A NOTE ON \( p \)-HARMONIC \( l \)-FORMS
ON COMPLETE MANIFOLDS

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Let \((M^m, g)\) be an \(m\)-dimensional complete noncompact manifold. We show that for all \(p > 1\) and \(l > 1\), any bounded set of \(p\)-harmonic \(l\)-forms in \(L^q(M)\), with \(0 < q < \infty\), is relatively compact with respect to the uniform convergence topology if the curvature operator of \(M\) is asymptotically nonnegative.

1. Introduction

Let \((M^m, g)\) be an \(m\)-dimensional complete oriented Riemannian manifold with associated Riemannian metric \(g\). Let \(d\) be the exterior differential operator and let

\[ \delta \equiv \ast d \ast \]

be the codifferential operator, where the linear operator \(\ast\) is defined pointwise by

\[ \ast (\omega_1 \wedge \cdots \wedge \omega_l) \equiv \omega_{l+1} \wedge \cdots \wedge \omega_m, \]

for a positively oriented orthonormal coframe \(\{\omega_1, \omega_2, \ldots, \omega_m\}\) at the point. The Hodge–Laplace–Beltrami operator \(\triangle\) acting on the space of smooth \(l\)-forms \(\Lambda^l(M)\) is defined by

\[ \triangle \equiv -(d\delta + \delta d). \]

**Definition 1.1.** An \(l\)-form \(\omega\) on \(M\) is a \(p\)-harmonic \(l\)-form if \(\omega\) satisfies \(d\omega = 0\) and \(\delta(|\omega|^{p-2}\omega) = 0\) for all \(p > 1\).

When \(p = 2\), the \(p\)-harmonic \(l\)-form \(\omega \in \Lambda^l(M)\) is called a harmonic \(l\)-form on \((M, g)\), that is,

\[ \triangle_g \omega = 0. \]

When \(l = 0\), let \(\Omega\) be a compact domain on the Riemannian manifold \((M, g)\), and let \(\omega\) be a real smooth function on \(M\). For \(p > 1\), the \(p\)-energy of \(\omega\) on \(\Omega\) is

\[ E_p(\Omega, \omega) \equiv \frac{1}{p} \int_{\Omega} |\nabla \omega|^p dV_g. \]
The function $\omega$ is said to be $p$-harmonic on $M$ if $\omega$ is a critical point of $E_p(\Omega, \cdot)$ for all $\Omega \subset M$, that is, if $\omega$ satisfies the Euler–Lagrange equation

$$\text{div}(|\nabla \omega|^{p-2}\nabla \omega) = 0.$$ 

A curvature operator $K_l$ on manifold $M^m$ is defined as follows:

$$K_l = \begin{cases} 
 \text{lower bound of the curvature operator on } M & \text{for } l > 1; \\
(m - 1)^{-1} \times \text{(lower bound of the Ricci curvature)} & \text{for } l = 1. 
\end{cases}$$

We call this curvature operator $K_l$ of $M$ asymptotically nonnegative if $K_l \geq -K(r)$, where

$$K(r) : [0, \infty) \to [0, \infty)$$

is a nonnegative and nonincreasing continuous function of distance $r$ to a fixed point $z \in M$, with

$$\int_0^\infty r K(r) < \infty.$$ 

Yau [1975] proved that any positive harmonic function on a manifold with nonnegative Ricci curvature must be constant. Much work has been done in the finite dimension of space of polynomial growth harmonic functions of growth order at most $d$ [Li 1997; Colding and Minicozzi 1997; Li and Tam 1995; Li and Wang 1999]. Concerning general harmonic $l$-forms, Li [1980] established a dimension estimate of the space of polynomial growth harmonic forms. In this paper, we study general $p$-harmonic $l$-forms and $p$-harmonic maps on complete noncompact manifolds, for $p > 1$ and $l \neq 0$. For $p = 2$, Chen and Sung [2007] considered the space consisting of all harmonic $l$-forms of polynomial growth for all $l \geq 1$, and gave a dimension estimate of such a space when $M$ has asymptotically nonnegative curvature. Since the set of $p$-harmonic $l$-forms is no longer linear, it is interesting to study the set of $p$-harmonic $l$-forms and to seek topological and geometrical links. Interestingly, Zhang [2001] proved that any $L^q(M)$ $p$-harmonic 1-forms must be zero on a manifold with nonnegative Ricci curvature for $p > 1$ and $0 < q < \infty$. Chang et al. [2010] generalized Zhang’s result to a complete manifold $M$ with asymptotically nonnegative curvature and finite first Betti number. They proved that a bounded set of $L^q(M)$ $p$-harmonic 1-forms on $(M, g)$ has a uniformly convergent subsequence.

