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ON COMPLETE MANIFOLDS

LIANG-CHU CHANG AND CHIUNG-JUE ANNA SUNG

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Let (M^m, g) be an m -dimensional complete noncompact manifold. We show that for all $p > 1$ and $l > 1$, any bounded set of p -harmonic l -forms in $L^q(M)$, with $0 < q < \infty$, is relatively compact with respect to the uniform convergence topology if the curvature operator of M is asymptotically non-negative.

1. Introduction

Let (M^m, g) be an m -dimensional complete oriented Riemannian manifold with associated Riemannian metric g . Let d be the exterior differential operator and let

$$\delta \equiv *d*$$

be the codifferential operator, where the linear operator $*$ is defined pointwise by

$$*(\omega_1 \wedge \cdots \wedge \omega_l) \equiv \omega_{l+1} \wedge \cdots \wedge \omega_m,$$

for a positively oriented orthonormal coframe $\{\omega_1, \omega_2, \dots, \omega_m\}$ at the point. The Hodge–Laplace–Beltrami operator Δ acting on the space of smooth l -forms $\Lambda^l(M)$ is defined by

$$\Delta \equiv -(d\delta + \delta d).$$

Definition 1.1. An l -form ω on M is a p -harmonic l -form if ω satisfies $d\omega = 0$ and $\delta(|\omega|^{p-2}\omega) = 0$ for all $p > 1$.

When $p = 2$, the p -harmonic l -form $\omega \in \Lambda^l(M)$ is called a harmonic l -form on (M, g) , that is,

$$\Delta_g \omega = 0.$$

When $l = 0$, let Ω be a compact domain on the Riemannian manifold (M, g) , and let ω be a real smooth function on M . For $p > 1$, the p -energy of ω on Ω is

$$E_p(\Omega, \omega) \equiv \frac{1}{p} \int_{\Omega} |\nabla \omega|^p dV_g.$$

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The function ω is said to be p -harmonic on M if ω is a critical point of $E_p(\Omega, \cdot)$ for all $\Omega \subset M$, that is, if ω satisfies the Euler–Lagrange equation

$$\operatorname{div}(|\nabla\omega|^{p-2}\nabla\omega) = 0.$$

A curvature operator K_l on manifold M^m is defined as follows:

$$K_l = \begin{cases} \text{lower bound of the curvature operator on } M & \text{for } l > 1; \\ (m - 1)^{-1} \times (\text{lower bound of the Ricci curvature}) & \text{for } l = 1. \end{cases}$$

We call this curvature operator K_l of M asymptotically nonnegative if $K_l \geq -K(r)$, where

$$K(r) : [0, \infty) \rightarrow [0, \infty)$$

is a nonnegative and nonincreasing continuous function of distance r to a fixed point $z \in M$, with

$$\int_0^\infty r K(r) < \infty.$$

Yau [1975] proved that any positive harmonic function on a manifold with nonnegative Ricci curvature must be constant. Much work has been done in the finite dimension of space of polynomial growth harmonic functions of growth order at most d [Li 1997; Colding and Minicozzi 1997; Li and Tam 1995; Li and Wang 1999]. Concerning general harmonic l -forms, Li [1980] established a dimension estimate of the space of polynomial growth harmonic forms. In this paper, we study general p -harmonic l -forms and p -harmonic maps on complete noncompact manifolds, for $p > 1$ and $l \neq 0$. For $p = 2$, Chen and Sung [2007] considered the space consisting of all harmonic l -forms of polynomial growth for all $l \geq 1$, and gave a dimension estimate of such a space when M has asymptotically nonnegative curvature. Since the set of p -harmonic l -forms is no longer linear, it is interesting to study the set of p -harmonic l -forms and to seek topological and geometrical links. Interestingly, Zhang [2001] proved that any $L^q(M)$ p -harmonic 1-forms must be zero on a manifold with nonnegative Ricci curvature for $p > 1$ and $0 < q < \infty$. Chang et al. [2010] generalized Zhang’s result to a complete manifold M with asymptotically nonnegative curvature and finite first Betti number. They proved that a bounded set of $L^q(M)$ p -harmonic 1-forms on (M, g) has a uniformly convergent subsequence.

