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Let F be a non-Archimedean local field whose residue field has order q and characteristic $p \neq 2$. We show that the Weil representations of the symplectic group $\mathrm{Sp}(2n, F)$ can be realized over the field

$$E_0 = \begin{cases} \mathbb{Q}(\sqrt{p}, \sqrt{-p}), & \text{if } q \text{ is not a square;} \\ \mathbb{Q}(\sqrt{-p}), & \text{if } q \text{ is a square and } p \equiv 1 \bmod 4; \\ \mathbb{Q}(\sqrt{-1}), & \text{if } q \text{ is a square and } p \equiv 3 \bmod 4. \end{cases}$$

Furthermore, the field E_0 is shown to be optimal if $q \equiv 1 \bmod 4$.

1. Introduction

Let F be a non-Archimedean local field whose residue field has order q and characteristic $p \neq 2$. Our main result is that the Weil representations of the symplectic group $\mathrm{Sp}(2n, F)$, can be realized over the number field

$$E_0 = \begin{cases} \mathbb{Q}(\sqrt{p}, \sqrt{-p}), & \text{if } q \text{ is not a square;} \\ \mathbb{Q}(\sqrt{-p}), & \text{if } q \text{ is a square and } p \equiv 1 \bmod 4; \\ \mathbb{Q}(\sqrt{-1}), & \text{if } q \text{ is a square and } p \equiv 3 \bmod 4. \end{cases}$$

This answers a question raised by D. Prasad [1998]. A consequence of this, also pointed out by Prasad, is that the local theta correspondence can be defined for representations which are realized over E_0.

Let λ be a nontrivial, continuous, complex, unitary character of the additive group of the field F. We shall use $\mathbb{Q}(\lambda)$ to denote the field obtained by adjoining all of the character values of λ to $\mathbb{Q}$, and set $E = \mathbb{Q}(\lambda)(\sqrt{-1})$. We observe that E is an algebraic extension of $\mathbb{Q}$. Indeed, if F has characteristic 0, E is the field obtained from $\mathbb{Q}$ by adjoining $\sqrt{-1}$ and all p-power roots of unity. On the other hand, if char $F = p$ then E is the number field obtained by adjoining a primitive $4p$-th root of unity to $\mathbb{Q}$.

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Let $F^{2n} = X \oplus Y$ be a decomposition of F^{2n} as a direct sum of totally isotropic F-subspaces, with respect to the alternating form on F^{2n} used to define the symplectic group $Sp(2n, F)$. Ranga Rao [1993] provided an explicit realization of the Weil representation W_λ of $\mathrm{Sp}(2n, F)$ associated with λ as integral operators acting on the Bruhat-Schwartz space $\mathcal{S}(X)$ of complex valued, locally constant functions on X of compact support. For a subfield L of $\mathbb{C}$, define $\mathcal{S}(X, L)$ to be the space of locally constant functions on X of compact support having values in L. Observing that the Haar measure $\mu_{\lambda,g}$ used to define the operators $W_\lambda(g)$ is $\mathbb{Q}(\sqrt{q})$-rational, we are able to show that the space $\mathcal{S}(X, E)$ is invariant under the Weil representation W_λ, hence provides a realization of the Weil representation over the algebraic extension E. In particular, this provides an affirmative answer to Prasad's question in the case char $F = p$.

The latter half of the paper is devoted to the construction a 1-cocycle δ on $\mathrm{Gal}\big(E/E_0\big)$ with values in $\mathrm{GL}\big(\mathcal{S}(X, E)\big)$ such that

$$\text{(I)} \qquad {}^{\sigma}W_\lambda(g) = \delta(\sigma)^{-1} W_\lambda(g)\delta(\sigma), \quad g \in \mathrm{Sp}(V).$$

Using Galois descent, we show that there exists $\alpha \in \mathrm{GL}(\mathcal{S}(X, E))$ such that $\delta(\sigma) = \alpha^{-1}\,{}^{\sigma}\alpha$ for $\sigma \in \mathrm{Gal}\big(E/E_0\big)$.

Main theorem. *The operators $\alpha W_\lambda(g)\alpha^{-1}$ leave $\mathcal{S}(X, E_0)$ invariant, and provide a form of the Weil representation realized over E_0.*

We should remark that we fail to provide an explicit description of the operator α. As such, the problem of finding an explicit realization of the Weil representation over E_0 remains open.

To indicate how we find the 1-cocycle satisfying (I), for the rest of the introduction we assume that F has characteristic 0. The Galois group of $\mathbb{Q}(\lambda)/\mathbb{Q}$ is isomorphic to the units $\mathbb{Z}_p^*$ of the p-adic integers. For an element s of $\mathbb{Z}_p^*$, we let σ_s denote the corresponding element of $\mathrm{Gal}(\mathbb{Q}(\lambda)/\mathbb{Q})$. For an element $t \in F^*$, we define the character $\lambda[t]$ of F by $\lambda[t](r) = \lambda(tr)$, $r \in F$.

For $t \in F^*$, let $g_t \in \mathrm{Sp}(2n, F)$ and $f_t \in \mathrm{GL}(2n, F)$ be defined by

$$(x + y)g_t = t^{-1}x + ty,$$
$$(x + y)f_t = x + ty,$$

where $x \in X$, $y \in Y$. In general, f_t is not in $\mathrm{Sp}(2n, F)$, but conjugation by f_t leaves $\mathrm{Sp}(2n, F)$ invariant. We have

$$\text{(II)} \qquad W_\lambda(g^{f_t}) = W_{\lambda[t]}(g), \quad g \in \mathrm{Sp}(V).$$

Furthermore, observing f_{t^2} is the composite $tI \circ g_t$, we show

$$\text{(III)} \qquad W_\lambda(g^{f_{t^2}}) = W_\lambda(g_t)^{-1} W_\lambda(g) W_\lambda(g_t).$$

On the other hand, restriction to $\mathbb{Q}(\lambda)$ identifies $\mathrm{Gal}(E/E_0)$ with $(F^*)^2 \cap \mathbb{Z}_p^*$. If $\sigma \in \mathrm{Gal}(E/E_0)$, we can write

$$\sigma|_{\mathbb{Q}(\lambda)} = \sigma_{t^2}$$

for some $t \in F^*$. We note

$$\text{(IV)} \qquad {}^{\sigma}W_\lambda(g) = W_{\lambda[t^2]}(g).$$

In light of (II) and (III), we deduce the fundamental identity

$${}^{\sigma}W_\lambda(g) = W_\lambda(g_t)^{-1} W_\lambda(g) W_\lambda(g_t).$$

The last equation is used to show that $\delta(\sigma) = W_\lambda(g_t)$ satisfies (I) and almost satisfies the one-cocycle condition. An actual one-cocycle is obtained by slightly modifying the operators $W_\lambda(g_t)$.

The paper concludes with an investigation of the optimality of the field E_0. Our main tool is the K-types associated with the compact subgroup $\mathrm{Sp}(\mathscr{L})$ of elements preserving a lattice $\mathscr{L}$ on which the symplectic form $\langle\ ,\ \rangle$ is nondegenerate. If $q \equiv 1 \mod 4$ then it is impossible to realize the K-types in a proper subfield of E_0, which allows us to deduce that E_0 is optimal for realizing W_λ. If $q \equiv 3 \mod 4$, the K-types can be realized over the proper subfield $\mathbb{Q}(\sqrt{-p})$ of E_0. In this case, the possibility of realizing the Weil representation over the smaller field is left open.

2. Preliminary remarks on local fields, characters and measures

We fix some notation and recall some elementary facts about the characters of the additive group of a local field. Further details can be found in the first two chapters of [Weil 1974].

Let F be a non-Archimedean local field, $\mathcal{O}$ its ring of integers, and $\mathfrak{m}$ the maximal ideal of $\mathcal{O}$. The order of the residue class field $\kappa = \mathcal{O}/\mathfrak{m}$ shall be denoted q; we note that q is power of $p = \operatorname{char} \kappa$. We assume throughout that p is different from 2; in particular, 2 is a unit of $\mathcal{O}$.

Given a fractional $\mathcal{O}$-ideal $\mathfrak{a}$, there exists an unique integer $v(\mathfrak{a})$, the *valuation of* $\mathfrak{a}$, such that

$$\mathfrak{a} = \mathfrak{m}^{v(\mathfrak{a})}.$$

If $s \in F$ is nonzero, the valuation of the ideal $s\mathcal{O}$ is referred to as the valuation of s, denoted $v(s)$. The absolute value on F is related to the valuation v on F by

$$|s| = q^{-v(s)}, \quad s \in F, s \neq 0.$$

Let λ be a nontrivial, continuous, unitary, complex linear character of F^+. The continuity of λ ensures that its kernel contains a fractional $\mathcal{O}$-ideal. The fact that λ is nontrivial allows one to deduce that the set of all such fractional $\mathcal{O}$-ideals has a unique maximal element $\mathfrak{i} = \mathfrak{i}_\lambda$, *the conductor of* λ. The *level of* λ is defined to be the valuation of $\mathfrak{i}_\lambda$.

