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# STRUCTURABLE ALGEBRAS OF SKEW-RANK 1 OVER THE AFFINE PLANE

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**Let $k$ be a field of characteristic not 2 or 3. Infinitely many mutually nonisomorphic structurable algebras of rank 20 over $k[X, Y]$ are constructed whose fiber is a given structurable algebra over $k$ of skew-rank 1.**

## Introduction

Let $R$ be a ring such that $\frac{1}{6} \in R$ and $k$ a field of characteristic not 2 or 3. Let $A$ be a unital nonassociative algebra over $R$ with an involution $^-$. The pair $(A, ^-)$ is called a *structurable algebra* if

$$\{x, y, \{z, w, q\}\} - \{z, w, \{x, y, q\}\} = \{\{x, y, z\}, w, q\} - \{z, \{y, x, w\}, q\}$$

for $x, y, z, w, q \in A$, where

$$\{x, y, z\} = (x\bar{y})z + (z\bar{y})x - (z\bar{x})y.$$

Structurable algebras were introduced in [Allison 1978]: an analogue of the Koecher–Kantor–Tits functor gives a correspondence between a structurable algebra and a Lie algebra. Using this functor all classical simple isotropic Lie algebras can be obtained [Allison 1979].

In [Parimala et al. 1999], nontrivial Albert algebra bundles over the affine plane were constructed whose associated principal $F_4$ bundle admits no reduction of the structure group to any proper connected reductive subgroup. (For an analogous result with the associated principal $G$ bundle being of type $G_2$, see [Parimala et al. 1997; 1999].) Over a field, every Albert algebra arises from the first or second Tits construction and the associated $F_4$ bundle admits reduction of the structure group to $\mathrm{SL}_1(B)$ for a central simple algebra $B$ either over $k$ or to $\mathrm{SU}(B, \sigma)$ for a central simple algebra $B$ over a quadratic field extension of $k$, $\sigma$ an involution of the second type. Hence the patched Albert algebras over the affine plane arise neither from a first nor a second Tits construction (and correspondingly, there are

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patched octonion algebras over the affine plane which do not arise from a Cayley-Dickson doubling process or are constructed involving a ternary hermitian form and a two-dimensional subalgebra).

In the present paper we employ the patching arguments from [Parimala et al. 1999] to obtain infinitely many structurable algebras $M_i$ of rank 20 over the affine plane $\mathbb{A}^2_k$, which are not extended from $k$ and mutually nonisomorphic and whose fiber is a given matrix algebra over $k$ (Theorem 4). In order to achieve this, we show that the matrix algebra $M(T, N, N^\vee)$ over $k[X, Y]$ admits a unique extension to a matrix algebra over $\mathbb{P}^2_k$ in Section 2. In Section 3, we look at forms of these matrix algebras. For a nonfree projective left $D[X, Y]$-module $P$ of rank one, the structurable algebra $S(D, \sigma, P, N)$ over $k[X, Y]$ admits a unique extension to a structurable algebra $S(\mathcal{D}, \sigma, \widetilde{P}, N)$ over $\mathbb{P}^2_k$, where $\widetilde{P}$ is an indecomposable vector bundle. We use this result to construct infinitely many mutually nonisomorphic structurable algebras $A^i$ over $\mathbb{A}^2_k$ such that

$$A^i \otimes_k K \cong M_i,$$

where $K$ is a separable quadratic field extension of $k$ (Theorem 9). In Section 4, some general results on extending structurable algebras from affine to projective space are obtained.

If a structurable algebra over $\mathbb{A}^2_k$ has rank 56, it corresponds to the structure group $E_7$. Such bundles were constructed in [Raghunathan 1989] for a connected reductive absolutely almost simple $k$-group $G$, which is $k$-anisotropic and is not of type $F_4$ or $G_2$ (for the type $G_2$ and $F_4$, see [Knus et al. 1994; Parimala et al. 1997; 1999]). The results show that although $GL(r)$-bundles over the affine plane $\mathbb{A}^n_k$ are trivial, this is not the case for a general reducible structure group.

It is also known that if $G$ is a $k$-anisotropic reductive absolutely almost simple algebraic $k$-group, there are infinite families of mutually nonisomorphic, nontrivial (sometimes indecomposable) principal $G$-bundles over $\mathbb{A}^2_k$, which do not admit a reduction of its structure group to any proper connected reductive subgroup of $G$.

The author is not able to say whether the new principal $G$-bundle constructed in this paper admit a reduction of their structure group to a proper reductive subgroup or not.

We use the results and terminology from [Achhammer 1995] (see also [Pumplün 2008; 2010a; 2010b) and [Parimala et al. 1999]. The approach in this last work is mostly functorial and formulated for base rings $R$ which are domains with $\frac{1}{6} \in R$, the one in [Achhammer 1995] works instead for arbitrary base rings. Both were originally developed to generalize the first and second Tits construction for Jordan algebras over rings.

For the standard terminology on Jordan algebras, see [McCrimmon 2004; Jacobson 1968; Schafer 1966].

## 1. Preliminaries

**1.1.** ***Algebras over*** $\boldsymbol{R}$**.** For $P \in \operatorname{Spec} R$, let $R_P$ be the local ring of $R$ at $P$ and $m_P$ the maximal ideal of $R_P$. The corresponding residue class field is denoted by $k(P) = R_P/m_P$. For an $R$-module $F$ the localization of $F$ at $P$ is denoted by $F_P$. The *rank* of $F$ is defined to be $\sup\{\operatorname{rank}_{R_P} F_P \mid P \in \operatorname{Spec} R\}$. The term "$R$-algebra" always refers to nonassociative $R$-algebras which are unital and finitely generated projective of finite constant rank as $R$-modules.

An antiautomorphism $\sigma : A \to A$ of order 2 is called an *involution* on $A$. Define $H(A,\sigma) = \{a \in A \mid \sigma(a) = a\}$ and $\mathrm{S}(A,\sigma) = \{a \in A \mid \sigma(a) = -a\}$. Then $A = H(A,\sigma) \oplus \mathrm{S}(A,\sigma)$.

**1.2.** ***Structurable algebras.*** An *algebra with involution* is a pair $(A, {}^-)$ consisting of an $R$-algebra $A$ and an involution ${}^- : A \to A$. A *structurable algebra* is an algebra with involution $(A, {}^-)$ satisfying

$$\{x, y, \{z, w, q\}\} - \{z, w, \{x, y, q\}\} = \{\{x, y, z\}, w, q\} - \{z, \{y, x, w\}, q\}$$

for all elements $x, y, z, w, q \in A$, where

$$\{x, y, z\} = (x\bar{y})z + (z\bar{y})x - (z\bar{x})y$$

[Allison 1978, (3) and Corollary 5]. If $B$ is an $R$-submodule if $A$ closed under multiplication, we call $B$ a subalgebra of $A$. If, additionally, $\overline{B} = B$ we call $(B, {}^-)$ a subalgebra of $(A, {}^-)$.

An *isotopy* from $(A, {}^-) \to (A', {}^{-\prime})$ is an $R$-linear bijective map $a : A \to A'$ such that

$$a\{x, y, z\} = \{ax, \hat{a}y, az\}$$

for all $x, y, z \in A$ and some $R$-linear map $\hat{a} : A \to A'$. Two structurable algebras $(A^-)$ and $(A, {}^-)$ are *isotopic* if there exists an isotopy from $A$ to $A'$. This is equivalent to $(A', {}^{-\prime}) \cong (A, {}^-)^{\langle u\rangle}$ for some invertible $u \in A$. Every isomorphism between structurable algebras is an isotopy.

In the following, we will only deal with structurable algebras $(A, {}^-)$ over $R$ whose residue class algebras $A(P) = A_P \otimes_{R_P} k(P)$ are central simple structurable algebras of skew-dimension 1.

