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A DIFFERENTIABLE SPHERE THEOREM INSPIRED BY RIGIDITY OF MINIMAL SUBMANIFOLDS

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We prove a vanishing theorem for the fundamental group of a compact submanifold in a space form, and then present a refined version of Ejiri's rigidity theorem for minimal submanifolds in a sphere. Inspired by the refined Ejiri theorem, we verify a new differentiable sphere theorem for compact submanifolds in space forms. We also show that our differentiable sphere theorem is sharp. We emphasize that our method of Ricci flow in the proof of the sphere theorem seems useful in the study of curvature and topology. Also, we obtain a differentiable pinching theorem for compact submanifolds in a Riemannian manifold.

1. Introduction

The investigation of geometrical and topological structures of manifolds plays an important role in global differential geometry. Since the first sphere theorem for Riemannian manifolds was proved by Rauch in 1951, there has been much progress in this field; see [Berger 2000; Brendle 2010; Shiohama 2000; Xu 2012]. Brendle and Schoen [2008] proved a remarkable classification theorem for compact manifolds with weakly $\frac{1}{4}$-pinched curvatures in the pointwise sense, implying this:

Theorem A. *Suppose that M is an n-dimensional complete and simply connected Riemannian manifold such that $\frac{1}{4} \leq K_M \leq 1$. Then M is either diffeomorphic to S^n or isometric to a compact rank-one symmetric space* (*CROSS*).

Petersen and Tao [2009] improved Brendle and Schoen's pinching constant in Theorem A to $\frac{1}{4} - \varepsilon_n$, with ε_n being a positive constant depending only on n. Motivated by Shen's topological sphere theorem [1989] for compact manifolds with positive Ricci curvature and Yau's open problem on the pinching theorem [Yau 1993, Problem 12], Gu and Xu [2011] obtained a differentiable sphere theorem for compact manifolds with positive scalar curvature. In particular, they proved that

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if M is an n-dimensional compact Riemannian manifold ($n \geq 3$) whose sectional curvature and scalar curvature satisfy $K_M \leq 1$ and $R > n(n-1)\delta_n$, then M is diffeomorphic to a spherical space form. Here

$$\delta_n = 1 - \frac{20 - 8\,\mathrm{sgn}(n-3)}{5n(n-1)}$$

and $\mathrm{sgn}(\,\cdot\,)$ is the standard sign function.

In addition, results on sphere theorems for Riemannian submanifolds have been obtained by various authors [Huisken 1987; Lawson and Simons 1973; Shiohama 2000; Shiohama and Xu 1997; Xu and Zhao 2009]. Let M^n be an n-dimensional submanifold in an $(n+p)$-dimensional Riemannian manifold N^{n+p}. Denote by H and S the mean curvature and the squared length of the second fundamental form of M, respectively. Motivated by Lawson, Simons, Shiohama, and Xu's topological sphere theorem [Lawson and Simons 1973; Shiohama and Xu 1997] for submanifolds, Xu and Zhao [2009] proved that for $n \geq 4$, if M is an n-dimensional complete submanifold in S^{n+p} and $S < 2\sqrt{2}$, then M is diffeomorphic to S^n. Let $F^{n+p}(c)$ be an $(n+p)$-dimensional simply connected space form with constant curvature c. Xu and Gu [2010] extended Huisken's sphere theorem [1987] for hypersurfaces in spheres to the case of submanifolds in space forms, and proved the following optimal differentiable sphere theorem:

Theorem B. *Suppose that M is an n-dimensional oriented complete submanifold in $F^{n+p}(c)$ with $c \geq 0$. If*

$$\lambda(M) := \sup_M \left(S - \frac{n^2 H^2}{n-1} - 2c \right) < 0,$$

then M is diffeomorphic to S^n.

After the pioneering rigidity theorem for closed minimal submanifolds in a sphere due to Simons [1968], several fundamental rigidity results for minimal submanifolds were proved by various authors [Chern et al. 1970; Ejiri 1979; Lawson 1969; An-Min and Jimin 1992; Yau 1974; 1975]. Ejiri [1979] obtained the following rigidity theorem for minimal submanifolds in spheres:

Theorem C. *Suppose that M is an n-dimensional simply connected compact orientable minimal submanifold in an $(n+p)$-dimensional unit sphere S^{n+p}, with $n \geq 4$. If the Ricci curvature of M satisfies*

$$\mathrm{Ric}_M \geq n-2,$$

then M is either the totally geodesic submanifold S^n, the Clifford torus $S^m(\sqrt{1/2}) \times S^m(\sqrt{1/2})$ in S^{n+1} with $n = 2m$, or $\mathbb{C}P^2(\frac{4}{3})$ in S^7. Here $\mathbb{C}P^2(\frac{4}{3})$ denotes the 2-dimensional complex projective space minimally immersed into S^7 with constant holomorphic sectional curvature $\frac{4}{3}$.