Given $n \geq 1$, let

$$\nu_{p^n} = \{\, z \in \mathbb{C} : z^{p^n} = 1 \,\}, \qquad \nu_{p^\infty} = \bigcup_{n=1}^{\infty} \nu_{p^n}.$$

(The more customary symbol μ will be used to denote a measure.)

Lemma 1. *We have*

$$\operatorname{im}\lambda = \begin{cases} \nu_p & \textit{if } \operatorname{char} F = p, \\ \nu_{p^\infty} & \textit{if } \operatorname{char} F = 0. \end{cases}$$

Proof. Take $x \in F$. If $\operatorname{char} F = p$ then

$$1 = \lambda(0) = \lambda(px) = \lambda(x)^p.$$

This shows $\operatorname{im}\lambda \subseteq \nu_p$. Equality follows from the fact $\operatorname{im}\lambda$ is a nontrivial subgroup of the simple abelian group ν_p.

If $\operatorname{char} F = 0$ then, since $p \in \mathfrak{m}$, there exists an $n \geq 0$ such that $p^n x \in \mathfrak{i}_\lambda$. For such n,

$$1 = \lambda(p^n x) = \lambda(x)^{p^n}.$$

Then $\operatorname{im}\lambda \subseteq \nu_{p^\infty}$. If the inclusion were proper then there would exist $m \geq 0$ such that $\operatorname{im}\lambda = \nu_{p^m}$. In this case, if $x \in F$ then

$$\lambda(x) = \lambda\left(p^m \cdot \frac{x}{p^m}\right) = \lambda\left(\frac{x}{p^m}\right)^{p^m} = 1$$

since $\lambda(x/p^m)$ is a p^m-th root of unity. As this would contradict the nontriviality of λ, $\operatorname{im}\lambda = \nu_{p^\infty}$. $\square$

Define $\mathbb{Q}(\lambda)$ to be the field obtained by adjoining to $\mathbb{Q}$ all the character values $\lambda(x)$, $x \in F$. Define

$$\mathcal{P} \simeq \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{if } \operatorname{char} F = p, \\ \mathbb{Z}_p & \text{if } \operatorname{char} F = 0. \end{cases}$$

Note that $\mathcal{P}$ is the topological closure of the prime ring of F.

Lemma 2. *There is a canonical topological isomorphism*

$$\operatorname{Gal}(\mathbb{Q}(\lambda)/\mathbb{Q}) \simeq \mathcal{P}^*.$$

Proof. The preceding lemma ensures that $\operatorname{im}\lambda$ is invariant under the action of Galois, hence restriction yields a homomorphism

$$\mathrm{Gal}(\mathbb{Q}(\lambda)/\mathbb{Q}) \to \mathrm{Aut}(\operatorname{im}\lambda) \simeq \begin{cases} (\mathbb{Z}/p\mathbb{Z})^* & \text{if } \operatorname{char} F = p, \\ \mathbb{Z}_p^* & \text{if } \operatorname{char} F = 0. \end{cases}$$

It is readily checked that this map is an isomorphism of topological groups. The proof is completed by appealing to the description of $\mathcal{P}$ given above. □

The pairing

$$(s,t) \to \lambda(st), \qquad s,t \in F,$$

is nondegenerate and leads to an identification of F^+ with its Pontryagin dual [Weil 1974, II.5]. The image of $s \in F$ in the dual shall be denoted $\lambda[s]$:

$$\lambda[s](t) = \lambda(st), \qquad t \in F.$$

Let $\mu = dt$ be a Haar measure on F^+. If ϕ is a locally constant, complex valued function on F of compact support, the Fourier transform $\mathcal{F}_\lambda\phi$ is the complex valued function on F defined by

$$\mathcal{F}_\lambda\phi(s) = \int_F \lambda[s](t)\phi(t)\,dt, \qquad s \in F.$$

It can be shown that $\mathcal{F}_\lambda\phi$ is locally constant and has compact support. Furthermore, the general theory of Fourier transforms asserts the existence of a positive constant c, depending only on the Haar measure dt, such that

$$(\mathcal{F}_\lambda\mathcal{F}_\lambda\phi)\,(t) = c\phi(-t), \qquad t \in F.$$

There is a unique Haar measure on F^+ for which $c = 1$; it shall be denoted $d_\lambda t$ and will be referred to as the *self-dual Haar measure associated with* λ [Weil 1974, VII.2].

Lemma 3. *If λ has level l then the associated self-dual Haar measure is characterized by the condition*

$$\int_{\mathcal{O}} d_\lambda t = q^{l/2}. \tag{1}$$

Proof. This follows from [Weil 1974, Corollary 3, VII.2]. □

Corollary. *If $s \in F^*$ then*

$$d_{\lambda[s]}t = |s|^{1/2} d_\lambda t.$$

Proof. Since $\mathrm{i}_\lambda = s\mathrm{i}_{\lambda[s]}$, the levels l_1 of λ and l_2 of $\lambda[s]$ satisfy the relation $l_1 = v(s) + l_2$. Therefore, Lemma 3 yields

$$\int_{\mathcal{O}} d_{\lambda[s]}t = q^{l_2/2} = q^{-v(s)/2}q^{l_1/2} = |s|^{1/2}\int_{\mathcal{O}} d_\lambda t.$$ □

3. The Schrödinger and Weil representations

Let $\langle\ ,\ \rangle$ be a nondegenerate, alternating, F-bilinear form on a finite dimensional F-vector space V. The *Heisenberg group* H is the group on $V \times F$ having multiplication

$$(v,t)(v',t') = \bigl(v+v', t+t'+\langle v,v'\rangle/2\bigr), \qquad t,\ t' \in F,\ v,\ v' \in V.$$

Let λ be a nontrivial, continuous, unitary, complex linear character of F^+. Since $Z(H) = 0 \times F \simeq F^+$, it may be viewed as a character of the center of the Heisenberg group H.

Theorem (Stone, von Neumann). *There exists a smooth, irreducible representation of H having central character λ. Such a representation is necessarily admissible, and is unique up to isomorphism.*

A proof of the Stone-von Neumann Theorem can be found in [Mœglin et al. 1987, 2.I]. The representation provided by the Stone-von Neumann Theorem is referred to as *the Schrödinger representation of type* λ.

The symplectic group

$$\mathrm{Sp}(V) = \{g \in \mathrm{GL}(V) : \langle vg, wg\rangle = \langle v, w\rangle,\ v, w \in V\}$$

acts on the Heisenberg group H as a group of automorphisms as follows: if $g \in \mathrm{Sp}(V)$ and $(t,v) \in H$ then

$$(t,v)g = (t, vg).$$

Given a Schrödinger representation S_λ of type λ and $g \in \mathrm{Sp}(V)$, consider the representation S_λ^g of H defined by

$$S_\lambda^g(h) = S_\lambda(hg), \quad h \in H.$$

It is readily verified that S_λ^g is a smooth, irreducible representation of H. Furthermore, observing that g acts trivially on $Z(H)$, S_λ^g has central character λ. The Stone-von Neumann Theorem allows us to conclude that the representation S_λ and S_λ^g are equivalent, hence the ambient space affording S_λ admits an operator $W_\lambda(g)$ for which

$$S_\lambda^g(h) = W_\lambda(g)^{-1} S_\lambda(h) W_\lambda(g), \quad h \in H.$$

In light of Schur's Lemma, the operator $W_\lambda(g)$ is uniquely defined up to multiplication by a nonzero constant. As a result, the map

$$g \mapsto W_\lambda(g), \quad g \in \mathrm{Sp}(V),$$

is a projective representation of $\mathrm{Sp}(V)$, called a *Weil representation of type* λ.