Next we introduce the Sobolev inequality. A geodesic ball $B_x(r)$ in a complete manifold $M$ is said to admit a Sobolev inequality $S(C, \nu)$ if there exist constants $C > 0$ and $\nu > 2$ such that for all $f \in C_0^\infty(B_x(r))$, we have

$$\left( \int_{B_x(r)} |f|^{2\nu/(\nu-2)} \right)^{(\nu-2)/\nu} \leq Cr^2 V_x^{-2/\nu}(r) \int_{B_x(r)} (|\nabla f|^2 + r^{-2} f^2),$$
where $V_x(r)$ is the volume of geodesic ball $B_x(r)$. Using the Bochner formula, the Moser iteration [1961] and the Sobolev inequality, Chang et al. [2010] showed that any bounded set of $p$-harmonic 1-forms in $L^q(M)$, with $0 < q < \infty$, is relatively compact with respect to the uniform convergence topology if $M$ has asymptotically nonnegative Ricci curvature and finite first Betti number. However, the Bochner formula does not work for $p$-harmonic $l$-forms for $l > 1$. We derive a new type of Bochner formula to overcome this obstacle. We study the set of $p$-harmonic $l$-forms, for $l > 1$, on a complete noncompact manifold $M$, and then study the set of $p$-harmonic maps from a complete manifold $M$ to a complete manifold $N$. In Section 2, we derive a different type of Bochner formula for $p$-harmonic $l$-forms and prove that any bounded set of $p$-harmonic $l$-forms in $L^q(M)$, with $0 < q < \infty$, must be relatively compact with respect to the uniform convergence topology if the curvature operator of $M$ is asymptotically nonnegative. Of course, this implies that the linear space of harmonic $l$-forms must be finite-dimensional when $p = 2$ and $l \geq 0$. Also, there is no nonzero $p$-harmonic $l$-form on $M$ in $L^q(M)$ if the curvature operator of $M$ is nonnegative. In Section 3, we also derive a different type of Bochner formula for $p$-harmonic maps from $M$ with asymptotically nonnegative Ricci curvature to $N$ with nonpositive sectional curvature. We prove that the set of such $p$-harmonic maps with finite $p$-energy on $M$ has a uniformly convergent subsequence. The $p$-harmonic map is constant if $M$ is compact with nonnegative Ricci curvature, which is an extension of the fact in the harmonic map case ($p = 2$).

2. $p$-harmonic $l$-forms

Any smooth $l$-form on an $m$-dimensional manifold $M$ satisfies the Kato inequality:

**Lemma 2.1** [Wan and Xin 2004; Calderbank et al. 2000; Herzlich 2000]. Let $\omega$ be a differentiable $l$-form on $M$. Then

$$|\nabla|\omega|^2| \leq 2|\omega||\nabla\omega|.$$

**Lemma 2.2** [Bochner 1946]. Let $\omega = \sum_I a_I \omega_I$ be an $l$-form on $M$. Then

$$\Delta|\omega|^2 = 2\langle\Delta\omega, \omega\rangle + 2|\nabla\omega|^2 + 2K_1\langle\omega, \omega\rangle.$$

Let $(M, g)$ be a complete noncompact manifold. We wish to study the set of $L^q$ $p$-harmonic $l$-forms on $M$ for $l > 1$ and $0 < q < \infty$. To prove the main theorem for all $l > 1$, we show a different type of Bochner formula for $p$-harmonic $l$-forms:

**Lemma 2.3** (Bochner-type formula for $p$-harmonic forms). Let $\omega$ be a $p$-harmonic $l$-form on an $m$-dimensional complete Riemannian $M^m$. Then

$$|\omega|\Delta|\omega|^{p-1} = \langle\Delta(|\omega|^{p-2}\omega), \omega\rangle + |\omega|^{2-p}\left(|\nabla(|\omega|^{p-2}\omega)|^2 - |\nabla|\omega|^{p-1}|^2\right) + K_1|\omega|^p,$$

in the sense of distributions.
Proof. The Bochner–Weitzenböck formula for $|\omega|^{p-2}\omega$ asserts that

\[ \frac{1}{2} \Delta |\omega|^{p-2}\omega |^2 = \left( \Delta (|\omega|^{p-2}\omega), |\omega|^{p-2}\omega \right) + |\nabla (|\omega|^{p-2}\omega)|^2 + K_l |\omega|^{p-2}\omega|^2. \]

