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EIGENVALUE ESTIMATES ON DOMAINS IN COMPLETE NONCOMPACT RIEMANNIAN MANIFOLDS

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In this paper, we obtain universal inequalities for eigenvalues of the Dirichlet eigenvalue problem of the Laplacian and the clamped plate problem on a bounded domain in an n-dimensional ($n \geq 3$) noncompact simply connected complete Riemannian manifold with sectional curvature Sec satisfying $-K^2 \leq \mathrm{Sec} \leq -k^2$, where $K \geq k \geq 0$ are constants. When M is $\mathbb{H}^n(-1)$ ($n \geq 3$), these inequalities become ones previously found by Cheng and Yang.

1. Introduction

Let M be an n-dimensional complete Riemannian manifold and $\Omega \subset M$ a bounded domain in M. The Dirichlet eigenvalue problem of the Laplacian is

$$\begin{cases} \Delta u = -\lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1-1}$$

It is well known that the spectrum of this problem is real and discrete:

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \nearrow \infty,$$

where each λ_i has finite multiplicity which is repeated according to its multiplicity.

A Dirichlet eigenvalue problem of the biharmonic operator or a clamped plate problem that describes the characteristic vibrations of a clamped plate is given by

$$\begin{cases} \Delta^2 u = \Gamma u & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1-2}$$

where Δ^2 is the biharmonic operator on M and ν denotes the outward normal derivative on $\partial\Omega$. We will denote eigenvalues and the corresponding real eigenfunctions by $\{\Gamma_i\}_{i=1}^{\infty}$ and $\{u_i\}_{i=1}^{\infty}$, respectively. The eigenvalues Γ_i satisfy

$$0 < \Gamma_1 \leq \Gamma_2 \leq \Gamma_3 \leq \cdots \nearrow \infty.$$

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When M is a Euclidean space $\mathbb{R}^n$, these are estimates for the eigenvalues (1-1) that do not involve domain dependencies [Protter 1988]; see also [Ashbaugh 1999; 2002]. The main developments were obtained by Payne, Pólya, and Weinberger [Payne et al. 1956], Hile and Protter [1980], and Yang [1991]. More recently, for the Dirichlet eigenvalue problems of the Laplacian on a bounded domain in the n-dimensional unit sphere, complex projective space, and compact homogeneous Riemannian manifolds, Cheng and Yang [2005; 2006b; 2007] obtained the Yang-type inequalities for eigenvalues. For a bounded domain Ω in a complete Riemannian manifold M, the first author and Cheng [Chen and Cheng 2008] proved a Yang-type inequality by using the Nash embedding theorem (compare [El Soufi et al. 2009; Harrell 2007]).

By making use of estimates for eigenvalues of the eigenvalue problem of the Schröinger like operator with a weight, Harrell and Michel [1994], Ashbaugh [2002], and Ashbaugh and Hermi [2007] have obtained several results. In fact, for $n = 2$, the Laplacian on $\mathbb{H}^2(-1)$ is like to the Laplacian on $\mathbb{R}^2$ with a weight. However, for $n > 2$, this property does not hold. Cheng and Yang [2009] found appropriate trial functions and obtained

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le 4\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\Big(\lambda_i-\frac{(n-1)^2}{4}\Big). \tag{1-3}$$

In this paper, we first treat the Dirichlet eigenvalue problem (1-1) of the Laplacian on a bounded domain of a complete noncompact Riemannian manifold M.

Theorem 1.1. *Assume that M^n $(n \ge 3)$ is a noncompact simply connected complete Riemannian manifold with sectional curvature* Sec *satisfying $-K^2 \le \mathrm{Sec} \le -k^2$, where $K \ge k \ge 0$ are constants.For a bounded domain Ω in M, let λ_i be the i-th eigenvalue of the eigenvalue problem* (1-1)*. Then we obtain*

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)(4\lambda_i-(n-1)^2k^2+2(n-1)(K^2-k^2)). \tag{1-4}$$

Remark. If $k = K = 1$, that is, M is a hyperbolic space $\mathbb{H}^n(-1)$, the eigenvalue inequality (1-4) agrees with (1-3) obtained by Cheng and Yang [2009].