**1.3.** Let $W$ and $W'$ be two finitely generated projective $R$-modules of constant rank with cubic forms $N : W \to R$ and $N' : W' \to R$, paired by a nondegenerate bilinear form $T : W \times W' \to R$. That is, $T$ induces $R$-module isomorphisms

$$T : W \to \operatorname{Hom}_R(W', R), \quad x \mapsto T(x, \cdot)$$

and

$$T : W' \to \operatorname{Hom}_R(W, R), \quad y' \mapsto T(\cdot, y').$$

We say that the triple $(T, N, N')$ *is defined on* $(W, W')$. Let $N(x, y, z)$ denote the trilinear form associated with $N$ and $N'(x', y', z')$ the trilinear form associated with $N'$. Let $x \in W$, $x' \in W'$ and define quadratic maps

$$\sharp : W \to W' \quad \text{and} \quad \sharp' : W' \to W$$

via

$$D_y N(x) = T(y, x^\sharp) \quad \text{and} \quad D_{y'} N'(x') = T(x'^{\sharp'}, y')$$

for all elements $x, y \in W$, $x', y' \in W'$; i.e.,

$$3N(x, x, y) = T(y, x^\sharp) \quad \text{and} \quad 3N'(x', x', y') = T(x'^{\sharp'}, y')$$

for all elements $x, y \in W$, $x', y' \in W'$. The triple $(T, N, N')$ *satisfies the adjoint identities* if

$$(x^\sharp)^{\sharp'} = N(x)x \quad \text{and} \quad (x'^{\sharp'})^\sharp = N'(x')x'.$$

If $N = 0$ and $N' = 0$ these identities are trivially satisfied. If $N \neq 0$ or $N' \neq 0$ then both $N$ and $N'$ are nonzero and $(T, N, N')$ is called *nontrivial*.

Let $(T, N, N')$ be a triple defined on $(W, W')$. Define symmetric bilinear maps $\times : W \times W \to W'$ and $\times' : W' \times W' \to W$ via

$$x \times y = (x + y)^\sharp - x^\sharp - y^\sharp, \quad x' \times' y' = (x' + y')^{\sharp'} - x'^{\sharp'} - y'^{\sharp'}.$$

Then

$$x^\sharp = \tfrac{1}{2} x \times x, \quad x'^{\sharp'} = \tfrac{1}{2} x' \times' x',$$
$$N(x, y, z) = T(x, y \times z), \quad N'(x', y', z') = T(x' \times' y', z').$$

If the triple $(T, N, N')$ satisfies the adjoint identities then the matrix algebra

$$A = M(T, N, N') = \begin{bmatrix} R & W \\ W' & R \end{bmatrix}$$

with multiplication

$$\begin{bmatrix} a & x \\ x' & b \end{bmatrix} \begin{bmatrix} c & y \\ y' & d \end{bmatrix} = \begin{bmatrix} ac + T(x, y') & ay + dx + x' \times' y' \\ cx' + by' + x \times y & bd + T(y, x') \end{bmatrix}$$

and involution

$$\overline{\begin{bmatrix} a & x \\ x' & b \end{bmatrix}} = \begin{bmatrix} b & x \\ x' & a \end{bmatrix}$$

is a structurable algebra [Allison and Faulkner 1984, p. 194; [Pumplün 2010b, Theorem 1]. We have $S(A, ^-) = s_0 R$ with

$$s_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

invertible and $s_0^2 = 1 \in R^\times$ and the residue class algebras $A(P) = A_P \otimes k(P)$ are central simple structurable algebras of skew-dimension 1 over $k(P)$ [Allison and Faulkner 1984; Pumplün 2010b]. Let

$$u = \begin{bmatrix} a & x \\ x' & b \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} c & y \\ y' & d \end{bmatrix}$$

with $a, b, c, d \in R$ and $x, y \in W$, $x', y' \in W'$. The (*conjugate*) norm

$$\nu : M(T, N, N') \to R$$

is given by

$$\nu(u) = 4aN(x) + 4bN'(x') - 4T(x'^{\sharp'}, x^\sharp) + (ab - T(x, x'))^2$$

and is isotropic since $\nu(u) = 0$ for

$$u = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The *trace* $\chi : M(T, N, N') \times M(T, N, N') \to R$ is defined by

$$\chi(u, v) = 2(ad + bc + T(x, y') + T(y, x')).$$

Note that $\chi(u, u) = 0$.

**1.4.** Let $B$ be an Azumaya algebra over $R$ of degree 3, $B^+ = (N_B, \sharp_B, 1)$ with $(N_B, \sharp_B, 1)$ a cubic form with adjoint and base point (see for instance [Pumplün 2010b, 1.4]). Let $\mathrm{Pic}_l B$ denote the set of isomorphism classes of locally free left $B$-modules of rank 1. Let $P \in \mathrm{Pic}_l B$ such that $N_B(P) \cong R$ and let $N : P \to R$ be a norm on $P$. Let $N^\vee : P^\vee \to R$ be the uniquely determined norm and $\sharp : P \to P^\vee$, $\check{\sharp} : P^\vee \to P$ be the uniquely determined adjoints satisfying

(1) $\langle w, w^\sharp \rangle = N(w)1$,

(2) $\langle \check{w}^{\check{\sharp}}, \check{w} \rangle = N^\vee(\check{w})1$, and

(3) $w^{\sharp\check{\sharp}} = N(w)w$

for all $w \in P$, $\check{w} \in P^\vee$ (these are identities (7), (8), (9) in [Pumplün 2010b]). Let $\times : P^\vee \times P^\vee \to P$ denote the bilinear map associated to the quadratic map $\sharp$ and $\check{\times} : P^\vee \times P^\vee \to P$ the bilinear map associated to the quadratic map $\check{\sharp}$ (see for instance [ibid., 3.2]). Define $T : P \times P^\vee \to R$ via

$$T(w, \check{w}) = T_B(\langle w, \check{w} \rangle).$$

For any $\mu \in R^\times$, the triple $(\mu T, \mu N, \mu^2 N^\vee)$ satisfies the adjoint identities [Pumplün 2010b, Theorem 6], hence

$$M = M(\mu T, \mu N, \mu^2 N^\vee) = \begin{bmatrix} R & P \\ P^\vee & R \end{bmatrix}$$

is a structurable algebra over $R$ with automorphism group isomorphic to the semidirect product of $\mathbb{Z}/2$ and the group of bijective norm isometries of $P$; see [ibid., Corollary 7 and Theorem 18].

The group $\mathrm{Inv}(M)$ defined in Section 4 is an absolutely almost simple linear algebraic group, which is connected except in the case that $M$ has rank 9. In that case its connected component is a subgroup of index 2 in $\mathrm{Inv}(M)$ [Krutelevich 2007, p. 941 ff.].

**1.5.** Let $R'$ be a ring and $B$ a unital separable associative algebra over $R'$. Let $* : R' \to R'$ be an involution on $R'$ and $*_B$ an involution on $B$ such that $*_B|_{R'} = *$. Let $(N_B, \sharp_B, 1)$ be a cubic form with adjoint and base point on $B$ such that $B^+ = J(N_B, \sharp_B, 1)$, with 1 the unit element in $B$, and that the conditions

$$xyx = T_B(x, y)x - x^{\sharp_B} \times_B y,$$
$$N_B(xy) = N_B(x)N_B(y),$$
$$N(x^{*_B}) = N(x)^{*_B}$$

are satisfied for all $x, y \in B$ (these are identities (1), (2), (3) in [Pumplün 2010b]). Let $(H(B, *_B), H(R', *_B))$ be a $B$-ample pair, and define $R = H(R', *_B)$. Let $P \in \mathrm{Pic}_l\, B$ be such that $N_B(P) \cong R'$ and such that there is a nondegenerate *hermitian* form $h : P \times P \longrightarrow B$ satisfying

$$h(w, w) \in H(B, *_B) \quad \text{and} \quad N_B(h(w, w)) = N(w)N(w)^{*_B}$$

for $w \in P$. Denote by $*$ the $H(B, *_B)$-admissible involution $j_h : P \to \overline{P}^\vee$ on $P$ induced by $h$. Let $N : P \to R'$ be a norm on $P$. Let $N^\vee : P^\vee \to R'$ be the uniquely determined norm and $\sharp : P \to P^\vee$, $\check{\sharp} : P^\vee \to P$ be the uniquely determined adjoints satisfying equations (1), (2), (3). We can also write