Next we introduce the Sobolev inequality. A geodesic ball $B_x(r)$ in a complete manifold M is said to admit a Sobolev inequality $S(C, \nu)$ if there exist constants $C > 0$ and $\nu > 2$ such that for all $f \in C_0^\infty(B_x(r))$, we have

$$\left(\int_{B_x(r)} |f|^{2\nu/(\nu-2)} \right)^{(\nu-2)/\nu} \leq Cr^2 V_x^{-2/\nu}(r) \int_{B_x(r)} (|\nabla f|^2 + r^{-2} f^2),$$

where $V_x(r)$ is the volume of geodesic ball $B_x(r)$. Using the Bochner formula, the Moser iteration [1961] and the Sobolev inequality, Chang et al. [2010] showed that any bounded set of p -harmonic 1-forms in $L^q(M)$, with $0 < q < \infty$, is relatively compact with respect to the uniform convergence topology if M has asymptotically nonnegative Ricci curvature and finite first Betti number. However, the Bochner formula does not work for p -harmonic l -forms for $l > 1$. We derive a new type of Bochner formula to overcome this obstacle. We study the set of p -harmonic l -forms, for $l > 1$, on a complete noncompact manifold M , and then study the set of p -harmonic maps from a complete manifold M to a complete manifold N . In Section 2, we derive a different type of Bochner formula for p -harmonic l -forms and prove that any bounded set of p -harmonic l -forms in $L^q(M)$, with $0 < q < \infty$, must be relatively compact with respect to the uniform convergence topology if the curvature operator of M is asymptotically nonnegative. Of course, this implies that the linear space of harmonic l -forms must be finite-dimensional when $p = 2$ and $l \geq 0$. Also, there is no nonzero p -harmonic l -form on M in $L^q(M)$ if the curvature operator of M is nonnegative. In Section 3, we also derive a different type of Bochner formula for p -harmonic maps from M with asymptotically nonnegative Ricci curvature to N with nonpositive sectional curvature. We prove that the set of such p -harmonic maps with finite p -energy on M has a uniformly convergent subsequence. The p -harmonic map is constant if M is compact with nonnegative Ricci curvature, which is an extension of the fact in the harmonic map case ($p = 2$).

2. p -harmonic l -forms

Any smooth l -form on an m -dimensional manifold M satisfies the Kato inequality:

Lemma 2.1 [Wan and Xin 2004; Calderbank et al. 2000; Herzlich 2000]. *Let ω be a differentiable l -form on M . Then*

$$|\nabla|\omega|^2| \leq 2|\omega||\nabla\omega|.$$

Lemma 2.2 [Bochner 1946]. *Let $\omega = \sum_I a_I \omega_I$ be an l -form on M . Then*

$$\Delta|\omega|^2 = 2\langle \Delta\omega, \omega \rangle + 2|\nabla\omega|^2 + 2K_l\langle \omega, \omega \rangle.$$

Let (M, g) be a complete noncompact manifold. We wish to study the set of L^q p -harmonic l -forms on M for $l > 1$ and $0 < q < \infty$. To prove the main theorem for all $l > 1$, we show a different type of Bochner formula for p -harmonic l -forms:

Lemma 2.3 (Bochner-type formula for p -harmonic forms). *Let ω be a p -harmonic l -form on an m -dimensional complete Riemannian M^m . Then*

$$|\omega|\Delta|\omega|^{p-1} = \langle \Delta(|\omega|^{p-2}\omega), \omega \rangle + |\omega|^{2-p} (|\nabla(|\omega|^{p-2}\omega)|^2 - |\nabla|\omega|^{p-1}|^2) + K_l|\omega|^p,$$

in the sense of distributions.

Proof. The Bochner–Weitzenböck formula for $|\omega|^{p-2}\omega$ asserts that

$$(2-1) \quad \frac{1}{2}\Delta\left|\omega|^{p-2}\omega\right|^2 = \langle\Delta(|\omega|^{p-2}\omega), |\omega|^{p-2}\omega\rangle + |\nabla(|\omega|^{p-2}\omega)|^2 + K_l\left|\omega|^{p-2}\omega\right|^2.$$

The left side of (2-1) is given by

$$\frac{1}{2}\Delta\left|\omega|^{p-2}\omega\right|^2 = \frac{1}{2}\Delta|\omega|^{2p-2} = \frac{1}{2}\Delta(|\omega|^{p-1})^2 = |\omega|^{p-1}\Delta|\omega|^{p-1} + |\nabla|\omega|^{p-1}|^2.$$