In this paper we consider the Schrödinger models of S_λ and W_λ [Kudla 1996, Lemma 2.2, Proposition 2.3; Mœglin et al. 1987, 2.I.4(a) and 2.II.6; Ranga Rao 1993, §3]. Let

$$V = X + Y$$

where X and Y are maximal, totally isotropic subspaces. The Schrödinger model is realized in the Bruhat-Schwartz space $\mathcal{S}(X)$ of locally constant functions $f : X \to \mathbb{C}$ of compact support: if $x \in X$, $y \in Y$ and $t \in F$ then $S_\lambda((x+y,t))$ is the operator defined by

$$[S_\lambda((x+y,t))\phi](x') = \lambda\left(t + \frac{\langle x, y\rangle}{2} + \langle x', y\rangle\right)\phi(x+x'), \quad \phi \in \mathcal{S}(X), x' \in X.$$

The description of the Weil representation requires some additional notation. Viewing $x + y \in V$ as a row vector (x, y), each $g \in \mathrm{Sp}(V)$ can be expressed in the matrix form

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{2}$$

where $a : X \to X$, $b : X \to Y$, $c : Y \to X$, and $d : Y \to Y$. With this notation, set

$$Y_g = Y/\ker c.$$

If μ_g is a Haar measure on Y_g then the action of $W_\lambda(g)$ on $\mathcal{S}(X)$ is given by

$$[W_\lambda(g)\phi](x) = \int_{Y_g} \lambda\left(\frac{\langle xa, xb\rangle - 2\langle xb, yc\rangle + \langle yc, yd\rangle}{2}\right)\phi(xa+yc)\, d\mu_g y, \tag{3}$$

for $\phi \in \mathcal{S}(X)$ and $x \in X$. Note that the integral appearing in (3) is well-defined, for the integrand is constant on the cosets of $\ker c$, hence can be viewed as a function on Y_g. The fact $\phi \in \mathcal{S}(X)$ can be used to show that the integrand belongs to $\mathcal{S}(Y_g)$, hence the integral converges, and that the resulting function $W_\lambda(g)\phi$ belongs to $\mathcal{S}(X)$.

We now recall a particular choice of Haar measures $\mu_{\lambda,g}$ on Y_g, $g \in \mathrm{Sp}(V)$ [Ranga Rao 1993, §3.3]. Fix a basis $x_1, \dots, x_n$ of X and let $y_1, \dots, y_n$ be the dual basis of Y defined by the conditions

$$\langle x_i, y_j\rangle = \delta_{ij}, \quad 1 \le i, j \le n.$$

Let τ_i, $0 \le i \le n$, be the element of $\mathrm{Sp}(V)$ defined by

$$x_j\tau_i = \begin{cases} -y_j & \text{if } j \le i, \\ x_j & \text{if } i < j, \end{cases}$$

$$y_j\tau_i = \begin{cases} x_j & \text{if } j \le i, \\ y_j & \text{if } i < j. \end{cases}$$

We note that Y_{τ_i} can be identified with the subspace of Y spanned by the elements $y_1, \ldots, y_i$. We define

$$d\mu_{\lambda,\tau_i} y = \prod_{k=1}^{i} d_\lambda y_k, \tag{4}$$

where $d_\lambda y_k$ is the self-dual Haar measure associated with λ.

Let

$$P = \{g \in \mathrm{Sp}(V) : Yg = g\},$$

the parabolic subgroup that leaves Y invariant. If $\dim Y_g = i$ then [Ranga Rao 1993, Theorem 2.14] ensures the existence of elements p_1 and p_2 of P such that

$$g = p_1 \tau_i p_2.$$

Observing that the operator p_1 induces an isomorphism $\overline{p}_1 : Y_g \to Y_{\tau_i}$, we set

$$\mu_{\lambda,g} = |\det(p_1 p_2|_Y)|^{-1/2}\, \overline{p}_1 \cdot \mu_{\lambda,\tau_i}. \tag{5}$$

Here, $\overline{p}_1 \cdot \mu_{\lambda,\tau_i}$ denotes the pullback of the Haar measure μ_{λ,τ_i} to Y_g via $\overline{p}_1$: if E is a measurable subset of Y_g then

$$\overline{p}_1 \cdot \mu_{\lambda,\tau_i}(O) = \mu_{\lambda,\tau_i}(O\,\overline{p}_1).$$

Theorem 4. *The measures $\mu_{\lambda,g}$, $g \in Sp(V)$, are well-defined. The projective representation W_λ of $\mathrm{Sp}(V)$ defined by (3) with the Haar measures $\mu_g = \mu_{\lambda,g}$ has the following properties.*

(i) *If $g \in \mathrm{Sp}(V)$ and $p_1, p_2 \in P$ then $W_\lambda(p_1 g p_2) = W_\lambda(p_1) W_\lambda(g) W_\lambda(p_2)$; in particular W_λ restricts to an ordinary representation of P.*

(ii) *If $\phi \in \mathscr{S}(X)$ and $p = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in P$ then*

$$[W_\lambda(p)\phi]\,(x) = |\det a|^{1/2} \lambda\left(\frac{\langle xa, xb\rangle}{2}\right) \phi(xa), \quad x \in X.$$

Proof. This follows from [Ranga Rao 1993, Theorem 3.5]. □

Lemma 5. *If $s \in F^*$ and $g \in \mathrm{Sp}(V)$ then $\mu_{\lambda[s],g} = |s|_{Y_g}^{1/2} \mu_{\lambda,g}$.*

Proof. In light of the Corollary to Lemma 3, (4) yields

$$d\mu_{\lambda[s],\tau_i} y = \prod_{k=1}^{i} d_{\lambda[s]} y_k = \prod_{k=1}^{i} \left[|s|^{1/2} d_\lambda y_k\right] = |s|^{i/2} \prod_{k=1}^{i} d_\lambda y_k = |s|^{i/2} d\mu_{\lambda,\tau_i} y.$$

Therefore, we obtain from (5) and the fact that Y_g has dimension i over F that

$$\begin{aligned}\mu_{\lambda[s],g} &= |\det(p_1 p_2|_Y)|^{-1/2}\ \overline{p}_1 \cdot \mu_{\lambda[s],\tau_i} = |s|^{i/2}\, |\det(p_1 p_2|_Y)|^{-1/2}\ \overline{p}_1 \cdot \mu_{\lambda,\tau_i} \\ &= |s|^{i/2} \mu_{\lambda,g} = |s|_{Y_g}^{1/2} \mu_{\lambda,g}\end{aligned}.$$ □

Let μ be a Haar measure on a totally disconnected topological group A. If O_1 and O_2 are nonempty compact open sets in A then the ratio

$$(O_1 : O_2) = \frac{\mu(O_1)}{\mu(O_2)}$$

is a rational number [Cartier 1979, I.1.1]. Hence, if $\mu(O)$ lies in a subfield L of $\mathbb{C}$ for some nonempty compact open set O then the same is true for all nonempty compact open sets. The measure μ is said to *L-rational* if this is the case.

Lemma 6. *The measures $\mu_{\lambda,g}$, $g \in \mathrm{Sp}(V)$, are $\mathbb{Q}(\sqrt{q})$-rational.*

Proof. If $t \in F^*$ then $|t|$ is a power of q. Therefore, (5) shows that it is sufficient to verify that the measures μ_{λ,τ_i} are $\mathbb{Q}(\sqrt{q})$-rational. Formulas (1) and (4) ensure that this is indeed the case: if $\mathcal{Y}_i = \sum_{k=1}^{i} \mathbb{O} y_k$ then

$$\int_{\mathcal{Y}_i} d\mu_{\lambda,\tau_i} y = q^{il/2}. \qquad \square$$

If L is a subfield of $\mathbb{C}$, let $\mathcal{S}(A, L)$ denote the space of locally constant, L-valued functions on A of compact support.

Lemma 7. *Let A be a totally disconnected topological group, $L \subseteq K$ an extension of fields, and μ a L-rational Haar measure on A. If $\phi \in \mathcal{S}(A, K)$ then $\int_A \phi \, d\mu$ belongs to K.*

Proof. Since $\phi \in \mathcal{S}(A, K)$, there exists compact open subsets $A_1, \ldots, A_k$ of A and scalars $c_1, \ldots, c_k$ in K such that

$$\phi = \sum_{i=1}^{k} c_i \chi_{A_i}.$$

Here, χ_{A_i} denotes the characteristic function of A_i. Since $\mu(A_i) \in L \subseteq K$, it follows that

$$\int_A \phi \, d\mu = \sum_{i=1}^{k} c_i \mu(A_i)$$

lies in K. $\square$

Let $\mathbb{Q}(\lambda)$ be the character field of λ and set

$$E = \mathbb{Q}(\lambda)(\sqrt{-1}).$$

Observe that Lemma 1 ensures that $\mathbb{Q}(\sqrt{q})$ is a subfield of E.