Shen [1992] proved that if M is a 3-dimensional compact minimal submanifold in a unit sphere S^{3+p}, and if the Ricci curvature of M is larger than 1, then M is the totally geodesic submanifold S^3.

Here we investigate differentiable sphere theorems for compact submanifolds of positive Ricci curvature. Using convergence results of Hamilton [1982] and Brendle [2008] for Ricci flow and the Lawson–Simons–Xin formula for the nonexistence of stable currents [Lawson and Simons 1973; Xin 1984], we prove such a theorem inspired by rigidity of minimal submanifolds. We first prove the following differentiable sphere theorem for submanifolds in a Riemannian manifold.

Theorem 1.1. *Suppose that M is an n-dimensional compact submanifold in an $(n+p)$-dimensional Riemannian manifold N^{n+p}, with $n \geq 4$. Denote by $\overline{K}(x,\pi)$ the sectional curvature of N for tangent 2-plane $\pi(\subset T_xN)$ at point $x \in N$. Set $\overline{K}_{\max}(x) := \max_{\pi\subset T_xN} \overline{K}(x,\pi)$ and $\overline{K}_{\min}(x) := \min_{\pi\subset T_xN} \overline{K}(x,\pi)$. If*

$$\mathrm{Ric}_M > (n-\tfrac{2}{3})\overline{K}_{\max} - \tfrac{4}{3}\overline{K}_{\min} + \tfrac{1}{8}n^2H^2,$$

then the normalized Ricci flow with initial metric g_0

$$\frac{\partial}{\partial t}g(t) = -2\,\mathrm{Ric}_{g(t)} + \frac{2}{n}r_{g(t)}g(t)$$

exists for all time and converges to a constant curvature metric as $t \to \infty$. Also, M is diffeomorphic to a spherical space form. In particular, if M is simply connected, then M is diffeomorphic to S^n.

We give a vanishing theorem for the fundamental group of a submanifold.

Theorem 1.2. *Suppose that M is an n-dimensional compact submanifold in an $(n+p)$-dimensional space form $F^{n+p}(c)$ with $c \geq 0$. If the Ricci curvature of M satisfies*

$$\mathrm{Ric}_M > \tfrac{1}{2}(n-1)c + \tfrac{1}{8}n^2H^2,$$

then $\pi_1(M) = 0$; that is, M is simply connected.

Applying Theorem 1.2 to Theorem C, we obtain a refined version of the Ejiri rigidity theorem for minimal submanifolds in spheres, without the assumption that M is simply connected. Finally, inspired by the refined Ejiri rigidity theorem for minimal submanifolds, we prove the following differentiable sphere theorem for submanifolds in space forms.

Theorem 1.3 (Main Theorem). *Suppose that M is an n-dimensional compact submanifold in an $(n+p)$-dimensional space form $F^{n+p}(c)$, with $c \geq 0$ and $n \geq 3$. If the Ricci curvature of M satisfies*

$$\mathrm{Ric}_M > (n-2)c + \tfrac{1}{8}n^2H^2,$$

then M is diffeomorphic to S^n.

We show in Example 3.4 that the pinching condition in Theorem 1.3 is sharp. Our method in the proof of the main theorem seems very useful in the study of curvature and topology.

2. Notation and fundamental tools

Throughout, let M^n (where $n \geq 2$) be an n-dimensional compact submanifold in an $(n+p)$-dimensional Riemannian manifold N^{n+p}. We use the following conventions on the range of indices:

$$1 \leq A, B, C, \ldots \leq n+p, \quad 1 \leq i, j, k, \ldots \leq n, \quad n+1 \leq \alpha, \beta, \gamma, \ldots \leq n+p.$$