Hence,

$$\begin{aligned} |\omega|^{p-1}\Delta|\omega|^{p-1} + |\nabla|\omega|^{p-1}|^2 &= \langle\Delta(|\omega|^{p-2}\omega), |\omega|^{p-2}\omega\rangle + |\nabla(|\omega|^{p-2}\omega)|^2 + K_l|\omega|^{2p-4}|\omega|^2. \end{aligned}$$

It follows that

$$\begin{aligned} |\omega|^{p-1}\Delta|\omega|^{p-1} &= |\omega|^{p-2}\langle\Delta(|\omega|^{p-2}\omega), \omega\rangle + (|\nabla(|\omega|^{p-2}\omega)|^2 - |\nabla|\omega|^{p-1}|^2) + K_l|\omega|^{2p-2}. \quad \square \end{aligned}$$

For l -forms with $l > 1$, the volume comparison property holds on M with asymptotically nonnegative curvature operator [Li and Tam 1995]. Therefore, inside geodesic ball $B_x(R)$ with $r(x) = 2R$, the volume doubling property holds [Li and Tam 1995]. Also, by [Saloff-Coste 1992], a local weak Poincaré inequality holds on geodesic ball $B_x(R)$, and hence we have the Sobolev inequality $S(C, \nu)$ on $B_x(R)$ [Hajlasz and Koskela 1995]; that is, there exists a real number $\nu > 2$ such that

$$\left(\int_{B_x(R)}|f|^{2\nu/(\nu-2)}dV\right)^{(\nu-2)/\nu} \leq C \cdot r^2 \cdot V^{-2/\nu}(B) \int_{B_x(R)}|\nabla f|^2dV,$$

for all $f \in C_0^\infty(B_x(r))$, where $r \leq R$.

Theorem 2.4 (main theorem). *Let M^m be an m -dimensional complete Riemannian manifold with asymptotically nonnegative curvature operator K_l , for $l > 1$. Then a bounded set of $L^q(M)$ p -harmonic l -forms on (M^m, g) has a uniformly convergent subsequence, for $1 < p < \infty$ and $0 < q < \infty$.*

Proof. Let ω be a p -harmonic l -form on M^m . Lemma 2.3 asserts that

$$\begin{aligned} |\omega|^{p-1}\Delta|\omega|^{p-1} &= |\omega|^{p-2}\langle\Delta(|\omega|^{p-2}\omega), \omega\rangle + (|\nabla(|\omega|^{p-2}\omega)|^2 - |\nabla|\omega|^{p-1}|^2) + K_l|\omega|^{2p-2}. \end{aligned}$$

By the Kato inequality, we have

$$|\nabla|\omega|^{p-1}| = |\nabla||\omega|^{p-2}\omega|| \leq |\nabla(|\omega|^{p-2}\omega)|.$$

Therefore,

$$|\omega|^{p-1} \Delta |\omega|^{p-1} \geq |\omega|^{p-2} \langle \Delta(|\omega|^{p-2} \omega), \omega \rangle - K(R) |\omega|^{2p-2},$$

where $-K(R)$ is the pointwise lower bound of the curvature operator. Let η be a compactly supported nonnegative smooth function on M .

$$\begin{aligned} \int_M \eta^2 |\omega|^{p-1} \Delta |\omega|^{p-1} &\geq \int_M \eta^2 |\omega|^{p-2} \langle \Delta(|\omega|^{p-2} \omega), \omega \rangle - K(R) \int_M \eta^2 |\omega|^{2p-2} \\ &= \int_M \eta^2 |\omega|^{p-2} \langle \delta d(|\omega|^{p-2} \omega), \omega \rangle - K(R) \int_M \eta^2 |\omega|^{2p-2} \\ &= -K(R) \int_M \eta^2 |\omega|^{2p-2}. \end{aligned}$$

Integration by parts yields

$$\begin{aligned} K(R) \int_M \eta^2 |\omega|^{2p-2} &\geq \int_M \nabla(\eta^2 |\omega|^{p-1}) \cdot \nabla |\omega|^{p-1} \\ &\geq \frac{(p-1)^2}{4} \int_M \eta^2 |\omega|^{2p-6} |\nabla |\omega|^2|^2 - (p-1) \int_M \eta |\nabla \eta| |\omega|^{2p-4} |\nabla |\omega|^2|. \end{aligned}$$

It follows that

$$(2-2) \quad \frac{(p-1)^2}{4} \int_M \eta^2 |\omega|^{2p-6} |\nabla |\omega|^2|^2 \leq (p-1) \int_M \eta |\nabla \eta| |\omega|^{2p-4} |\nabla |\omega|^2| + K(R) \int_M \eta^2 |\omega|^{2p-2},$$

for all $p > 1$.