The left side of (2-1) is given by

\[ \frac{1}{2} \Delta |\omega|^{p-2}\omega |^2 = \frac{1}{2} \Delta |\omega|^{2p-2} = \frac{1}{2} \Delta (|\omega|^{p-1})^2 = |\omega|^{p-1} \Delta |\omega|^{p-1} + |\nabla |\omega|^{p-1}|^2. \]

Hence,

\[ |\omega|^{p-1} \Delta |\omega|^{p-1} + |\nabla |\omega|^{p-1}|^2 = \left( \Delta (|\omega|^{p-2}\omega), |\omega|^{p-2}\omega \right) + |\nabla (|\omega|^{p-2}\omega)|^2 + K_l |\omega|^{2p-4}|\omega|^2. \]

It follows that

\[ |\omega|^{p-1} \Delta |\omega|^{p-1} = |\omega|^{p-2}(\Delta (|\omega|^{p-2}\omega), \omega) + (|\nabla (|\omega|^{p-2}\omega)|^2 - |\nabla |\omega|^{p-1}|^2) + K_l |\omega|^{2p-2}. \]

For $l$-forms with $l > 1$, the volume comparison property holds on $M$ with asymptotically nonnegative curvature operator [Li and Tam 1995]. Therefore, inside geodesic ball $B_x(R)$ with $r(x) = 2R$, the volume doubling property holds [Li and Tam 1995]. Also, by [Saloff-Coste 1992], a local weak Poincaré inequality holds on geodesic ball $B_x(R)$, and hence we have the Sobolev inequality $S(C, v)$ on $B_x(R)$ [Hajłasz and Koskela 1995]; that is, there exists a real number $v > 2$ such that

\[ \left( \int_{B_x(R)} |f|^{2v/(v-2)} \, dV \right)^{(v-2)/v} \leq C \cdot r^2 \cdot V^{-2/v}(B) \int_{B_x(R)} |\nabla f|^2 \, dV, \]

for all $f \in C_0^\infty(B_x(r))$, where $r \leq R$.

Theorem 2.4 (main theorem). Let $M^m$ be an $m$-dimensional complete Riemannian manifold with asymptotically nonnegative curvature operator $K_l$, for $l > 1$. Then a bounded set of $L^q(M)$ $p$-harmonic $l$-forms on $(M^m, g)$ has a uniformly convergent subsequence, for $1 < p < \infty$ and $0 < q < \infty$.

Proof. Let $\omega$ be a $p$-harmonic $l$-form on $M^m$. Lemma 2.3 asserts that

\[ |\omega|^{p-1} \Delta |\omega|^{p-1} = |\omega|^{p-2}(\Delta (|\omega|^{p-2}\omega), \omega) + (|\nabla (|\omega|^{p-2}\omega)|^2 - |\nabla |\omega|^{p-1}|^2) + K_l |\omega|^{2p-2}. \]

By the Kato inequality, we have

\[ |\nabla |\omega|^{p-1}| = |\nabla ||\omega|^{p-2}\omega|| \leq |\nabla(|\omega|^{p-2}\omega)|. \]
Therefore,

\[ |\omega|^{p-1} \Delta |\omega|^{p-1} \geq |\omega|^{p-2}(\Delta (|\omega|^{p-2} \omega), \omega) - K(R)|\omega|^{2p-2}, \]

where \(-K(R)\) is the pointwise lower bound of the curvature operator. Let \(\eta\) be a compactly supported nonnegative smooth function on \(M\).

\[
\int_M \eta^2 |\omega|^{p-1} \Delta |\omega|^{p-1} \geq \int_M \eta^2 |\omega|^{p-2}(\Delta (|\omega|^{p-2} \omega), \omega) - K(R) \int_M \eta^2 |\omega|^{2p-2} \\
= \int_M \eta^2 |\omega|^{p-2} \delta d(|\omega|^{p-2} \omega), \omega) - K(R) \int_M \eta^2 |\omega|^{2p-2} \\
= -K(R) \int_M \eta^2 |\omega|^{2p-2}.
\]