$$\langle u, v^* \rangle = h(u, v), \quad v^* = j_h(v) \quad \text{and} \quad \check{v}^{\check{*}} = j_h^{-1}(\check{v})$$

for $j_h : P \to \overline{P}^\vee$ induced by $h$. The $R$-module $S(B, *_B, P, N, h) = R' \oplus P$ together with the multiplication

$$(a, u)(b, v) = (ab + T_B(\langle u, v^* \rangle), b^{*_B} u + av + (u \times v)^{\check{*}})$$

and the involution

$$\overline{(a, u)} = (\bar{a}, u)$$

for $a, b \in R'$, $u, v \in P$ is a structurable algebra over $R$, which is a form of the structurable algebra $M(T, N, N^\vee)$ [Pumplün 2010b, Theorem 20]. We define the (*conjugate*) *norm* $\nu : S(B, *_B, P, N, h) \to R$ of $S(B, *_B, P, N, h)$ via

$$\nu((\lambda, w)) = N_B(\lambda\lambda^* - h(w, w)).$$

If $R'$ is a field this definition coincides with [Allison and Faulkner 1992, Theorem 6.1]. $\nu$ is a quartic form. Even if $B$ is a division algebra and $R'$ is a field, the norm is isotropic: then $\nu((\lambda, w)) = 0$ if and only if $(\lambda, w)$ is an admissible scalar; i.e., $\mu \in R'^\times$, $w \in H(B, *_B)^\times$ and $N_B(w) = \mu\mu^*$.

If $R'$ is a quadratic étale ring extension of the ring $R$ then $R' = \mathrm{Cay}(R, P, N)$ with $L \in \mathrm{Pic}\, R$ of order 2, since $2 \in R^\times$. For $A = S(B, *_B, P, N, h)$ this means $S(A, {}^-) = \{(r, 0) \mid r \in S(R', *)\} = L$. If $R$ is a domain and $R' = \mathrm{Cay}(R, c) = R(\sqrt{c})$ then $S(A, {}^-) = (\sqrt{c}, 0)R$ and $s_0 = (\sqrt{c}, 0)$ satisfies $s_0^2 = (c, 0) = c1_A$ with $c \in R^\times$. This means we can define the (conjugate) norm $\nu : A \to R$ also by

$$\nu(x) = \frac{1}{12c}\chi(s_0 x, \{x, s_0 x, x\}),$$

and also a *trace* $\chi : A \times A \to R$ on $A$ by

$$\chi(x, y) = \frac{2}{c}\psi(s_0 x, y)s_0 = \frac{2}{c}(V^\delta_{y,x} s_0)s_0,$$

analogously as in [Allison and Faulkner 1984; 1992], where $\psi(x, y) = x\bar{y} - y\bar{x}$ [Allison and Faulkner 1992, 5.4]. $\chi$ is a nondegenerate symmetric bilinear form independent of the choice of $s_0$ and $\chi(1, 1) = 4$. (Nondegeneracy follows from [Allison and Faulkner 1984, Proposition 2.5] applied to the residue class forms.)

## 2. Nontrivial structurable algebras over the affine plane which locally are matrix algebras

**2.1.** We mostly use the results and notation of [Parimala et al. 1999, Section 4]. Occasionally, we also use the notation of [Pumplün 2008]: in the notation of [Parimala et al. 1999], the map $\times$ in [Pumplün 2008] or Section 1.4 is denoted by $\phi$ and the map $\check{\times}$ in Section 1.4 by $\phi_*$. There is the obvious notion of a structurable algebra over a locally ringed space; see [Pumplün 2010b]. We identify structurable algebras over $k[X, Y]$ and over $\mathbb{A}^2_k$ using the canonical equivalence described in [ibid., 6.2]. Let $X = \mathbb{P}^2_k$.

**Remark 1.** Let $D$ be a central simple algebra over $k$ of degree 3. Once we have picked a locally free left $D[X, Y]$-module of rank 1 with $N_{D[X,Y]}(P) \cong k[X, Y]$, the choice of a norm $N : P \to k[X, Y]$ automatically determines $N^\vee$ and the adjoints $\sharp$ and $\check{\sharp}$; see [ibid., 3.2]. This fact is expressed in [Parimala et al. 1999] by explicitly choosing a trivialization $\tilde{\mu} : N_{D[X,Y]}(P) \to k[X, Y]$ which in turn determines uniquely the choice of $N$, hence of $N^\vee$, $\sharp$ and $\check{\sharp}$. Recall that the norm $N$ is uniquely determined up to a scalar $\mu \in k^\times$. For any $\mu \in k^\times$, the adjoint belonging to $\mu N$ is $\mu\,\sharp$ and $(\mu N)^\vee = \mu^2 N^\vee$, $(\mu\,\sharp)^\vee = \mu^2\,\sharp$.

**2.2.** Let $D$ be a central division algebra over $k$ of degree 3. Let $De$ be a free module of rank 1 over $D$ with $e$ as a basis element such that $N_D(De) \cong k$ and let $\mu_0 : N_D(De) \to k$ be such an isomorphism. Let $\{g_i\}$ be an infinite family of mutually coprime polynomials in $k[X]$. Then there exist nonfree projective left modules $P_i$ of rank 1 over $D[X,Y]$ and polynomials $f_i \in k[X]$ with $(f_i, f_j) = 1$ for $i \neq j$, $(f_i, g_j) = 1$ for all $i, j$, such that $P_i \otimes k[X]_{f_i}[Y]$ is free for each $i$. Further, there exists

$$\widetilde{\mu}_i : N_{D[X,Y]}(P_i) \to k[X,Y]$$

such that $(P_i, \widetilde{\mu}_i)$ modulo $Y$ is $(De, \mu_0) \otimes_k k[X]$; see [Parimala et al. 1999, 4.1]. The $P_i$ are mutually nonisomorphic $D[X,Y]$-modules [ibid., 4.2].

**2.3.** Let $P$ be a nonfree projective $D[X,Y]$-module such that $N_{D[X,Y]}(P) \cong k[X,Y]$, the isomorphism given by the trivialization $\widetilde{\mu} : N_{D[X,Y]}(P) \to k[X,Y]$ of the reduced norm. Then the pair $(P, \widetilde{\mu})$ is a principal $\mathrm{SL}_1(D)$-bundle over $\mathbb{A}^2_k$ which admits an extension $(\widetilde{P}, \widetilde{\mu})$ to $\mathbb{P}^2_k$; the bundle $\widetilde{P}$ is simply an extension of the $D[X,Y]$-module $P$ [ibid., p. 31] (by abuse of notation, we denote both $\widetilde{\mu}$ and its extension by the same name). Let $N : P \to k[X,Y]$ be the norm on $P$ determined by the choice of the trivialization $\widetilde{\mu}$. The choice of $\widetilde{\mu}$ also determines the maps $\times : P \times P \to P^\vee$, $\check{\times} : P^\vee \times P^\vee \to P$, and $N^\vee$, and therefore also $\sharp$ and $\check{\sharp}$; see [ibid., p. 16]. Take $T(u, \check{v}) = T_{D[X,Y]}(\langle u, \check{v}\rangle)$. The adjoints satisfy the adjoint identities [ibid., 1.2].

Analogously, $\widetilde{\mu}$ determines extensions

$$(\dagger) \qquad \widetilde{N} : \widetilde{P} \to \mathbb{O}_X, \quad \widetilde{N}^\vee : \widetilde{P}^\vee \to \mathbb{O}_X, \quad \widetilde{\sharp}, \quad \text{and} \quad \widetilde{\check{\sharp}}$$

of $N$, $N^\vee$, $\sharp$ and $\check{\sharp}$, respectively, which satisfy the adjoint identities. Let $\widetilde{T}(u, \check{v}) = T_{D\otimes\mathbb{O}_X}(\langle u, \check{v}\rangle)$.