By Young's inequality, we have

$$(p-1) \eta |\nabla \eta| |\omega|^{2p-4} |\nabla |\omega|^2| \leq \frac{(p-1)^2}{8} \eta^2 |\omega|^{2p-6} |\nabla |\omega|^2|^2 + 2 |\nabla \eta|^2 |\omega|^{2p-2}.$$

Since

$$|\omega|^{2p-6} |\nabla |\omega|^2|^2 = \frac{4}{(p-1)^2} |\nabla |\omega|^{p-1}|^2,$$

then (2-2) can be written as

$$(2-3) \quad \int_M \eta^2 |\nabla |\omega|^{p-1}|^2 \leq 4 \int_M |\nabla \eta|^2 |\omega|^{2p-2} + 2K(R) \int_M \eta^2 |\omega|^{2p-2},$$

for all $p > 1$.

For $R > 0$ and $x \in \partial B_z(2R)$, let $\eta \in \mathcal{C}_0^\infty(B_x(R))$ be a cut-off function satisfying

$$\eta(y) = \begin{cases} 1 & \text{if } y \in B_x(\rho R), \\ 0 & \text{if } y \in M \setminus B_x(\gamma R). \end{cases}$$

Note that $\eta \in [0, 1]$ on M and $|\nabla\eta| \leq 2/((\gamma - \rho)R)$, for $0 < \rho < \gamma \leq 1$.

By the Sobolev inequality and (2-3),

$$\begin{aligned} \left(\int_{B_x(\rho R)} (|\omega|^{p-1})^{2\alpha} \right)^{1/\alpha} &\leq \left(\int_{B_x(\gamma R)} (\eta|\omega|^{p-1})^{2\alpha} \right)^{1/\alpha} \\ &\leq c_s(\nu) V_x(R)^{-2/\nu} R^2 16 \left(\frac{1}{(\gamma - \rho)^2 R^2} + K(R) \right) \int_{B_x(\gamma R)} |\omega|^{2p-2}, \end{aligned}$$

where $\alpha = \nu/(\nu - 2)$, and $c_s(\nu)$ is the Sobolev constant.

By the assumption on function $K(R)$, it is easy to see that

$$K(R) \leq \frac{c}{R^2}$$

on ball $B_x(R)$. Therefore,

$$(2-4) \quad \left(\int_{B_x(\rho R)} |\omega|^{2(p-1)\alpha} \right)^{1/\alpha} \leq c_s(\nu) V_x(R)^{-2/\nu} 4^2 \left(\frac{1}{(\gamma - \rho)^2} \right) \int_{B_x(\gamma R)} |\omega|^{2(p-1)},$$

where $\alpha = \nu/(\nu - 2)$.

Define

$$p = q_0 \alpha^i + 1 \quad \text{and} \quad R_i = (\rho + 2^{-i}(\gamma - \rho))R,$$

for $i = 0, 1, 2, 3, \dots$. Observe that $\lim_{i \rightarrow \infty} R_i = \rho R$. Let $\rho R = R_{i+1}$ and $\gamma R = R_i$ in inequality (2-4) and iterate the inequality; then

$$(2-5) \quad \sup_{B_x(\rho R)} |\omega|^{2q_0} \leq C V_x(R)^{-1} \left(\frac{1}{\gamma - \rho} \right)^\nu \int_{B_x(\gamma R)} |\omega|^{2q_0}.$$

When $q \geq 2q_0$, by (2-5), we have

$$|\omega|(x) \leq C \left(V_x(R)^{-1} \int_{B_x(R)} |\omega|^q \right)^{1/q},$$

for some constant C .

