LOCAL COMPARISON THEOREMS FOR KÄHLER MANIFOLDS

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We establish a sharp relative volume comparison theorem for small balls on Kähler manifolds with lower bound on Ricci curvature, assuming real analyticity of the metric. The model spaces being compared to are complex space forms, that is, Kähler manifolds with constant holomorphic sectional curvature. Moreover, we give an example showing that on Kähler manifolds, the pointwise Laplacian comparison theorem does not hold when the Ricci curvature is bounded from below.

1. Introduction

Comparison theorems are fundamental tools in geometric analysis. They are vital in estimates of spectra, heat kernels and the Sobolev constants. The classical Bishop–Gromov relative volume comparison theorem [Bishop and Crittenden 1964; Gromov 1981; Li 1993] in Riemannian geometry is this:

**Theorem 1.1.** Let $M^n$ be a complete Riemannian manifold of dimension $n$ such that $\text{Ric} \geq (n-1)K$. For any $p \in M$ and $0 < a < b$, the volume of geodesic balls satisfies

$$\frac{\text{Vol} B_p(b)}{\text{Vol} B_p(a)} \leq \frac{\text{Vol} B_{M_K}(b)}{\text{Vol} B_{M_K}(a)},$$

where $M_K$ is the simply connected real space form with sectional curvature $K$ and $\text{Vol} B_{M_K}(r)$ is the volume of the geodesic ball in $M_K$ with radius $r$. Equality holds if and only if $B_p(b)$ is isometric to $B_{M_K}(b)$.

The key ingredient in Theorem 1.1 is the Laplacian comparison theorem [Cheeger and Ebin 2008; Schoen and Yau 1994]:

**Theorem 1.2.** Let $M^n$ be a complete Riemannian manifold with $\text{Ric} \geq (n-1)K$. Let $M_k$ be the simply connected real space form with sectional curvature $K$. Denote by $r_M(x)$ the distance function from $p$ to $x$ in $M$. Let $r_{M_k}$ be the distance function on $M_k$. Then for any $x \in M$ and $y \in M_k$ with $r_M(x) = r_{M_k}(y)$,

$$\Delta r_M(x) \leq \Delta r_{M_k}(y).$$

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The model spaces in the theorems above are real space forms. In the Kähler category, a natural question is whether we can replace the model spaces by Kähler models, that is, complex space forms which are Kähler manifolds with constant holomorphic sectional curvature. Li and Wang [2005] showed that when the bi-sectional curvature has a lower bound, both of the theorems above hold with Kähler models. So the question left is: what can we get if we only assume the lower bound of the Ricci curvature? This note addresses the local case. Our main theorem is:

**Theorem 1.3.** Let \( M^n \) be a Kähler manifold of complex dimension \( n \) with a real analytic metric. Assume \( \text{Ric} \geq K \), where \( K \) is any real number. Given any point \( p \in M \), there exists \( r = r(p, M) > 0 \) such that for any \( 0 < a < b < r \), the volume of geodesic balls satisfies

\[
\frac{\text{Vol} B_{M^n}(p, b)}{\text{Vol} B_{M^n}(p, a)} \leq \frac{\text{Vol} B_{N_K}(b)}{\text{Vol} B_{N_K}(a)},
\]

where \( N_K \) denotes the rescaled complex space form with \( \text{Ric} = K \) and \( \Delta_{N_K} r \) is the Laplacian of distance function on \( N_K \). Equality holds if and only if \( M \) is locally isometric to \( N_K \).

**Remark.** Theorem 1.3 is a local version of the Bishop–Gromov relative volume comparison theorem on Kähler manifolds. However, one cannot directly extend the result to any radius. A simple counterexample is a product of several \( \mathbb{P}^1 \) with the standard product metric: the diameter is greater than that of the complex space form. Thus, when \( r \) is large, the inequality in Theorem 1.3 does not hold.

We will prove a slightly stronger result:

**Theorem 1.4.** Under the hypotheses of Theorem 1.3, there exists \( r_0 = r_0(p, M) > 0 \) such that for any \( r < r_0 \), the average Laplacian comparison holds,

\[
\int_{\partial B_p(r)} \frac{\Delta r}{A(\partial B_p(r))} \leq \Delta_{N_K} r(r),
\]

where \( \Delta_{N_K} r \) is the Laplacian of distance function on \( N_K \). Moreover, the equality holds if and only if \( M \) is locally isometric to \( N_K \).

**Remark.** Theorem 1.4 is a local version of Theorem 1.2 in the average sense. However, on Kähler manifolds with lower bound on Ricci curvature, the pointwise Laplacian comparison does not even hold locally (see Section 6).

The idea of the proof of Theorem 1.4 is very simple. We shall expand the area of the geodesic sphere \( A(\partial B_p(r)) \) as a power series, then compare the coefficients with those of the rescaled complex space form. The computation is complicated since it involves the covariant derivatives of the curvature tensor of arbitrary order.

This note is organized as follows:
In Section 2, we state two propositions which demonstrate the relation between the derivatives of $A(\partial B_p(r))$ and the covariant derivatives of the curvature tensor at $p$. Section 3 is the first part of the proof of Proposition 2.1. We shall estimate the derivatives of $A(\partial B_p(r))$ up to order 4. In the estimate of the 4-th derivative, the Kähler condition is employed. The most important part is Section 4. We use induction to prove Proposition 2.1. Besides the routine computation, there are two technical lemmas (Lemma 4.4 and Lemma 4.6) which simplify the computation of higher order covariant derivatives of the curvature tensor significantly. One should note that the Kähler condition is essential in these two lemmas. We complete the proof of Proposition 2.2 and Theorem 1.4 in Section 5. The last section is devoted to giving an example showing that the pointwise Laplacian comparison with the complex space form does not necessarily hold if the complex dimension is greater or equal to 2.