For an arbitrary fixed point $x \in M \subset N$, we choose an orthonormal local frame field $\{e_A\}$ in N^{n+p} such that e_i's are tangent to M. Let $\{\omega_A\}$ be the dual frame field of $\{e_A\}$. Let Rm, h and ξ denote the Riemannian curvature tensor, second fundamental form and mean curvature vector of M, respectively, and let $\overline{Rm}$ be the Riemannian curvature tensor of N. Then

$$Rm = \sum_{i,j,k,l} R_{ijkl}\omega_i \otimes \omega_j \otimes \omega_k \otimes \omega_l,$$

$$\overline{Rm} = \sum_{A,B,C,D} \bar{R}_{ABCD}\omega_A \otimes \omega_B \otimes \omega_C \otimes \omega_D,$$

$$h = \sum_{\alpha,i,j} h^\alpha_{ij}\omega_i \otimes \omega_j \otimes e_\alpha, \quad \xi = \frac{1}{n}\sum_{\alpha,i} h^\alpha_{ii} e_\alpha,$$

$$R_{ijkl} = \bar{R}_{ijkl} + \sum_\alpha \big(h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}\big), \tag{2-1}$$

and the mean curvature H of M is given by $H := \big|(1/n)\sum_{\alpha,i} h^\alpha_{ii}e_\alpha\big|$. Denote by $\mathrm{Ric}(u)$ the Ricci curvature of M in the direction of $u \in UM$. From the Gauss equation (2-1), we have

$$\mathrm{Ric}(e_i) = \sum_j \bar{R}_{ijij} + \sum_{\alpha,j} \big(h^\alpha_{ii}h^\alpha_{jj} - h^\alpha_{ij}h^\alpha_{ij}\big).$$

Denote by A_α the shape operator of M with respect to e_α. Choose $\{e_\alpha\}$ such that e_{n+1} is parallel to ξ and

$$\operatorname{tr} A_{n+1} = nH, \quad \operatorname{tr} A_\alpha = 0, \quad \alpha \neq n+1. \tag{2-2}$$

Thus the mean curvature vector ξ is equal to He_{n+1}, and

$$\sum_j h^\alpha_{jj} = \begin{cases} nH, & \alpha = n+1, \\ 0, & \alpha \neq n+1. \end{cases}$$

Hence

$$\mathrm{Ric}(e_i) = \sum_j \bar{R}_{ijij} + nHh_{ii}^{n+1} - \sum_{\alpha,j} h_{ij}^{\alpha} h_{ij}^{\alpha}. \tag{2-3}$$

Denote by $K(x,\pi)$ the sectional curvature of M for tangent 2-plane $\pi(\subset T_x M)$ at point $x \in M$, and by $\bar{K}(x,\pi)$ the sectional curvature of N for tangent 2-plane $\pi(\subset T_x N)$ at point $x \in N$. Set

$$\bar{K}_{\max}(x) := \max_{\pi \subset T_x N} \bar{K}(x,\pi) \quad \text{and} \quad \bar{K}_{\min}(x) := \min_{\pi \subset T_x N} \bar{K}(x,\pi).$$

By Berger's inequality [Brendle 2010, Proposition 1.9],

$$|\bar{R}_{ABCD}| \le \tfrac{2}{3}(\bar{K}_{\max} - \bar{K}_{\min}) \tag{2-4}$$

for all distinct indices A, B, C, D.

Theorem 2.1 [Hamilton 1982]. *Let (M, g_0) be a compact Riemannian manifold of dimension* 3. *If $Ric_M > 0$, then the normalized Ricci flow with initial metric g_0*

$$\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}_{g(t)} + \tfrac{2}{3} r_{g(t)} g(t)$$

exists for all time and converges to a constant curvature metric as $t \to \infty$. Here $r_{g(t)}$ denotes the mean value of the scalar curvature of $g(t)$.

Next we quote an important convergence result for the Ricci flow in higher dimensions.

Theorem 2.2 [Brendle 2008; Brendle and Schoen 2009]. *Suppose (M, g_0) is a compact Riemannian manifold of dimension $n \ge 4$. Assume that*

$$R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} > 0 \tag{2-5}$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$ and all $\lambda \in [-1, 1]$. Then the normalized Ricci flow with initial metric g_0

$$\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}_{g(t)} + \frac{2}{n} r_{g(t)} g(t)$$

exists for all time and converges to a constant curvature metric as $t \to \infty$. Here $r_{g(t)}$ denotes the mean value of the scalar curvature of $g(t)$.