When $0 < q < 2q_0$, let $h_i = \sum_{j=1}^{i+1} 2^{-j}$, $\rho = h_i$, and $\gamma = h_{i+1}$, for all $i = 0, 1, 2, 3, \dots$. By (2-5), we have

$$(2-6) \quad \sup_{B_x(h_i R)} |\omega|^{2q_0} \leq C V_x(R)^{-1} 2^{(i+2)\nu} \int_{B_x(h_{i+1} R)} |\omega|^q \cdot \sup_{B_x(h_{i+1} R)} |\omega|^{2q_0 - q}.$$

Write $M(i) = \sup_{B_x(h_i R)} |\omega|^{2q_0}$. Inequality (2-6) becomes

$$(2-7) \quad M(i) \leq C V_x(R)^{-1} 2^{(i+2)\nu} \int_{B_x(R)} |\omega|^q M(i+1)^{(2q_0 - q)/2q_0}.$$

Let $\lambda = 1 - q/2q_0 \in (0, 1)$; iterating inequality (2-7), we have

$$M(0) \leq \prod_{i=0}^{j-1} \tilde{c}^{\lambda^i} M^{\lambda^j}(j) = \prod_{i=0}^{j-1} \left(C V_x(R)^{-1} 2^{\nu(i+1)} \int_{B_x(R)} |\omega|^q \right)^{\lambda^i} M^{\lambda^j}(j).$$

Let $j \rightarrow \infty$; we have

$$M(0) \leq (C)^{2q_0/q} V_x(R)^{-2q_0/q} \left(\int_{B_x(R)} |\omega|^q \right)^{2q_0/q}.$$

Hence,

$$|\omega|(x) \leq (C)^{1/q} V_x(R)^{-1/q} \left(\int_{B_x(R)} |\omega|^q \right)^{1/q} \leq C V_x(R)^{-1/q} \left(\int_{B_x(R)} |\omega|^q \right)^{1/q},$$

for some constant C .

For ω a p -harmonic l -form on M , and $x \in \partial B_z(2R)$, we have

$$|\omega|(x) \leq C \left(V_x(R)^{-1} \int_{B_x(R)} |\omega|^q \right)^{1/q}.$$

When the $L^q(M)$ norm of ω is assumed to be bounded by a fixed constant, since we also have $V_x(R) \geq cR$, we conclude that for any given $\epsilon > 0$, by taking R to be sufficiently large, $|\omega| < \epsilon$ on $M \setminus B_z(R)$. On the other hand, using the standard elliptic PDE theory, on ball $B_z(R)$, the length of ω and all its covariant derivatives can be bounded by the $L^q(M)$ norm of ω . In particular, we conclude that any bounded sequence of such ω admits a uniformly convergent subsequence on M . This finishes the proof of the theorem. \square

An immediate corollary is obtained from the proof of Theorem 2.4.

Corollary 2.5. *Let (M^m, g) be a complete noncompact manifold with nonnegative curvature operator. Then any bounded $L^q(M)$ p -harmonic l -forms on (M, g) must be zero.*

3. p -Harmonic maps

Here we derive a different type of Bochner formula for p -harmonic maps and study the set of p -harmonic maps with finite p -energy. Let (M^m, g) be a complete Riemannian manifold (without boundary) of dimension m with metric g , and let (N^n, g') be a complete manifold of dimension n with metric g' . For any smooth map $f : M \rightarrow N$ and compact domain $\Omega \subset M$, we define the p -energy of f on Ω :

$$E_p(\Omega, f) \equiv \frac{1}{p} \int_{\Omega} |df(x)|^p dV_g,$$

where $|df(x)|$ is the norm of the differential $df(x)$ of f at $x \in \Omega$, dV_g is the volume element of M , and $1 < p < \infty$ is a fixed number. Let $f^{-1}TN$ be the induced vector bundle by f over M . Then df can be viewed as a section of the bundle $\Lambda^1(f^{-1}TN) = T^*M \otimes f^{-1}TN$. We denote by $|df(x)|$ its norm at a point x of M , induced by the metrics g and g' .

A map f is called p -harmonic if it is a critical point of p -energy functional $E_p(\Omega, \cdot)$ for any compact domain $\Omega \subset M$. That is, f is a p -harmonic map if and only if

$$\frac{dE_p(f_s)}{ds} = 0$$

at $s = 0$ for any one-parameter family of maps $f_s : M \rightarrow N$ with $f_0 = f$ and $f_s(x) = f(x)$ if $x \in M \setminus \Omega$. We define the p -tension field $\tau_p(f)$ of f by

$$\tau_p(f) = -\delta(|df|^{p-2}df),$$

where $\delta : \Lambda^1(f^{-1}TN) \rightarrow \Lambda^0(f^{-1}TN)$ is the codifferential operator. Equivalently, a smooth map $f : M \rightarrow N$ is p -harmonic if and only if $\tau_p(f) = 0$.

