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KEITH JONES

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Given an action $G \overset{\rho}{\curvearrowright} T$ by a finitely generated group on a locally finite tree, we view points of the visual boundary ∂T as directions in T and use ρ to lift this sense of direction to G. For each point $E \in \partial T$, this allows us to ask whether G is $(n-1)$-connected "in the direction of E." Then the invariant $\Sigma^n(\rho) \subseteq \partial T$ records the set of directions in which G is $(n-1)$-connected. We introduce a family of actions for which $\Sigma^1(\rho)$ can be calculated through analysis of certain quotient maps between trees. We show that for actions of this sort, under reasonable hypotheses, $\Sigma^1(\rho)$ consists of no more than a single point. By strengthening the hypotheses, we characterize precisely when a given end point lies in $\Sigma^n(\rho)$ for any n.

1. Introduction

Let G be a group having type F_n,[1] and let M be a proper CAT(0) metric space.[2] Let $\rho : G \to \mathrm{Isom}(M)$ be an action by isometries. Bieri and Geoghegan [2003a] introduce a collection of geometric Σ-*invariants*, $\Sigma^n(\rho)$, $n \geq 0$. These arise naturally from the study of the Bieri–Neumann–Strebel–Renz (BNSR) invariants $\Sigma^n(G)$, which can then be viewed as a special case. Σ-invariants give topological insight into ρ and algebraic information about G. In particular, if ρ has discrete orbits and G is finitely generated, then $\Sigma^1(\rho) = \partial M$ if and only if the point stabilizers under ρ are finitely generated; more generally, if G has type F_n, then $\Sigma^n(\rho) = \partial M$ if and only if the point stabilizers under ρ have type F_n.[3]

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[1] By definition, G has type F_n if and only if there exists a $K(G, 1)$-complex with finite n-skeleton. This is equivalent to saying that there is an n-dimensional $(n-1)$-connected CW-complex on which G acts freely and cocompactly by permuting cells. All groups have type F_0, while type F_1 is equivalent to finitely generated and type F_2 is equivalent to finitely presented [Geoghegan 2008, §7.2].

[2] A CAT(0) space is a geodesic metric space whose geodesic triangles are no fatter than the corresponding "comparison triangles" in the Euclidean plane, and a metric space is proper if every closed ball is compact [Bridson and Haefliger 1999, Chapter II.1].

[3] See [Bieri and Geoghegan 2003a, Theorem A and the Boundary Criterion]; the required condition "almost geodesically complete" is ensured by cocompactness [Ontaneda 2005, Theorem B].

The invariant $\Sigma^n(\rho)$ depends on a notion of *controlled connectivity*, which we describe briefly here.[4] The action ρ can be used to impose a sense of direction on G as follows. The space M has a CAT(0) boundary ∂M, which is in one-to-one correspondence with the collection of geodesic rays emanating from any particular point of M. Thus ∂M encompasses the set of directions in M in which one can go to infinity. For an end point $E \in \partial M$, there is a nested sequence of subsets of M (called horoballs about E). This nested sequence provides a filtration of M. Because G has type F_n, there is an n-dimensional $(n-1)$-connected CW-complex X on which G acts freely and cocompactly by permuting cells. One can then choose a G-equivariant "control" map $h : X \to M$. With $E \in \partial M$ fixed, h allows us to lift the sense of direction from M up to X (and therefore G by proxy) by taking the preimages of horoballs about E. If, roughly speaking, the preimages of the horoballs about E are $(n-1)$-connected, the action ρ is said to be *controlled $(n-1)$-connected* or CC^{n-1} over E.[5] The precise definition ensures that this is independent of choice of X or h, and is in fact a property of ρ [Bieri and Geoghegan 2003a, §3.2].

For $n \geq 0$, the invariant $\Sigma^n(\rho)$ consists of all those end points over which ρ is CC^{n-1}. These form a nested family

$$\Sigma^0(\rho) \supseteq \Sigma^1(\rho) \supseteq \Sigma^2(\rho) \dots .$$

The action ρ induces a topological action by G on ∂M, under which $\Sigma^n(\rho)$ is invariant. Those familiar with the BNSR invariant $\Sigma^n(G)$ may recall that the BNSR invariant is an open subset of the boundary, which in the original case is a sphere. The Bieri–Geoghegan invariant $\Sigma^n(\rho)$ is in general not open in ∂M.

Bieri and Geoghegan calculate Σ^n for the modular group acting on the hyperbolic plane [2003b], and provide information about Σ^n for actions on trees by metabelian groups of finite Prüfer rank [2003a, Chapter 10, Example C]. Rehn [2007] provides calculations for the natural action by $\mathrm{SL}_n(\mathbb{Z}[1/k])$ on the symmetric space for $\mathrm{SL}_n(\mathbb{R})$.