2. Basic set up

Throughout this note, we implicitly evaluate derivatives of functions of $r$ at $r = 0$. Given a point $p$ on a Kähler manifold $M^n$, fix a unit vector $e_0 \in T_p M$. Along the geodesic $l$ from $p$ with initial direction $e_0$, consider the Jacobian equation $J'' = R(e_0, J) e_0$. Set up an orthonormal frame $\{e_k\}$ at $p$ such that

$$Je_{2i} = e_{2i+1} \quad \text{and} \quad Je_{2i+1} = -e_{2i}$$

for $0 \leq i \leq n - 1$. Parallel transport the frame along the geodesic $l$. Consider the Jacobian field $J_u$ with initial value $J_u(0) = 0$, $J'_u(0) = e_u$. We may write

$$J_u = J_u(r, e_0) = \sum_{i=1}^{2n-1} \sum_{v=0}^{i-2} r^i C^v_{u,i} e_v$$

where $C^v_{u,i}$ are constants independent of $r$. Denote $R_{e_0 e_u e_0 e_v}$ by $R_{u v}$ when $e_0$ is fixed. Plugging (2-1) into the Jacobian equation, we get

$$\sum_i \sum_v i(i - 1) r^{i-2} C^v_{u,i} e_v = \sum_k \sum_w r^k C^w_{u,k} R(e_0, e_w) e_0.$$  

Along the geodesic $l$,

$$R(e_0, e_w) e_0 = \sum_{s=0}^{2n-1} \sum_{j=0}^{\infty} \frac{R_{s w}^{(j)}}{j!} e_s r^j$$

where $R_{s w}^{(j)}$ denotes the $j$-th covariant derivative of $R_{s w}$ along $e_0$ at $p$.  


Inserting this into (2-2), we get
\[ \sum_{i,v} i(i - 1)r^{i-2}C^v_{u,i} e_v = \sum_{k,j,w,s} r^{k+j} C^w_{u,k} \frac{R^{(j)}_{sw}}{j!} e_s. \]
Comparing coefficients, we obtain
\[ (2-3) \quad C^v_{u,i} = \sum_{k+j=i-2,w} C^w_{u,k} \frac{R^{(j)}_{uw}}{j!i(i-1)}. \]

A simple iteration now gives the constants $C^v_{u,i}$. First we have $C^v_{u,1} = \delta^v_u$ and $C^v_{u,2} = 0$. Then we get
\[ C^v_{u,3} = \sum_w C^w_{u,1} \frac{R_{uv}}{6} = \frac{R_{uv}}{6}, \quad C^v_{u,4} = \sum_w C^w_{u,1} \frac{R'_{uv}}{12} = \frac{R'_{uv}}{12}, \]
\[ C^v_{u,5} = \sum_w \left( C^w_{u,1} \frac{R''_{uv}}{40} + C^w_{u,3} \frac{R_{uv}}{20} \right) = \frac{1}{120} \left( \sum_s R_{us} R_{sv} + 3R''_{uv} \right). \]

Plugging these values into (2-1), we have
\[ (2-4) \quad J_u = re_u + \frac{r^3}{6} R_{uv} e_v + \frac{r^4}{12} R'_{uv} e_v + \frac{r^5}{120} \left( \sum_s R_{us} R_{sv} + 3R''_{uv} \right) e_v + O(r^6). \]

We write $dA$ for the standard measure of the unit tangent bundle $UT_p(M)$ at $p$, and we write $\int_{\partial B(p,r)} dA$ as $\int$. We define
\[ W = \frac{\int \sqrt{\det(J_u, J_v)}}{r^{2n-1}}, \]
and introduce two propositions:

**Proposition 2.1.** Assume the hypotheses of Theorem 1.3. Let the derivatives of $W$ of order from 1 to $2m - 1$ for $m \geq 1$ be the same as that of the complex space form.

1. If $m = 1, 2$, then $\text{Ric} = K$ at $p$.
   If $m \geq 3$, then
   \[ R_{ijkl} = \frac{K}{n+1} (\delta_{ij} \delta_{kl} + \delta_{il} \delta_{jk}) \]
   at $p$. Moreover, if $u, v, e_0 \in UT_p(M)$ are any unit vectors, then $R^{(2)}_{uv} = 0$ for $1 \leq \lambda \leq m - 3$ and $\text{Ric}^{(l)}(e_0, e_0) = 0$ for $1 \leq l \leq 2m - 4$. The superscripts are the orders of covariant derivatives along the direction $e_0$.

2. In either case, $W^{(2m)}$ is less than or equal to that of the complex space form.

**Proposition 2.2.** Under the same conditions as in Theorem 1.3, if the derivatives of $W$ of order 1 to $2m$ for $m \geq 1$ are the same as the complex space form, then $W^{(2m+1)} = 0$.

We divide the proof of Proposition 2.1 into two parts: $m = 1, 2$ and $m \geq 3$. 
3. The proof of Proposition 2.1, case \( m = 1, 2 \)

By (2-1), we have
\[
\frac{\langle J_u, J_v \rangle}{r^2} = \sum_{i,j,w} r^{i+j-w} C_{u,i}^w C_{v,j}^w.
\]

By (2-4),
\[
\frac{\langle J_u, J_u \rangle}{r^2} = 1 + \frac{1}{3} R_{uu} r^2 + \frac{1}{6} R'_{uu} r^3 + \left( \frac{2}{45} \sum_s R_{us}^2 + \frac{1}{20} R''_{uu} \right) r^4 + O(r^5).
\]

If \( u \neq v \),
\[
\frac{\langle J_u, J_v \rangle}{r^2} = \frac{1}{3} R_{uv} r^2 + \frac{1}{6} R'_{uv} r^3 + \left( \frac{2}{45} \sum_s R_{us} R_{vs} + \frac{1}{20} R''_{uv} \right) r^4 + O(r^5).
\]

