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STRUCTURE OF SOLUTIONS OF 3D AXISYMMETRIC NAVIER–STOKES EQUATIONS NEAR MAXIMAL POINTS

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Let v be a solution of the axially symmetric Navier–Stokes equation. We determine the structure of a certain (possible) maximal singularity of v in the following sense. Let (x_0, t_0) be a point where the flow speed $Q_0 = |v(x_0, t_0)|$ is comparable with the maximum flow speed at and before time t_0. We show, after a space-time scaling with the factor Q_0 and the center (x_0, t_0), that the solution is arbitrarily close in $C^{2,1,\alpha}_{\text{local}}$ norm to a nonzero constant vector in a fixed parabolic cube, provided that $r_0 Q_0$ is sufficiently large. Here r_0 is the distance from x_0 to the z axis. Similar results are also shown to be valid if $|r_0 v(x_0, t_0)|$ is comparable with the maximum of $|r v(x, t)|$ at and before time t_0. This mirrors a numerical result of Hou for the Euler equation: there exists a certain "calm spot" or depletion of vortex stretching in a region of high flow speed.

1. Introduction

We study the structure, in a space-time region with maximum flow speed, of solutions to the three-dimensional incompressible Navier–Stokes equations

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla p = \mu \Delta v, \\ \nabla \cdot v = 0, \end{cases} \qquad t \geq 0, \quad x \in \mathbb{R}^3, \tag{1-1}$$

with the axially symmetric initial data

$$v_0(x) = a^r(r, z, t) e_r + a^\theta(r, z, t) e_\theta + a^z(r, z, t) e_z. \tag{1-2}$$

In cylindrical coordinates, the solution $v = v(x, t)$ is of the form

$$v(x, t) = v^r(r, z, t) e_r + v^\theta(r, z, t) e_\theta + v^z(r, z, t) e_z. \tag{1-3}$$

Here $x = (x_1, x_2, z)$ and $r = \sqrt{x_1^2 + x_2^2}$, while

$$e_r = \begin{pmatrix} x_1/r \\ x_2/r \\ 0 \end{pmatrix}, \quad e_\theta = \begin{pmatrix} -x_2/r \\ x_1/r \\ 0 \end{pmatrix}, \quad e_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{1-4}$$

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are the three orthogonal unit vectors along the radial, angular, and axial directions. Also, the angular, swirl and axial components v^r, v^θ and v^z of the velocity field are solutions of the axially symmetric Navier–Stokes equations (or ASNS)

$$\begin{cases} \partial_t v^r + b \cdot \nabla v^r - \dfrac{(v^\theta)^2}{r} + \partial_r p = \Big(\Delta - \dfrac{1}{r^2}\Big) v^r, \\ \partial_t v^\theta + b \cdot \nabla v^\theta + \dfrac{v^r v^\theta}{r} = \Big(\Delta - \dfrac{1}{r^2}\Big) v^\theta, \\ \partial_t v^z + b \cdot \nabla v^z + \partial_z p = \Delta v^z, \\ b = v^r e_r + v^z e_z, \quad \nabla \cdot b = \partial_r v^r + \dfrac{v^r}{r} + \partial_z v^z = 0. \end{cases} \tag{1-5}$$

Here, without loss of generality, we set the viscosity constant μ equal to 1.

Although the axially symmetric case is a special instance of the full Navier–Stokes equations, the main regularity problem is just as wide open. Let us briefly discuss some interesting results on the axially symmetric Navier–Stokes equations. When $v^\theta = 0$, that is, in the no swirl case, Ladyzhenskaya [1968] and Ukhovskii and Iudovich [1968] proved that weak solutions are regular for all time. See also [Leonardi et al. 1999]. More recent activity, in the presence of swirl, includes [Chen et al. 2008; 2009], where it is proven that suitable axially symmetric solutions bounded by $Cr^{-\alpha}\sqrt{|t|}^{-1+\alpha}$ $(0 \le \alpha \le 1)$ are smooth. Here, r is the distance from a point to the z axis, and t is time. See also [Koch et al. 2009] and its local version using different methods, [Seregin and Šverák 2009]. Also in the presence of swirl, there is [Neustupa and Pokorný 2000], proving that regularity of one component (either v^r or v^θ) implies regularity of the other components of the solution. Also proving regularity, under an assumption of sufficiently small zero-dimension scaled norms, is [Jiu and Xin 2003].