**Proposition 2.** *The matrix algebra*

$$M(T, N, N^\vee) = \begin{bmatrix} k[X,Y] & P \\ P^\vee & k[X,Y] \end{bmatrix}$$

*over $k[X,Y]$ admits a unique extension to a matrix algebra*

$$M(\widetilde{T}, \widetilde{N}, \widetilde{N}^\vee) = \begin{bmatrix} \mathbb{O}_X & \widetilde{P} \\ \widetilde{P}^\vee & \mathbb{O}_X \end{bmatrix}$$

*over $\mathbb{P}^2_k$. The vector bundles $\widetilde{P}$ and $\widetilde{P}^\vee$ are indecomposable and $\widetilde{P}$ and $\widetilde{P}^\vee$ are not isomorphic as vector bundles on $X$.*

*Proof.* There is a unique extension $\widetilde{P}$ over $X = \mathbb{P}^2_k$ of $P$ of norm one that is a locally free right $D \otimes \mathscr{O}_X$-module: by [Parimala et al. 1999, p. 29], $P$ extends to a vector bundle $\widetilde{P}$, which is unique up to a line bundle $\mathscr{L}$. Since we require $\widetilde{P}$ to be of norm one this implies $\mathscr{L}^3 \cong \mathscr{O}_X$, hence $\mathscr{L} = \mathscr{O}_X$ and the extension is unique. Let $N : P \to k[X, Y]$ be the norm on $P$ determined by the choice of the trivialization $\widetilde{\mu}$. Two extensions $\widetilde{N} : \widetilde{P} \to \mathscr{O}_X$ and, say $\widetilde{N}' : \widetilde{P} \to \mathscr{O}_X$ of $N$, can only differ by a scalar $\lambda \in k^\times$. Being its extension, the algebra $M(\widetilde{T}, \widetilde{N}, \widetilde{N}^\vee)$ restricts to the structurable matrix algebra

$$M(T, N, N^\vee) = \begin{bmatrix} k[X, Y] & P \\ P^\vee & k[X, Y] \end{bmatrix}$$

over $\mathbb{A}^2_k$. Therefore $\widetilde{N}|_{\mathbb{A}^2_k} = N = \widetilde{N}'|_{\mathbb{A}^2_k}$ implies that $\lambda = 1$. Thus the maps listed in (†), which are the extensions of $N$, $\check{N}^\vee$, $\sharp$ and $\check{\sharp}$ from $\mathbb{A}^2_k$ to $\mathbb{P}^2_k$ determined by the trivializations $\widetilde{\mu}$ and $\mu$, are uniquely determined as well.[-2pt]

The proof of the second statement follows from [Parimala et al. 1999, 3.2]. □

More precisely, by [ibid., Remark] and [Arason et al. 1992], $\widetilde{P} \cong tr_{l/k}(\mathscr{P}_0)$ for some cubic field extension $l/k$ and a suitable vector bundle $\mathscr{P}_0$ over $\mathbb{P}^2_l$ that is absolutely indecomposable and of rank 3.

**2.4.** Let $J$ be an Albert algebra over $k$ that is a first Tits construction and a division algebra. Choose two cyclic division algebras $D_1, D_2$ of degree 3 over $k$ such that the Jordan algebras $D_1^+$ and $D_2^+$ are subalgebras of $J$ with $D_1^+ \cap D_2^+ = k$. By [Parimala et al. 1999, 4.3], these can be even chosen such that $D_2^+ = \Phi(D_1^+)$ for a suitable automorphism $\Phi$ of $J$; that is, we can and will assume that additionally we have $D_1^+ \cong D_2^+$. Then $J = J(D_1, e_1, \mu_1) = J(D_2, e_2, \mu_2)$ for some $e_i \in J$ and isomorphisms $\mu_i : N(D_i e_i) \to k$. Again, the choice of $\mu_i$ determines a norm $N_i : D_i \to k$, (a scalar multiple of $N_{D_i}$) and an adjoint $\sharp_i : D_i \to D_i$ (a scalar multiple of $\sharp_{D_i}$), so with $T_i(a, b) = T_{D_i}(ab)$ we obtain the structurable algebra

$$M = M(T_1, N_1, N_1) \cong M(T_2, N_2, N_2)$$

over $k$. By 2.2, for every $i \geq 1$ there exists a pair $(P_i^1, \widetilde{\mu}_i^1)$, where $P_i^1$ is a nonfree projective $D_1[X, Y]$-module of rank 1 and $\widetilde{\mu}_i^1$ a trivialization of its reduced norm and a polynomial $f_i \in k[X]$ such that:

(4) The polynomials $f_i$ and $f_j$ are coprime for $i \neq j$ and $(P_i^1)_{f_i}$ is free.

(5) The reduction of $(P_i^1, \widetilde{\mu}_i^1)$ modulo $Y$ is $(D_1 e_1, \mu_1) \otimes k[X]$.

Similarly, for every $i \geq 1$, there is a pair $(P_i^2, \widetilde{\mu}_i^2)$, where $P_i^2$ is a nonfree projective $D_2[X, Y]$-module of rank 1 and $\widetilde{\mu}_i^2$ a trivialization of its reduced norm and a polynomial $g_i \in k[X]$ such that:

(6) The polynomials $g_i$ and $g_j$ are coprime for $i \neq j$, the polynomials $f_i$ and $g_j$ are coprime for all $i$, $j$, and $(P_i^2)_{g_i}$ is free.

(7) The reduction of $(P_i^2, \widetilde{\mu}_i^2)$ modulo $Y$ is $(D_2 e_2, \mu_2) \otimes k[X]$.

For each pair $(P_i^j, \widetilde{\mu}_i^j)$, $j = 1, 2$, let

$$N_i^j : P_i^j \to k[X, Y]$$

be the norm on $P_i^j$ induced by $\widetilde{\mu}_i^j$, let

$$T_i^j : P_i^j \times (P_i^j)^\vee \to k[X, Y]$$

be the usual trace, given by $T_i^j(u, \check{v}) = T_{D_j}(\langle u, \check{v} \rangle)$, and let $\sharp_i^j$ be the induced adjoint.

Define matrix algebras

$$M_i^1 = M(T_i^1, N_i^1, N_i^{1\vee}) = \begin{bmatrix} k[X, Y] & P_i^1 \\ (P_i^1)^\vee & k[X, Y] \end{bmatrix}$$

and

$$M_i^2 = M(T_i^2, N_i^2, N_i^{2\vee}) = \begin{bmatrix} k[X, Y] & P_i^2 \\ (P_i^2)^\vee & k[X, Y] \end{bmatrix}$$

of rank 20. Then $\{M_i^j \mid j = 1, 2, i \geq 1\}$ is a family of structurable algebras over $k[X, Y]$ such that $M_i^j = M \otimes k[X]$ modulo $Y$ and

$$M_i^1 \otimes k[X]_{f_i}[Y] \cong M \otimes k[X]_{f_i}[Y], \quad M_i^2 \otimes k[X]_{g_i}[Y] \cong M \otimes k[X]_{g_i}[Y],$$

with $(f_i, f_j) = 1 = (g_i, g_j)$ for $i \neq j$, $(f_i, g_j) = 1$ for all $i, j$. As in [Parimala et al. 1997, 4.5] we can then conclude:

**Proposition 3.** *The matrix algebras $M_i^1$, respectively $M_i^2$, over $k[X, Y]$ are mutually nonisomorphic.*

*Proof.* Suppose there are $i \neq j$ such that $M_i^1 \cong M_j^1$. Since $M_i^1$ and $M_j^1$ are extended after inverting $f_i$ and $f_j$, respectively, and since $(f_i, f_j) = 1$, $M_i^1$ is extended from $M \otimes k[X]$. Let $\tau : X \to k$ be the structure morphism. Since the extension $\widetilde{M}_i^1$ of $M_i^1$ to $\mathbb{P}_k^2$ is unique, it must be thus isomorphic to $\tau^*(M)$. Therefore, the underlying vector bundles must be isomorphic; i.e.,

$$\mathbb{O}_X^2 \oplus \widetilde{P}_i^1 \oplus (\widetilde{P}_i^1)^\vee \cong \mathbb{O}_X^{20}.$$