Now use the above two expressions to see that
\[
\det \frac{\langle J_u, J_v \rangle}{r^{4n-2}} = 1 + \frac{1}{3} \sum_u R_{uu} r^2 + \frac{1}{6} \sum_u R'_{uu} r^3
\]
\[\quad + \left( \frac{2}{45} \sum_{u,s} R_{us}^2 + \frac{1}{20} \sum_u R''_{uu} + \frac{1}{9} \sum_{u < v} (R_{uu} R_{vv} - R_{uv}^2) \right) r^4 + O(r^5).
\]

Considering the identity \( \sqrt{1+x} = 1 + \frac{1}{2} x - \frac{1}{8} x^2 + O(x^3) \), we get
\[
\frac{\sqrt{\det \langle J_u, J_v \rangle}}{r^{2n-1}} = 1 + \frac{1}{6} \sum_u R_{uu} r^2 + \frac{1}{12} \sum_u R'_{uu} r^3
\]
\[\quad + \left( \frac{1}{45} \sum_{u,s} R_{us}^2 + \frac{1}{40} \sum_u R''_{uu} + \frac{1}{18} \sum_{u < v} (R_{uu} R_{vv} - R_{uv}^2) - \frac{1}{72} \sum_u (R_{uu})^2 \right) r^4 + O(r^5).
\]

Since \( W = \frac{1}{r^{2n-1}} \int \sqrt{\det \langle J_u, J_v \rangle} \), we find
\[
W'(0) = 0 \quad \text{and} \quad W''(0) = -c s,
\]
where \( c \) is a positive constant depending only on \( n \), and \( s \) is the scalar curvature at \( p \). Therefore \( W''(0) \) is less than or equal to that of the complex space form. This proves Proposition 2.1 for \( m = 1 \).

Now we consider \( m = 2 \). According to the assumption of Proposition 2.1, \( W'' \) is the same as that of the complex space form. Therefore \( s = nK \) at \( p \). Since the Ricci curvature is bounded from below by \( K \), \( \text{Ric} = Kg \) at \( p \). By (3-3), it is simple to see that the \( r^3 \) coefficient of \( W \) is 0 by symmetry. Thus to complete the proof for \( m = 2 \), we just need to show that the 4th derivative of \( W \) is less than or equal to that of the complex space form.
We keep in mind that $\text{Ric} = Kg$ at $p$. The $r^4$ coefficient of $W$ is

\[
c_4 = \frac{1}{360} \int \left( \frac{1}{45} \sum_{u,s} R_{us}^2 + \frac{1}{40} \sum_u R''_{uu} + \frac{1}{18} \sum_{u<v} (R_{uu} R_{vv} - R_{uv}^2) - \frac{1}{12} \left( \sum_u R_{uu} \right)^2 \right)
\]

\[
= \frac{1}{360} \int \left( 8 \sum_u R_{uu}^2 + 16 \sum_{u<v} R_{uv}^2 + 9 \sum_u R''_{uu} 
+ 20 \sum_{u<v} R_{uu} R_{vv} - 20 \sum_{u<v} R_{uv}^2 - 5 \left( \sum_u R_{uu} \right)^2 \right)
\]

\[
= \frac{1}{360} \int \left( -2 \sum_u R_{uu}^2 + 10 \left( \sum_u R_{uu} \right)^2 - 4 \sum_{u<v} R_{uv}^2 + 9 \sum_u R''_{uu} - 5 \left( \sum_u R_{uu} \right)^2 \right)
\]

\[
= \frac{1}{360} \int \left( 9 \sum_u R''_{uu} - 4 \sum_{u<v} R_{uv}^2 - 2 \sum_u R_{uu}^2 + 5 \left( \sum_u R_{uu} \right)^2 \right).
\]

Note that the Ricci curvature attains the minimum $K$ at $p$, so

\[
\sum_u R''_{uu} = -\text{Ric}''(e_0, e_0) \leq 0.
\]

Therefore we have

\[
(3-4) \quad c_4 = \frac{1}{360} \int \left( 9 \sum_u R''_{uu} - 4 \sum_{u<v} R_{uv}^2 - 2 \sum_u R_{uu}^2 + 5K^2 \right)
\]

\[
\leq -\frac{1}{360} \int \left( 2 \sum_u R_{uu}^2 - 5K^2 \right)
\]

\[
= -\frac{1}{360} \int \left( 2 \sum_{u \neq 1} R_{uu}^2 + 2R_{11}^2 - 5K^2 \right)
\]

\[
\leq -\frac{1}{360} \int \left( \frac{1}{n-1} \left( \sum_{u \neq 1} R_{uu} \right)^2 + 2R_{11}^2 - 5K^2 \right)
\]

\[
= -\frac{1}{360} \int \left( \frac{1}{n-1} (\text{Ric}(e_0, e_0) + R_{11})^2 + 2R_{11}^2 - 5K^2 \right)
\]

\[
= -\frac{1}{360} \int \left( \frac{1}{n-1} K^2 + \frac{2}{n-1} KR_{11} + \left( \frac{1}{n-1} + 2 \right) R_{11}^2 - 5K^2 \right)
\]

\[
\leq -\frac{1}{360} \left( \int \frac{1}{n-1} K^2 + \frac{2}{n-1} K \int R_{11} + C_1 \left( \int R_{11} \right)^2 - \int 5K^2 \right)
\]

\[
= C_2 K^2,
\]

where $C_1, C_2$ are constants depending only on $n$.

We explain the inequalities above. In the first inequality, we drop the two terms $\sum_{u<v} R_{uv}^2$ and $\sum_u R''_{uu}$. In the second inequality, we apply the Schwartz inequality for directions $e_u$ that are perpendicular to $e_1, e_0$. In the third inequality we use the Schwartz inequality $\int R_{11}^2 \geq C \left( \int R_{11} \right)^2$. We make use of the Kähler condition to
obtain \( \int R_{11} = C_3 s = nC_3 K \), where \( C_3 \) is a constant depending only on \( n \). This explains the last equality.