We also wish to mention the regularity results of Chae and Lee [2002], who prove regularity results assuming finiteness of another zero-dimensional integral. On the other hand, Tian and Xin [1998] constructed a family of singular axis symmetric solutions with singular initial data, and Hou and Li [2008] found a special class of global smooth solutions. See also the recent extension [Hou et al. 2008].

In this paper, we take another approach to ASNS, seeking to understand the local structure of solutions when the flow velocity is very high. This is akin to the approach taken by Hamilton and Perelman in the study of Ricci flow. We can reach understanding when the flow speed $|v(x_0, t_0)|$ at a space-time point (x_0, t_0) is comparable with the maximum flow speed, or $r_0|v(x_0, t_0)|$ at a space-time point (x_0, t_0) is comparable with the maximum of $r|v(x, t)|$, at and before time t_0.

In order to present the result, we introduce some notations. Let $v = v(x, t)$ be a solution to ASNS. Here (x, t) is a point in space-time. Given a number $a > 0$ and a point in space-time (x_0, t_0), we define the parabolic cube

$$P(x_0, t_0, a) \equiv \{(x, t) \ : \ |x_0 - x| < a, \ t_0 - a^2 \le t \le t_0\}.$$

Unless stated otherwise, we use r, r_0, r_k to denote the distance between points x, x_0, x_k in space and the z-axis, respectively.

Now we are ready to state the main result of the paper.

Theorem 1.1. *Let $v = v(x, t)$, with $(x, t) \in \mathbb{R}^3 \times [0, T_0)$ and $T_0 > 0$, be a smooth solution to the three-dimensional ASNS, with initial condition v_0 satisfying*

$$\|v_0\|_{L^\infty(\mathbb{R}^3)} \le N_0, \quad \|v_0\|_{L^2(\mathbb{R}^3)} \le N_0, \quad |r v_0| \le N_0. \tag{1-6}$$

Here N_0 is any positive number. For any sufficiently small constant $\epsilon > 0$ and two other constants $\sigma_0 > 0$ and $0 < \alpha < 1$, there exists some $\rho_0 = \rho_0(\epsilon, N_0, \sigma_0, \alpha) > 0$ with the following properties.

(a) *Suppose*

$$r_0 |v(x_0, t_0)| \ge \rho_0^{-2}$$

at some point (x_0, t_0), where $x_0 \in \mathbb{R}^3$ and $t_0 \in (0, T_0)$. Suppose also that (x_0, t_0) is an almost maximal point in the sense that

$$|v(x_0, t_0)| \ge \tfrac{1}{4} \sup_{x \in \mathbb{R}^3, \ t \le t_0} |v(x, t)|.$$

Then the velocity v in the cube

$$P(x_0, t_0, (\sigma_0 \epsilon Q)^{-1}), \qquad Q \equiv |v(x_0, t_0)|,$$

after scaling by the factor Q, that is, $Q^{-1} v(Q^{-1} x + x_0, \, Q^{-2} t + t_0)$, is ϵ-close in $C^{2,1,\alpha}_{\text{local}}$ norm to a nonzero constant vector.

