e^{-3r}), 1 + O(e^{-2r}), 0) - \Theta(0, Ce^{-2r} + O(e^{-3r}), -1 + O(e^{-2r}), 0) = \frac{2}{3}f'''(0) + O(e^{-2r}).$$

Therefore, formula (50) leads to

$$(57) \quad \nabla_{c'(0)}c'(0) = c''(0) + c'(0)(\Theta(c(0), c'(0)) - \Theta(c(0), -c'(0))) = \left(\frac{2}{3}f'''(0), -\frac{1}{2} \right) + O(e^{-2r}).$$

Using (56) and (57) we get

$$\frac{\mathbf{g}_{c'(0)}(\nabla_{c'(0)}c'(0), c'(0))}{F(c(0), c'(0))^2} = \frac{\frac{2}{3}f'''(0)g_{11} - \frac{1}{2}g_{12}}{F(0, Ce^{-2r} + O(e^{-3r}), 1 + O(e^{-2r}), 0)^2}.$$

Here g_{ij} are calculated at the point $(0, Ce^{-2r} + O(e^{-3r}), 1 + O(e^{-2r}), 0)$. After substituting the values from (30) and (44), we obtain

$$\frac{\mathbf{g}_{c'(0)}(\nabla_{c'(0)}c'(0), c'(0))}{F(c(0), c'(0))^2} = -\frac{f'''(0)}{3} + O(e^{-2r}).$$

Therefore,

$$(58) \quad \nabla_{\dot{c}(0)}\dot{c}(0) = \frac{(f'''(0), -\frac{1}{2}) + (1, 1)O(e^{-2r})}{F(c(0), c'(0))^2}.$$

Taking into account (30),

$$\nabla_{\dot{c}(0)}\dot{c}(0) = (4f'''(0), -2)e^{-2r} + (1, 1)O(e^{-3r}).$$

Calculate the Rund curvature (4) using the formulae (56) and (58).

$$\begin{aligned} \mathbf{k}_R(r)^2 &= F(c(0), \nabla_{\dot{c}(0)}\dot{c}(0)) \\ &= \frac{F(0, Ce^{-2r} + O(e^{-3r}), -f'''(0) + O(e^{-2r}), \frac{1}{2} + O(e^{-2r}))}{F(0, Ce^{-2r} + O(e^{-3r}), 1 - Ce^{-2r} + O(e^{-3r}), 0)^2}. \end{aligned}$$

From (30) and (49) we get

$$(59) \quad \mathbf{k}_R(r)^2 = 1 + C\left(\frac{2}{L} - \frac{8f'''(0)^2}{9}\right)e^{-2r} + O(e^{-3r}).$$

Here $L > 0$ is the length of the chord ℓ of ∂U in the direction $(f'''(0), -1/2)$. Proposition 2.2 gives the uniform bounds on the curvature of ∂U . Proposition 2.3 claims that the angle between the chord ℓ and x_2 is uniformly separated from $\pi/2$. Thus we conclude that $2/L$ is bounded from above.

Calculate the Finsler curvature (3) using the formulae (56) and (58).

$$\begin{aligned} \mathbf{k}_F(r)^2 &= \mathbf{g}_{\dot{c}(0)}(\nabla_{\dot{c}(0)}\dot{c}(0), \nabla_{\dot{c}(0)}\dot{c}(0)) \\ &= \frac{f'''(0)^2g_{11} - f'''(0)g_{12} + \frac{1}{4}g_{22}}{F(0, Ce^{-2r} + O(e^{-3r}), 1 - Ce^{-2r} + O(e^{-3r}), 0)^4}. \end{aligned}$$

Here g_{ij} are considered at the point $(0, Ce^{-2r} + O(e^{-3r}), 1 + O(e^{-2r}), 0)$. Finally, from (30) and (44) we obtain that

$$(60) \quad \mathbf{k}_F(r)^2 = 1 + C\left(-\frac{8}{9}f'''(0)^2 + 4f^{(4)}(0)\right)e^{-2r} + O(e^{-3r}).$$

Proposition 2.3 gives the uniform bounds on the derivatives of f . Theorem 1.1 is proved.

Note that the normal curvature $\mathbf{g}_n(\nabla_{\dot{c}(s)}\dot{c}(s), \mathbf{n})$ of a hypersurface at the point x depends only on the tangent vector to the curve $c(s)$ at x [Shen 2001]. So in order to obtain the normal curvature of the Hilbert hypersphere S_r centered at o at the point p in the tangent direction w , we consider the normal curvature of the circle $S_r \cap \Pi$ which lies in the plane $\Pi = \text{span}(w, \vec{o}\vec{p})$.

From (57) we get the normal curvature of the circle of radius r :

$$(61) \quad \mathbf{k}_n(r) = \mathbf{g}_n(\nabla_{\dot{c}(0)}\dot{c}(0), \mathbf{n}) = \frac{\mathbf{g}_n(c''(0), \mathbf{n})}{F(c(0), c'(0))^2}.$$

Since $g_{12}(0, x_2, 0, 1) = 0$ by (48), it follows that the unit normal vector \mathbf{n} to the circle at $(0, x_2)$ is exactly

$$\frac{1}{F(0, x_2, 0, 1)}(0, -1).$$

Finally, taking into account (30), (56), (55), (45), and (48):

$$(62) \quad \mathbf{k}_n(r) = \frac{\frac{1}{2}g_{22}(0, Ce^{-2r} + O(e^{-3r}), 0, 1)}{F(0, Ce^{-2r} + O(e^{-3r}), 1 - Ce^{-2r} + O(e^{-3r}), 0)^2 F(0, Ce^{-2r} + O(e^{-3r}), 0, 1)} = 1 + C\left(\frac{1}{H} - \frac{8f'''(0)^2}{9}\right)e^{-2r} + O(e^{-3r}).$$

If it is the case that the Euclidean normal curvatures of the hypersurface ∂U are bounded ($k_2 \leq k_n \leq k_1$) then the curvature of the curve $\partial U' = \partial U \cap \Pi$ is bounded as well. Consider the point $x \in \partial U' \subset \partial U$. Then the curvature $k(x)$ of $\partial U'$ and the normal curvature $k_n(x)$ of ∂U are related as

$$k(x) = \frac{k_n(x)}{\cos \beta}.$$

Here β is the angle between the radial and normal direction to ∂U at x . Using Lemma 2.1 we find that $\omega_0 k_2 \leq \cos \beta \leq 1$. Hence the curvature of $\partial U'$ is uniformly bounded for all y . Applying Proposition 2.2 for the Hilbert geometry based on U' , we get the uniformity of the series expansion (62) which ends the proof of Theorem 1.2.

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