Proposition 8. *The operators $W_\lambda(g)$, $g \in \mathrm{Sp}(V)$, leave the subspace $\mathcal{S}(X, E)$ invariant.*

Proof. If $\phi \in \mathcal{S}(X, E)$ then the integrand in (3) lies in $\mathcal{S}(Y_g, E)$, since $\mathbb{Q}(\lambda) \subseteq E$. In light of Lemma 6, Lemma 7 applied in the case $A = Y_g$, $K = E$, $L = \mathbb{Q}(\sqrt{q})$, and $\mu = \mu_{\lambda,g}$ allows us to deduce that the integral (3) lies in E. It follows immediately that $W_\lambda(g)\phi \in \mathcal{S}(X, E)$. □

In particular, if F has odd characteristic p, the preceding result allows one to conclude that the Weil representation W_λ can be realized over the number field $\mathbb{Q}(\nu_{4p})$.

4. Galois action

By Lemma 1, E is a Galois extension of $\mathbb{Q}$. Its Galois group acts on $\mathcal{S}(X, E)$: if $\sigma \in \mathrm{Gal}(E/\mathbb{Q})$ and $\phi \in \mathcal{S}(X, E)$ then

(6) $$(\sigma(\phi))(x) = \sigma(\phi(x)), \quad x \in X.$$

There is an associated Galois action on $\mathrm{End}\,\mathcal{S}(X, E)$: if $\sigma \in G$ and $T \in \mathrm{End}\,\mathcal{S}(X, E)$ then

(7) $${}^\sigma T(\phi) = \sigma\left[T\left(\sigma^{-1}(\phi)\right)\right], \quad \phi \in \mathcal{S}(X, E).$$

The Galois group also permutes the unitary characters of F^+: if $\sigma \in \mathrm{Gal}(E/\mathbb{Q})$ and λ is a unitary character of F^+ then ${}^\sigma\lambda$ is the character defined by

$${}^\sigma\lambda(t) = \sigma(\lambda(t)), \quad t \in F^+.$$

Let $\mathcal{P}$ be the topological closure of the prime ring of F. The image of $s \in \mathcal{P}^*$ in $\mathrm{Gal}(\mathbb{Q}(\lambda)/\mathbb{Q})$ under the canonical isomorphism of Lemma 2 will be denoted σ_s.

Lemma 9. *Let $\sigma \in \mathrm{Gal}(E/\mathbb{Q})$. If $\sigma|_{\mathbb{Q}(\lambda)} = \sigma_s$ then ${}^\sigma\lambda = \lambda[s]$.*

Proof. (char $F = 0$) Let $\mathfrak{i}$ be the conductor of λ. Given $t \in F$, fix $n \geq 1$ such that $t \in p^{-n}\mathfrak{i}$. Since $p^n t \in \mathfrak{i}$,

$$1 = \lambda(p^n t) = \lambda(t)^{p^n},$$

thus $\lambda(t) \in \nu_{p^n}$. Fixing $r \in \mathbb{Z}$ such that $s \equiv r \mod p^n\mathcal{P}$,

$$({}^\sigma\lambda)(t) = \sigma(\lambda(t)) = \lambda(t)^r = \lambda(rt) = \lambda(st),$$

the last equality following from the fact $rt \equiv st \mod \mathfrak{i}$. □

Given $\sigma \in \mathrm{Gal}(E/\mathbb{Q})$, let ${}^\sigma W_\lambda$ be the projective representation defined by

$$({}^\sigma W_\lambda)(g) = {}^\sigma(W_\lambda(g)), \quad g \in \mathrm{Sp}(V).$$

Proposition 10. *Let $\sigma \in \mathrm{Gal}(E/\mathbb{Q}(\sqrt{q}))$. If $\sigma|_{\mathbb{Q}(\lambda)} = \sigma_s$ then ${}^\sigma W_\lambda(g) = W_{\lambda[s]}(g)$.*

The proof of Proposition 10 is based on the integral formula (3) and the following result:

Lemma 11. *Let A be a totally disconnected topological group, $L \subseteq K$ an extension of fields, and μ a L-rational Haar measure on A. If σ is an L-automorphism of K then, for all $\phi \in \mathcal{S}(A, K)$,*

$$\int_A \sigma(\phi)\, d\mu = \sigma\left(\int_A \phi\, d\mu\right).$$

Proof. Using the notation introduced in the proof of Lemma 7, if $\phi = \sum_{i=1}^{k} c_i \chi_{A_i}$ then

$$\sigma(\phi) = \sum_{i=1}^{k} \sigma(c_i)\chi_{A_i}.$$

Therefore, since $\mu(A_i) \in L$ is fixed by σ,

$$\begin{aligned}\int \sigma(\phi)\, d\mu &= \sum_{i=1}^{k} \sigma(c_i)\mu(A_i) = \sum_{i=1}^{k} \sigma(c_i)\sigma(\mu(A_i)) \\ &= \sigma\left(\sum_{i=1}^{k} c_i \mu(A_i)\right) = \sigma\left(\int_A \phi\, d\mu\right). \qquad \square\end{aligned}$$

Proof of Proposition 10. Let $g \in \mathrm{Sp}(V)$, $\phi \in \mathcal{S}(X, E)$, and $x \in X$. We assume g has the matrix representation (2). Lemma 6 asserts that the measure $\mu_{\lambda,g}$ is $\mathbb{Q}(\sqrt{q})$-rational. Applying Lemma 11 to the case $A = Y_g$, $L = \mathbb{Q}(\sqrt{q})$, $K = E$, and $\mu = \mu_{\lambda,g}$, the definition of ${}^{\sigma}W_\lambda$, the formula (3), and Lemma 9 yield

$$\begin{aligned}&\left[{}^{\sigma}W_\lambda(g)\phi\right](x) \\ &\quad= \sigma\left[W_\lambda(g)(\sigma^{-1}\phi)(x)\right] \\ &\quad= \sigma\left[\int_{Y_g} \lambda\left(\frac{\langle xa, xb\rangle - 2\langle xb, yc\rangle + \langle yc, yd\rangle}{2}\right)(\sigma^{-1}\phi)(xa+yc)\, d\mu_{\lambda,g}y\right] \\ &\quad= \int_{Y_g} {}^{\sigma}\lambda\left(\frac{\langle xa, xb\rangle - 2\langle xb, yc\rangle + \langle yc, yd\rangle}{2}\right)\phi(xa+yc)\, d\mu_{\lambda,g}y \\ &\quad= \int_{Y_g} \lambda[s]\left(\frac{\langle xa, xb\rangle - 2\langle xb, yc\rangle + \langle yc, yd\rangle}{2}\right)\phi(xa+yc)\, d\mu_{\lambda,g}y.\end{aligned}$$

Observing $s \in \mathcal{P}^* \subseteq \mathcal{O}^*$, Lemma 5 implies that $\mu_{\lambda[s],g} = \mu_{\lambda,g}$. The preceding calculation thus gives

$$\begin{aligned}&\left[{}^{\sigma}W_\lambda(g)\phi\right](x) \\ &\qquad= \int_{Y_g}\left[\lambda[s]\left(\frac{\langle xa, xb\rangle - 2\langle xb, yc\rangle + \langle yc, yd\rangle}{2}\right)\phi(xa+yc)\right] d\mu_{\lambda[s],g}y \\ &\qquad= \left[W_{\lambda[s]}(g)\phi\right](x). \qquad \square\end{aligned}$$

5. Action of symplectic similitudes

In the previous section, we described the action of Galois on the projective representations W_λ. Here, we discuss an action of the group of symplectic similitudes on the Weil representations.

Given $s \in F^*$, let f_s be the element of $\mathrm{GL}(V)$ defined by

$$(x+y)f_s = x+sy, \quad x \in X,\ y \in Y.$$

Conjugation by f_s leaves the symplectic group $\mathrm{Sp}(V)$ invariant. In fact, if $g \in \mathrm{Sp}(V)$ is expressed in the matrix form (2) then

$$g^{f_s} = \begin{pmatrix} a & sb \\ s^{-1}c & d \end{pmatrix}. \tag{8}$$

In particular, we note that the spaces Y_g and $Y_{g^{f_s}}$ are equal, since $\ker c = \ker s^{-1}c$.