The other purpose of this paper is to investigate estimates for eigenvalues of the clamped plate problem (1-2) on bounded domains Ω in a complete Riemannian manifold M^n.

For the universal inequalities for eigenvalues of the clamped plate problem in a bounded domain in $\mathbb{R}^n$, Payne et al. [1955; 1956] proved that

$$\Gamma_{k+1}-\Gamma_k \le \frac{8(n+2)}{n^2k}\sum_{i=1}^{k}\Gamma_i, \quad k=1,2,\dots. \tag{1-5}$$

Hile and Yeh [1984] obtained

$$\sum_{i=1}^{k} \frac{\Gamma_i^{\frac{1}{2}}}{\Gamma_{k+1}-\Gamma_i} \geq \frac{n^2 k^{3/2}}{8(n+2)} \Big(\sum_{i=1}^{k} \Gamma_i\Big)^{-\frac{1}{2}}, \quad k=1,2,\dots. \tag{1-6}$$

Hook [1990] and Chen and Qian [1990] independently proved

$$\frac{n^2k^2}{8(n+2)} \leq \Big(\sum_{i=1}^{k} \frac{\Gamma_i^{\frac{1}{2}}}{\Gamma_{k+1}-\Gamma_i}\Big)\Big(\sum_{i=1}^{k} \Gamma_i^{\frac{1}{2}}\Big), \quad k=1,2,\dots. \tag{1-7}$$

Cheng and Yang [2006a] gave an affirmative answer for a problem on universal inequalities for eigenvalues proposed by Ashbaugh [1999]: they proved that

$$\Gamma_{k+1} - \frac{1}{k}\sum_{i=1}^{k} \Gamma_i \leq \Big(\frac{8(n+2)}{n^2}\Big)^{\frac{1}{2}} \frac{1}{k} \Big(\sum_{i=1}^{k} \Gamma_i(\Gamma_{k+1}-\Gamma_i)\Big)^{\frac{1}{2}}, \quad k=1,2,\dots. \tag{1-8}$$

For domains in a unit sphere, Wang and Xia [2007] gave a universal inequality for the clamped plate problem (1-2). They proved

$$\sum_{i=1}^{k} (\Gamma_{k+1}-\Gamma_i)^2 \leq \frac{8(n+2)}{n^2} \sum_{i=1}^{k} (\Gamma_{k+1}-\Gamma_i)\Big(\Gamma_i^{\frac{1}{2}} + \frac{n^2}{2n+4}\Big)\Big(\Gamma_i^{\frac{1}{2}} + \frac{n^2}{4}\Big). \tag{1-9}$$

For an n-dimensional complete manifold M, Cheng, Ichikawa, and Mametsuka [Cheng et al. 2010] obtained

$$\begin{aligned} &\sum_{i=1}^{k} (\Gamma_{k+1}-\Gamma_i)^2 \\ &\qquad \leq \frac{8(n+2)}{n^2} \sum_{i=1}^{k} (\Gamma_{k+1}-\Gamma_i)\Big(\Gamma_i^{\frac{1}{2}} + \frac{n^2}{2n+4} \sup_{\Omega} |H|^2\Big)\Big(\Gamma_i^{\frac{1}{2}} + \frac{n^2}{4} \sup_{\Omega} |H|^2\Big). \end{aligned} \tag{1-10}$$

For the real hyperbolic space $\mathbb{H}^n(-1)$, Cheng and Yang [2011] proved that

$$\sum_{i=1}^{k} (\Gamma_{k+1}-\Gamma_i)^2 \leq 24 \sum_{i=1}^{k} (\Gamma_{k+1}-\Gamma_i)\Big(\Gamma_i^{\frac{1}{2}} - \frac{(n-1)^2}{4}\Big)\Big(\Gamma_i^{\frac{1}{2}} - \frac{(n-1)^2}{6}\Big). \tag{1-11}$$

That paper motivated the present one, where we treat the clamped plate problem on a bounded domain of a noncompact simply connected complete Riemannian manifold M^n.