The right hand side of (3-4) is exactly the case of the complex space form. Therefore when \( W' \) and \( W'' \) are the same as the complex space form, \( W^{(3)} = 0 \) and \( W^{(4)} \) is less than or equal to that of the complex space form. Equation (3-4) becomes an equality if and only if the holomorphic sectional curvature is constant at \( p \) and \( \text{Ric}''(e_0, e_0) = 0 \) for any \( e_0 \in UT_p M \). This completes the proof for \( m = 2 \).

4. The proof of Proposition 2.1, case \( m \geq 3 \)

Denote \( \text{Ric}^{(l)}(e_0, e_0) \) by \( \text{Ric}^{(l)} \). According to the assumption of Proposition 2.1, the derivatives of \( W \) of order 1 to \( 2m - 1 \) are the same as the complex space form. From results in the last section, the holomorphic sectional curvature is constant at \( p \) and \( \text{Ric}'' = 0 \) for any \( e_0 \). That is to say,

\[
R_{ijkl} = \frac{K}{n+1} (\delta_{ij} \delta_{kl} + \delta_{il} \delta_{jk}) \quad \text{and} \quad \text{Ric}'' = 0
\]

at \( p \). Therefore, we have proved part (1) of Proposition 2.1 for \( m = 3 \).

Now we use induction. Assuming that part (1) of Proposition 2.1 holds for \( k = m \), we shall prove that it holds for \( k = m + 1 \).

**Claim 4.1.** Let \( C_{u,i}^v \) be the coefficients defined in (2-1) for \( i \leq m \). Under the hypothesis of the induction above, \( C_{u,i}^v \) are constants independent of the direction \( e_0 \). In fact, they are the same as that of the complex space form.

**Proof.** The claim follows if we insert the induction hypothesis into (2-3). \( \square \)

Let us write

\[
\frac{\det(J_u, J_v)}{r^{4n-2}} = 1 + \sum_{i=1}^{m-1} a_i r^i + \sum_{j=m}^{2m} b_j r^j + O(r^{2m+1}). \tag{4-1}
\]

Combining Claim 4.1 with (3-1), we find that \( a_i \) are constants independent of the direction \( e_0 \). Equation (3-1) also yields \( C_{u,m+1}^v = C_{v,m+1}^u \) for all \( u, v \). Direct expansion of the determinant via (3-1) gives

\[
b_{2m} = \sum_{u,v} (C_{u,m+1}^v)^2 + 4 \sum_{u < v} C_{u,m+1}^u C_{v,m+1}^v + 2 \sum_u C_{u,2m+1}^{u} - 4 \sum_{u < v} C_{u,m+1}^v C_{v,m+1}^u + \sum_{i=1}^{m} C_{u,m+i}^v C_{i,m,u,v} + C_{0,m} \tag{4-2}
\]

where \( C_{i,m,u,v} \) and \( C_{0,m} \) are all constants independent of the direction \( e_0 \).

Note also

\[
b_{m} = 2 \sum_u C_{u,m+1}^u + \text{Constant}. \tag{4-3}
\]
Let us set $\gamma_m = \sum_{i=1}^{m-1} a_i r^i + \sum_{j=m}^{2m} b_j r^j$ for $m \geq 1$. Applying the Taylor series expansion

$$\sqrt{1 + x} = 1 + \frac{1}{2} x - \frac{1}{8} x^2 + \sum_{k=3}^{\infty} \lambda_k x^k$$

for $|x| < 1$, we obtain

$$\frac{\sqrt{\det(J_u, J_v)}}{r^{2n-1}} = 1 + \frac{1}{2} \gamma_m - \frac{1}{8} \gamma_m^2 + \sum_{k=3}^{\infty} \lambda_k \gamma_m^k + O(r^{2m+1}).$$

**Lemma 4.2.** The $2m$-th order coefficient of the expansion of $W$ is

\begin{equation}
(4-5) \quad c_{2m} = \int \left( \frac{1}{2} \sum_{u,v} (C_{u,m+1}^v)^2 + 2 \sum_{u<v} C_{u,m+1}^u C_{v,m+1}^v + \sum_{u} C_{u,2m+1}^u 
- 2 \sum_{u<v} C_{u,m+1}^v C_{u,m+1}^u - \frac{1}{2} \left( \sum_u C_{u,m+1}^u \right)^2 + \sum_{i=1}^{m} C_{u,m+i}^v \tilde{C}_{i,m,u,v} \right) + \tilde{C}_{0,m}
\end{equation}

where $\tilde{C}_{i,m,u,v}$ and $\tilde{C}_{0,m}$ are constants independent of the direction $e_0$.

**Proof.** It suffices to find out the contribution of each term in (4-4) to $c_{2m}$. We keep in mind that coefficients $a_i$ in (4-1) are independent of $e_0$.

By (4-2), the contribution of the term $1 + \frac{1}{2} \gamma_m$ to $c_{2m}$ is

\begin{equation}
(4-6) \quad \int \frac{1}{2} \sum_{u,v} (C_{u,m+1}^v)^2 + 2 \sum_{u<v} C_{u,m+1}^u C_{v,m+1}^v + \sum_{u} C_{u,2m+1}^u 
- 2 \sum_{u<v} C_{u,m+1}^v C_{u,m+1}^u + \frac{1}{2} \left( \sum_i^{m} C_{u,m+i}^v C_{i,m,u,v} + C_{0,m} \right).
\end{equation}

The contribution of the term $-\frac{1}{8} \gamma_m^2$ to $c_{2m}$ is

\begin{equation}
(4-7) \quad - \int \left( \frac{1}{8} b_m^2 + \sum_{i=1}^{m} C_{u,m+i}^v P_{i,m,u,v} \right) + p_{0,m}.
\end{equation}