The following nonexistence theorem for stable currents in a compact Riemannian manifold M isometrically immersed into $F^{n+p}(c)$ was proved for $c > 0$ in [Lawson and Simons 1973] and was extended to $c = 0$ in [Xin 1984]. It is used to eliminate the homology groups $H_q(M; \mathbb{Z})$ for $1 \le q \le n-1$ and the fundamental group $\pi_1(M)$.

Theorem 2.3. *Let M be an n-dimensional compact submanifold in $F^{n+p}(c)$ with $c \geq 0$. Assume that*

$$\sum_{k=q+1}^{n} \sum_{i=1}^{q} \left[2|h(e_i, e_k)|^2 - \langle h(e_i, e_i), h(e_k, e_k)\rangle \right] < q(n-q)c \tag{2-6}$$

for any orthonormal basis $\{e_i\}$ of $T_x M$ at any point $x \in M$, where q is an integer satisfying $1 \leq q \leq n-1$. Then there do not exist any stable q-currents. Also,

$$H_q(M; \mathbb{Z}) = H_{n-q}(M; \mathbb{Z}) = 0,$$

where $H_i(M; \mathbb{Z})$ is the i-th homology group of M with integer coefficients.

3. Proofs of the theorems

To complete the proof of the main theorem (Theorem 1.3), we need to prove the differentiable pinching theorem for submanifolds (Theorem 1.1) and the vanishing theorem for the fundamental group of a submanifold (Theorem 1.2).

Proof of Theorem 1.1. It is sufficient to show that inequality (2-5) in Theorem 2.2 holds for all $\lambda \in \mathbb{R}$. Suppose that $\{e_1, e_2, e_3, e_4\}$ is an orthonormal four-frame and that $\lambda \in \mathbb{R}$. From the Gauss equation, we have

$$\begin{aligned}
&R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} \\
&= \bar{R}_{1313} + \lambda^2 \bar{R}_{1414} + \bar{R}_{2323} + \lambda^2 \bar{R}_{2424} - 2\lambda \bar{R}_{1234} \\
&\quad + \textstyle\sum_\alpha \left(h^\alpha_{11} h^\alpha_{33} - (h^\alpha_{13})^2 + h^\alpha_{22} h^\alpha_{33} - (h^\alpha_{23})^2 \right) \\
&\quad + \lambda^2 \textstyle\sum_\alpha \left(h^\alpha_{11} h^\alpha_{44} - (h^\alpha_{14})^2 + h^\alpha_{22} h^\alpha_{44} - (h^\alpha_{24})^2 \right) \\
&\quad - 2\lambda \textstyle\sum_\alpha \left(h^\alpha_{13} h^\alpha_{24} - h^\alpha_{14} h^\alpha_{23} \right) \\
&= \bar{R}_{1313} + \lambda^2 \bar{R}_{1414} + \bar{R}_{2323} + \lambda^2 \bar{R}_{2424} - 2\lambda \bar{R}_{1234} \\
&\quad + \textstyle\sum_\alpha \big(-(h^\alpha_{13})^2 - (h^\alpha_{23})^2 - \lambda^2 (h^\alpha_{14})^2 - \lambda^2 (h^\alpha_{24})^2 \\
&\qquad - 2\lambda h^\alpha_{13} h^\alpha_{24} + 2\lambda h^\alpha_{14} h^\alpha_{23} + h^\alpha_{11} h^\alpha_{33} + h^\alpha_{22} h^\alpha_{33} + \lambda^2 h^\alpha_{11} h^\alpha_{44} + \lambda^2 h^\alpha_{22} h^\alpha_{44} \big).
\end{aligned} \tag{3-1}$$

It follows from Berger's inequality that $\bar{R}_{1234} \leq \frac{2}{3}(\bar{K}_{\max} - \bar{K}_{\min})$. This together with (3-1) implies

$$\begin{aligned}
&R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} \\
&\geq 2(1+\lambda^2)\bar{K}_{\min} - \tfrac{4}{3}|\lambda|(\bar{K}_{\max} - \bar{K}_{\min}) \\
&\quad + \textstyle\sum_\alpha \big(-(h^\alpha_{13})^2 - (h^\alpha_{23})^2 - \lambda^2 (h^\alpha_{14})^2 - \lambda^2 (h^\alpha_{24})^2 \\
&\qquad - |\lambda|(h^\alpha_{13})^2 - |\lambda|(h^\alpha_{24})^2 - |\lambda|(h^\alpha_{23})^2 - |\lambda|(h^\alpha_{14})^2 \\
&\qquad - \tfrac{1}{2}(1+\lambda^2)(h^\alpha_{11})^2 - \tfrac{1}{2}(1+\lambda^2)(h^\alpha_{22})^2 - (h^\alpha_{33})^2 - \lambda^2 (h^\alpha_{44})^2 \\
&\qquad + \tfrac{1}{2}(h^\alpha_{11} + h^\alpha_{33})^2 + \tfrac{1}{2}(h^\alpha_{22} + h^\alpha_{33})^2 + \tfrac{1}{2}\lambda^2 (h^\alpha_{11} + h^\alpha_{44})^2 + \tfrac{1}{2}\lambda^2 (h^\alpha_{22} + h^\alpha_{44})^2 \big)
\end{aligned} \tag{3-2}$$