In the case where M is a locally finite simplicial tree, Bieri and Geoghegan [2003a] ask whether $\Sigma^1(\rho)$ is always either empty, a singleton, or the entire boundary of the tree. (The "entire boundary" case has been discussed above.) Lehnert [2009] gives an example for which this is not the case. However, here we illustrate that there does exist a class of actions for which Σ^n is either empty or a singleton.

Main result. All trees are assumed to be simplicial trees viewed as CAT(0) metric spaces, by giving each edge a length of 1. All actions under consideration are by simplicial automorphisms, and therefore are by isometries. Also, we assume

[4]The technical definition is provided in Section 2.

[5]For $n = 0$, we take (-1)-connected to mean nonempty.

that actions are without inversions — that is, an edge is stabilized if and only if it is fixed pointwise — since we can simply pass to the barycentric subdivision otherwise. Any tree exhibiting such an action by a group G is called a G-tree. We assume that all G-trees are infinite and that G is always finitely generated.

A group action on a tree is *minimal* if there exists no proper invariant subtree. A cocompact action on an infinite tree is minimal if and only if the tree has no leaves. We define a *morphism of trees* to mean a map between two trees that sends vertices to vertices and edges to edges and that preserves adjacency. All maps between G-trees are assumed to be G-equivariant morphisms of trees, and therefore continuous. The *star* of a vertex is the set of edges adjacent to that vertex, and a morphism is *locally surjective* or *locally injective* if, for each vertex of the domain tree, the corresponding map between stars is surjective or injective. See [Bass 1993] for further discussion. In the context of morphisms of trees (as opposed to graphs), local injectivity is equivalent to injectivity, and local surjectivity implies surjectivity. A tree is *locally finite* if the star of each vertex is finite; such trees are proper metric spaces.

Theorem 1.1 (Main Theorem). *Let G be a finitely generated group, T a locally finite tree, and $G \overset{\rho}{\curvearrowright} T$ a cocompact action by isometries. If there exists a minimal G-tree $\tilde{T}$ and a G-morphism $q : \tilde{T} \to T$ that is locally surjective, but not locally injective, then $\Sigma^1(\rho)$ consists of at most a single point of ∂T.*

We do not require $\tilde{T}$ to be locally finite, because it is irrelevant to us whether $\tilde{T}$ is proper as a metric space. Also, the map $q : \tilde{T} \to T$ does not generally extend to a map $\partial\tilde{T} \to \partial T$, because geodesic rays may be collapsed to finite paths by q.

As mentioned in the introduction, $\Sigma^1(\rho)$ is a G-invariant subset of ∂T. Hence, if the conditions of the Main Theorem apply and there does exist a point $E_0 \in \Sigma^1(\rho)$, then E_0 is necessarily fixed by ρ. In some cases, this allows us to easily determine that $\Sigma^1(\rho)$ is empty, as in the following examples.

Example 1.2. Let G be the group given by the presentation

$$G = \langle a, s, t \mid a^s = a^2, a^t = a^3 \rangle.$$

As is clear from this presentation, G can be realized as a fundamental group of a graph of groups, where the graph is a 2-rose (a single vertex with two loops). The Bass–Serre tree $\tilde{T}$ associated with this graph of groups decomposition is a regular 7-valent tree. Let N be the normal closure of a. Then N consists of all elements of G that stabilize a vertex in $\tilde{T}$. The quotient group G/N is free on two generators and acts freely on $T = N\backslash\tilde{T}$ with quotient a 2-rose of circles, so T is a regular 4-valent tree. Figure 1 demonstrates the collapsing on a neighborhood of a vertex in $\tilde{T}$. (One can take T to be the Cayley graph of G/N.) The natural quotient map

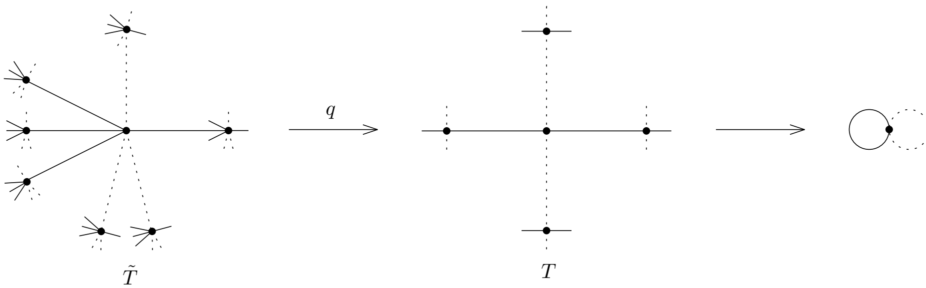

Figure 1. G admits a normal subgroup N, whose action on $\tilde{T}$ collapses $\tilde{T}$ to T.

$\tilde{T} \to T$ satisfies the conditions of the Main Theorem, and no end point $E \in \partial T$ is fixed by ρ. Hence $\Sigma^1(\rho) = \varnothing$.