Theorem 1.2. *Assume that M^n ($n \geq 3$) is a noncompact simply connected complete Riemannian manifold with sectional curvature* Sec *satisfying $-K^2 \leq \mathrm{Sec} \leq -k^2$, where $K \geq k \geq 0$ are constants. For a bounded domain Ω in M, let Γ_i be the i-th* *eigenvalue of the eigenvalue problem (1-2). Then we have*

$$\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2 \le 24\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\Big(\Gamma_i^{\frac{1}{2}}-\frac{n-1}{4}((n-1)k^2-2(K^2-k^2))\Big) \times\Big(\Gamma_i^{\frac{1}{2}}-\frac{n-1}{6}((n-1)k^2-2(K^2-k^2))\Big). \tag{1-12}$$

Remark. If $k = K = 1$, that is, M^n is a hyperbolic space $\mathbb{H}^n(-1)$, then the eigenvalue inequality (1-12) agrees with (1-11) obtained by Cheng and Yang. Wang and Xia [2011] generalized (1-11) under the assumption that there exists some function whose norm of gradient is 1 and whose Laplacian is a constant.

From Theorem 1.2, we can immediately obtain the following.

Corollary 1.3. *Let Γ_i be the i-th eigenvalue of the eigenvalue problem (1-2). Then we have*

$$\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2 \le 24\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\Big(\Gamma_i-\frac{(n-1)^2}{16}((n-1)k^2-2(K^2-k^2))^2\Big).$$

2. Preliminaries

Let B and C be $(n-1)\times(n-1)$ real symmetrical matrixes. If all the eigenvalues of B are equal or greater than all the ones of C, then we write $B \succ C$.

Let (M, g) be an n-dimensional Riemannian manifold and D the Riemannian connection. The curvature tensor is a (1,3)-tensor defined by

$$R(X, Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z \tag{2-1}$$

for all $X, Y, Z \in \Gamma(\mathrm{TM})$. Let $\gamma : [0, b) \longrightarrow M$ be the minimal normal geodesic and $\{e_i(t)\}_{i=1}^n$ parallel orthonormal frame fields along $\gamma(t)$ such that $e_n(t) = \gamma(t)$. Let

$$J_i(t) = \sum_{j=1}^{n-1} f_{ij}(t)e_j(t), i = 1, \dots, n-1,$$

be the normal Jacobi fields along the geodesic $\gamma(t)$; that is

$$\ddot{f}_{ij} - f_{il}R_{njnl} = 0, \quad f_{ij}(0) = 0, \quad \dot{f}_{ij}(0) = \delta_{ij}, \tag{2-2}$$

where

$$\dot{f}_{ij} = \frac{\mathrm{d}}{\mathrm{d}t}f_{ij}(t), \quad \ddot{f}_{ij} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}f_{ij}(t), \quad R_{njnl} = g(R(e_n, e_l)e_n, e_j) = R_{nlnj}.$$

Set

$$f(t) = (f_{ij}(t))_{(n-1)\times(n-1)}, \quad K(t) = (R_{nlnj}(\gamma(t)))_{(n-1)\times(n-1)},$$

where $f_{ij}(t)$ is on column j and row i. Then (2-2) can be written as

$$
\begin{cases}
\ddot{f}(t) - f(t)K(t) = 0, & 0 < t < b, \\
f(0) = 0, \\
\dot{f}(0) = I_{n-1},
\end{cases}
\tag{2-3}
$$

where I_{n-1} is the $(n-1) \times (n-1)$ unit matrix.

Define the distance function $r(x) = \text{distance}(x, \gamma(0))$. Then

$$
\text{Hess}\, r(\gamma(t)) = f(t)^{-1}\dot{f}(t), \quad \Delta r(\gamma(t)) = \text{tr}(f(t)^{-1}\dot{f}(t)). \tag{2-4}
$$