$$
\begin{aligned}
&\geq 2(1+\lambda^2)\overline{K}_{\min} - \tfrac{2}{3}(1+\lambda^2)(\overline{K}_{\max} - \overline{K}_{\min}) \\
&\quad + \textstyle\sum_\alpha \big(-(h_{31}^\alpha)^2 - (h_{32}^\alpha)^2 - (h_{33}^\alpha)^2 - \lambda^2 (h_{41}^\alpha)^2 - \lambda^2 (h_{42}^\alpha)^2 - \lambda^2 (h_{44}^\alpha)^2 \\
&\qquad\qquad - \tfrac{1}{2}(1+\lambda^2)(h_{11}^\alpha)^2 - |\lambda|(h_{13}^\alpha)^2 - |\lambda|(h_{14}^\alpha)^2 \\
&\qquad\qquad - \tfrac{1}{2}(1+\lambda^2)(h_{22}^\alpha)^2 - |\lambda|(h_{23}^\alpha)^2 - |\lambda|(h_{24}^\alpha)^2 \\
&\qquad\qquad + \tfrac{1}{2}(h_{11}^\alpha + h_{33}^\alpha)^2 + \tfrac{1}{2}(h_{22}^\alpha + h_{33}^\alpha)^2 + \tfrac{1}{2}\lambda^2 (h_{11}^\alpha + h_{44}^\alpha)^2 + \tfrac{1}{2}\lambda^2 (h_{22}^\alpha + h_{44}^\alpha)^2 \big) \\
&\geq 2(1+\lambda^2)\overline{K}_{\min} - \tfrac{2}{3}(1+\lambda^2)(\overline{K}_{\max} - \overline{K}_{\min}) \\
&\quad + \textstyle\sum_\alpha \Big(- \sum_j \big((h_{3j}^\alpha)^2 + \lambda^2 (h_{4j}^\alpha)^2 + \tfrac{1}{2}(1+\lambda^2)(h_{1j}^\alpha)^2 + \tfrac{1}{2}(1+\lambda^2)(h_{2j}^\alpha)^2 \big) \\
&\qquad\qquad + \tfrac{1}{2}(h_{11}^\alpha + h_{33}^\alpha)^2 + \tfrac{1}{2}(h_{22}^\alpha + h_{33}^\alpha)^2 + \tfrac{1}{2}\lambda^2 (h_{11}^\alpha + h_{44}^\alpha)^2 + \tfrac{1}{2}\lambda^2 (h_{22}^\alpha + h_{44}^\alpha)^2 \Big).
\end{aligned}
$$