(b) *The conclusion in (a) still holds if*

$$r_0 |v(x_0, t_0)| \ge \rho_0^{-2}$$

at (x_0, t_0) and

$$r_0 |v(x_0, t_0)| \ge \tfrac{1}{4} \sup_{x \in \mathbb{R}^3, \ t \le t_0} r |v(x, t)|.$$

Remark 1.2. According to [Seregin and Šverák 2009] and [Chen et al. 2009], if a smooth solution blows up in finite time, then the scaling invariant quantity $r|v(x, t)|$ must also blow up in finite time near singularity. So the condition in (b) can always be satisfied if the solution develops finite time singularity.

Remark 1.3. The factor $\frac{1}{4}$ in the statement of the theorem can be replaced by any fixed positive number smaller than or equal to 1. In particular, the statement is true if (x_0, t_0) is a point such that $r_0|v(x_0, t_0)| = \sup_{x \in \mathbb{R}^3, \, t \le t_0} r|v(x, t)|$.

An important open question is to generalize the current result in (a) to the case when $|v(x_0, t_0)|$ is very large but still much smaller than maximum.

Another question is: what happens when $r_0|v(x_0, t_0)|$ is not large, but $|v(x_0, t_0)|$ is large at almost maximal point (x_0, t_0)?

Remark 1.4. The result and parameters in the theorem depend only on the norms of the initial value in (1-6). They do not depend on individual solutions.

We end the introduction by stating the main result in a more intuitive manner.

Definition 1.5 (calm spot). Let v be a solution of (1-5), and $\epsilon > 0$. We say the ball $B(x, 1/s)$ is an ϵ-calm spot of speed s if

$$\sup_{B(x,1/s)} |v| = s \quad \text{and} \quad \sup_{B(x,1/s)} |\nabla v| \le \epsilon s^2.$$

When ϵ is small, the gradient of the velocity is much smaller than the speed in an ϵ-calm spot, after scaling by s.

Corollary 1.6. *Let v be a solution of* (1-5) *whose initial value satisfies* (1-6). *If the flow becomes turbulent*, *that is*, *the speed becomes arbitrarily large*, *then there exist ϵ-calm spots of arbitrarily high speed. Here ϵ is any given positive number.*

By axial symmetry, there is a ring of very small vorticity. In [Hou 2009], one can find a related numerical result for the Euler equation, which is called deletion of vortex stretching. As an application, the method in this paper has helped to prove regularity of solutions in the BMO^{-1} class for axially symmetric Navier–Stokes equations. See [Lei and Zhang 2011].

2. Proof of Theorem 1.1

Let us prove part (a) first, after which the proof of (b) follows easily.

Proof. From the condition

$$\|v_0\|_{L^\infty(\mathbb{R}^3)} \le N_0, \quad \|v_0\|_{L^2(\mathbb{R}^3)} \le N_0, \quad |r v_0| \le N_0,$$

by standard theory (see [Koch et al. 2009, Proposition 4.1], for example), there exists a time h_0 such that

$$\|v(\cdot\,, t)\|_{L^\infty(\mathbb{R}^3)} \le 2N_0, \quad t \le h_0. \tag{2-1}$$

The proof is divided into several steps and uses the method of contradiction.

Step 1 (setting up a limit solution). Suppose part (a) of the theorem is false. Then for some $\epsilon > 0$ and $\sigma_0 > 0$, there exists a sequence of solutions v_k, with associated pressure $p_k = (-\Delta)^{-1}\nabla\cdot(v_k \cdot \nabla v_k)$ and initial condition satisfying (1-6), defined on the time interval $[0, T_k)$ for some $T_k > h_0$, which satisfies the following conditions:

(i) There exist sequences of positive numbers $\rho_k \to 0$, points $x_k \in \mathbb{R}^3$, and times $t_k \in [0, T_k)$ such that

$$r_k |v_k(x_k, t_k)| \ge \rho_k^{-2}.$$

(ii) For each k, the solution v_k in the parabolic region

$$P(x_k, t_k, [cQ_k]^{-1}) \equiv \{(x,t) \in [0, T_k) \,:\, |x_k - x| < (cQ_k)^{-1},\ t_k - (cQ_k)^{-2} \le t \le t_k\}$$

is not, after scaling by the factor Q_k, ϵ close, in $C^{2,1,\alpha}$ norm, to a nonzero constant vector. Here $c = \sigma_0 \epsilon$ and also

$$Q_k = |v_k(x_k, t_k)| \ge \tfrac{1}{4} \sup_{t \in [0, t_k],\ x \in R^3} |v_k(x,t)|.$$