Lemma 12. *If $s \in F^*$ then $\mu_{\lambda,g^{f_s}} = |s|_{Y_g}^{-1/2}\mu_{\lambda,g}$.*

Proof. Let $p_{i,s}$, $0 \le i \le n$, be the elements of $\mathrm{Sp}(V)$ defined by

$$x_j p_{i,s} = \begin{cases} s^{-1}x_j & \text{if } j \le i, \\ x_j & \text{if } i < j, \end{cases}$$
$$y_j p_{i,s} = \begin{cases} sy_j & \text{if } j \le i, \\ y_j & \text{if } i < j. \end{cases}$$

Note that $p_{i,s} \in P$ and

$$\det(p_{i,s}|_Y) = s^i.$$

Moreover, one readily verifies that

$$\tau_i^{f_s} = \tau_i p_{i,s}.$$

Let $g \in G$. If $g = p_1\tau_i p_2$, p_1, $p_2 \in P$, then

$$g^{f_s} = (p_1\tau_i p_2)^{f_s} = p_1^{f_s}\tau_i^{f_s}p_2^{f_s} = p_1^{f_s}\tau_i(p_{i,s}p_2^{f_s}).$$

Observing that both $p_1^{f_s}$ and $p_{i,s}p_2^{f_s}$ belong to P, (5) yields

$$\mu_{\lambda,g^{f_s}} = |\det(p_1^{f_s}p_{i,s}p_2^{f_s}|_Y)|^{-1/2}\,\overline{p_1^{f_s}}\cdot\mu_{\lambda,\tau_i}.$$

Using (8), if $p \in P$ then $p^{f_s}|_Y = p|_Y$. As a consequence,

$$\overline{p_1^{f_s}} = \overline{p}_1 : Y_g \to Y_{\tau_i}.$$

In light of these observations,

$$\det(p_1^{f_s}p_{i,s}p_2^{f_s}|_Y) = \det(p_1p_{i,s}p_2|_Y) = \det(p_{i,s}|_Y)\cdot\det(p_1p_2|_Y) = s^i\det(p_1p_2|_Y);$$

hence

$$\mu_{\lambda,g^{f_s}} = |s^i \det(p_1 p_2|_Y)|^{-1/2}\overline{p}_1 \cdot \mu_{\lambda,\tau_i} = |s|^{-i/2}\mu_{\lambda,g} = |s|_{Y_g}^{-1/2}\mu_{\lambda,g},$$

since Y_g has dimension i over F. □

Let $W_\lambda^{f_s}$ be the projective representation of $\mathrm{Sp}(V)$ defined by

$$W_\lambda^{f_s}(g) = W_\lambda\left(g^{f_s}\right).$$

For the proof of the next result, let $|\alpha|_V$ denote the module of an automorphism α of an F-vector space V [Weil 1974, I.2]. We have

$$|\alpha|_V = |\det \alpha|.$$

In particular, the module of left multiplication by $s \in F^*$ on V satisfies

$$|s|_V = |s|^{\dim V}.$$

Proposition 13. *If $s \in F^*$ then $W_\lambda^{f_s} = W_{\lambda[s]}$.*

Proof. Let $g \in \mathrm{Sp}(V)$. We assume that g has the matrix representation (2), hence that of g^{f_s} is given by (8). If $\phi \in \mathcal{S}(X)$ and $x \in X$ then the integral formula (3) and Lemma 12 yield

$$\begin{aligned}
&\left[W_\lambda(g^{f_s})\phi\right](x)\\
&= \int_{Y_{g^{f_s}}} \lambda\left(\frac{\langle xa, sxb\rangle - 2\langle sxb, s^{-1}yc\rangle + \langle s^{-1}yc, yd\rangle}{2}\right)\phi(xa+s^{-1}yc)\, d\mu_{\lambda,g^{f_s}}\, y\\
&= |s|_{Y_g}^{-1/2}\int_{Y_g} \lambda\left(\frac{\langle xa, sxb\rangle - 2\langle sxb, s^{-1}yc\rangle + \langle s^{-1}yc, yd\rangle}{2}\right)\phi(xa+s^{-1}yc)\, d\mu_{\lambda,g}\, y.
\end{aligned}$$

Replacing y by sy, the definition of $|s|_{Y_g}$ and Lemma 5 yield

$$\begin{aligned}
&\left[W_\lambda(g^{f_s})\phi\right](x)\\
&\quad= |s|_{Y_g}^{-1/2}|s|_{Y_g}\int_{Y_g} \lambda\left(\frac{\langle xa, sxb\rangle - 2\langle sxb, yc\rangle + \langle yc, syd\rangle}{2}\right)\phi(xa+yc)\, d\mu_{\lambda g}\, y\\
&\quad= |s|_{Y_g}^{1/2}\int_{Y_g} \lambda\left(s\cdot\frac{\langle xa, xb\rangle - 2\langle xb, yc\rangle + \langle yc, yd\rangle}{2}\right)\phi(xa+yc)\, d\mu_{\lambda,g}\, y\\
&\quad= |s|_{Y_g}^{1/2}\int_{Y_g} \lambda[s]\left(\frac{\langle xa, xb\rangle - 2\langle xb, yc\rangle + \langle yc, yd\rangle}{2}\right)\phi(xa+yc)\, d\mu_{\lambda,g}\, y\\
&\quad= \int_{Y_g} \lambda[s]\left(\frac{\langle xa, xb\rangle - 2\langle xb, yc\rangle + \langle yc, yd\rangle}{2}\right)\phi(xa+yc)\, d\mu_{\lambda[s],g}\, y\\
&\quad= [W_{\lambda[s]}(g)\phi](x).
\end{aligned}$$

This completes the proof of the proposition. □

6. The fundamental identity

Let

$$\mathfrak{G} = \{\sigma \in \mathrm{Gal}(E/\mathbb{Q}(\sqrt{q})) : \exists s \in \mathbb{O}^* \text{ such that } \sigma|_{\mathbb{Q}(\lambda)} = \sigma_{s^2}\}.$$

Note that $\mathfrak{G}$ is a subgroup of $\mathrm{Gal}(E/\mathbb{Q}(\sqrt{q}))$. Given $s \in F^*$, let $g_s \in \mathrm{Sp}(V)$ be the map defined by

$$(x+y)g_s = s^{-1}x + sy, \quad x \in X, y \in Y.$$

We observe that g_s lies in the parabolic subgroup P that leaves Y invariant and is related to the operator f_{s^2} defined earlier by the identity

$$f_{s^2} = sI \circ g_s.$$

Proposition 14. *Let $\sigma \in \mathfrak{G}$ and $g \in \mathrm{Sp}(V)$. If $\sigma|_{\mathbb{Q}(\lambda)} = \sigma_{s^2}$, $s \in \mathbb{O}^*$, then*

$${}^{\sigma}W_\lambda(g) = W_\lambda(g_s)^{-1} W_\lambda(g) W_\lambda(g_s).$$

Proof. In light of Propositions 10 and 13,

$${}^{\sigma}W_\lambda(g) = W_{\lambda[s^2]}(g) = W_\lambda^{f_{s^2}}(g) = W_\lambda(g^{f_{s^2}}) = W_\lambda(g^{g_s}).$$

Applying Theorem 4(i) with $p_1^{-1} = p_2 = g_s$,

$$W_\lambda(g^{g_s}) = W_\lambda(g_s^{-1}) W_\lambda(g) W_\lambda(g_s) = W_\lambda(g_s)^{-1} W_\lambda(g) W_\lambda(g_s).$$

This completes the proof of the proposition. □

Corollary. *If $t \in F^*$ and $\sigma \in \mathfrak{G}$ then ${}^{\sigma}W_\lambda(g_t) = W_\lambda(g_t)$.*

Proof. Fix $s \in \mathbb{O}^*$ such that $\sigma|_{\mathbb{Q}(\lambda)} = \sigma_{s^2}$. Observing that g_s and g_t are commuting elements of P, the preceding proposition combines with Theorem 4(i) to yield

$${}^{\sigma}W_\lambda(g_t) = W_\lambda(g_s)^{-1} W_\lambda(g_t) W_\lambda(g_s) = W_\lambda(g_s^{-1} g_t g_s) = W_\lambda(g_t),$$

as required. □

7. The cocycle

Our aim in this section is the construction of a 1-cocycle δ on

$$\mathfrak{H} = \mathrm{Gal}(E/E_0)$$

with values in $\mathrm{GL}(\mathcal{S}(X, E))$ satisfying the identity (I):

$${}^{\sigma}W_\lambda(g) = \delta(\sigma)^{-1} W_\lambda(g) \delta(\sigma), \quad g \in \mathrm{Sp}(V), \ \sigma \in \mathfrak{H}.$$

When combined with restriction to $\mathbb{Q}(\lambda)$, the canonical isomorphism of Lemma 2 yields

$$\mathfrak{H} \simeq \mathrm{Gal}\,(\mathbb{Q}(\lambda)/E_0 \cap \mathbb{Q}(\lambda)) \simeq (F^*)^2 \cap \mathcal{P}^*. \tag{9}$$

Let

$$o = \begin{cases} 2(p-1) & \text{if } q \text{ is a square,} \\ p-1 & \text{if } q \text{ is not a square,} \end{cases}$$

and fix a primitive o-th root of unity $\epsilon \in F^*$. Furthermore, let

$$U_1 = \begin{cases} \{1\} & \text{if } \operatorname{char} F = p, \\ \{r \in \mathcal{P} : r \equiv 1 \bmod p\} & \text{if } \operatorname{char} F = 0. \end{cases}$$

Since p is odd, the map $r \mapsto r^2$ is an automorphism of the pro-p group U_1. This allows us to conclude that

$$(F^*)^2 \cap \mathcal{P}^* = \langle \epsilon^2 \rangle \times U_1.$$

The isomorphism (9) identifies U_1 with $\operatorname{Gal}\big(E/\mathbb{Q}(\nu_p, \sqrt{-1})\big)$, where ν_p is the group of complex p-th roots of unity. This in turn leads to an identification of $\langle \epsilon^2 \rangle$ with

$$\mathfrak{H}/\operatorname{Gal}\big(E/\mathbb{Q}(\nu_p, \sqrt{-1})\big) \simeq \operatorname{Gal}\big(\mathbb{Q}(\nu_p, \sqrt{-1})/E_0\big).$$

In particular, the element η of $\mathfrak{H}$ characterized by

$$\eta|_{\mathbb{Q}(\lambda)} = \sigma_{\epsilon^2} \tag{10}$$

has order $o/2$ and restricts to a generator of $\operatorname{Gal}\big(\mathbb{Q}(\nu_p, \sqrt{-1})/E_0\big)$.