Substituting (2-3) into (3-2), we get

$$
\begin{aligned}
&R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} \\
&\geq 2(1+\lambda^2)\overline{K}_{\min} - \tfrac{2}{3}(1+\lambda^2)(\overline{K}_{\max} - \overline{K}_{\min}) \\
&\quad + \operatorname{Ric}(e_3) - nHh_{33}^{n+1} - \textstyle\sum_j \overline{R}_{3j3j} \\
&\quad + \lambda^2 \big(\operatorname{Ric}(e_4) - nHh_{44}^{n+1} - \textstyle\sum_j \overline{R}_{4j4j} \big) \\
&\quad + \tfrac{1}{2}(1+\lambda^2) \big(\operatorname{Ric}(e_1) - nHh_{11}^{n+1} - \textstyle\sum_j \overline{R}_{1j1j} \big) \\
&\quad + \tfrac{1}{2}(1+\lambda^2) \big(\operatorname{Ric}(e_2) - nHh_{22}^{n+1} - \textstyle\sum_j \overline{R}_{2j2j} \big) \\
&\quad + \tfrac{1}{2}(h_{11}^{n+1} + h_{33}^{n+1})^2 + \tfrac{1}{2}(h_{22}^{n+1} + h_{33}^{n+1})^2 \\
&\quad + \tfrac{1}{2}\lambda^2 (h_{11}^{n+1} + h_{44}^{n+1})^2 + \tfrac{1}{2}\lambda^2 (h_{22}^{n+1} + h_{44}^{n+1})^2 \\
&\geq 2(1+\lambda^2)\overline{K}_{\min} - \tfrac{2}{3}(1+\lambda^2)(\overline{K}_{\max} - \overline{K}_{\min}) - 2(1+\lambda^2)(n-1)\overline{K}_{\max} \\
&\quad + \operatorname{Ric}(e_3) + \lambda^2 \operatorname{Ric}(e_4) + \tfrac{1}{2}(1+\lambda^2)\operatorname{Ric}(e_1) + \tfrac{1}{2}(1+\lambda^2)\operatorname{Ric}(e_2) \\
&\quad + \tfrac{1}{2}(h_{11}^{n+1} + h_{33}^{n+1})^2 - \tfrac{1}{2}nH(h_{11}^{n+1} + h_{33}^{n+1}) \\
&\quad + \tfrac{1}{2}(h_{22}^{n+1} + h_{33}^{n+1})^2 - \tfrac{1}{2}nH(h_{22}^{n+1} + h_{33}^{n+1}) \\
&\quad + \tfrac{1}{2}\lambda^2 (h_{11}^{n+1} + h_{44}^{n+1})^2 - \tfrac{1}{2}nH\lambda^2 (h_{11}^{n+1} + h_{44}^{n+1}) \\
&\quad + \tfrac{1}{2}\lambda^2 (h_{22}^{n+1} + h_{44}^{n+1})^2 - \tfrac{1}{2}nH\lambda^2 (h_{22}^{n+1} + h_{44}^{n+1}) \\
&= 2(1+\lambda^2)\overline{K}_{\min} - \tfrac{2}{3}(1+\lambda^2)(\overline{K}_{\max} - \overline{K}_{\min}) - 2(1+\lambda^2)(n-1)\overline{K}_{\max} \\
&\quad + \operatorname{Ric}(e_3) + \lambda^2 \operatorname{Ric}(e_4) + \tfrac{1}{2}(1+\lambda^2)\operatorname{Ric}(e_1) + \tfrac{1}{2}(1+\lambda^2)\operatorname{Ric}(e_2) \\
&\quad + \tfrac{1}{2}\big(h_{11}^{n+1} + h_{33}^{n+1} - \tfrac{1}{2}nH\big)^2 + \tfrac{1}{2}\big(h_{22}^{n+1} + h_{33}^{n+1} - \tfrac{1}{2}nH\big)^2 \\
&\quad + \tfrac{1}{2}\lambda^2\big(h_{11}^{n+1} + h_{44}^{n+1} - \tfrac{1}{2}nH\big)^2 + \tfrac{1}{2}\lambda^2\big(h_{22}^{n+1} + h_{44}^{n+1} - \tfrac{1}{2}nH\big)^2 \\
&\quad - \tfrac{1}{4}(1+\lambda^2)n^2H^2.
\end{aligned}
\tag{3-3}
$$

By assumption, we have

$$\operatorname{Ric}(e_i) > \big((n-\tfrac{2}{3})\overline{K}_{\max} - \tfrac{4}{3}\overline{K}_{\min}\big) + \tfrac{1}{8}n^2H^2. \tag{3-4}$$

Substituting (3-4) into (3-3), we have

$$\begin{aligned} &R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} \\ &\quad > 2(1+\lambda^2)\overline{K}_{\min} - \tfrac{2}{3}(1+\lambda^2)(\overline{K}_{\max} - \overline{K}_{\min}) - 2(1+\lambda^2)(n-1)\overline{K}_{\max} \\ &\qquad + 2(1+\lambda^2)\big((n-\tfrac{2}{3})\overline{K}_{\max} - \tfrac{4}{3}\overline{K}_{\min} + \tfrac{1}{8}n^2H^2\big) - (1+\lambda^2)\tfrac{1}{4}n^2H^2 \\ &\quad = 2(1+\lambda^2)\big(-(n-\tfrac{2}{3}) + (n-\tfrac{2}{3})\big)\overline{K}_{\max} \\ &\qquad + 2(1+\lambda^2)(1+\tfrac{1}{3}-\tfrac{4}{3})\overline{K}_{\min} = 0. \end{aligned} \tag{3-5}$$