This is a contradiction, since $\widetilde{P}_i^1$ is an indecomposable vector bundle by [Parimala et al. 1999, 3.2]. □

**2.5.** Let
$$\pi_i^1 : (P_i^1, \widetilde{\mu}_i^1) \otimes k[X]_{f_i}[Y] \to (D_1 e_1, \mu_1) \otimes k[X]_{f_i}[Y]$$
and
$$\pi_i^2 : (P_i^2, \widetilde{\mu}_i^2) \otimes k[X]_{g_i}[Y] \to (D_1 e_2, \mu_2) \otimes k[X]_{g_i}[Y]$$
be isomorphisms such that $\overline{\pi}_i^j = \mathrm{id}$, $j = 1, 2$ (we may assume this by [Parimala et al. 1997, 6.1]). These canonically induce isomorphisms
$$M(\pi_i^1) : M_i^1 \otimes k[X]_{f_i}[Y] \to M \otimes k[X]_{f_i}[Y]$$
and
$$M(\pi_i^2) : M_i^2 \otimes k[X]_{g_i}[Y] \to M \otimes k[X]_{g_i}[Y]$$
with $\overline{M(\pi_i^j)} = \mathrm{id}$, $j = 1, 2$. Let $M_i$ be the structurable algebra obtained by patching $M_i^1$ on $k[X]_{g_i}[Y]$ and $M_i^2$ on $k[X]_{f_i}[Y]$ over $k[X]_{f_i g_i}[Y]$ by $\phi_i = M(\pi_i^2)^{-1} M(\pi_i^1)$.

We obtain an involution $^-: M_i \to M_i$ by analogously patching the involutions of $M_i^1$ on $k[X]_{g_i}[Y]$ and of $M_i^2$ on $k[X]_{f_i}[Y]$ over $k[X]_{f_i g_i}[Y]$ by $\phi_i = M(\pi_i^2)^{-1} M(\pi_i^1)$.

Since $\overline{M}_i^j = M$ modulo $Y$ and $\overline{M(\pi_i^j)} = \mathrm{id}$, we get $\overline{\phi}_i = \mathrm{id}$ and $\overline{M}_i = M \otimes k[X]$ modulo $Y$. By construction,
$$M_i \otimes k[X]_{f_i g_i}[Y] \cong M \otimes k[X]_{f_i g_i}[Y]$$
and the polynomials $r_i := f_i g_i$ are mutually coprime. The algebras $M_i$ are mutually nonisomorphic by the same argument as given in [Parimala et al. 1999, p. 33], and thus we can conclude:

**Theorem 4.** *The structurable algebras $M_i$ on $\mathbb{A}_k^2$ have the following properties*:

(i) $\overline{M}_i = M \otimes k[X]$ *modulo* $Y$.

(ii) *There are mutually coprime polynomials $r_i \in k[X]$ such that $M_i \otimes k[X]_{r_i}[Y] \cong M \otimes k[X]_{r_i}[Y]$.*

(iii) *The algebras $M_i$ are nonextended and mutually nonisomorphic.*

*Proof.* By construction, we have
$$M_i \otimes k[X]_{f_i g_i}[Y] \cong M \otimes k[X]_{f_i g_i}[Y]$$
and the polynomials $r_i = f_i g_i$ are mutually coprime. To show that the algebras $M_i$ are mutually nonisomorphic, suppose that there are $i \neq j$ such that $M_i \cong M_j$. Then both $(M_i)_{r_i}$ and $(M_i)_{r_j}$ are extended from $M$. Since $(r_i, r_j) = 1$, $M_i \cong M \otimes k[X, Y]$. Restrict $M_i$ to $k[X]_{g_i}[Y]$. This yields that $M_i^1 \otimes k[X]_{g_i}[Y]$ and $M_i^1 \otimes k[X]_{f_i}[Y]$ are extended. Since $(f_i, g_i) = 1$, $M_i^1$ is extended from $M$. This contradicts Proposition 3. □

Note that all the ingredients for these proofs have been provided in [Parimala et al. 1999, Section 4].

It is not clear that these structurable algebras are again matrix algebras. We are not able to say if the corresponding principal $G$-bundle $P_{M_i}$ admits reduction of the structure group to a proper reductive subgroup of $G$ or not. They are subalgebras of a 56-dimensional matrix algebra:

**2.6.** Let $J_i^1$ and $J_i^2$ be the infinitely many mutually nonisomorphic Albert algebras over $k[X,Y]$ used in [Parimala et al. 1999, Proposition 4.5]. They give rise to infinitely many matrix algebras

$$M(J_i^1)=\begin{bmatrix} k[X,Y] & J_i^1 \\ J_i^1 & k[X,Y]\end{bmatrix} \quad \text{and} \quad M(J_i^2)=\begin{bmatrix} k[X,Y] & J_i^2 \\ J_i^2 & k[X,Y]\end{bmatrix}$$

over $k[X,Y]$ of rank 56 which contain the mutually nonisomorphic subalgebras

$$M_i^1=M(T_i^1,N_i^1,N_i^{1\vee})=\begin{bmatrix} k[X,Y] & P_i^1 \\ (P_i^1)^\vee & k[X,Y]\end{bmatrix}$$

and

$$M_i^2=M(T_i^2,N_i^2,N_i^{2\vee})=\begin{bmatrix} k[X,Y] & P_i^2 \\ (P_i^2)^\vee & k[X,Y]\end{bmatrix}$$

of rank 20, which are stable under the involution $^{-}$. They also contain the subalgebras

$$M(D_1)=M(T_{D_1},N_{D_1},N_{D_1})=\begin{bmatrix} k[X,Y] & D_1 \\ D_1 & k[X,Y]\end{bmatrix}$$

and

$$M(D_2)=M(T_{D_2},N_{D_2},N_{D_2})=\begin{bmatrix} k[X,Y] & D_2 \\ D_2 & k[X,Y]\end{bmatrix}$$

of rank 20, which are again stable under the involution $^{-}$ [Pumplün 2010b, Theorem 10].

Let $J_i$ be the Jordan algebra we get if we patch $J_i^1$ on $k[X]_{g_i}[Y]$ and $J_i^2$ on $k[X]_{f_i}[Y]$ over $k[X]_{f_i g_i}[Y]$ using the isomorphisms $J(\pi_i^1)$ and $J(\pi_i^2)$ respectively, that are canonically induced by the $\pi_i^j$, $j=1,2$, as described in [Parimala et al. 1999, p. 32]. The algebras $J_i$ are nonextended, mutually nonisomorphic and no longer a first Tits construction starting with some Azumaya algebra of degree 3 [Parimala et al. 1999, 6.3]. The matrix algebra

$$M(J_i)=\begin{bmatrix} k[X,Y] & J_i \\ J_i & k[X,Y]\end{bmatrix}$$

can then be also viewed as obtained from the matrix algebras

$$M(J_i^1)=\begin{bmatrix} k[X,Y] & J_i^1 \\ J_i^1 & k[X,Y]\end{bmatrix} \quad \text{and} \quad M(J_i^2)=\begin{bmatrix} k[X,Y] & J_i^2 \\ J_i^2 & k[X,Y]\end{bmatrix}$$

by patching them using the obvious induced isomorphisms. Call them $S(\pi_i^j)$, $j = 1, 2$.

By construction, $M_i$ is then clearly a subalgebra of the matrix algebra $M(J_i)$ (the isomorphisms used to patch it are restrictions of the $S(\pi_i^j)$) and there are mutually coprime polynomials $r_i \in k[X]$ with $M(J_i) \otimes k[X]_{r_i}[Y] \cong M(J) \otimes k[X]_{r_i}[Y]$ and $M_i \otimes k[X]_{r_i}[Y] \cong M \otimes k[X]_{r_i}[Y]$, where $M \cong M(D_1^+) \cong M(D_2^+) \subset M(J)$.