Assume that (M, g) is a complete noncompact manifold with asymptotically nonnegative Ricci curvature, and that (N, g') is a complete manifold with non-positive sectional curvature. We denote the Ricci tensor of (M, g) by Ricci_M , and the curvature tensor of (N, g') by R_N . Let $\{e_1, \dots, e_m\}$ be a local orthonormal frame on M ; by the Weitzenböck formula [Eells and Lemaire 1983], we have

$$\begin{aligned} (3-1) \quad \frac{1}{2} \Delta |df|^2 &= \langle \Delta df, df \rangle + |\nabla df|^2 + \sum_{i=1}^m \langle df(\text{Ricci}_M(e_i)) \cdot df(e_i) \rangle \\ &\quad - \sum_{i,j=1}^m \langle R_N(df(e_j), df(e_i))df(e_i), df(e_j) \rangle \\ &\geq \langle \Delta df, df \rangle + |\nabla df|^2 - K|df|^2. \end{aligned}$$

Lemma 3.1 (Bochner-type formula for p -harmonic maps). *Let $u : M \rightarrow N$ be a smooth p -harmonic map and $\{e_i\}_{i=1}^m$ be an orthonormal basis of the tangent space of M . Then*

$$\begin{aligned} (3-2) \quad |du|^{p-1} \Delta |du|^{p-1} &= |du|^{p-2} \langle \Delta(|du|^{p-2}du), du \rangle \\ &\quad + (|\nabla(|du|^{p-2}du)|^2 - |\nabla |du|^{p-1}|^2) \\ &\quad + |du|^{2p-4} \sum_i^m \langle \text{Ricci}_M(du(e_i)), du(e_i) \rangle \\ &\quad - |du|^{2p-4} \sum_{i,j=1}^n \langle R_N(du(e_i), du(e_j))du(e_i), du(e_j) \rangle, \end{aligned}$$

in the sense of distributions. Also, if $\text{Ricci}_M \geq 0$ and $K_N \leq 0$, then

$$\begin{aligned} &|du|^{p-1} \Delta |du|^{p-1} \\ &\geq |du|^{p-2} \langle \Delta(|du|^{p-2} du), du \rangle + (|\nabla(|du|^{p-2} du)|^2 - |\nabla |du|^{p-1}|^2). \end{aligned}$$

Proof. The Bochner–Weitzenböck formula for $|du|^{p-1}$ asserts that

$$\begin{aligned} \frac{1}{2} \Delta |du|^{2p-2} &= \frac{1}{2} \Delta |du|^{p-2} du|^2 \\ &= \langle \Delta(|du|^{p-2} du), |du|^{p-2} du \rangle + |\nabla(|du|^{p-2} du)|^2 \\ &\quad + \sum_i^m \langle |du|^{p-2} (\text{Ricci}_M(du(e_i)), |du|^{p-2} du(e_i)) \rangle \\ &\quad - \sum_{i,j=1}^n \langle |du|^{p-2} R_N(du(e_i), du(e_j)) du(e_i), |du|^{p-2} du(e_j) \rangle \\ &= \langle \Delta(|du|^{p-2} du), |du|^{p-2} du \rangle + |\nabla(|du|^{p-2} du)|^2 \\ &\quad + |du|^{2p-4} \sum_i^m \langle \text{Ricci}_M(du(e_i)), du(e_i) \rangle \\ &\quad - |du|^{2p-4} \sum_{i,j=1}^n \langle R_N(du(e_i), du(e_j)) du(e_i), du(e_j) \rangle. \end{aligned}$$

On the other hand,

$$\frac{1}{2} \Delta |du|^{2p-2} = \frac{1}{2} \Delta (|du|^{p-1})^2 = |du|^{p-1} \Delta |du|^{p-1} + |\nabla |du|^{p-1}|^2.$$