Assume that Ω is a bounded domain in an n-dimensional noncompact simply connected complete Riemannian manifold (M, g) with section curvature Sec satisfying $-K^2 \le \text{Sec} \le -k^2$, where $0 \le k \le K$ are constants. For $p \notin \overline{\Omega}$ fixed, define the distance function $r(x) = \text{distance}(x, p)$. Then from the Hessian comparison theorem (cf. [Wu et al. 1989]), we have

$$
K\frac{\cosh Kr}{\sinh Kr} I_{n-1} \succ \text{Hess}\, r \succ k\frac{\cosh kr}{\sinh kr} I_{n-1}. \tag{2-5}
$$

From (2-4) and (2-5), we have

$$
(n-1)k\frac{\cosh kr}{\sinh kr} \le \Delta r \le (n-1)K\frac{\cosh Kr}{\sinh Kr}. \tag{2-6}
$$

Since $\partial_r \Delta r = -|\text{Hess}\, r|^2 - \text{Ric}(\partial_r, \partial_r)$ (cf. [Petersen 1998]), we have

$$
-\partial_r \Delta r \le (n-1)K^2\frac{\cosh^2 Kr}{\sinh^2 Kr} - (n-1)k^2. \tag{2-7}
$$

3. Proof of Theorem 1.1

Theorem 3.1 [Cheng and Yang 2006b]. *Let λ_i be the i-th eigenvalue of the above eigenvalue problem (1-1) and u_i the orthonormal eigenfunction corresponding to λ_i; that is, u_i satisfies*

$$
\begin{cases}
u_i = -\lambda u_i & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega, \\
\int_\Omega u_i u_j = \delta_{ij} & \text{for all } i, j = 1, 2, \dots.
\end{cases}
$$

Then for any $f \in C^3(\Omega) \cap C^2(\partial\Omega)$, we have

$$
\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \int_\Omega |\nabla f|^2 u_i^2 \le \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \int_\Omega (2\nabla f \cdot \nabla u_i + u_i \Delta f)^2. \tag{3-1}
$$

Proof of Theorem 1.1. Taking $f = r$ in the formula (3-1), we have

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_\Omega |\nabla r|^2 u_i^2 \le \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\int_\Omega (2\nabla r\cdot\nabla u_i + u_i\Delta r)^2.$$

Since $|\nabla r| = 1$, we have

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\int_\Omega (2\nabla r\cdot\nabla u_i + u_i\Delta r)^2.$$

From (2-6) and (2-7), we obtain

$$\begin{aligned}
\text{(3-2)}\quad & \int_\Omega \Big(2\nabla r\cdot\nabla u_i + u_i\Delta r\Big)^2 \\
&= 4\int_\Omega (\nabla r\cdot\nabla u_i)^2 + 4\int_\Omega u_i\Delta r\nabla r\cdot\nabla u_i + \int_\Omega (u_i\Delta r)^2 \\
&\le 4\int_\Omega |\nabla u_i|^2 - \int_\Omega u_i^2(\Delta r)^2 - 2\int_\Omega u_i^2\nabla r\cdot\nabla\Delta r \\
&= 4\int_\Omega |\nabla u_i|^2 - \int_\Omega u_i^2(\Delta r)^2 - 2\int_\Omega u_i^2\partial_r\Delta r \\
&= 4\int_\Omega |\nabla u_i|^2 - \int_\Omega u_i^2(\Delta r)^2 + 2\int_\Omega u_i^2(\mathrm{Ric}(\partial_r,\partial_r) + |\mathrm{Hess}\, r|^2) \\
&\le 4\lambda_i - (n-1)^2k^2\int_\Omega u_i^2\frac{\cosh^2 kr}{\sinh^2 kr} \\
&\qquad\qquad - 2(n-1)k^2 + 2(n-1)K^2\int_\Omega u_i^2\frac{\cosh^2 Kr}{\sinh^2 Kr} \\
&= 4\lambda_i - (n-1)^2k^2 + 2(n-1)(K^2-k^2) \\
&\qquad\qquad - (n-1)^2\int_\Omega \frac{k^2}{\sinh^2 kr}u_i^2 + 2(n-1)\int_\Omega \frac{K^2}{\sinh^2 Kr}u_i^2.
\end{aligned}$$