Integration by parts yields

\[
K(R) \int_M \eta^2 |\omega|^{2p-2} \\
\geq \int_M \nabla(\eta^2 |\omega|^{p-1}) \cdot \nabla |\omega|^{p-1} \geq \frac{(p-1)^2}{4} \int_M \eta^2 |\omega|^{2p-6} |\nabla |\omega|^2|^2 - (p-1) \int_M \eta |\nabla \eta| |\omega|^{2p-4} |\nabla |\omega|^2|.
\]

It follows that

\[
(2-2) \quad \frac{(p-1)^2}{4} \int_M \eta^2 |\omega|^{2p-6} |\nabla |\omega|^2|^2 \\
\leq (p-1) \int_M \eta |\nabla \eta| |\omega|^{2p-4} |\nabla |\omega|^2| + K(R) \int_M \eta^2 |\omega|^{2p-2},
\]

for all \(p > 1\).

By Young’s inequality, we have

\[
(p-1)\eta |\nabla \eta| |\omega|^{2p-4} |\nabla |\omega|^2| \leq \frac{(p-1)^2}{8} \eta^2 |\omega|^{2p-6} |\nabla |\omega|^2|^2 + 2 |\nabla \eta|^2 |\omega|^{2p-2}.
\]

Since

\[
|\omega|^{2p-6} |\nabla |\omega|^2|^2 = \frac{4}{(p-1)^2} |\nabla |\omega|^p|^2,
\]

then (2-2) can be written as

\[
(2-3) \quad \int_M \eta^2 |\nabla |\omega|^p|^2 \leq 4 \int_M |\nabla \eta|^2 |\omega|^{2p-2} + 2K(R) \int_M \eta^2 |\omega|^{2p-2},
\]

for all \(p > 1\).

For \(R > 0\) and \(x \in \partial B_x(2R)\), let \(\eta \in \mathcal{C}^\infty_0(B_x(R))\) be a cut-off function satisfying

\[
\eta(y) = \begin{cases} 
1 & \text{if } y \in B_x(\rho R), \\
0 & \text{if } y \in M \setminus B_x(\gamma R).
\end{cases}
\]
Note that \( \eta \in [0, 1] \) on \( M \) and \( |\nabla \eta| \leq 2/((\gamma - \rho) R) \), for \( 0 < \rho < \gamma \leq 1 \).

By the Sobolev inequality and (2-3),

\[
\left( \int_{B_x(\rho R)} (|\omega|^{p-1})^\alpha \right)^{1/\alpha} \leq \left( \int_{B_x(\gamma R)} (\eta|\omega|^{p-1})^\alpha \right)^{1/\alpha} \leq c_s(\nu) V_x(R)^{-2/\nu} R^2 16 \left( \frac{1}{(\gamma - \rho)^2 R^2} + K(R) \right) \int_{B_x(\gamma R)} |\omega|^{2p-2},
\]

where \( \alpha = \nu/(\nu - 2) \), and \( c_s(\nu) \) is the Sobolev constant.

By the assumption on function \( K(R) \), it is easy to see that

\[
K(R) \leq \frac{c}{R^2}
\]

on ball \( B_x(R) \). Therefore,

(2-4) \[
\left( \int_{B_x(\rho R)} |\omega|^{2(p-1)\alpha} \right)^{1/\alpha} \leq c_s(\nu) V_x(R)^{-2/\nu} 4^2 4 \left( \frac{1}{(\gamma - \rho)^2} \right) \int_{B_x(\gamma R)} |\omega|^{2p-2},
\]

where \( \alpha = \nu/(\nu - 2) \).

Define

\[
p = q_0 \alpha^i + 1 \quad \text{and} \quad R_i = (\rho + 2^{-i}(\gamma - \rho)) R,
\]

for \( i = 0, 1, 2, 3, \ldots \). Observe that \( \lim_{i \to \infty} R_i = \rho R \). Let \( \rho R = R_{i+1} \) and \( \gamma R = R_i \) in inequality (2-4) and iterate the inequality; then

(2-5) \[
\sup_{B_x(\rho R)} |\omega|^{2q_0} \leq CV_x(R)^{-1} \left( \frac{1}{\gamma - \rho} \right)^\nu \int_{B_x(\gamma R)} |\omega|^{2q_0}.
\]

When \( q \geq 2q_0 \), by (2-5), we have

\[
|\omega|(x) \leq C \left( V_x(R)^{-1} \int_{B_x(R)} |\omega|^q \right)^{1/q},
\]

for some constant \( C \).