By (4-3), it could be written as

\begin{equation}
(4-8) \quad - \int \left( \frac{1}{2} \left( \sum_u C_{u,m+1}^u \right)^2 + \sum_{i=1}^{m} C_{u,m+i}^v P_{i,m,u,v} \right) + p_{0,m}.
\end{equation}

The contribution of $\sum_{k=3}^{\infty} \lambda_k \gamma_m^k$ to $c_{2m}$ is

\begin{equation}
(4-9) \quad \int \sum_{i=1}^{m} C_{u,m+i}^v q_{i,m,u,v} + q_{0,m}.
\end{equation}

In (4-7), (4-8), (4-9), $P_{i,m,u,v}$, $q_{i,m,u,v}$, $p_{0,m}$ and $q_{0,m}$ are all constants independent of the direction $e_0$. Lemma 4.2 follows if we combine (4-6), (4-7), (4-8) and (4-9). \qed
Lemma 4.3. There is a negative definite quadratic form $Q$, constants $h_{m,i}$ and $C$ and a negative constant $C_m$ such that

\[(4-10) \quad c_{2m} = \int Q(R_{uv}^{(m-2)}) + \sum_{i=-2}^{m-4} h_{m,i} \int R_{11}^{(m+i)} + C_m \int \text{Ric}^{(2m-2)} + C.\]

Proof. By the induction hypothesis and (2-3), we have

\[(4-11) \quad C_{u,2m+1}^v = \sum_{k+j=2m-1,w} \frac{C_{u,k} R_{jw}^{(j)}}{j! (2m+1) 2m} \]

\[= \frac{1}{(2m+1) 2m} \left( \sum_w \left( \frac{R_{uw}^{(m-2)} C_{u,m+1}^w}{(m-2)!} + \sum_{j=m-1}^{2m-2} B_{j,m,w,u} R_{u}^{(j)} + R_{uu} C_{u,2m-1}^u \right) \right)\]

where $B_{j,m,w,u}$ are constants. For $i \leq m$, we have

\[(4-12) \quad C_{u,m+i}^v = \sum_{j=m-2}^{m+i-3} d_{m,i,j,w,u} R_{u}^{(j)} + C\]

where $C$ and $d_{m,i,j,w,u}$ are constants. In particular, we have

\[(4-13) \quad C_{u,m+1}^v = \sum_{k+j=m-1,w} \frac{C_{u,k} R_{jw}^{(j)}}{j! (m+1)} = \frac{1}{m+1} \left( \frac{R_{vu}^{(m-2)}}{m-2)!} + C_{u,m-1}^v R_{vv} \right).\]

By the induction hypothesis,

\[(4-14) \quad \sum_u R_{uu}^{(m-2)} = -\text{Ric}^{(m-2)} = 0.\]

Therefore

\[(4-15) \quad \sum_u \left( R_{uu}^{(m-2)} \right)^2 = \left( \sum_u R_{uu}^{(m-2)} \right)^2 - 2 \sum_{u<v} R_{uu}^{(m-2)} R_{vv}^{(m-2)} = -2 \sum_{u<v} R_{uu}^{(m-2)} R_{vv}^{(m-2)}.\]

Inserting (4-11), (4-12), (4-13) into (4-5), we find

\[(4-16) \quad c_{2m} = \int Q(R_{uv}^{(m-2)}) + \sum_{i=-2}^{m-2} h_{m,i,u,v} R_{uv}^{(m+i)} + C.\]

Now we prove that $Q$ is negative definite. Let us check each term in (4-5). By (4-13), the term $\frac{1}{2} \sum_{u,v} (C_{u,m+1}^v)^2$ in (4-5) contributes to the quadratic term

\[(4-17) \quad \sum_{u,v} \frac{1}{2m^2(m+1)^2 ((m-2)!)^2} (R_{uv}^{(m-2)})^2.\]
The term \(2 \sum_{u < v} C_{u,m+1}^{u} C_{v,m+1}^{v}\) contributes to the quadratic term

\[(4-18) \quad \sum_{u < v} \frac{2}{m^2(m+1)^2((m-2)!)^2} R_{uu}^{(m-2)} R_{vv}^{(m-2)}.\]

By (4-15), it could be written as

\[(4-19) \quad -\frac{1}{m^2(m+1)^2((m-2)!)^2} \sum_u (R_{uu}^{(m-2)})^2.\]

By (4-11) and (4-13), the term \(\sum_{u} C_{u,2m+1}^{u}\) contributes to the quadratic term

\[(4-20) \quad \frac{1}{2m^2(m+1)(2m+1)((m-2)!)^2} (R_{uv}^{(m-2)})^2.\]

The term \(-2 \sum_{u < v} C_{u,m+1}^{u} C_{v,m+1}^{v}\) contributes to the quadratic term

\[(4-21) \quad -\sum_{u < v} \frac{2}{m^2(m+1)^2((m-2)!)^2} (R_{uv}^{(m-2)})^2.\]

The term \(-\frac{1}{2}(\sum_{u} C_{u,m+1}^{u})^2\) is obviously negative semidefinite.

By combining (4-17), (4-18), (4-19), (4-20) and (4-21), it follows that the quadratic form in (4-10) is negative definite.

Consider the linear terms in (4-16). By the induction hypothesis, the coefficients \(h_{m,i,u,v}\) are unchanged if we take a unitary transformation keeping the direction \(e_0\) fixed. Comparing the coefficients of the linear order terms, we see that \(h_{m,i,u,v} = 0\) if \(u \neq v\), and \(h_{m,i,u,u} = h_{m,i,v,v}\) if \(u \neq e_1\) and \(v \neq e_1\). Therefore, the linear terms \(h_{m,i,u,u} R_{uu}^{(m+i)}\) could be absorbed into \(\text{Ric}^{(m+i)}\) with the terms \(-h_{m,i} R_{11}^{(m+i)}\) left.