Given $\sigma \in \mathfrak{H}$, there is a unique integer i, $1 \le i \le o/2$, and a unique element $s \in U_1$, such that

$$\sigma|_{\mathbb{Q}(\lambda)} = \sigma_{\epsilon^{2i} s^2}.$$

If τ is a second element of $\mathfrak{H}$, say

$$\tau|_{\mathbb{Q}(\lambda)} = \sigma_{\epsilon^{2j} t^2}, \quad 1 \le j \le o/2, \quad t \in U_1,$$

then

$$\sigma\tau|_{\mathbb{Q}(\lambda)} = \sigma_{\epsilon^{2k} (st)^2},$$

where $st \in U_1$ and

$$k = \begin{cases} i+j & \text{if } i+j \le o/2, \\ i+j-o/2 & \text{if } i+j > o/2. \end{cases}$$

Our initial attempt at the construction of the cocycle is to define

$$D(\sigma) = W_\lambda(g_{\epsilon^i s}), \quad \sigma|_{\mathbb{Q}(\lambda)} = \sigma_{\epsilon^{2i} s^2}, \quad 1 \le i \le o/2, \quad s \in U_1.$$

Proposition 14 ensures that

$$^{\sigma}W_\lambda(g) = D(\sigma)^{-1} W_\lambda(g) D(\sigma), \quad g \in \operatorname{Sp}(V), \sigma \in \mathfrak{H}. \tag{11}$$

Assuming σ and τ are as above, the definition of D yields

$$D(\sigma\tau) = W_\lambda(g_{\epsilon^k st}).$$

On the other hand, the Corollary to Proposition 14 gives

$${}^\sigma D(\tau) = {}^\sigma W_\lambda(g_{\epsilon^j t}) = W_\lambda(g_{\epsilon^j t}),$$

hence Theorem 4(i) yields

$$D(\sigma)^\sigma D(\tau) = W_\lambda(g_{\epsilon^i s}) W_\lambda(g_{\epsilon^j t}) = W_\lambda(g_{\epsilon^{i+j} st}).$$

If $i + j \le o/2$ then

$$W_\lambda(g_{\epsilon^{i+j} st}) = W_\lambda(g_{\epsilon^k st}).$$

If $i + j > o/2$ then, observing $\epsilon^{o/2} = -1$, Theorem 4(i) yields

$$W_\lambda(g_{\epsilon^{i+j} st}) = W_\lambda(g_{-\epsilon^k st}) = W_\lambda(\iota) W_\lambda(g_{\epsilon^k st}),$$

where $\iota = g_{-1}$ is the central involution of $\mathrm{Sp}(V)$ that maps $v \in V$ to $-v$. In summary,

$$D(\sigma)^\sigma D(\tau) = \begin{cases} D(\sigma\tau) & \text{if } i + j \le o/2, \\ W_\lambda(\iota) D(\sigma\tau) & \text{if } i + j > o/2. \end{cases} \tag{12}$$

In particular, D is not a 1-cocycle; to get one we must account for the factor $W_\lambda(\iota)$.

Since $\iota \in P$, Theorem 4(ii) implies that if ϕ belongs to $\mathcal{S}(X, E)$ then

$$[W_\lambda(\iota)\phi](x) = \phi(-x), \quad x \in X.$$

In particular, $W_\lambda(\iota)$ is an involution, hence the operators

$$\rho_e = \frac{1}{2}(I + W_\lambda(\iota)) \quad \text{and} \quad \rho_o = \frac{1}{2}(I - W_\lambda(\iota))$$

are orthogonal idempotents. Furthermore, recalling $\iota = g_{-1}$, the Corollary to Proposition 14 shows that both ρ_e and ρ_o are fixed by the action of Galois. Finally, since $I = \rho_e + \rho_o$, it is easily verified that the operators

$$\rho_e + c\rho_o, \quad c \in E, c \ne 0,$$

are invertible.

Lemma 15. *The norm equation*

$$N(u) = -1, \quad N : \mathbb{Q}(\nu_p, \sqrt{-1}) \to E_0$$

has a solution.

Proof. The case $p \equiv 1 \bmod 4$ is covered by [Cliff et al. 2004, Lemma 24], an application of the Hasse Norm Theorem. Suppose $p \equiv 3 \bmod 4$. If q is not a square then the extension $\mathbb{Q}(\nu_p, \sqrt{-1})/E_0$ has odd degree $(p-1)/2$, hence -1 is a solution of the norm equation. If q is square then the extension has degree $p-1 \equiv 2 \bmod 4$. In this case, $\sqrt{-1} \in E_0$ is a solution. $\square$

Let u be a solution of the norm equation of the preceding lemma. Given $\sigma \in \mathfrak{H}$, set

$$A(\sigma) = \rho_e + \left(\prod_{l=0}^{i-1} \eta^l(u)\right)\rho_0, \quad \text{where } \sigma|_{\mathbb{Q}(\lambda)} = \sigma_{\epsilon^{2i}s^2}, \quad 1 \le i \le o/2, \quad s \in U_1$$

where η satisfies (10). The remarks preceding Lemma 15 ensure that $A(\sigma) \in \mathrm{GL}(\mathcal{S}(X, E))$. With the notation introduced earlier, if σ and τ belong to $\mathfrak{H}$ then

$$A(\sigma\tau) = \rho_e + \left(\prod_{l=0}^{k-1} \eta^l(u)\right)\rho_0.$$

On the other hand, observing

$$\sigma\eta^{-i}|_{\mathbb{Q}(\lambda)} = \sigma_{\epsilon^{2i}s^2}\sigma_{\epsilon^2}^{-i} = \sigma_{\epsilon^{2i}s^2}\sigma_{\epsilon^{-2i}} = \sigma_{s^2},$$

the fact (9) identifies U_1 with $\mathrm{Gal}\big(E/\mathbb{Q}(\nu_p, \sqrt{-1})\big)$ allows us to deduce that the restrictions of σ and η^i to $\mathbb{Q}(\nu_p, \sqrt{-1})$ coincide. Therefore,

$$\begin{aligned} {}^{\sigma}A(\tau) &= {}^{\sigma}\left[\rho_e + \left(\prod_{l=0}^{j-1} \eta^l(u)\right)\rho_0\right] = \rho_e + {}^{\sigma}\left(\prod_{l=0}^{j-1} \eta^l(u)\right)\rho_0 \\ &= \rho_e + {}^{\eta^i}\left(\prod_{l=0}^{j-1} \eta^l(u)\right)\rho_0 = \rho_e + \left(\prod_{l=i}^{i+j-1} \eta^l(u)\right)\rho_0; \end{aligned}$$

hence

$$\begin{aligned} A(\sigma)\,{}^{\sigma}A(\tau) &= \left[\rho_e + \left(\prod_{l=0}^{i-1} \eta^l(u)\right)\rho_0\right]\left[\rho_e + \left(\prod_{l=i}^{i+j-1} \eta^l(u)\right)\rho_0\right] \\ &= \left[\rho_e + \left(\prod_{l=0}^{i+j-1} \eta^l(u)\right)\rho_0\right]. \end{aligned}$$