When \( 0 < q < 2q_0 \), let \( h_i = \sum_{j=1}^{i+1} 2^{-j} \), \( \rho = h_i \), and \( \gamma = h_{i+1} \), for all \( i = 0, 1, 2, 3 \ldots \). By (2-5), we have

(2-6) \[
\sup_{B_x(h_i R)} |\omega|^{2q_0} \leq CV_x(R)^{-1} 2^{(i+2)\nu} \int_{B_x(h_{i+1} R)} |\omega|^q \sup_{B_x(h_{i+1} R)} |\omega|^{2q_0-q}.
\]

Write \( M(i) = \sup_{B_x(h_i R)} |\omega|^{2q_0} \). Inequality (2-6) becomes

(2-7) \[
M(i) \leq CV_x(R)^{-1} 2^{(i+2)\nu} \int_{B_x(R)} |\omega|^q M(i + 1)^{(2q_0-q)/2q_0}.
\]
Let $\lambda = 1 - q/2q_0 \in (0, 1)$; iterating inequality (2-7), we have

$$M(0) \leq \prod_{i=0}^{j-1} \tilde{c}^{\lambda^i} M^{\lambda^i}(j) = \prod_{i=0}^{j-1} \left(C V_x(R)^{-1} 2^{i(j+1)} \int_{B_x(R)} |\omega|^q \right)^{\lambda^i} M^{\lambda^i}(j).$$

Let $j \to \infty$; we have

$$M(0) \leq (C)^{2q_0/q} V_x(R)^{-2q_0/q} \left(\int_{B_x(R)} |\omega|^q \right)^{2q_0/q}.$$

Hence,

$$|\omega|(x) \leq (C)^{1/q} V_x(R)^{-1/q} \left(\int_{B_x(R)} |\omega|^q \right)^{1/q} \leq C V_x(R)^{-1/q} \left(\int_{B_x(R)} |\omega|^q \right)^{1/q},$$

for some constant $C$.

For $\omega$ a $p$-harmonic $l$-form on $M$, and $x \in \partial B_z(2R)$, we have

$$|\omega|(x) \leq C \left(V_x(R)^{-1} \int_{B_x(R)} |\omega|^q \right)^{1/q}.$$

When the $L^q(M)$ norm of $\omega$ is assumed to be bounded by a fixed constant, since we also have $V_x(R) \geq cR$, we conclude that for any given $\epsilon > 0$, by taking $R$ to be sufficiently large, $|\omega| < \epsilon$ on $M \setminus B_z(R)$. On the other hand, using the standard elliptic PDE theory, on ball $B_z(R)$, the length of $\omega$ and all its covariant derivatives can be bounded by the $L^q(M)$ norm of $\omega$. In particular, we conclude that any bounded sequence of such $\omega$ admits a uniformly convergent subsequence on $M$. This finishes the proof of the theorem.

An immediate corollary is obtained from the proof of Theorem 2.4.

**Corollary 2.5.** Let $(M^m, g)$ be a complete noncompact manifold with nonnegative curvature operator. Then any bounded $L^q(M)$ $p$-harmonic $l$-forms on $(M, g)$ must be zero.

### 3. $p$-Harmonic maps

Here we derive a different type of Bochner formula for $p$-harmonic maps and study the set of $p$-harmonic maps with finite $p$-energy. Let $(M^m, g)$ be a complete Riemannian manifold (without boundary) of dimension $m$ with metric $g$, and let $(N^n, g')$ be a complete manifold of dimension $n$ with metric $g'$. For any smooth map $f : M \to N$ and compact domain $\Omega \subset M$, we define the $p$-energy of $f$ on $\Omega$:

$$E_p(\Omega, f) \equiv \frac{1}{p} \int_{\Omega} |df(x)|^p \, dV_g,$$
where \(|df(x)|\) is the norm of the differential \(df(x)\) of \(f\) at \(x \in \Omega\), \(dV_g\) is the volume element of \(M\), and \(1 < p < \infty\) is a fixed number. Let \(f^{-1}TN\) be the induced vector bundle by \(f\) over \(M\). Then \(df\) can be viewed as a section of the bundle \(\Lambda^1(f^{-1}TN) = T^*M \otimes f^{-1}TN\). We denote by \(|df(x)|\) its norm at a point \(x\) of \(M\), induced by the metrics \(g\) and \(g'\).