Since $K \ge k \ge 0$ and $r > 0$, we have

$$\frac{K}{\sinh Kr} \le \frac{k}{\sinh kr}. \tag{3-3}$$

Since $n \ge 3$, we have

$$(n-1)^2\frac{k^2}{\sinh^2 kr} - 2(n-1)\frac{K^2}{\sinh^2 Kr} \ge (n-1)(n-3)\frac{k^2}{\sinh^2 kr} \ge 0. \tag{3-4}$$

Finally, we have

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\big(4\lambda_i - (n-1)^2k^2 + 2(n-1)(K^2-k^2)\big). \qquad \square$$

4. Proof of Theorem 1.2

Let u_i be the i-th orthonormal eigenfunction corresponding to the eigenvalue Γ_i, $i = 1, \dots, k$; that is,

$$\begin{cases} \Delta^2 u_i = \Gamma_i u_i & \text{in } \Omega, \\ u_i = \dfrac{\partial u_i}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ \displaystyle\int_\Omega u_i u_j = \delta_{ij} & \text{for any } i, j. \end{cases} \tag{4-1}$$

Defining the functions

$$\phi_i = r u_i - \sum_{j=1}^{k} a_{ij} u_j,$$

where

$$a_{ij} = \int_\Omega r u_i u_j,$$

we have

$$\phi_i|_{\partial\Omega} = \frac{\partial \phi_i}{\partial \nu}\Big|_{\partial\Omega} = 0 \quad \text{and} \quad \int_\Omega \phi_i u_j = 0 \qquad \text{for all } i, j = 1, \dots, k. \tag{4-2}$$

Therefore, we know that ϕ_is are trial functions. From the Rayleigh–Ritz inequality [Chavel 1984], we have

$$\Gamma_{k+1} \le \frac{1}{\|\phi_i\|^2} \int_\Omega (\Delta \phi_i)^2, \tag{4-3}$$

where

$$\|\phi_i\|^2 = \int_\Omega \phi_i^2.$$

From (4-1) and (4-2), we have

$$\begin{aligned} \Gamma_{k+1} \int_\Omega \phi_i^2 \le \int_\Omega (\Delta\phi_i)^2 = \int_\Omega \phi_i \Delta^2 \phi_i = \int_\Omega \phi_i \Delta^2 \Big(r u_i - \sum_{j=1}^{k} a_{ij} u_j \Big) \\ = \int_\Omega \phi_i \Delta^2 (r u_i) = \int_\Omega \phi_i (\Delta^2 (r u_i) - \Gamma_i r u_i) + \Gamma_i \int_\Omega \phi_i^2, \end{aligned}$$

that is,

$$(\Gamma_{k+1} - \Gamma_i) \|\phi_i\|^2 \le \int_\Omega \phi_i (\Delta^2 (r u_i) - \Gamma_i r u_i).$$

From the definition of ϕ_i and (4-2), we have

$$
\begin{aligned}
(4\text{-}4)\quad &(\Gamma_{k+1}-\Gamma_i)\|\phi_i\|^2\\
&\le \int_\Omega (ru_i-\sum_{j=1}^k a_{ij}u_j)(\Delta^2(ru_i)-\Gamma_i ru_i)\\
&= \int_\Omega ru_i(\Delta^2(ru_i)-\Gamma_i ru_i)+\sum_{j=1}^k a_{ij}^2(\Gamma_i-\Gamma_j)\\
&= \int_\Omega ru_i(\Delta(u_i\Delta r)+2\Delta(\nabla r\cdot\nabla u_i)+2\nabla r\cdot\nabla\Delta u_i+\Delta r\Delta u_i)\\
&\qquad +\sum_{j=1}^k a_{ij}^2(\Gamma_i-\Gamma_j).
\end{aligned}
$$