Also note that by induction hypothesis, \(\text{Ric}^{(l)} = 0\) for \(0 < l \leq 2m - 3\) (the term \(\text{Ric}^{(2m-3)}\) vanishes as the Ricci curvature attains its minimum at \(p\)). Finally, one verifies that \(\sum_{u} C_{u,2m+1}^{u}\) is the only term in (4-5) that has contribution to \(R_{uv}^{(2m-2)}\). Therefore the linear terms in (4-16) could be written as

\[m-4 \sum_{i=-2}^{m-4} h_{m,i} \int R_{11}^{(m+i)} + C_m \int \text{Ric}^{(2m-2)}.\]

From (4-11), it is simple to check that \(C_m\) is negative. \(\square\)

By the induction hypothesis and that the Ricci curvature attains its minimum at \(p\), we have \(\text{Ric}^{(2m-2)} \geq 0\). It follows from Lemma 4.3 that

\[(4-22) \quad c_{2m} \leq \sum_{i=-2}^{m-4} h_{m,i} \int R_{11}^{(m+i)} + \text{Constant}.\]

We would like to prove that the linear terms \(\int R_{11}^{(m+i)}\) vanish for \(-2 \leq i \leq m-4\). Note that by symmetry, if \(m+i\) is odd, the integral equals 0. Let us deal with case
when \( m + i \) is even. We shall check when \( i = m - 4 \). The other cases are similar. Let

\[
A = -\frac{1}{4} \int R_{11}^{(2m-4)}.
\]

Set up an orthonormal frame \( \{f_i\} \) at \( p \) such that \( Jf_{2j} = f_{2j+1} \) and \( Jf_{2j+1} = -f_{2j} \) for \( 0 \leq j \leq n - 1 \). Letting \( \beta_j = \frac{1}{2}(f_{2j} - \sqrt{-1}f_{2j+1}) \), in a small neighborhood of \( p \), we parallel transport the frame along each geodesic through \( p \). Suppose that

\[
e_0 = \sum_{j=0}^{n-1} (z_j \beta_j + \bar{z_j} \beta_j).
\]

**Lemma 4.4.** Under the assumption of the induction in Proposition 2.1, \( Rm^{(\lambda)} = 0 \) at \( p \) for \( 1 \leq \lambda \leq m - 3 \), where \( Rm^{(\lambda)} \) denotes any covariant derivative of the curvature tensor of order \( \lambda \) at \( p \).

**Proof.** We use induction. If \( \lambda = 0 \), the result automatically holds since there is nothing to prove. Suppose the result holds for \( k < \lambda \). For \( k = \lambda \), we plug (4-24) in \( R_{uv}^{(\lambda)} \).

**Claim 4.5.** We can commute the covariant derivatives of \( R_{uv}^{(\lambda)} \).

**Proof.** To prove the claim, we only need to consider the case \( \lambda \geq 2 \). By the induction hypothesis of Lemma 4.4, the covariant derivatives of the curvature tensor vanish up to order \( \lambda - 1 \) at \( p \). If \( \lambda > 3 \), the claim follows from the Ricci identity. Now suppose \( \lambda = 2 \). By the Ricci identity, the difference of commuting the covariant derivatives is a function of the curvature tensor. Note that the curvature tensor at \( p \) is the same as of the complex space form. This completes the proof for \( \lambda = 2 \). \( \square \)

We insert (4-24) into \( R_{j\epsilon_0J\epsilon_0}^{(\lambda)} \). By Claim 4.5 and the Bianchi identities, \( R_{j\epsilon_0J\epsilon_0}^{(\lambda)} \) becomes a polynomial in the variables \( z_j, \bar{z}_j \). The coefficients of the polynomial are exactly all the covariant derivatives of \( Rm \) at \( p \) of order \( \lambda \). According to the assumption of Lemma 4.4, \( R_{j\epsilon_0J\epsilon_0}^{(\lambda)} \) is identically 0 for all \( \epsilon_0 \). Therefore, the coefficients of the polynomial are all 0. This completes the induction of Lemma 4.4. \( \square \)

**Lemma 4.6.** Under the assumption of the induction in Proposition 2.1, \( A \) could be written as \( \sum_{i=1}^{m-2} g_{i,m} \Delta^i s \), where \( s \) denotes the scalar curvature, and \( g_{i,m} \) are constants depending only on \( n, m \) and \( i \).

**Proof.** Define \( X = \frac{1}{2}(\epsilon_0 - \sqrt{-1}J\epsilon_0) \), then \( A = \int R_{X\bar{X}XX,\epsilon_0\epsilon_0...\epsilon_0} \), where the number of \( \epsilon_0 \) is \( 2m - 4 \). Integrating and plugging (4-24) into the result, we find

\[
A = \sum_{\alpha_1\alpha_2...\alpha_2m} \left( \int \alpha_1\alpha_2...\alpha_2m \right) R_{\alpha_1\alpha_2...\alpha_2m}
\]
where each $\alpha_i$ is either $z_j$ or $\bar{z}_k$ for $0 \leq j, k \leq n - 1$, with the further condition that $\alpha_1, \alpha_3 \in \{z_j\}$, and $\alpha_2, \alpha_4 \in \{\bar{z}_k\}$. Under the subscript of $R$, $z_j$ stands for $\beta_j$, and $\bar{z}_k$ stands for $\beta_k$.

From the expression of (4-25), we see that $z_j, \bar{z}_j$ must all go in pairs in the sequence $\alpha_1 \alpha_2 \ldots \alpha_{2m}$, otherwise the integral $\int \alpha_1 \alpha_2 \ldots \alpha_{2m}$ would equal 0. Using the Kähler identities, we can switch the covariant derivatives in (4-25) and rearrange it as

$$A = \sum_{I_1, I_2, \ldots, I_n} C_{I_1 I_2 \ldots I_n} R_{I_1 I_2 \ldots I_n} + B.$$  

(4-26)

Here the symbol $I_j$ denotes $z_j \bar{z}_j \ldots z_j \bar{z}_j$; we have $\sum_j |I_j| = 2m$; subscripts after the fourth subscript of $R$ denote covariant derivatives; $C_{I_1 I_2 \ldots I_n}$ are the coefficients in (4-25); and $B$ is a combination of covariant derivatives of $Rm$ of lower order.