Hence,

$$\begin{aligned} &|du|^{p-1} \Delta |du|^{p-1} + |\nabla |du|^{p-1}|^2 \\ &= \langle \Delta(|du|^{p-2} du), |du|^{p-2} du \rangle + |\nabla(|du|^{p-2} du)|^2 \\ &\quad + |du|^{2p-4} \sum_i^m \langle (\text{Ricci}_M(du(e_i)), du(e_i)) \rangle \\ &\quad - |du|^{2p-4} \sum_{i,j=1}^n \langle R_N(du(e_i), du(e_j)) du(e_i), du(e_j) \rangle. \end{aligned}$$

It follows that

$$\begin{aligned} &|du|^{p-1} \Delta |du|^{p-1} \\ &= |du|^{p-2} \langle \Delta(|du|^{p-2} du), du \rangle + (|\nabla(|du|^{p-2} du)|^2 - |\nabla |du|^{p-1}|^2) \\ &\quad + |du|^{2p-4} \sum_i^m \langle (\text{Ricci}_M(du(e_i)), du(e_i)) \rangle \\ &\quad - |du|^{2p-4} \sum_{i,j=1}^n \langle R_N(du(e_i), du(e_j)) du(e_i), du(e_j) \rangle. \end{aligned}$$

If $\text{Ricci}_M \geq 0$ and $K_N \leq 0$, then

$$|du|^{p-1} \Delta |du|^{p-1} \geq |du|^{p-2} \langle \Delta(|du|^{p-2} du), du \rangle + (|\nabla(|du|^{p-2} du)|^2 - |\nabla |du|^{p-1}|^2). \quad \square$$

Theorem 3.2. *Let (M, g) be a complete noncompact manifold with asymptotically nonnegative Ricci curvature, and let (N, g') be a complete Riemannian manifold with nonpositive sectional curvature. Then the set of p -harmonic maps u from M to N with $\int_M |du|^p dV_g \leq C$, for some $C > 0$ and $1 < p < \infty$, has a uniformly convergent subsequence.*

Proof. Let u be a p -harmonic map; if $K_N < 0$, the Bochner type formula (3-2) asserts that

$$|du|^{p-1} \Delta |du|^{p-1} \geq |du|^{p-2} \langle \Delta(|du|^{p-2} du), du \rangle + (|\nabla(|du|^{p-2} du)|^2 - |\nabla |du|^{p-1}|^2) - |du|^{2p-2} K(R).$$

By the Kato inequality, we have

$$|\nabla |du|^{p-1}| = |\nabla |du|^{p-2} du| \leq |\nabla(|du|^{p-2} du)|.$$

Thus,

$$(3-3) \quad |du|^{p-1} \Delta |du|^{p-1} \geq |du|^{p-2} \langle \Delta(|du|^{p-2} du), du \rangle - |du|^{2p-2} K(R).$$

Dividing both sides of (3-3) by $|du|^{p-2}$, we get

$$|du| \Delta |du|^{p-1} \geq \langle \Delta(|du|^{p-2} du), du \rangle - |du|^p K(R).$$

Let η be a compactly supported nonnegative smooth function on M ; then

$$\begin{aligned} \int_M \eta^2 |du| \Delta |du|^{p-1} &\geq \int_M \eta^2 \langle (d\delta + \delta d) |du|^{p-2} du, du \rangle - \int_M \eta^2 |du|^p K(R) \\ &= \int_M \eta^2 \langle d |du|^{p-2} du, d(du) \rangle - \int_M \eta^2 |du|^p K(R) \\ &= - \int_M \eta^2 |du|^p K(R). \end{aligned}$$

On the other hand, by integration by parts,

$$\begin{aligned}
 (3-4) \quad & - \int_M \eta^2 |du|^p K(R) \leq \int_M \eta^2 |du| \Delta |du|^{p-1} \\
 & = - \int_M \nabla(\eta^2 |du|) \cdot \nabla |du|^{p-1} \\
 & = - \int_M (\eta^2 \nabla |du| + |du| 2\eta \cdot \nabla \eta) \cdot ((p-1) |du|^{p-2} \nabla |du|) \\
 & = -(p-1) \int_M \eta^2 |du|^{p-2} |\nabla |du||^2 \\
 & \quad - 2(p-1) \int_M \eta \cdot \nabla \eta |du|^{p-1} \cdot \nabla |du|.
 \end{aligned}$$