From (2-6), (2-7), and Stokes' theorem, by a direct calculation, we have

$$
\begin{aligned}
(4\text{-}5)\quad &\int_\Omega ru_i(\Delta(u_i\Delta r)+2\Delta(\nabla r\cdot\nabla u_i)+2\nabla r\cdot\nabla\Delta u_i+\Delta r\Delta u_i)\\
&= \int_\Omega (\Delta(ru_i)(u_i\Delta r+2\nabla r\cdot\nabla u_i)+u_i\nabla r^2\cdot\nabla\Delta u_i+u_i r\Delta r\Delta u_i)\\
&= \int_\Omega \big((u_i\Delta r+2\nabla r\cdot\nabla u_i+r\Delta u_i)(u_i\Delta r+2\nabla r\cdot\nabla u_i)\\
&\qquad +u_i\nabla r^2\cdot\nabla\Delta u_i+u_i r\Delta r\Delta u_i\big)\\
&= \int_\Omega \big((\Delta r)^2u_i^2+2\nabla r\cdot\nabla u_i^2\Delta r+4(\nabla r\cdot\nabla u_i)^2+2ru_i\Delta r\Delta u_i+\nabla r^2\cdot\nabla u_i\Delta u_i\big)\\
&\qquad +\int_\Omega u_i\nabla r^2\cdot\nabla\Delta u_i\\
&= \int_\Omega \big((\Delta r)^2u_i^2+2\nabla r\cdot\nabla u_i^2\Delta r+4(\nabla r\cdot\nabla u_i)^2+2ru_i\Delta r\Delta u_i+\nabla r^2\cdot\nabla(u_i\Delta u_i)\big)\\
&= \int_\Omega ((\Delta r)^2u_i^2+2\nabla r\cdot\nabla u_i^2\Delta r+4(\nabla r\cdot\nabla u_i)^2+(2r\Delta r-\Delta r^2)u_i\Delta u_i)\\
&= \int_\Omega (-(\Delta r)^2u_i^2-2u_i^2\nabla r\cdot\nabla\Delta r+4(\nabla r\cdot\nabla u_i)^2-2u_i\Delta u_i)\\
&\le \int_\Omega (4|\nabla u_i|^2-2u_i\Delta u_i)-\int_\Omega u_i^2(2\nabla r\cdot\nabla\Delta r+(\Delta r)^2)\\
&\le \int_\Omega u_i^2\Big(-(n-1)^2k^2\frac{\cosh^2 kr}{\sinh^2 kr}-2(n-1)k^2+2(n-1)K^2\frac{\cosh^2 Kr}{\sinh^2 Kr}\Big)\\
&\qquad +\int_\Omega (4|\nabla u_i|^2+2u_i(-\Delta u_i))
\end{aligned}
$$

Since $n \geq 3$, from (3-4), we have

$$\text{(4-6)} \quad \int_\Omega ru_i(\Delta(u_i\Delta r)+2\Delta(\nabla r\cdot\nabla u_i)+2\nabla r\cdot\nabla\Delta u_i+\Delta r\Delta u_i) \\ \leq 6\int_\Omega u_i(-\Delta u_i)-\int_\Omega u_i^2((n-1)^2k^2-2(n-1)(K^2-k^2)) \\ \leq 6\Big(\int_\Omega(\Delta u_i)^2\Big)^{\frac{1}{2}}-(n-1)((n-1)k^2-2(K^2-k^2)) \\ =6\Big(\Gamma_i^{\frac{1}{2}}-\frac{n-1}{6}((n-1)k^2-2(K^2-k^2))\Big).$$

From (4-4) and (4-6), we deduce

$$\text{(4-7)} \quad (\Gamma_{k+1}-\Gamma_i)\|\phi_i\|^2 \\ \leq 6\Big(\Gamma_i^{\frac{1}{2}}-\frac{n-1}{6}((n-1)k^2-2(K^2-k^2))\Big)+\sum_{j=1}^k a_{ij}^2(\Gamma_i-\Gamma_j).$$