If $i + j \le o/2$ then

$$\prod_{l=0}^{i+j-1} \eta^l(u) = \prod_{l=0}^{k-1} \eta^l(u),$$

hence

$$A(\sigma)^{\sigma}A(\tau) = A(\sigma\tau).$$

If $i + j > o/2$ then the choice of η and u yield

$$\prod_{l=0}^{i+j-1} \eta^l(u) = \left(\prod_{l=0}^{(o-2)/2} \eta^l(u)\right)\left(\prod_{l=o/2}^{i+j-1} \eta^l(u)\right) = N(u)\prod_{l=0}^{k-1} \eta^l(u) = -\prod_{l=0}^{k-1} \eta^l(u).$$

Observing that $\rho_e = \rho_e W_\lambda(\iota)$ and $-\rho_o = \rho_o W_\lambda(\iota)$,

$$A(\sigma)^{\sigma}A(\tau) = \rho_e - \left(\prod_{l=0}^{k-1} \eta^l(u)\right)\rho_0 = \left[\rho_e + \left(\prod_{l=0}^{k-1} \eta^l(u)\right)\rho_0\right]W_\lambda(\iota) = A(\sigma\tau)W_\lambda(\iota).$$

In summary,

$$A(\sigma)^{\sigma}A(\tau) = \begin{cases} A(\sigma\tau) & \text{if } i + j \le o/2, \\ A(\sigma\tau)W_\lambda(\iota) & \text{if } i + j > o/2. \end{cases} \tag{13}$$

Consider the map $\delta : \mathfrak{H} \to \mathrm{GL}(\mathcal{S}(X, E))$ given by

$$\delta(\sigma) = A(\sigma)D(\sigma).$$

If $\sigma, \tau \in \mathfrak{H}$ are as above

$$\delta(\sigma)^{\sigma}\delta(\tau) = (A(\sigma)D(\sigma))^{\sigma}(A(\tau)D(\tau)) = A(\sigma)D(\sigma)^{\sigma}A(\tau)^{\sigma}D(\tau).$$

By Theorem 4(ii), ${}^{\sigma}A(\tau) \in E[W_\lambda(i)]$ commutes with $D(\sigma) = W_\lambda(g_{\epsilon^i s})$, hence

$$A(\sigma)D(\sigma)^{\sigma}A(\tau)^{\sigma}D(\tau) = A(\sigma)^{\sigma}A(\tau)D(\sigma)^{\sigma}D(\tau).$$

If $i + j > o/2$ then (12) and (13) yield

$$A(\sigma)^{\sigma}A(\tau)D(\sigma)^{\sigma}D(\tau) = A(\sigma\tau)W_\lambda(\iota)W_\lambda(\iota)D(\sigma\tau) = A(\sigma\tau)D(\sigma\tau).$$

Since this is trivially true if $i + j \le o/2$, we conclude

$$\delta(\sigma)^{\sigma}\delta(\tau) = A(\sigma\tau)D(\sigma\tau) = \delta(\sigma\tau).$$

This shows that δ is a 1-cocycle. Furthermore, if $g \in \mathrm{Sp}(V)$ then Theorem 4(i) shows that $A(\sigma) \in E[W_\lambda(\iota)]$ commutes with $W_\lambda(g)$, hence (11) yields

$$\begin{aligned}\delta(\sigma)^{-1}W_\lambda(g)\delta(\sigma) &= (A(\sigma)D(\sigma))^{-1}W_\lambda(g)A(\sigma)D(\sigma) \\ &= D(\sigma)^{-1}A(\sigma)^{-1}W_\lambda(g)A(\sigma)D(\sigma) \\ &= D(\sigma)^{-1}W_\lambda(g)D(\sigma) \\ &= {}^{\sigma}W_\lambda(g),\end{aligned}$$

which verifies that (I) is satisfied.

8. The triviality of the cocycle

Let $\delta : \mathfrak{H} \to \mathrm{GL}(\mathscr{S}(X, E))$ be the 1-cocycle satisfying (I) constructed above.

Lemma 16. *If $\phi \in \mathscr{S}(X, E)$ then there exists an open subgroup $\mathfrak{K}$ of $\mathfrak{H}$ such that*

$$\delta(\sigma)\phi = \phi, \quad \sigma \in \mathfrak{K}.$$

Proof. If char $F = p$ then $\mathfrak{H}$ is a finite discrete group, so one may take $\mathfrak{K}$ to be the trivial subgroup.

Assume char $F = 0$. If $\mathfrak{X}$ is a lattice in X then the subgroups

$$p^k \mathfrak{X}, \quad k \in \mathbb{Z},$$

form a local base at the origin. Therefore, given $x \in X$, there exist $i_x \in \mathbb{Z}$ such that ϕ is constant on the coset $x + p^{i_x}\mathfrak{X}$. As the family $\left\{x + p^{i_x}\mathfrak{X} : x \in X\right\}$ is an open cover of X, there exists $x_1, \ldots, x_m$ in X such that

$$\operatorname{supp}\phi \subseteq \bigcup_{j=1}^{m} x_j + p^{i_{x_j}}\mathfrak{X}.$$

Set

$$i = \max\left\{i_{x_1}, \ldots, i_{x_m}\right\}$$

and consider $x + p^i\mathfrak{X} \cap \operatorname{supp}\phi$, $x \in X$. If it is empty then the restriction of ϕ to the coset $x + p^i\mathfrak{X}$ is identically 0. If not, there exists j such that $x + p^i\mathfrak{X} \cap x_j + p^{i_{x_j}}\mathfrak{X}$ is nonempty, hence

$$x + p^i\mathfrak{X} \subseteq x_j + p^{i_{x_j}}\mathfrak{X}$$

by choice of i. The choice of i_{x_j} thus ensures that the restriction of ϕ to $x + p^i\mathfrak{X}$ is the constant function with value $\phi(x_j)$. We conclude that ϕ is constant on the $p^i\mathfrak{X}$-cosets of X.

Let $\sigma \in \mathfrak{H}$. If $\sigma|_{\mathbb{Q}(\lambda)} = \sigma_{r^2}$, $r \in U_1$, then by construction $\delta(\sigma) = W_\lambda(g_r)$. Observing

$$g_r = \begin{pmatrix} r^{-1} \cdot 1_X & 0 \\ 0 & r \cdot 1_Y \end{pmatrix} \in P,$$

if $x \in X$ then Theorem 4(i) yields

$$(\delta(\sigma)\phi)(x) = (W_\lambda(g_r)\phi)(x) = |r|^{-\dim X/2}\lambda\left(\frac{\langle r^{-1}x, rx\rangle}{2}\right)\phi(r^{-1}x) = \phi(r^{-1}x),$$

since r is a unit and $\langle\ ,\ \rangle$ is F-bilinear and alternating. Fix $j \in \mathbb{Z}$ such that $i > j$ and

$$\operatorname{supp}\phi \subseteq p^j\mathfrak{X}.$$

If $x \notin p^j\mathfrak{X}$ then neither is $r^{-1}x$, so the choice of j ensures that

$$(\delta(\sigma)\phi)(x) = \phi(r^{-1}x) = 0 = \phi(x).$$

On the other hand, suppose $x \in p^j \mathfrak{X}$. In this case, if $r \equiv 1 \mod p^{i-j}$ then

$$r^{-1}x + p^i\mathfrak{X} = x + p^{i-j}p^j\mathfrak{X} + p^i\mathfrak{X} = x + p^i\mathfrak{X},$$

hence the choice of i ensures that

$$(\delta(\sigma)\phi)(x) = \phi(r^{-1}x) = \phi(x).$$

In light of the preceding discussion,

$$\mathfrak{K} = \left\{\sigma \in \mathfrak{H} : \sigma|_{\mathbb{Q}(\lambda)} = \sigma_{r^2}, r \equiv 1 \mod p^{i-j}\right\} = \mathrm{Gal}\big(E/\mathbb{Q}\big(\nu_{p^{i-j}}, \sqrt{-1}\big)\big)$$

has the required properties. □

Let K/k be a Galois extension and M a K-vector space equipped with an semilinear action of the Galois group $\mathrm{Gal}(K/k)$: if $\sigma \in \mathrm{Gal}(K/k)$, $m \in M$ and $e \in K$ then

$$\sigma(em) = \sigma(e)\sigma(m).$$

For such an action, the fixed-point set

$$M^{\mathrm{Gal}(K/k)} = \{m \in M : m = \sigma(m) \text{ for all } \sigma \in \mathrm{Gal}(K/k)\}$$

is a k-vector space. The canonical action of $\mathrm{Gal}(K/k)$ on K yields a semilinear action on the tensor product $K \otimes_k M^{\mathrm{Gal}(K/k)}$:

$$\sigma(e \otimes m) = \sigma(e) \otimes m, \quad \sigma \in \mathrm{Gal}(K/k),\ e \in E,\ m \in M^{\mathrm{Gal}(K/k)}.$$

The action of Galois on M is said to be *smooth* if the stabilizer of each $m \in M$ is open in $\mathrm{Gal}(K/k)$.