A map \(f\) is called \(p\)-harmonic if it is a critical point of \(p\)-energy functional \(E_p(\Omega, \cdot)\) for any compact domain \(\Omega \subset M\). That is, \(f\) is a \(p\)-harmonic map if and only if

\[
\frac{dE_p(f_s)}{ds} = 0
\]

at \(s = 0\) for any one-parameter family of maps \(f_s : M \to N\) with \(f_0 = f\) and \(f_s(x) = f(x)\) if \(x \in M \setminus \Omega\). We define the \(p\)-tension field \(\tau_p(f)\) of \(f\) by

\[
\tau_p(f) = -\delta(|df|^{p-2}df),
\]

where \(\delta : \Lambda^1(f^{-1}TN) \to \Lambda^0(f^{-1}TN)\) is the codifferential operator. Equivalently, a smooth map \(f : M \to N\) is \(p\)-harmonic if and only if \(\tau_p(f) = 0\).

Assume that \((M, g)\) is a complete noncompact manifold with asymptotically nonnegative Ricci curvature, and that \((N, g')\) is a complete manifold with nonpositive sectional curvature. We denote the Ricci tensor of \((M, g)\) by \(\text{Ricci}_M\), and the curvature tensor of \((N, g')\) by \(R_N\). Let \(\{e_1, \ldots, e_m\}\) be a local orthonormal frame on \(M\); by the Weitzenböck formula [Eells and Lemaire 1983], we have

\[
\frac{1}{2} \Delta|df|^2 = \langle \Delta df, df \rangle + |\nabla df|^2 + \sum_{i=1}^{m} \left\langle df(\text{Ricci}_M(e_i)), df(e_i) \right\rangle
\]

\[
- \sum_{i,j=1}^{m} \left\langle R_N(df(e_j), df(e_i))df(e_i), df(e_j) \right\rangle
\]

\[
geq \langle \Delta df, df \rangle + |\nabla df|^2 - K|df|^2.
\]

**Lemma 3.1** (Bochner-type formula for \(p\)-harmonic maps). Let \(u : M \to N\) be a smooth \(p\)-harmonic map and \(\{e_i\}_{i=1}^{m}\) be an orthonormal basis of the tangent space of \(M\). Then

\[
|du|^{p-1} \Delta|du|^{p-1} = |du|^{p-2} \left( \Delta(|du|^{p-2} du), du \right)
\]

\[
+ \left( |\nabla(|du|^{p-2} du)|^2 - |\nabla|du|^{p-1}|^2 \right)
\]

\[
+ |du|^{2p-4} \sum_{i} \left\langle \text{Ricci}_M(du(e_i)), du(e_i) \right\rangle
\]

\[
- |du|^{2p-4} \sum_{i,j} \left\langle R_N(du(e_i), du(e_j))du(e_i), du(e_j) \right\rangle,
\]

in the sense of distributions. Also, if \(\text{Ricci}_M \geq 0\) and \(K_N \leq 0\), then
\[ |du|^{p-1} \Delta |du|^{p-1} \]
\[ \geq |du|^{p-2} \{ \Delta (|du|^{p-2} du), \ du \} + \left( |\nabla (|du|^{p-2} du)|^2 - |\nabla |du|^{p-1}|^2 \right). \]

**Proof.** The Bochner–Weitzenböck formula for $|du|^{p-1}$ asserts that
\[
\frac{1}{2} \Delta |du|^{2p-2} = \frac{1}{2} \Delta \left( |du|^{p-2} du \right)^2
\]
\[ = \{ \Delta (|du|^{p-2} du), |du|^{p-2} du \} + |\nabla (|du|^{p-2} du)|^2
\]
\[ + \sum_{i}^{m} |du|^{p-2} (\text{Ricci}_M(du(e_i)), |du|^{p-2} du(e_i))
\]
\[ - \sum_{i,j=1}^{n} |du|^{p-2} R_N(du(e_i), du(e_j)) du(e_i), |du|^{p-2} du(e_j))
\]
\[ = \{ \Delta (|du|^{p-2} du), |du|^{p-2} du \} + |\nabla (|du|^{p-2} du)|^2
\]
\[ + |du|^{2p-4} \sum_{i}^{m} \{ \text{Ricci}_M(du(e_i)), du(e_i) \}
\]
\[ - |du|^{2p-4} \sum_{i,j=1}^{n} \{ R_N(du(e_i), du(e_j)) du(e_i), du(e_j) \}. \]

On the other hand,
\[
\frac{1}{2} \Delta |du|^{2p-2} = \frac{1}{2} \Delta (|du|^{p-1})^2 = |du|^{p-1} \Delta |du|^{p-1} + |\nabla |du|^{p-1}|^2.
\]