Since

$$\frac{4}{p^2} |\nabla |du|^{p/2}|^2 = \frac{4}{p^2} \left| \frac{p}{2} |du|^{(p/2)-1} \nabla |du| \right|^2 = |du|^{p-2} |\nabla |du||^2$$

and

$$\frac{2}{p} |du|^{p/2} \nabla |du|^{p/2} = \frac{2}{p} |du|^{p/2} \frac{p}{2} |du|^{(p/2)-1} \nabla |du| = |du|^{p-1} \nabla |du|,$$

inequality (3-4) can be rewritten as

$$\begin{aligned}
 & - \int_M \eta^2 |du|^p K(R) \\
 & \leq - \frac{4(p-1)}{p^2} \int_M \eta^2 |\nabla |du|^{p/2}|^2 - \frac{4(p-1)}{p} \int_M |du|^{p/2} \cdot \nabla \eta \cdot \eta \cdot \nabla |du|^{p/2}.
 \end{aligned}$$

By Young's inequality,

$$\begin{aligned}
 & - \int_M \eta^2 |du|^p K(R) \\
 & \leq - \frac{4(p-1)}{p^2} \int_M \eta^2 |\nabla |du|^{p/2}|^2 + \left(\zeta \int_M \eta^2 |\nabla |du|^{p/2}|^2 + \frac{c_1}{\zeta} \int_M |\nabla \eta|^2 |du|^p \right),
 \end{aligned}$$

for some positive constants c_1 and $0 < \zeta < 1$. Therefore,

$$\begin{aligned}
 (3-5) \quad & \left(\frac{4(p-1)}{p^2} - 2\zeta \right) \int_M \eta^2 |\nabla |du|^{p/2}|^2 \\
 & \leq \frac{c_2}{\zeta} \left(\int_M |\nabla \eta|^2 |du|^p + \int_M \eta^2 |du|^p K(R) \right).
 \end{aligned}$$

For $R > 0$ and $x \in \partial B_z(2R)$, let $\eta \in C_0^\infty(B_x(R))$ be a cut-off function such that

$$\eta(y) = \begin{cases} 1 & \text{if } y \in B_x(\rho R), \\ 0 & \text{if } y \in M \setminus B_x(\gamma R). \end{cases}$$

Note that $\eta \in [0, 1]$ on M and $|\nabla\eta| \leq c_3/R$, for $0 < \rho < \gamma \leq 1$ and some positive constant c_3 .

By the curvature assumption on function $K(R)$, we have

$$K(R) \leq \frac{c_4}{R^2},$$

for some constant c_4 . Let $\zeta = (p-1)/p^2$; then inequality (3-5) becomes

$$\int_{B_x(R)} |\nabla|du|^{p/2}|^2 \leq \frac{c_5}{R^2} \int_{B_x(R)} |du|^p + \int_{B_x(R)} \frac{c_6}{R^2} |du|^p \leq \frac{C}{R^2} \int_{B_x(R)} |du|^p.$$

Therefore, for u a p -harmonic map from M to N and $x \in \partial B_z(2R)$, we have

$$\int_{B_x(R)} |\nabla|du|^{p/2}|^2 \leq \frac{C}{R^2} \int_M |du|^p.$$

When $\int_M |du|^p$ is assumed to be bounded by a fixed constant, by taking R to be sufficiently large, for any $\epsilon > 0$, we have $|\nabla|du|^{p/2}| < \epsilon$ on $M \setminus B_z(R)$. On the other hand, $|\nabla|du|^{p/2}|$ can be bounded by the finite energy of u on ball $B_z(R)$. We conclude that the set of such p -harmonic maps admits a uniformly convergent subsequence. If M is a compact manifold with nonnegative Ricci curvature, then the p -harmonic map is constant, which is an extension of the fact in the harmonic map case ($p = 2$) [Eells and Sampson 1964]. \square

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LIANG-CHU CHANG
CENTER FOR GENERAL EDUCATION
NATIONAL FORMOSA UNIVERSITY
HUWEI, YUNLIN COUNTY 632
TAIWAN
lachu4@gmail.com

CHIUNG-JUE ANNA SUNG
DEPARTMENT OF MATHEMATICS
NATIONAL TSING HUA UNIVERSITY
HSINCHU 30013
TAIWAN
cjsung@math.nthu.edu.tw

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pacific@math.ucla.edu

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Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Darren Long
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
long@math.ucsb.edu

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Robert Finn
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
finn@math.stanford.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Alexander Merkurjev
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
merkurev@math.ucla.edu

Jonathan Rogawski
Department of Mathematics
University of California
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