It follows from Theorem 2.2 that the normalized Ricci flow with initial metric g_0,

$$\frac{\partial}{\partial t} g(t) = -2\operatorname{Ric}_{g(t)} + \frac{2}{n} r_{g(t)} g(t),$$

exists for all time and converges to a constant curvature metric as $t \to \infty$. Also, M is diffeomorphic to a spherical space form. In particular, if M is simply connected, then M is diffeomorphic to S^n. This completes the proof of Theorem 1.1. □

Corollary 3.1. *Suppose M is an n-dimensional compact submanifold $(n \geq 4)$ in an $(n+p)$-dimensional pinched Riemannian manifold N^{n+p} whose sectional curvature satisfies $b \leq \overline{K}_N \leq c$. If the Ricci curvature of M satisfies*

$$\operatorname{Ric}_M > \big(n-\tfrac{2}{3}\big)c - \tfrac{4}{3}b + \tfrac{1}{8}n^2H^2,$$

then M is diffeomorphic to a spherical space form. In particular, if M is simply connected, then M is diffeomorphic to S^n.

Corollary 3.2. *Suppose M is an n-dimensional compact submanifold $(n \geq 4)$ in an $(n+p)$-dimensional pointwise δ-pinched Riemannian manifold N^{n+p}, where $\delta > 0$. If the Ricci curvature of M satisfies*

$$\operatorname{Ric}_M > \big(n - \tfrac{2}{3} - \tfrac{4}{3}\delta\big)\overline{K}_{\max} + \tfrac{1}{8}n^2H^2,$$

then M is diffeomorphic to a spherical space form. In particular, if M is simply connected, then M is diffeomorphic to S^n.

Proof of Theorem 1.2. It follows from the Gauss equation (2-1) that

$$\operatorname{Ric}(e_i) = (n-1)c + \sum_{\alpha,k}\big(h^\alpha_{ii}h^\alpha_{kk} - (h^\alpha_{ik})^2\big). \tag{3-6}$$

This, together with the assumption, implies that

$$\begin{aligned}\text{(3-7)}\quad &\sum_{k=2}^{n}\big(2|h(e_1,e_k)|^2-\langle h(e_1,e_1),h(e_k,e_k)\rangle\big)\\ &=2\sum_{\alpha}\sum_{k=2}^{n}(h_{1k}^{\alpha})^2-\sum_{\alpha}\sum_{k=2}^{n}h_{11}^{\alpha}h_{kk}^{\alpha}\\ &=-2\,\mathrm{Ric}(e_1)+2(n-1)c-\sum_{\alpha}(h_{11}^{\alpha})^2+nHh_{11}^{n+1}\\ &=-2\,\mathrm{Ric}(e_1)+2(n-1)c-\sum_{\alpha\neq n+1}(h_{11}^{\alpha})^2-\big(h_{11}^{n+1}-\tfrac{1}{2}nH\big)^2+\tfrac{1}{4}n^2H^2\\ &\leq -2\,\mathrm{Ric}(e_1)+2(n-1)c+\tfrac{1}{4}n^2H^2<(n-1)c,\end{aligned}$$

for any orthonormal basis $\{e_i\}$ of T_xM at any point $x\in M$.

Suppose that $\pi_1(M)\neq 0$. Because M is compact, it follows from a classical theorem due to Cartan and Hadamard that there exists a minimal closed geodesic in any nontrivial homotopy class in $\pi_1(M)$. However, by Theorem 2.3, we know that there do not exist any stable integral 1-currents on M. This contradicts the hypothesis. Therefore, $\pi_1(M)=0$. This proves Theorem 1.2. □

Applying Theorem 1.2 to Theorem C, we get a refined version of Ejiri's rigidity theorem.

Theorem 3.3. *Suppose M is an n-dimensional compact orientable minimal submanifold in an $(n+p)$-dimensional unit sphere S^{n+p}, where $n\geq 4$. If the Ricci curvature of M satisfies*

$$\mathrm{Ric}_M\geq n-2,$$

then M is either

- *the totally geodesic submanifold S^n, or*
- *the Clifford torus $S^m(\sqrt{1/2})\times S^m(\sqrt{1/2})$ in S^{n+1} with $n=2m$, or*
- *$\mathbb{C}P^2(\frac{4}{3})$, the 2-dimensional complex projective space minimally immersed in S^7 with constant holomorphic sectional curvature $\frac{4}{3}$.*

Proof of Theorem 1.3. When $n=3$, we have

$$\text{(3-8)}\qquad \mathrm{Ric}_M>c+\tfrac{9}{8}H^2\geq 0.$$

From Hamilton's convergence theorem [1982] for Ricci flow in dimension 3, it follows that M is diffeomorphic to a 3-dimensional spherical space form. When $n\geq 4$, it follows from Theorem 1.1 that M is diffeomorphic to a spherical space form.