Write $\alpha_k = r_k Q_k = r_k |v_k(x_k, t_k)|$. We consider v_k in the space-time cube

$$P\left(x_k, t_k, \frac{r_k}{\sqrt{\alpha_k}}\right) \equiv B\left(x_k, \frac{r_k}{\sqrt{\alpha_k}}\right) \times \left[t_k - \left(\frac{r_k}{\sqrt{\alpha_k}}\right)^2, t_k\right].$$

Note that

$$\beta_k \equiv \frac{r_k}{\sqrt{\alpha_k}} = \frac{r_k}{\sqrt{r_k Q_k}} = o(r_k), \tag{2-2}$$

$$Q_k \beta_k = \sqrt{r_k Q_k} \to \infty, \quad k \to \infty.$$

Define the scaled function

$$\tilde{v}_k = Q_k^{-1} v_k(Q_k^{-1}\tilde{x} + x_k,\, Q_k^{-2}\tilde{t} + t_k). \tag{2-3}$$

Then $\tilde{v}_k$ is a solution of the Navier–Stokes equation in the slab $\mathbb{R}^3 \times [-(Q_k\beta_k)^2, 0]$. By the assumption on Q_k, we know that $|\tilde{v}_k| \le 4$ whenever it is defined. Since $\tilde{v}_k$ is a bounded mild solution, [Koch et al. 2009, Proposition 4.1] gives, for example, that the $C^{2,1,\alpha}$ norm of $\tilde{v}_k$ are uniformly bounded in $\mathbb{R}^3 \times [-(Q_k\beta_k)^2 + 1, 0]$. The pressure

$$P_k = Q_k^{-2} p_k(Q_k^{-1}\tilde{x} + x_k,\, Q_k^{-2}\tilde{t} + t_k),$$

satisfying $\Delta P_k = \operatorname{div}(\tilde{v}_k \cdot \nabla \tilde{v}_k)$, also has uniformly bounded $C^{2,1,\alpha}_{\text{local}}$ norm, by virtue of standard Schauder theory. Indeed, all $C^{p,p/2}$ norms are bounded for $p \ge 1$, though we do not need this fact here.

Let us restrict the solution $\tilde{v}_k$ to the cube

$$P(0, 0, Q_k\beta_k) = \{(\tilde{x}, \tilde{t}) \,:\, |\tilde{x}| \le Q_k\beta_k, -(Q_k\beta_k)^2 \le \tilde{t} \le 0\}.$$

By the uniform bounds on the $C^{2,1,\alpha}_{\text{local}}$ norm and the fact that $Q_k\beta_k \to \infty$, we know there exists a subsequence, still called $\{\tilde{v}_k\}$, that converges to an ancient solution of the Navier–Stokes equation in $C^{2,1,\alpha}_{\text{local}}$ sense. Let us call this ancient solution $\tilde{v}$. Note that $\tilde{v}$ has length 1 at $(0, 0)$ and is hence nontrivial. In the next step, we show that it is a spatial 2-dimensional solution, one dimension being the z-dimension.