Hence,
\[
|du|^{p-1} \Delta |du|^{p-1} + |\nabla |du|^{p-1}|^2
\]
\[ = \{ \Delta (|du|^{p-2} du), |du|^{p-2} du \} + |\nabla (|du|^{p-2} du)|^2
\]
\[ + |du|^{2p-4} \sum_{i}^{m} \{ \text{Ricci}_M(du(e_i)), du(e_i) \}
\]
\[ - |du|^{2p-4} \sum_{i,j=1}^{n} \{ R_N(du(e_i), du(e_j)) du(e_i), du(e_j) \}. \]

It follows that
\[
|du|^{p-1} \Delta |du|^{p-1}
\]
\[ = |du|^{p-2} \{ \Delta (|du|^{p-2} du), du \} + \left( |\nabla (|du|^{p-2} du)|^2 - |\nabla |du|^{p-1}|^2 \right)
\]
\[ + |du|^{2p-4} \sum_{i}^{m} \{ \text{Ricci}_M(du(e_i)), du(e_i) \}
\]
\[ - |du|^{2p-4} \sum_{i,j=1}^{n} \{ R_N(du(e_i), du(e_j)) du(e_i), du(e_j) \}. \]
If $\operatorname{Ricci}_M \geq 0$ and $K_N \leq 0$, then
\[
|\partial^p| \partial|\partial|^{p-1}
\geq |\partial|^{p-2}\{\partial(|\partial|^{p-2}\partial), \partial\} + \left(|\nabla(|\partial|^{p-2}\partial)|^2 - |\nabla|\partial|^{p-1}|^2\right).
\]

\[\square\]

**Theorem 3.2.** Let $(M, g)$ be a complete noncompact manifold with asymptotically nonnegative Ricci curvature, and let $(N, g')$ be a complete Riemannian manifold with nonpositive sectional curvature. Then the set of $p$-harmonic maps $u$ from $M$ to $N$ with
\[
\int_M |\partial|^p dV_g \leq C,
\]
for some $C > 0$ and $1 < p < \infty$, has a uniformly convergent subsequence.

**Proof.** Let $u$ be a $p$-harmonic map; if $K_N < 0$, the Bochner type formula (3-2) asserts that
\[
|\partial|^{p-1}\partial|\partial|^{p-1} \geq |\partial|^{p-2}\{\partial(|\partial|^{p-2}\partial), \partial\} + \left(|\nabla(|\partial|^{p-2}\partial)|^2 - |\nabla|\partial|^{p-1}|^2\right) - |\partial|^{2p-2}K(R).
\]

By the Kato inequality, we have
\[
|\nabla|\partial|^{p-1}| = |\nabla|\partial|^{p-2}\partial| \leq |\nabla(|\partial|^{p-2}\partial)|.
\]

Thus,
\[
|\partial|^{p-1}\partial|\partial|^{p-1} \geq |\partial|^{p-2}\{\partial(|\partial|^{p-2}\partial), \partial\} - |\partial|^{2p-2}K(R).
\]

Dividing both sides of (3-3) by $|\partial|^{p-2}$, we get
\[
|\partial|\partial|\partial|^{p-1} \geq \{\partial(|\partial|^{p-2}\partial), \partial\} - |\partial|^pK(R).
\]

Let $\eta$ be a compactly supported nonnegative smooth function on $M$; then
\[
\int_M \eta^2|\partial|\partial|\partial|^{p-1} \geq \int_M \eta^2((d\eta + \delta d)|\partial|^{p-2}\partial, \partial) - \int_M \eta^2|\partial|^pK(R)
\]
\[
= \int_M \eta^2(d|\partial|^{p-2}\partial, d(\partial)) - \int_M \eta^2|\partial|^pK(R)
\]
\[
= - \int_M \eta^2|\partial|^pK(R).
\]

On the other hand, by integration by parts,
\begin{align*}
(3-4) \quad - & \int_M \eta^2 |d\eta|^{p} K(R) \leq \int_M \eta^2 |d\eta| \Delta |d\eta|^{p-1} \\
& = - \int_M \nabla (\eta^2 |d\eta|) \cdot \nabla |d\eta|^{p-1} \\
& = - \int_M (\eta^2 \nabla |d\eta| + |d\eta| 2\eta \cdot \nabla \eta) \cdot ((p-1)|d\eta|^{p-2} \nabla |d\eta|) \\
& = -(p-1) \int_M \eta^2 |d\eta|^{p-2} \nabla |d\eta|^{2} \\
& \quad - 2(p-1) \int_M \eta \cdot \nabla \eta |d\eta|^{p-1} \cdot \nabla |d\eta|.
\end{align*}