On the other hand, it follows from the assumption that

$$\text{(3-9)}\qquad \mathrm{Ric}_M>(n-2)c+\tfrac{1}{8}n^2H^2\geq \tfrac{1}{2}(n-1)c+\tfrac{1}{8}n^2H^2\quad\text{for } n\geq 3.$$

This together with Theorem 1.2 implies that M is simply connected. Therefore, M is diffeomorphic to S^n. This completes the proof of Theorem 1.3. □

The following example shows that the pinching condition in Theorem 1.3 is sharp.

Example 3.4. (i) Let $M = \mathbb{C}P^2\big(\frac{4}{3}(c+H^2)\big)$ be a 2-dimensional complex projective space minimally immersed into $S^7\big(1/\sqrt{c+H^2}\big)$ with constant holomorphic sectional curvature $\frac{4}{3}(c+H^2)$, and let $S^7\big(1/\sqrt{c+H^2}\big)$ be the totally umbilical sphere in $F^{4+p}(c)$. Here H is a nonnegative constant and $c+H^2>0$. Then M is a compact submanifold in $F^{4+p}(c)$ with constant mean curvature H and constant Ricci curvature $\mathrm{Ric}_M \equiv 2c+2H^2$. M is not a topological sphere.

(ii) Let $M = S^2\big(1/\sqrt{2(c+H^2)}\big) \times S^2\big(1/\sqrt{2(c+H^2)}\big)$ be a minimal Clifford hypersurface in $S^5\big(1/\sqrt{c+H^2}\big)$, and let $S^5\big(1/\sqrt{c+H^2}\big)$ be the totally umbilical sphere in $F^{4+p}(c)$. Here H is a nonnegative constant and $c+H^2>0$. Then M is a compact submanifold in $F^{4+p}(c)$ with constant mean curvature H and constant Ricci curvature $\mathrm{Ric}_M \equiv 2c+2H^2$ that is not a topological sphere.

(iii) Let $M = S^m\big(1/\sqrt{2c}\big) \times S^m\big(1/\sqrt{2c}\big)$ be a minimal Clifford hypersurface in $F^{n+1}(c)$ with $c>0$ and $n=2m\geq 6$, and let $F^{n+1}(c)$ be the totally geodesic submanifold in $F^{n+p}(c)$. Then M is a compact minimal submanifold in $F^{n+p}(c)$ with $\mathrm{Ric}_M \equiv (n-2)c$, and is not homeomorphic to S^n.

Remark 3.5. Using the nonexistence theorem for stable currents on submanifolds in hyperbolic spaces [Fu and Xu 2008] and Theorem 1.1, one can also extend Theorem 1.3 to the case of compact submanifolds in hyperbolic spaces.

Motivated by Theorem 1.3 and the convergence results for mean curvature flow in higher codimensions [Andrews and Baker 2010; Liu et al. 2011], we would like to propose the following conjecture on mean curvature flow in higher codimensions.

Conjecture 3.6. *Let $M_0 = F_0(M)$ be an n-dimensional compact submanifold in an $(n+p)$-dimensional space form $F^{n+p}(c)$, with $c+H^2>0$. If the Ricci curvature of M_0 satisfies*

$$\mathrm{Ric}_{M_0} > (n-2)c + \tfrac{1}{8}n^2H^2,$$

then there exists for the mean curvature flow

$$\begin{cases} \dfrac{\partial}{\partial t}F(x,t) = n\xi(x,t), \quad x\in M, \quad t\geq 0,\\ F(\,\cdot\,,0) = F_0(\,\cdot\,) \end{cases} \tag{3-10}$$

a unique smooth solution $F_t(\,\cdot\,)$, and either $F_t(\,\cdot\,)$ converges to a round point in finite time, or $c>0$ and $F_t(\,\cdot\,)$ converges to a totally geodesic sphere as $t\to\infty$.

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