Defining

$$b_{ij}=\int_\Omega\big(\nabla r\cdot\nabla u_i+\tfrac{1}{2}u_i\Delta r\big)u_j,$$

we have

$$b_{ij}=-b_{ji}.$$

From the definitions of b_{ij} and ϕ_i, we obtain

$$\text{(4-8)} \quad -2\int_\Omega\phi_i(\nabla r\cdot\nabla u_i+\tfrac{1}{2}\Delta ru_i) \\ =-2\int_\Omega\Big(ru_i-\sum_{j=1}^k a_{ij}u_j\Big)\big(\nabla r\cdot\nabla u_i+\tfrac{1}{2}\Delta ru_i\big) \\ =-2\int_\Omega ru_i\big(\nabla r\cdot\nabla u_i+\tfrac{1}{2}\Delta ru_i\big)+2\sum_{j=1}^k a_{ij}b_{ij} \\ =-\int_\Omega\big(\tfrac{1}{2}\nabla r^2\cdot\nabla u_i^2+r\Delta ru_i^2\big)+2\sum_{j=1}^k a_{ij}b_{ij} \\ =1+2\sum_{j=1}^k a_{ij}b_{ij}.$$

From (4-2), (4-8), and the Cauchy–Schwartz inequality, we have

$$
\begin{aligned}
1+2\sum_{j=1}^{k} a_{ij}b_{ij} &= -2\int_\Omega \phi_i\big(\nabla r\cdot\nabla u_i+\tfrac12 u_i\Delta r\big)\\
&= -2\int_\Omega \phi_i\Big(\nabla r\cdot\nabla u_i+\tfrac12 u_i\Delta r-\sum_{j=1}^{k} b_{ij}u_j\Big)\\
&\le \alpha_i\|\phi_i\|^2+\frac{1}{\alpha_i}\Big\|\nabla r\cdot\nabla u_i+\tfrac12 u_i\Delta r-\sum_{j=1}^{k} b_{ij}u_j\Big\|^2\\
&= \alpha_i\|\phi_i\|^2+\frac{1}{\alpha_i}\Big(\|\nabla r\cdot\nabla u_i+\tfrac12 u_i\Delta r\|^2-\sum_{j=1}^{k} b_{ij}^2\Big),
\end{aligned}
\tag{4-9}
$$

where $\alpha_i>0$ is a positive constant.

If $\Gamma_{k+1}-\Gamma_i>0$, defining

$$\alpha_i=(\Gamma_{k+1}-\Gamma_i)\beta_i \quad \text{for } \beta_i>0,$$

we infer that

$$
\begin{aligned}
(\Gamma_{k+1}-\Gamma_i)^2\Big(1+2\sum_{j=1}^{k} a_{ij}b_{ij}\Big)
&\le (\Gamma_{k+1}-\Gamma_i)^3\beta_i\|\phi_i\|^2\\
&\quad +m\frac{1}{\beta_i}(\Gamma_{k+1}-\Gamma_i)\Big(\|\nabla r\cdot\nabla u_i+\tfrac12 u_i\Delta r\|^2-\sum_{j=1}^{k} b_{ij}^2\Big).
\end{aligned}
\tag{4-10}
$$

From (2-6), (2-7), and (3-4), we obtain

$$
\begin{aligned}
\int_\Omega (2\nabla r\cdot\nabla u_i+u_i\Delta r)^2
&= 4\int_\Omega(\nabla r\cdot\nabla u_i)^2+4\int_\Omega u_i\Delta r\nabla r\cdot\nabla u_i+\int_\Omega(u_i\Delta r)^2\\
&\le 4\int_\Omega|\nabla u_i|^2-\int_\Omega u_i^2(\Delta r)^2-2\int_\Omega u_i^2\nabla r\cdot\nabla\Delta r\\
&\le 4\Gamma_i^{\frac12}-(n-1)\big((n-1)k^2-2(K^2-k^2)\big)\\
&\le 4\Big(\Gamma_i^{\frac12}-\frac{n-1}{4}((n-1)k^2-2(K^2-k^2))\Big).
\end{aligned}
\tag{4-11}
$$

Therefore, from (4-7), (4-9), (4-10), and (4-11), we obtain

$$(4\text{-}12)\quad (\Gamma_{k+1}-\Gamma_i)^2\Big(1+2\sum_{j=1}^{k}a_{ij}b_{ij}\Big)$$
$$\le(\Gamma_{k+1}-\Gamma_i)^2\beta_i\Big(6\Big(\Gamma_i^{\frac{1}{2}}-\frac{n-1}{6}((n-1)k^2-2(K^2-k^2))\Big)+\sum_{j=1}^{k}a_{ij}^2(\Gamma_i-\Gamma_j)\Big)$$
$$+\frac{1}{\beta_i}(\Gamma_{k+1}-\Gamma_i)\Big(\Gamma_i^{\frac{1}{2}}-\frac{n-1}{4}((n-1)k^2-2(K^2-k^2))\Big)$$
$$-\frac{1}{\beta_i}(\Gamma_{k+1}-\Gamma_i)\sum_{j=1}^{k}b_{ij}^2.$$