Step 2 (proving $\tilde{v}$ is a 2D solution). Denote by v_k^θ the angular component of v_k. For the given initial value, it is known that

$$|v_k^\theta(x,t)| \le \frac{N_0}{r}.$$

For $x \in B(x_k, \beta_k)$, we have, by (2-2),

$$|v_k^\theta(x,t)| \le \frac{2N_0}{r_k}$$

when k is sufficiently large. Therefore,

$$Q_k^{-1}|v_k^\theta(x,t)| \le \frac{2N_0}{Q_k r_k} \to 0, \quad k \to \infty. \tag{2-4}$$

In the standard basis for $\mathbb{R}^3$, put $x_k = (x_{k,1}, x_{k,2}, x_{k,3})$, with the third component being the z-component, and let $\xi_k = (0, 0, x_{k,3})$. Since $(x_k - \xi_k)/|x_k - \xi_k|$ are unit vectors, there exists a subsequence, still labeled by k, that converges to a unit vector $\zeta = (\zeta_1, \zeta_2, 0)$. We use

$$\zeta,\ \zeta' = (-\zeta_2, \zeta_1, 0),\ (0, 0, 1)$$

as the basis of a new coordinate. Since this basis is obtained by a rotation around the z-axis, we know v_k is invariant. From now on, when we mention the coordinates of a point, we use the new basis with the same origin. We still use (θ, r, z) to denote the variables for the cylindrical system corresponding to this new basis.

For $x \in B(x_k, \beta_k)$, we recall that θ is the angle between x and ζ. Then

$$\cos\theta = \frac{(x - \xi_k)\cdot\zeta}{|x - (0,0,x_3)^T|} = \frac{(x_k - \xi_k)\cdot\zeta}{|x_k - \xi_k|} + \frac{O(\beta_k)}{r_k} \to 1, \quad k \to \infty. \tag{2-5}$$

For $v_k = v_k(x,t)$ in $B(x_k, \beta_k) \times [t_k - \beta_k^2, t_k]$, we have defined

$$\tilde{v}_k = \tilde{v}_k(\tilde{x}, \tilde{t}) = Q_k^{-1} v_k(Q_k^{-1}\tilde{x} + x_k,\ Q_k^{-2}\tilde{t} + t_k),$$

where $x = Q_k^{-1}\tilde{x} + x_k$ and $t = Q_k^{-2}\tilde{t} + t_k$. Then for $x = (x^{(1)}, x^{(2)}, x^{(3)})$ and $\tilde{x} = (\tilde{x}^{(1)}, \tilde{x}^{(2)}, \tilde{x}^{(3)})$, we have

$$\begin{cases} \partial_r v_k(x,t) = \partial_{x^{(1)}} v_k(x,t)\cos\theta + \partial_{x^{(2)}} v_k(x,t)\sin\theta \\ \qquad\qquad\quad = Q_k^2 \partial_{\tilde{x}^{(1)}} \tilde{v}_k(\tilde{x},\tilde{t})\cos\theta + Q_k^2 \partial_{\tilde{x}^{(2)}} \tilde{v}_k(\tilde{x},\tilde{t})\sin\theta, \\ \partial_r^2 v_k(x,t) = Q_k^3 \partial^2_{\tilde{x}^{(1)}} \tilde{v}_k(\tilde{x},\tilde{t})\cos^2\theta + 2Q_k^3 \partial^2_{\tilde{x}^{(1)}\tilde{x}^{(2)}} \tilde{v}_k(\tilde{x},\tilde{t})\sin\theta\cos\theta \\ \qquad\qquad\qquad\qquad\qquad\qquad + Q_k^3 \partial^2_{\tilde{x}^{(2)}} \tilde{v}_k(\tilde{x},\tilde{t})\sin^2\theta, \\ \partial_z^2 v_k(x,t) = Q_k^3 \partial^2_{\tilde{x}^{(3)}} \tilde{v}_k(\tilde{x},\tilde{t}), \\ \partial_t^2 v_k(x,t) = Q_k^3 \partial^2_{\tilde{t}} \tilde{v}_k(\tilde{x},\tilde{t}). \end{cases} \tag{2-6}$$

For the pressure $p_k = p_k(x, t)$, recall that

$$\tilde{p}_k = \tilde{p}_k(\tilde{x}, \tilde{t}) = Q_k^{-2} p_k(Q_k^{-1}\tilde{x} + x_k,\, Q_k^{-2}\tilde{t} + t_k).$$