**Remark 5.** We observe independently of this that be the infinitely many mutually nonisomorphic reduced Albert algebras $A_i$ over $k[X, Y]$ constructed in [Parimala et al. 1997, Step I and 6.2], also give rise to matrix algebras

$$H_i = \begin{bmatrix} k[X,Y] & A_i \\ A_i & k[X,Y] \end{bmatrix}$$

over $k[X, Y]$ of rank 56 which are mutually nonisomorphic, which is proved analogously to [Parimala et al. 1997, 6.2].

## 3. Structurable algebras over $\mathbb{A}_k^2$ which are forms of matrix algebras

**Remark 6.** Let $T$ be a quadratic étale algebra over $k[X, Y]$ with anisotropic norm. As in [Parimala et al. 1997, 4.6], one can see that $T$ extends uniquely to a quadratic étale algebra $\mathcal{T} = \mathrm{Cay}(\mathcal{O}_X, \mathcal{L}, N)$ over $X = \mathbb{P}_k^2$. Since $\mathrm{Pic}\, X = \mathbb{Z}$, $\mathcal{L} \cong \mathcal{O}_X$ and $\mathcal{T}$ is defined over $k$, thus so is $T$. We conclude that every quadratic étale algebra over $k[X, Y]$ with anisotropic norm is of the kind $K \otimes_k k[X, Y] \cong K[X, Y]$ with $K = k(\sqrt{c})$ a separable quadratic field extension. As a consequence, every quadratic étale ring extension $R'$ of $k[X, Y]$ satisfies $R' = k(\sqrt{c})[X, Y]$ and every form of a matrix algebra of the type $S(B, *, P, N, h)$, $B$ a central simple algebra over $R'$ has $S(A, ^-) = (\sqrt{c}, 0)R$.

**3.1.** Let $K$ be a separable quadratic field extension of $k$. Let $D$ be a central division algebra over $K$ of degree 3 with an involution $\sigma$ of the second kind over $K/k$. Let $X = \mathbb{P}_k^2$, $X' = X \otimes_k K = \mathbb{P}_K^2$ and $\mathcal{D} = D \otimes_K \mathcal{O}_{X'}$.

**Proposition 7.** *Let $P$ be a nonfree projective left $D[X, Y]$-module of rank one. The structurable algebra $S(D, \sigma, P, N) = K[X, Y] \oplus P$ over $k[X, Y]$ admits a unique extension to a structurable algebra $S(\mathcal{D}, \sigma, \widetilde{P}, N) = \mathcal{O}_{X'} \oplus \widetilde{P}$ over $X = \mathbb{P}_k^2$. The vector bundle $\widetilde{P}$ over $X'$ is indecomposable.*

*Proof.* There is a unique extension of the quadratic étale algebra $K[X, Y]$ over $k[X, Y]$ to a quadratic étale algebra $\mathcal{O}_{X'} = K \otimes_k \mathcal{O}_X$ over $X$. There is a unique extension $\widetilde{P}$ over $X' = \mathbb{P}_K^2$ of $P$ of norm one that is a locally free left $\mathcal{D}$-module: by [Parimala et al. 1999, p. 29], $P$ extends to a vector bundle $\widetilde{P}$ over $X'$ that is unique up to a line bundle $\mathcal{L} \in \mathrm{Pic}\, X'$. Since we require $\widetilde{P}$ to be of norm one this implies $\mathcal{L}^3 \cong \mathcal{O}_{X'}$, hence $\mathcal{L} = \mathcal{O}_{X'}$ and the extension is unique. More precisely, by [ibid., Remark] and [Arason et al. 1992], $\widetilde{P} \cong tr_{L'/K'}(\mathcal{P}_0)$ for some cubic field extension $L'/K'$ and a suitable vector bundle $\mathcal{P}$ over $\mathbb{P}^2_{L'}$ that is absolutely indecomposable and must have rank 3. In particular, $N$ and $h$ can be extended as well.

The algebra $S(\mathcal{D}, \sigma, \widetilde{P}, N) = \mathcal{O}_{X'} \oplus \widetilde{P}$ restricts to the structurable algebra

$$S(D, \sigma, P, N) = K[X, Y] \oplus P$$

over $\mathbb{A}^2_k$. The second statement follows from [Parimala et al. 1999, 3.2]. $\square$

**3.2.** Let $K$ be a separable quadratic field extension of $k$. Let $D$ be a central division algebra over $K$ of degree 3 with an involution $\sigma$ of the second kind over $K/k$. Let $(u, \mu)$ be an admissible scalar; i.e., $\mu \in K^\times$, $c \in H(B, *_B)^\times$ and $N_B(c) = \mu\mu^*$. By [ibid., p. 33], there exists a projective left $D[X, Y]$-module $P$ of rank 1 together with a nondegenerate hermitian form $h : P \times P \to D[X, Y]$ and a trivialization $\widetilde{\mu} : \mathrm{disc}(h) \to (K[X, Y], \langle 1\rangle)$ such that:

(8) The reduction of $(P, h, \widetilde{\mu})$ modulo $Y$ is isomorphic to $(D, \langle u\rangle, \mu)$, where $\langle u\rangle$ denotes the hermitian form $a \to au\sigma(a)$ and $\mu$ is treated as a trivialization of the discriminant of $\langle u\rangle$. Moreover, $(De, u_e, \mu_e) \otimes k[X] = (P, h, \widetilde{\mu})$ modulo $Y$, where $De$ is the free module of rank one over $D$ with $e$ a basis element, $u_e$ the hermitian form on $De$ given by $u_e(xe, ye) = xu\sigma(y)$ and $\mu_e N_D(e) = \mu$.

(9) There exists $f \in k[X]$, $f(0) \neq 0$, such that

$$(P, h, \widetilde{\mu}) \otimes k[X]_f[Y] \cong (D, \langle u\rangle, \mu) \otimes k[X]_f[Y].$$

(10) The principal $SU(D, \sigma)$-bundle on $\mathbb{A}^2_k$ associated to $(P, h, \widetilde{\mu})$ admits no reduction of the structure group to any proper connected reductive subgroup of $SU(D, \sigma)$. In particular, $(P, h, \widetilde{\mu})$ is not extended from $(D, \langle u\rangle, \mu)$.

Now let $J$ be an Albert division algebra over $k$ that is a second Tits construction but not a first one. We may write

$$J = J(D^1 e_1, u_{e_1}, \mu_{e_1}) = J(D^2 e_2, u_{e_2}, \mu_{e_2})$$

where $D^1$, $D^2$ are two isomorphic central simple algebras of degree 3 over a quadratic extension $F/k$ with involution $\sigma^1, \sigma^2$ of the second kind and norms $N_1$ and $N_2$, such that $H(D^1, \sigma^1) \cap H(D^2, \sigma^2) = k$; see [ibid., 5.2].

Define the structurable algebra

$$A = S(D^1, \sigma^1, D^1, N_1, u_{e_1}) \cong S(D^2, \sigma^2, D^2, N_2, u_{e_2}).$$

By [Parimala et al. 1999, p. 35], there exist nontrivial hermitian spaces $(P^i_1, h^i_1, \widetilde{\mu}^i_1)$ over $(D^1[X, Y], \sigma^1)$ and $(P^i_2, h^i_2, \widetilde{\mu}^i_2)$ over $(D^2[X, Y], \sigma^2)$ of rank 1, and $f_i, g_i \in k[X]$ such that:

(11) $(P_1^i, h_1^i, \widetilde{\mu}_1^i)$ modulo $Y$ reduces to $(D^1 e_1, u_{e_1}, \mu_{e_1})$, $(P_2^i, h_2^i, \widetilde{\mu}_2^i)$ modulo $Y$ reduces to $(D^2 e_2, u_{e_2}, \mu_{e_2})$.