Since
\[ \frac{4}{p^2} |\nabla |d\eta|^{p/2}|^2 = \frac{4}{p^2} \left( \frac{p}{2} |d\eta|^{(p/2)-1} \nabla |d\eta| \right)^2 = |d\eta|^{p-2} \nabla |d\eta|^{2} \]
and
\[ \frac{2}{p} |d\eta|^{p/2} \nabla |d\eta|^{p/2} = \frac{2}{p} |d\eta|^{p/2} \left( \frac{p}{2} |d\eta|^{(p/2)-1} \nabla |d\eta| \right) = |d\eta|^{p-1} \nabla |d\eta| , \]
inequality (3-4) can be rewritten as
\begin{align*}
- & \int_M \eta^2 |d\eta|^{p} K(R) \\
\leq & - \frac{4(p-1)}{p^2} \int_M \eta^2 |\nabla |d\eta|^{p/2}|^2 \quad - \frac{4(p-1)}{p} \int_M |d\eta|^{p/2} \cdot \nabla \eta \cdot \eta \cdot \nabla |d\eta|^{p/2}.
\end{align*}

By Young’s inequality,
\begin{align*}
- & \int_M \eta^2 |d\eta|^{p} K(R) \\
\leq & - \frac{4(p-1)}{p^2} \int_M \eta^2 |\nabla |d\eta|^{p/2}|^2 \quad + \left( \zeta \int_M \eta^2 |\nabla |d\eta|^{p/2}|^2 \quad + \frac{c_1}{\zeta} \int_M |\nabla \eta|^2 |d\eta|^{p} \right),
\end{align*}
for some positive constants \(c_1\) and \(0 < \zeta < 1\). Therefore,
\begin{align*}
(3-5) \quad \left( \frac{4(p-1)}{p^2} - 2\zeta \right) & \int_M \eta^2 |\nabla |d\eta|^{p/2}|^2 \\
\leq & \frac{c_2}{\zeta} \left( \int_M |\nabla \eta|^2 |d\eta|^{p} + \int_M \eta^2 |d\eta|^{p} K(R) \right).
\end{align*}

For \(R > 0\) and \(x \in \partial B_x(2R)\), let \(\eta \in C_0^\infty (B_x(R))\) be a cut-off function such that
\[ \eta(y) = \begin{cases} 
1 & \text{if } y \in B_x(\rho R), \\
0 & \text{if } y \in M \setminus B_x(\gamma R).
\end{cases} \]
Note that $\eta \in [0, 1]$ on $M$ and $|\nabla \eta| \leq c_3/R$, for $0 < \rho < \gamma \leq 1$ and some positive constant $c_3$.

By the curvature assumption on function $K(R)$, we have

$$K(R) \leq \frac{c_4}{R^2},$$

for some constant $c_4$. Let $\zeta = (p-1)/p^2$; then inequality (3-5) becomes

$$\int_{B_x(R)} |\nabla|du|^{p/2}|^2 \leq \frac{c_5}{R^2} \int_{B_x(R)} |du|^p + \int_{B_x(R)} \frac{c_6}{R^2} |du|^p \leq \frac{C}{R^2} \int_{B_x(R)} |du|^p.$$

Therefore, for $u$ a $p$-harmonic map from $M$ to $N$ and $x \in \partial B_z(2R)$, we have

$$\int_{B_x(R)} |\nabla|du|^{p/2}|^2 \leq \frac{C}{R^2} \int_{M} |du|^p.$$

When $\int_{M} |du|^p$ is assumed to be bounded by a fixed constant, by taking $R$ to be sufficiently large, for any $\epsilon > 0$, we have $|\nabla|du|^{p/2}| < \epsilon$ on $M \setminus B_z(R)$. On the other hand, $|\nabla|du|^{p/2}|$ can be bounded by the finite energy of $u$ on ball $B_z(R)$. We conclude that the set of such $p$-harmonic maps admits a uniformly convergent subsequence. If $M$ is a compact manifold with nonnegative Ricci curvature, then the $p$-harmonic map is constant, which is an extension of the fact in the harmonic map case ($p = 2$) [Eells and Sampson 1964].

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References


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