From (4-23), we see that the coefficients $C_{I_1 I_2 \ldots I_n}$ in (4-26) are unitary invariants. For fixed $I_1, I_2, \ldots, I_n$, let $d = |I_1| + |I_2|$. Denote the coefficient $C_{I_1 I_2 \ldots I_n}$ by $C_p$, where $0 \leq |I_1| = p \leq d$. We want to find relations between the different $C_p$. Define a unitary transformation by setting $\tilde{\beta}_i = \beta_i$ for $i \neq 1, 2$ and let

$$\beta_1 = \cos \theta \tilde{\beta}_1 + \sin \theta \tilde{\beta}_2 \quad \text{and} \quad \beta_2 = -\sin \theta \tilde{\beta}_1 + \cos \theta \tilde{\beta}_2.$$

Insert the unitary transformation above in (4-26). Then the new coefficient $\tilde{C}_d$ becomes $\sum_{p=0}^{d} C_p \cos^{2p} \theta \sin^{2(d-p)} \theta$. Therefore we have:

$$\sum_{p=0}^{d} C_p \cos^{2p} \theta \sin^{2(d-p)} \theta = C_d = C_d (\cos^2 \theta + \sin^2 \theta)^d.$$  

(4-27)

Claim 4.7. $C_p = C_d \binom{d}{p}$.

Proof. Divide by $\cos^{2d} \theta$ on both sides, then (4-27) becomes

$$\sum_{p=0}^{d} C_p \tan^{2(d-p)} \theta = C_d = C_d (1 + \tan^2 \theta)^d.$$

Since $\theta$ is arbitrary, the claim follows. $\square$

By Claim 4.7, $C_p / C_d = \binom{d}{p}$. Since we can substitute any index $u, v$ for 1, 2, the ratios of all coefficients in (4-26) are determined. Note that to get the relations between $C_p$, we only use the condition that the form (4-23) is unitary invariant. Since $\Delta^{m-2}s$ is also unitary invariant with respect to the frame, we can write it in the same form as (4-26). By the same argument, the ratios of coefficients of $\Delta^{m-2}s$ are the same as of coefficients in (4-26). It follows that the term $\sum_{I_1, I_2, \ldots, I_n} C_{I_1 I_2 \ldots I_n} R_{I_1 I_2 \ldots I_n}$ in (4-26) equals $C(m, n) \Delta^{(m-2)}s$ modulo lower order covariant derivatives, where $C(m, n)$ is a constant depending only on $m, n$. 


Now we make an important observation. From the Ricci identity,
\[
R_{i_1i_2...i_p}i_{p+3}...i_{2m} - R_{i_1i_2...i_p}i_{p+3}...i_{2m}
\]
is the sum of \((RmRm^{(p-4)}),i_{p+3}...i_{2m}\). By Lemma 4.4, \(Rm^{(\lambda)} = 0\) for \(1 \leq \lambda \leq m - 3\).

It follows that \((RmRm, i_{p+3}...i_{2m})\) can be expanded as a linear combination of the covariant derivatives of curvature tensor. Therefore \(A - C(m, n)\Delta^{(m-2)}s\) can be written as a linear combination of the covariant derivatives of the curvature tensor with the highest order \(2m - 6\). Furthermore it is unitary invariant since the curvature tensor is unitary invariant at \(p\). By induction, we have completed the proof of Lemma 4.6.

From the induction in Proposition 2.1, \(\text{Ric}^{(l)} = 0\) for \(1 \leq l \leq 2m - 4\). Integrating over the unit sphere in \(T_pM\) we find, by similar arguments as in the proof of Lemma 4.6, that for \(l\) even
\[
0 = \int \text{Ric}_{e_0, e_0, ..., e_0} = \sum_{k=1}^{1/2} C_{l,k} \Delta^k s
\]
where the order of the covariant derivative above is \(l\). It is straightforward to check that the highest order coefficient \(C_{l,l/2}\) is not equal to 0. Then, by induction, \(\Delta^k s = 0\) at \(p\) for \(1 \leq k \leq m - 2\). Combining this with Lemma 4.6, it follows that \(A = 0\). Similarly all linear terms in (4-10) vanish. Therefore, under the induction hypothesis in Proposition 2.1, in order that \(c_{2m}\) in (4-10) achieves the maximum, we must have \(\text{Ric}^{(2m-2)} = 0\) and \(R^{(\lambda)}_{uv} = 0\) for \(1 \leq \lambda \leq m - 2\). This is exactly the case of the complex space form. Therefore we have completed the induction step for part (1) in Proposition 2.1 and, as a byproduct, we have proved part (2) as well. The proof is thus complete.

5. The proof of Theorem 1.4

Proof of Proposition 2.2. Using the same argument as in the last section, we find that \(W^{(2m+1)}\) is a linear combination of \(\int R^{(m+i)}_{11}\) for \(1 \leq i \leq m - 3\) (the terms of order greater than \(2m - 3\) can be absorbed into \(\text{Ric}^{(m+i)}\)). Then \(W^{(2m+1)}\) is equal to 0 by similar arguments as in the proof of Lemma 4.6.

Proof of Theorem 1.4. Consider the two cases below:

1. All coefficients of the power series of \(W\) are equal to that of the complex space form. From Proposition 2.1, all covariant derivatives of the curvature tensor at \(p\) are the same as the complex space form. Since the metric is real analytic, we conclude that near \(p\), the manifold is isometric to the complex space form.