(12) $(P_1^i, h_1^i, \widetilde{\mu}_1^i) \otimes k[X]_{f_i}[Y]$ is isomorphic to $(D^1 e_1, u_{e_1}, \mu_{e_1}) \otimes k[X]_{f_i}[Y]$ and $(P_2^i, h_2^i, \widetilde{\mu}_2^i) \otimes k[X]_{g_i}[Y]$ is isomorphic to $(D^2 e_2, u_{e_2}, \mu_{e_2}) \otimes k[X]_{g_i}[Y]$ with $(f_i, f_j) = 1 = (g_i, g_j)$ for all $i \neq j$ and $(f_i, g_j) = 1$ for all $i, j$.

(13) The vector bundles $(P_1^i, h_1^i)$ and $(P_2^i, h_2^i)$ are not extended from $D^1$ and $D^2$, respectively.

Let $N_j^i : P_j^i \to D_j[X, Y]$ denote the norm on $P_j^i$ determined by the choice of $\widetilde{\mu}_j^i$, $j = 1, 2$. We define two families of structurable algebras

$$A_1^i = S(D^1, \sigma^1, P_1^i, N_1^i, h_1^i) \quad \text{and} \quad A_2^i = S(D^2, \sigma^2, P_2^i, N_2^i, h_2^i)$$

over $k[X, Y]$ with underlying modules structures

$$A_1^i \cong K[X, Y] \oplus P_1^i \quad \text{and} \quad A_2^i \cong K[X, Y] \oplus P_2^i.$$

Let

$$\pi_1^i : (P_1^i, h_1^i, \widetilde{\mu}_1^i)_{f_i} \to (D^1 e_1, u_{e_1}, \mu_{e_1}) \otimes k[X]_{f_i}[Y],$$
$$\pi_2^i : (P_2^i, h_2^i, \widetilde{\mu}_2^i)_{g_i} \to (D^2 e_2, u_{e_2}, \mu_{e_2}) \otimes k[X]_{g_i}[Y]$$

be isometries such that $\overline{\pi}_j^i = \mathrm{id}$ for $j = 1, 2$. These isometries induce isomorphisms

$$A(\pi_1^i) : A_1^i \otimes k[X]_{f_i}[Y] \to A \otimes k[X]_{f_i}[Y],$$
$$A(\pi_1^i) : A_2^i \otimes k[X]_{g_i}[Y] \to A \otimes k[X]_{g_i}[Y],$$

which reduce to the identity map modulo $Y$.

**Proposition 8.** *The structurable algebras $A_1^i$ and $A_2^i$ over $k[X, Y]$ have the following properties*:

(i) *$A_1^i$ and $A_2^i$ modulo $Y$ reduce to $A$.*

(ii) *$A_1^i \otimes k[X]_{f_i}[Y]$ is extended from $A \otimes k[X]_{f_i}[Y]$ and $A_2^i \otimes k[X]_{g_i}[Y]$ is extended from $A \otimes k[X]_{g_i}[Y]$, with $(f_i, f_j) = 1 = (g_i, g_j)$ for $i \neq j$, $(f_i, g_j) = 1$ for all $i, j$.*

(iii) *$A_j^i \otimes_k K \cong M(T_j^i, N_j^i, N_j^{i\vee}) = M_j^i$ for $j = 1, 2$, where the matrix algebras $M_j^i$ are the ones constructed in Section 2.4.*

(iv) *All the $A_1^i$ are mutually nonisomorphic and all the $A_2^i$ are mutually nonisomorphic.*

*Proof.* Properties (i) and (ii) are immediate consequences of the properties of $(P^i_j, h^i_j, \tilde{\mu}^i_j)$. Property (iii) follows from the construction of the algebras. We use the identification from the proof of [Pumplün 2010b, Theorem 20].

(iv) Since $A^i_1 \otimes_k K \cong M(T^i_j, N^i_j, N^i_j)$ is not extended from $M = M(T_{D^1}, N^1, N^1)$ and $P^i_1$ is not free, it follows that $A^i_1$ is not extended from $A$ by [Bass et al. 1977]. Thus the algebras $A^i_1$ are mutually nonisomorphic. The same argument holds for the $A^i_2$. □

We now patch the structurable algebras $(A^i_1)_{g_i}$ over $k[X]_{g_i}[Y]$ and $(A^i_2)_{f_i}$ over $k[X]_{f_i}[Y]$ over $k[X]_{f_i g_i}[Y]$ and their involutions using the isomorphism

$$\psi_i : A^i_1 \otimes k[X]_{f_i g_i}[Y] \to A^i_2 \otimes k[X]_{f_i g_i}[Y], \quad \psi_i = A(\pi^i_2)^{-1} A(\pi^i_1).$$

This way we obtain a structurable algebra $A^i$ over $k[X, Y]$.

**Theorem 9.** *The structurable algebras $A^i$ over $\mathbb{A}^2_k$ have the following properties*:

1. $\overline{A}^i = A \otimes k[X]$ *modulo* $Y$.
2. *There exists* $\pi^i : A^i \otimes k[X]_{s_i}[Y] \to A \otimes k[X]_{s_i}[Y]$ *such that* $\overline{\pi}^i = \mathrm{id}$, *for some* $s_i \in k[X]$ *with* $(s_i, s_j) = 1$ *for* $i \neq j$.
3. *The* $A^i$ *are mutually nonisomorphic.*
4. $A^i \otimes_k K \cong M_i$ *with the* $M_i$ *as constructed in Section 2.5.*

*Proof.* Since $A^i_j$ reduces modulo $Y$ to $A$ and $\overline{\psi}_i = \mathrm{id}$, $A^i$ reduces modulo $Y$ to $A$. By construction,

$$A^i \otimes k[X]_{f_i g_i}[Y] \cong A \otimes k[X]_{f_i g_i}[Y]$$

and the polynomials $s_i := f_i g_i$ satisfy $(s_i, s_j) = 1$ for $i \neq j$. As in the proof of Theorem 4, it follows that the $A^i$ are mutually nonisomorphic. □

Again, the ingredients for the results were provided in [Parimala et al. 1999, Section 5].

## 4. On extending structurable algebras from the affine to the projective plane

We conclude with some general results about extending structurable algebras from the affine to the projective plane, imitating the techniques used in [Parimala et al. 1997, 4.1, 4.2, 4.3]. Let $R$ be a domain with $\frac{1}{6} \in R$.

**4.1.** For a structurable algebra $(A, ^-)$, an *isotopy* from $(A, ^-)$ to $(A, ^-)$ is an element $\alpha \in \mathrm{GL}(A)$ such that

$$\alpha\{x, y, z\} = \{\alpha(x), \widehat{\alpha}(y), \alpha(z)\}$$

for all $x, y, z \in A$ and some $\widehat{\alpha} \in \mathrm{GL}(A)$. $\widehat{\alpha}$ is uniquely determined by $\alpha$. The *structure group* $\Gamma(A, ^-)$ of $(A, ^-)$ is the subgroup of $\mathrm{GL}(A)$ which consists of all isotopies of $(A, ^-)$ onto itself.