Proposition 17. *[Galois Descent] If M is a K-vector space equipped with a semilinear, smooth action of* $\mathrm{Gal}(K/k)$ *then the canonical map*

$$\psi : K \otimes_k M_k \to M$$

is a K-linear isomorphism of $\mathrm{Gal}(K/k)$*-modules.*

Proof. The case $K = k_s$, the separable closure of k, is proved in [Borel 1991, AG.14.2]. The general case is proved using the same argument, *mutatis mutandis*. □

Proposition 18. *There exists* $\alpha \in \mathrm{GL}\,(\mathcal{S}(X, E))$ *such that*

$$\delta(\sigma) = \alpha^{-1\sigma}\alpha, \quad \sigma \in \mathfrak{H}. \tag{14}$$

Proof. The canonical action (7) of $\mathfrak{H}$ on $\mathcal{S}(X, E)$ is clearly semilinear. It is furthermore smooth, since each element of $\mathcal{S}(X, E)$ takes only finitely many values in E.

On the other hand, since δ is a 1-cocycle, then

$$(\sigma, \phi) \mapsto \delta(\sigma)\sigma(\phi), \quad \sigma \in \mathfrak{H}, \phi \in \mathscr{S}(X, E),$$

is also an action of $\mathfrak{H}$ on $\mathscr{S}(X, E)$, referred to as the twisted action by δ. It is semi-linear, since δ takes values in $\mathrm{GL}\,(\mathscr{S}(X, E))$. Since the original action is smooth, if $\phi \in \mathscr{S}(X, E)$ then there exists an open subgroup $\mathfrak{H}_1$ such that

$$\sigma(\phi) = \phi, \quad \sigma \in \mathfrak{H}_1.$$

Furthermore, Lemma 16 asserts that there is an open subgroup $\mathfrak{K}$ of $\mathfrak{H}$ such that

$$\delta(\sigma)\phi = \phi, \quad \sigma \in \mathfrak{K}.$$

Therefore, if $\sigma \in \mathfrak{H}_1 \cap \mathfrak{K}$ then

$$\delta(\sigma)\sigma(\phi) = \delta(\sigma)\phi = \phi.$$

This shows that the stabilizer of ϕ under the twisted action contains the open subgroup $\mathfrak{H}_1 \cap \mathfrak{K}$. Since it is the union of its $\mathfrak{H}_1 \cap \mathfrak{K}$-cosets, it follows that the stabilizer of ϕ under the twisted action is open. We conclude that the twisted action is smooth.

Using $\mathscr{S}(X, E)$ and ${}_\delta\mathscr{S}(X, E)$ to denote the $\mathfrak{H}$-modules defined by the natural and twisted actions, respectively, Galois Descent asserts the existence of E-linear, $\mathfrak{H}$-equivariant isomorphisms

$${}_\delta\mathscr{S}(X, E) \simeq E \otimes_{E_0} {}_\delta\mathscr{S}(X, E)^{\mathfrak{H}} \quad \text{and} \quad E \otimes_{E_0} \mathscr{S}(X, E)^{\mathfrak{H}} \simeq \mathscr{S}(X, E).$$

In particular,

$$\dim_{E_0} {}_\delta\mathscr{S}(X, E)^{\mathfrak{H}} = \dim_E \mathscr{S}(X, E) = \dim_{E_0} \mathscr{S}(X, E)^{\mathfrak{H}},$$

so ${}_\delta\mathscr{S}(X, E)^{\mathfrak{H}}$ and $\mathscr{S}(X, E)^{\mathfrak{H}}$ are E_0-isomorphic. As any such isomorphism extends by scalars to a E-linear, $\mathfrak{H}$-equivariant isomorphism

$$E \otimes_{E_0} {}_\delta\mathscr{S}(X, E)^{\mathfrak{H}} \simeq E \otimes_{E_0} \mathscr{S}(X, E)^{\mathfrak{H}},$$

we conclude that

$${}_\delta\mathscr{S}(X, E) \simeq \mathscr{S}(X, E).$$

Let $\alpha \in \mathrm{GL}\,(\mathscr{S}(X, E))$ be a $\mathfrak{H}$-equivariant isomorphism ${}_\delta\mathscr{S}(X, E) \to \mathscr{S}(X, E)$. If $\sigma \in \mathfrak{H}$ and $\phi \in \mathfrak{H}$ then the definition of the twisted action ensures that

$$\alpha\delta(\sigma)\sigma(\phi) = \sigma(\alpha\phi);$$

hence

$$\delta(\sigma)\phi = \alpha^{-1}\alpha\delta(\sigma)\sigma(\sigma^{-1}(\phi)) = \alpha^{-1}\sigma\big(\alpha(\sigma^{-1}(\phi))\big) = \alpha^{-1\sigma}\alpha(\phi). \qquad \square$$

9. Proof of the main theorem

Fix $\alpha \in \mathrm{GL}(\mathcal{S}(X, E))$ satisfying the conclusion of Proposition 18. In light of (9) and (14), if $\sigma \in \mathfrak{H}$ and $g \in \mathrm{Sp}(V)$ then

$$^{\sigma}\left(\alpha W_\lambda(g)\alpha^{-1}\right) = {}^{\sigma}\alpha^{\sigma} W_\lambda(g)({}^{\sigma}\alpha)^{-1} = {}^{\sigma}\alpha\delta(\sigma)^{-1} W_\lambda(g)\delta(\sigma)({}^{\sigma}\alpha)^{-1} = \alpha W_\lambda(g)\alpha^{-1}.$$

The compatibility of the Galois actions (6) and (7) allows us to deduce that the operators

$$\alpha W_\lambda(g)\alpha^{-1}, \quad g \in \mathrm{Sp}(V),$$

leave

$$\mathcal{S}(X, E)^{\mathfrak{H}} = \mathcal{S}(X, E^{\mathfrak{H}}) = \mathcal{S}(X, E_0)$$

invariant, hence provide a projective Weil representation realized over E_0.

10. Optimality of the field E_0

It is natural to ask if the field E_0 is optimal in the sense that the Weil representation W_λ may not be realized over a proper subfield. To investigate this, fix a lattice $\mathcal{L}$ of V on which the symplectic form $\langle\ ,\ \rangle$ is nondegenerate and consider the K-types of the Weil representation W_λ obtained by restricting to the compact subgroup $\mathrm{Sp}(\mathcal{L})$ [Prasad 1998].

Given a natural number k, let Γ_k denote that normal subgroup of $\mathrm{Sp}(\mathcal{L})$ consisting of those elements g for which

$$vg \equiv v \bmod \mathfrak{m}^k\mathcal{L}, \quad v \in \mathcal{L},$$

and let Fix_k be the space of Γ_k-fixed points in the Weil representation. The nontrivial K-types of W_λ associated with $\mathrm{Sp}(\mathcal{L})$ can be realized as the ± 1-eigenspaces of ι, the central involution of $\mathrm{Sp}(V)$, acting on the quotients $\mathrm{Fix}_{2i+2}/\mathrm{Fix}_{2i}$, $i = 0, 1, \ldots$. Indeed, in light of Proposition 13 and the remarks preceding Proposition 14, it is sufficient to verify this when λ has level 0 and -1. The first case is an immediate consequence of the description of the K-types provided by [Prasad 1998, Theorem 2], while the second case follows from the analogous result for representations arising from characters of odd level. In particular, if W_λ can be realized over a field L then its K-types can also be realized over L.

The nontrivial K-types of W_λ can be shown to coincide with the irreducible representations $\mathrm{Top}^{\pm}$ studied in [Cliff et al. 2004]. If $q \equiv 1 \bmod 4$ then Top^{-} has Schur index 2, by Theorem 26 of that reference. Since Theorem 17 of the same work asserts that its character field is $\mathbb{Q}$ (respectively, $\mathbb{Q}(\sqrt{p})$) if q is square (respectively, not square), Top^{-} may not be realized over a proper subfield of E_0. The remarks made above allow us to conclude that E_0 is an optimal field for realizing W_λ.

In the case $q \equiv 3 \bmod 4$, the representations $\mathrm{Top}^{\pm}$ all have Schur index 1 and character fields $\mathbb{Q}(\sqrt{-p})$ [Cliff et al. 2004, Theorems 17 and 26]. As a result, the restriction of W_λ to the compact group $\mathrm{Sp}(\mathscr{L})$ can be realized over the subfield $\mathbb{Q}(\sqrt{-p})$ of E_0. The possibility of realizing the entire Weil representation over the field $\mathbb{Q}(\sqrt{-p})$ is left open.

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