Therefore,

$$\partial_r p_k(x, t) = Q_k^3 \partial_{\tilde{x}^{(1)}} \tilde{p}_k(\tilde{x}, \tilde{t}) \cos\theta + Q_k^3 \partial_{\tilde{x}^{(2)}} \tilde{p}_k(\tilde{x}, \tilde{t}) \sin\theta. \tag{2-7}$$

Writing $v_k = v_k^r e_r + v_k^\theta e_\theta + v_k^z e_z$, we get

$$\begin{aligned} v_k^r \partial_r v_k^r + v_k^z \partial_z v_k^r = Q_k^3 \big(& v_k^r(\tilde{x}, \tilde{t}) \partial_{\tilde{x}^{(1)}} \tilde{v}_k^r(\tilde{x}, \tilde{t}) \cos\theta \\ & + v_k^r(\tilde{x}, \tilde{t}) \partial_{\tilde{x}^{(2)}} \tilde{v}_k^r(\tilde{x}, \tilde{t}) \sin\theta + \tilde{v}_k^z \partial_{\tilde{x}^{(3)}} \tilde{v}_k^r(\tilde{x}, \tilde{t}) \big). \end{aligned} \tag{2-8}$$

Substituting these identities into the equation for v_k^r,

$$\Big(\partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\Big) v_k^r - (b \cdot \nabla) v_k^r + \frac{(v_k^\theta)^2}{r} - \frac{\partial p_k}{\partial r} - \frac{\partial v_k^r}{\partial t} = 0,$$

we arrive at

$$\begin{aligned} &(\partial_{\tilde{x}^{(1)}}^2 + \partial_{\tilde{x}^{(3)}}^2)\tilde{v}_k^r - (\tilde{v}_k^r \partial_{\tilde{x}^{(1)}} + \tilde{v}_k^z \partial_{\tilde{x}^{(3)}})\tilde{v}_k^r - \partial_{\tilde{x}^{(1)}} \tilde{p}_k - \partial_{\tilde{t}} v_k^r \\ &\quad + \frac{1}{Q_k r}\big(\partial_{\tilde{x}^{(1)}} \tilde{v}_k(\tilde{x}, \tilde{t}) \cos\theta + \partial_{\tilde{x}^{(2)}} \tilde{v}_k(\tilde{x}, \tilde{t}) \sin\theta\big) - \frac{1}{(Q_k r)^2} \tilde{v}_k^r + \frac{(r v_k^\theta)^2}{(Q_k r)^3} + O(\theta) = 0. \end{aligned}$$

Here $O(\theta)$ represents all the terms that vanish when $\theta \to 0$ as $k \to \infty$. In particular, all terms involving the derivative with respect to $\tilde{x}^{(2)}$ are included in $O(\theta)$.

Recall that $Q_k r$ is comparable to $Q_k r_k$, which goes to ∞. Letting $k \to \infty$ and noting that v_k and derivatives are uniformly bounded, we know that $\tilde{v}^1$, the limit of $\tilde{v}_k^r$, satisfies

$$(\partial_{\tilde{x}^{(1)}}^2 + \partial_{\tilde{x}^{(3)}}^2)\tilde{v}^{(1)} - (\tilde{v}^{(1)} \partial_{\tilde{x}^{(1)}} + \tilde{v}^{(3)} \partial_{\tilde{x}^{(3)}})\tilde{v}^{(1)} - \partial_{\tilde{x}^{(1)}} \tilde{p} - \partial_{\tilde{t}} v^{(1)} = 0.$$