Let $(A, ^-)$ be a structurable algebra of skew-rank one such that $S(A, ^-) = s_0 R$ for some $s_0 \in S(A, ^-)$ that is conjugate invertible, which means that left multiplication $L_{s_0}$ with $s_0$ is invertible. Since $\widehat{s}_0 \in S(A, ^-)$ for its conjugate inverse $\widehat{s}_0$, there is $\beta \in R$, $\beta \neq 0$, such that $\widehat{s}_0 = \beta s_0$ and since $s_0 \widehat{s}_0 = -1_A$ we obtain $\beta s_0^2 = -1_A$. Assume that $\beta \in R^\times$ and denote $c = \beta^{-}1$. Then $s_0^2 = c 1_A$ with $c \in R^\times$. Suppose in addition that the invertible elements in $(A, ^-)$ are Zariski dense in $A$. Then we can define a (*conjugate*) norm $\nu : A \to R$ on $A$ via

$$\nu(x) = \frac{1}{12c} \chi(s_0 x, \{x, s_0 x, x\}),$$

a *trace* $\chi : A \times A \to R$ on $A$ by

$$\chi(x, y) = \frac{2}{c} \psi(s_0 x, y) s_0 = \frac{2}{c} (V^\delta_{y,x} s_0) s_0,$$

and a nondegenerate skew-symmetric bilinear form on $A$

$$\langle x, y \rangle = \psi(x, y) s_0 = \tfrac{1}{2} \chi(s_0 x, y)$$

analogously as in [Allison and Faulkner 1984; 1992], where $\psi(x, y) = x\bar{y} - y\bar{x}$ [Allison and Faulkner 1992, 5.4]. (The nondegeneracy of $\langle\ ,\ \rangle$ follows from [Allison and Faulkner 1984, p. 192], applied to the residue class forms.) $\nu$ is a quartic form such that $\nu(1_A) = 1$. $\chi$ is a nondegenerate symmetric bilinear form independent of the choice of $s_0$ and $\chi(1_A, 1_A) = 4$. (Nondegeneracy follows from [ibid., Proposition 2.5], applied to the residue class forms.) Note that if desired, $A$ can be viewed as a Freudenthal triple system as explained in [ibid., 2.18], in this setting. An element $x \in A$ is conjugate invertible if and only if $\nu(x) \neq 0$ [ibid., 4.4]. So if the norm is anisotropic, every nonzero element of $A$ is conjugate invertible and, if $R$ is a field, $(A, ^-)$ a conjugate division algebra [Allison and Faulkner 1984, 2.11]. The norm $\nu$ is a semi-invariant for the structure group $\Gamma(A, ^-)$, which is proved analogously as in [Allison and Faulkner 1992, 4.7]. Denote the group of all invertible linear transformations on $A$ that preserve the norm and the skew-symmetric bilinear form $\langle\ ,\ \rangle$ by $\mathrm{Inv}(A)$.

**Theorem 10.** *Let $(A_1, ^-)$ and $(A_2, ^-)$ be structurable algebras of skew-rank one over $R$. Suppose that $(A_2, ^-)$ satisfies all of the criteria in Section 4.1 (i.e., it carries a conjugate norm), and that the conjugate norm of $(A_2 \otimes R/(p), ^-)$ is anisotropic. Let*

$$\alpha : (A_1 \otimes R[1/p], ^-) \to (A_2 \otimes R[1/p], ^-)$$

*be an isotopy of structurable algebras. Then $\alpha$ extends uniquely to an isotopy*

$$\tilde{\alpha} : (A_1, {}^-) \to (A_2, {}^-).$$

*In particular, every isomorphism $\alpha : (A_1 \otimes R[1/p], {}^-) \to (A_2 \otimes R[1/p], {}^-)$ of the structurable algebras $(A_1, {}^-)$ and $(A_2, {}^-)$ extends uniquely to an isomorphism $\tilde{\alpha} : (A_1, {}^-) \to (A_2, {}^-)$.*

*Proof.* We show that $\alpha(A_1) = A_2$, which is sufficient: let $x \in A_1$ and assume that $\alpha(x) \notin A_2$. Let $n$ be the least integer such that $y = p^n \alpha(x) \in A_2$ and $p^{n-1}\alpha(x) \notin A_2$. Then $n \geq 1$. $\nu$ is a semi-invariant for the structure group of $(A, {}^-)$, i.e., there is $0 \neq r \in R$ such that $\nu(\alpha(x)) = r\nu(x)$ for all $x \in A_1$. Thus we obtain $\nu(y) = rp^{4n}\nu(x)$. Hence $\nu(y) = 0$ modulo $p$ and $y \neq 0$ modulo $p$. This contradicts the assumption that the norm $\nu \otimes R/(p)$ of $(A_2 \otimes R/(p), {}^-)$ is anisotropic. □

**4.2.** There is an obvious notion of a structurable algebra over a locally ringed space [Pumplün 2010b, Section 6]. Let $(A, {}^-)$ be a structurable algebra of skew-rank one over $X = \mathbb{P}^n_k$ such that $S(A, {}^-) = s_0 \mathcal{O}_X$ for some $s_0 \in H^0(X, S(A, {}^-)) = k$ which is conjugate invertible, which means that left multiplication $L_{s_0}$ with $s_0$ is invertible. Since $\widehat{s}_0 \in H^0(X, S(A, {}^-))$ for its conjugate inverse $\widehat{s}_0$, there is $c \in k^\times$, such that $\widehat{s}_0 = -c^{-1} s_0$ and since $s_0 \widehat{s}_0 = -1_A$ we obtain $s_0^2 = c 1_A$. Suppose in addition that the invertible elements in $H^0(U, (A, {}^-))$ are Zariski dense in $H^0(U, A)$ for every open subset $U \subset X$. Then we can define a (*conjugate*) norm $\nu : A \to \mathcal{O}_X$ via

$$\nu(x) = \frac{1}{12c} \chi(s_0 x, \{x, s_0 x, x\}),$$

a *trace* $\chi : A \times A \to \mathcal{O}_X$ on $A$ by

$$\chi(x, y) = \frac{2}{c} \psi(s_0 x, y) s_0 = \frac{2}{c} (V^{\delta}_{y,x} s_0) s_0$$

and a nondegenerate skew-symmetric bilinear form $\nu : A \times A \to \mathcal{O}_X$

$$\langle x, y \rangle = \psi(x, y) s_0 = \frac{1}{2} \chi(s_0 x, y)$$

analogously as in 4.1, $\psi(x, y) = x\bar{y} - y\bar{x}$. $\nu$ is a quartic form such that $\nu(1_A) = 1$. $\chi$ is a nondegenerate symmetric bilinear form independent of the choice of $s_0$ and $\chi(1_A, 1_A) = 4$.

Theorem 10 now implies:

**Corollary 11.** *Let $(\mathcal{A}_1, {}^-{}_1)$, $(\mathcal{A}_2, {}^-{}_2)$ be two structurable algebras of skew-rank one over $\mathbb{P}^n_k$ which satisfy the assumptions of* 4.2*. Suppose that the restrictions $(\mathcal{A}_1)_\xi$ and $(\mathcal{A}_2)_\xi$ to the generic point $\xi$ have anisotropic norms. Then every isotopy $\alpha : \mathcal{A}_1 \to \mathcal{A}_2$ over $\mathbb{A}^n_k$ extends uniquely to an isotopy $\widetilde{\alpha} : \mathcal{A}_1 \to \mathcal{A}_2$ over $\mathbb{P}^n_k$.*

*In particular, every isomorphism* $\alpha : \mathcal{A}_1 \to \mathcal{A}_2$ *over* $\mathbb{A}^n_k$ *extends uniquely to an isomorphism* $\tilde{\alpha} : \mathcal{A}_1 \to \mathcal{A}_2$ *over* $\mathbb{P}^n_k$.

The proof is verbatim to the proof of [Parimala et al. 1997, 4.3], substituting "isotopy" (and "isomorphism") for "isometry" throughout.

From Corollary 11 and [Parimala et al. 1997, 4.5], we obtain:

**Corollary 12.** *Let $k$ have characteristic* 0*. Let* $(\mathcal{A}, ^-)$ *be a structurable algebra of skew-rank one over* $\mathbb{A}^2_k$ *satisfying the conditions of Section 4.1, such that its restriction* $\mathcal{A}_\xi$ *to the generic point* $\xi$ *has an anisotropic norm. Then* $(\mathcal{A}, ^-)$ *extends uniquely to an algebra* $(\mathcal{A}, ^-)$ *over* $\mathbb{P}^2_k$.

*If* $H = \mathrm{Inv}(A)$ *is a connected reductive algebraic group defined over $k$ then every $H$-bundle over* $\mathbb{A}^2_k$ *extends to* $\mathbb{P}^2_k$ *as an $H$-bundle.*

*If the structurable algebra bundle has rank* 56 *and admits a reduction of the structure group to a proper connected reductive subgroup of* $E_7$*, its corresponding extension to* $\mathbb{P}^2_k$ *has the same property.*

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