From the antisymmetry of b_{ij} and the Cauchy–Schwartz inequality, we have

$$2\sum_{i,j=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2a_{ij}b_{ij}-\sum_{i,j=1}^{k}(\Gamma_{k+1}-\Gamma_i)(\Gamma_i-\Gamma_j)^2\beta_ia_{ij}^2-\sum_{i,j=1}^{k}\frac{1}{\beta_i}(\Gamma_{k+1}-\Gamma_i)b_{ij}^2\le 0.$$

From the above inequality and (4-12), we obtain

$$(4\text{-}13)\quad \sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2$$
$$\le 6\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\beta_i\Big(\Gamma_i^{\frac{1}{2}}-\frac{n-1}{6}((n-1)k^2-2(K^2-k^2))\Big)$$
$$+\sum_{i=1}^{k}\frac{1}{\beta_i}(\Gamma_{k+1}-\Gamma_i)\Big(\Gamma_i^{\frac{1}{2}}-\frac{n-1}{4}((n-1)k^2-2(K^2-k^2))\Big)$$
$$+\sum_{i,j=1}^{k}(\Gamma_{k+1}-\Gamma_i)(\Gamma_{k+1}-\Gamma_j)(\Gamma_i-\Gamma_j)\beta_ia_{ij}^2.$$

From the variational principle, we can prove that

$$\Gamma_i\ge\lambda_i^2,$$

where λ_i denotes the i-th eigenvalue of the Dirichlet eigenvalue problem of the Laplacian on the same domain Ω. Since $4\lambda_1\ge(n-1)^2k^2-2(K^2-k^2)$ from (3-2), setting

$$\beta_i=\beta\Big(\Gamma_i^{\frac{1}{2}}-\frac{n-1}{6}((n-1)k^2-2(K^2-k^2))\Big)^{-1}\quad\text{for }\beta>0$$

gives us

$$\begin{aligned}\sum_{i,j=1}^{k}(\Gamma_{k+1}-\Gamma_i)(\Gamma_{k+1}-\Gamma_j)(\Gamma_i-\Gamma_j)\beta_i a_{ij}^2 \\ &= \frac{1}{2}\sum_{i,j=1}^{k}(\Gamma_{k+1}-\Gamma_i)(\Gamma_{k+1}-\Gamma_j)(\Gamma_i-\Gamma_j)(\beta_i-\beta_j)a_{ij}^2 \\ &= -\frac{1}{2}\beta\sum_{i,j=1}^{k}\frac{(\Gamma_{k+1}-\Gamma_i)(\Gamma_{k+1}-\Gamma_j)(\Gamma_i-\Gamma_j)(\Gamma_i^{\frac{1}{2}}-\Gamma_j^{\frac{1}{2}})}{\big(\Gamma_i^{\frac{1}{2}}-\frac{n-1}{6}((n-1)k^2-2(K^2-k^2))\big) \times\big(\Gamma_i^{\frac{1}{2}}-\frac{n-1}{6}((n-1)k^2-2(K^2-k^2))\big)}a_{ij}^2 \\ &\le 0.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2 \le 6\beta\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2 \\ +\frac{1}{\beta}\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\Big(\Gamma_i^{\frac{1}{2}}-\frac{n-1}{4}((n-1)k^2-2(K^2-k^2))\Big)\cdot \\ \times\Big(\Gamma_i^{\frac{1}{2}}-\frac{n-1}{6}((n-1)k^2-2(K^2-k^2))\Big).\end{aligned}$$

Finally, taking $\beta=\frac{1}{12}$, we deduce (1-12). This completes the proof of Theorem 1.2.

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