Here $\tilde{v}^{(3)}$ is the limit of v_k^z, for which we have, in a similar manner,

$$(\partial_{\tilde{x}^{(1)}}^2 + \partial_{\tilde{x}^{(3)}}^2)\tilde{v}^{(3)} - (\tilde{v}^{(1)} \partial_{\tilde{x}^{(1)}} + \tilde{v}^{(3)} \partial_{\tilde{x}^{(3)}})\tilde{v}^{(3)} - \partial_{\tilde{x}^{(3)}} \tilde{p} - \partial_{\tilde{t}} v^{(3)} = 0.$$

Note that $\tilde{v}_k$ and its derivatives are uniformly bounded in the region of concern. When $k \to \infty$, then $\theta \to 0$ in the region of concern. Hence $\tilde{v}_k^\theta$ and derivatives all vanish when $k \to \infty$.

Finally, we need to show that $\tilde{v}^{(1)}$ and $\tilde{v}^{(3)}$ are independent of the variable $\tilde{x}^{(2)}$. To prove this, let us recall that $\partial_\theta v_k^r = \partial_\theta v_k^z = 0$. Hence

$$-\partial_{x^{(1)}} v_k^r \sin\theta + \partial_{x^{(2)}} v_k^r \cos\theta = -\partial_{x^{(1)}} v_k^z \sin\theta + \partial_{x^{(2)}} v_k^z \cos\theta = 0.$$

This implies

$$\partial_{\tilde{x}^{(2)}} \tilde{v}_k = \partial_{\tilde{x}^{(1)}} \tilde{v}_k \tan\theta.$$

Taking $k \to \infty$ (and therefore $\theta \to 0$), we see the desired result.

Step 3. Here we use a regularity result already cited [Koch et al. 2009, Proposition 4.1] and the fact that 2-dimensional ancient (mild) solutions are constants [Koch et al. 2009] to conclude that $\tilde{v}_k$, with k large, is ϵ-close to a nonzero constant vector in $C^{2,1,\alpha}_{\text{local}}$ sense. This contradiction with the condition (ii) completes the proof.

Now we prove part (b). Suppose it is false. Then for some $\epsilon > 0$, there exists a sequence of solutions v_k with normalized initial condition as above, defined on the time interval $[0, T_k)$ for some $T_k \in [h_0, T_0]$, that satisfies the following conditions.

(i) There exist sequences of positive numbers $\rho_k \to 0$, points $x_k \in \mathbb{R}^3$, and times $t_k \in [0, T_k)$ such that

$$r_k |v_k(x_k, t_k)| \ge \rho_k^{-2}.$$

(ii) For each k, the solution v_k in the parabolic region

$$P(x_k, t_k, [cQ_k]^{-1}) \equiv \{(x,t) \in [0, T_k) : |x - x_k| < (cQ_k)^{-1},\ t_k - (cQ_k)^{-2} \le t \le t_k\}$$

is not, after scaling by the factor Q_k, ϵ-close, in $C^{2,1,\alpha}_{\text{local}}$ norm, to a nonzero constant vector. Here $c = \sigma_0 \epsilon$ and also

$$r_k |v_k(x_k, t_k)| \ge \frac{1}{4} \sup_{t \in [0, t_k],\ x \in R^3} r |v_k(x, t)|.$$

As before, define $Q_k = |v(x_k, t_k)|$. Suppose k is large. Then for $x \in B(x_k, \beta_k)$ with $\beta_k = r_k / \sqrt{r_k Q_k} = o(r_k)$, there holds, for $t \le t_k$,

$$r|v(x,t)| \le r_k |v(x_k, t_k)| = r_k Q_k$$

and $\frac{1}{2} r_k \le r \le 2 r_k$, when k is large. This shows, in the ball $B(x_k, \beta_k)$ and for $t \le t_k$, that

$$|v(x,t)| \le 2 Q_k.$$

Now we can scale by Q_k^{-1} in the above ball again, as in the proof of part (a). By [Seregin and Šverák 2009, Theorem 2.8], the limit of scaled solutions is again a bounded, mild, ancient solution. Similar arguments as in part (a) lead to a contradiction, proving part (b). □

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