2. There is a \(i_0 \geq 1\) such that for all \(i < i_0\), the coefficients of the power series of \(W\) are equal to that of the complex space form, but the \(i_0\)-th coefficient is less than that of the complex space form. Checking the power series of \(W'/W\) at \(p\),
we find that for sufficiently small \( r \), \( W'/W \) is less than that of the complex space form. From the definition of \( W \) we have, for small \( r \),

\[
\frac{\int_{\partial B_p(r)} \Delta r}{A(\partial B_p(r))} = \frac{\int \sqrt{\det(J_u, J_v)}'}{\int \sqrt{\det(J_u, J_v)}} < \Delta_N r(r). \tag*{\square}
\]

6. An example

In this section we give an example showing that the analogous Laplacian comparison theorem is not true on Kähler manifolds when the Ricci curvature is bounded from below by a nonzero constant. The example is in dimension 2. For higher dimensions, the construction is similar.

Identify \( \mathbb{R}^4 \) with \( \mathbb{C}^2 \) in the usual way. The corresponding almost complex structure \( J \) is given by

\[
J \frac{\partial}{\partial x_1} = \frac{\partial}{\partial x_2}, \quad J \frac{\partial}{\partial x_2} = -\frac{\partial}{\partial x_1}, \quad J \frac{\partial}{\partial x_3} = \frac{\partial}{\partial x_4} \quad \text{and} \quad J \frac{\partial}{\partial x_4} = -\frac{\partial}{\partial x_3}.
\]

Given a small ball near the origin of \( \mathbb{C}^2 \), define a function \( f \) to be

\[
f = |z_1|^2 + |z_2|^2 + a|z_1|^4 + 8a|z_1|^2|z_2|^2 + a|z_2|^4 + \frac{8}{3}a^2|z_1|^6 + 28a^2|z_1|^4|z_2|^2 + 28a^2|z_1|^2|z_2|^4 + \frac{8}{3}a^2|z_2|^6 + p(|z_1|, |z_2|),
\]

where \( a \) is a nonzero constant and \( p \) is a homogeneous polynomial of degree 8 to be determined later. We define

\[
\omega = \sqrt{-1} \partial \bar{\partial} f = \sqrt{-1} \sum_{i,j} g_{ij} dz_i \wedge d\bar{z}_j.
\]

It is straightforward to check that \( \omega \) defines a Kähler metric \( g \) if the ball is sufficiently small (note that the metric is not complete).

Direct computation gives

\[
g_{\bar{1}1} = 1 + 4a|z_1|^2 + 8a|z_2|^2 + 24a^2|z_1|^4 + 112a^2|z_1|^2|z_2|^2 + 28a^2|z_2|^4 + O((|z_1| + |z_2|)^6),
\]

\[
g_{\bar{1}\bar{2}} = 8a \bar{z}_1 z_2 + 56a^2 z_1 \bar{z}_1^2 z_2 + 56a^2 \bar{z}_1 z_2^2 \bar{z}_2 + O((|z_1| + |z_2|)^6),
\]

and \( g_{\bar{2}\bar{2}} = g_{\bar{1}\bar{1}} \). Therefore

\[
\det(g_{ij}) = g_{\bar{1}\bar{1}} g_{\bar{2}\bar{2}} - |g_{\bar{1}\bar{2}}|^2
\]

\[
= \left(1 + 4a|z_1|^2 + 8a|z_2|^2 + 24a^2|z_1|^4 + 112a^2|z_1|^2|z_2|^2 + 28a^2|z_2|^4\right)^2
\]

\[
- \left(8a \bar{z}_1 z_2 + 56a^2 z_1 \bar{z}_1^2 z_2 + 56a^2 \bar{z}_1 z_2^2 \bar{z}_2 + O((|z_1| + |z_2|)^6)\right)^2
\]

\[
= 1 + 12a(|z_1|^2 + |z_2|^2) + 84a^2(|z_1|^4 + |z_2|^4)
\]

\[
+ 240a^2|z_1|^2|z_2|^2 + O((|z_1| + |z_2|)^6).\]
Using $\log(1+x) = x - \frac{1}{2}x^2 + O(x^3)$, we have
$$\text{Ric} + 12ag = \partial \bar{\partial}(- \log(\det g_{ij}) + 12af) = \partial \bar{\partial} O((|z_1| + |z_2|)^6).$$
Therefore $\text{Ric} + 12ag$ vanishes up to order 3 at the origin. If we choose the function $p$ to be $-\lambda(|z_1|^8 + |z_2|^8 + 8(|z_1|^6|z_2|^2 + |z_1|^2|z_2|^6))$, a direct computation gives
$$\text{Ric} + 12ag = \partial \bar{\partial}(24\lambda(|z_1|^2 + |z_2|^2)^3 + O((|z_1| + |z_2|)^6))$$
where the term $O((|z_1| + |z_2|)^6)$ does not depend on $\lambda$. If $\lambda$ is sufficiently large, $\text{Ric} + 12ag \geq 0$ near the origin. Set $K = -12a$. Thus, near the origin, $\text{Ric} \geq K$. By direct computation $R_{1212} = R_{1313} = R_{1414} = 4a$ and $R_{1u1v} = 0$ at the origin if $u \neq v$. Combining this with the fact that the second derivatives of the Ricci tensor vanish at the origin we find, after a slight computation, that the fourth order term of (3-3) is greater than that of the complex space form if $e_0 = \partial/\partial x_1$. So when $r$ is very small, $\sqrt{\det(J_u, J_v)}$ is greater than that of the complex space form along the geodesic with initial direction $\partial/\partial x_1$ at the origin. Since
$$\Delta r = \frac{\partial \log \sqrt{\det(J_u, J_v)}}{\partial r},$$
it follows that the pointwise Laplacian comparison with the complex space forms is not true for Kähler manifolds.

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