This example can be generalized to any nonfree group with a graph of groups decomposition over a graph containing a single vertex. Such a group always has a free quotient obtained by collapsing the normal closure of the subgroup associated with the vertex, and as above, the Cayley graph of this free group can be viewed as the quotient of the original Bass–Serre tree.

Example 1.3. One of Lehnert's counterexamples to the question of whether Σ^1 must be either $\varnothing$, a singleton, or ∂T in the case of simplicial trees is closely related to the group G discussed in Example 1.2. Let $H = \mathbb{Z}[\frac{1}{6}] \rtimes F_2(x, y)$, where $F_2(x, y)$ is a free group generated by the letters x and y. One obtains H from G by adding relations corresponding to the commutator subgroup of N. The semidirect product structure is given by $t^x = t/2$ and $t^y = t/3$ for $t \in \mathbb{Z}[\frac{1}{6}]$. This group acts on the same tree T, by viewing it as the Cayley graph of its factor $F_2(x, y)$, and one can represent points in ∂T by infinite reduced words in $F_2(x, y)$. Any point represented by an infinite word eventually consisting of only x or only y does not lie in Σ^1 [Lehnert 2009]; this is a consequence of the interplay between the actions by $F_2(x, y)$ on $\mathbb{Z}[\frac{1}{6}]$ and on T. The author has a proof of this result in a paper currently in preparation, which is based on the topological construction of the Bass–Serre tree [Scott and Wall 1979; Geoghegan 2008, Chapter 6] and distinct in flavor from both the contents of this paper and the proof in [Lehnert 2009].

Evidently, for the action $H \curvearrowright T$, there exists no $\tilde{T}$ and $q : \tilde{T} \to T$ as described in the Main Theorem.

Example 1.4. Here is an example where $\tilde{T}$ is not locally finite. Let $K_4 = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ be the Klein 4-group, and let $D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$ be the infinite dihedral group. Take the quotient map $\pi : D_\infty * D_\infty \twoheadrightarrow K_4 * K_4$, induced by performing the abelianization map $D_\infty \twoheadrightarrow K_4$ on each free factor of $D_\infty * D_\infty$. There is an action $\tilde{\rho} : D_\infty * D_\infty \to \mathrm{Aut}(\tilde{T})$, where $\tilde{T}$ (a regular ∞-valent tree) is the Bass–Serre tree corresponding to the given free product decomposition. There is also an action $\rho : D_\infty * D_\infty \to \mathrm{Aut}(T)$, where T, a regular 4-valent tree, is the Bass–Serre tree for $K_4 * K_4$; this action factors through π. We can realize T as a quotient of $\tilde{T}$ satisfying the conditions of the Main Theorem. Again, because no point of ∂T is fixed by ρ, it follows that $\Sigma^1(\rho)$ is empty. This example is of a kind initially pointed out to the author by Mike Mihalik.

This, too, can be generalized: if A_1 and A_2 are two finitely generated infinite groups that admit finite quotients Q_1 and Q_2, respectively, then $G = A_1 * A_2$ admits a quotient map $\pi : G \to Q_1 * Q_2$. While G acts on the Bass–Serre tree $\tilde{T}$ corresponding to the decomposition $A_1 * A_2$, it also acts on $\ker \pi \backslash \tilde{T}$, which is isomorphic to the Bass–Serre tree corresponding to $Q_1 * Q_2$.

Example 1.5. More generally, there is a notion of a *morphism of graphs of groups* (essentially, a morphism of graphs together with a collection of homomorphisms of vertex and edge groups that ensure that certain squares commute), which lifts to an equivariant morphism between the corresponding Bass–Serre trees [Bass 1993, Proposition 2.4], and one can determine whether the lift will be locally surjective and not locally injective [Bass 1993, Corollary 2.5]. This can be used to produce maps satisfying the conditions of the Main Theorem. For example, consider the Baumslag–Solitar groups $\mathrm{BS}(m, n) = \langle a, t \mid ta^m t^{-1} = a^n \rangle$. There is a projection map $\mathrm{BS}(2, 4) \twoheadrightarrow \mathrm{BS}(1, 2)$ obtained by adding the relation $tat^{-1}a^{-2}$. One can show that this corresponds to a morphism of graphs of groups that lifts to a map between the corresponding Bass–Serre trees and has the desired properties.

Applying [Bieri and Geoghegan 2003a, Theorems A and H], we have:

Corollary 1.6. *If $G \overset{\rho}{\curvearrowright} T$ satisfies the conditions of the Main Theorem, then for any point $z \in T$, the stabilizer G_z of z under the action ρ is not finitely generated.*

Collapsing pairs. Recall that, in the language of [Serre 1980, Chapter I.2], each geometric edge of T corresponds to two oriented edges, one pointing in either direction.

Remark 1.7. We use the lowercase e to refer to edges of T, oriented or not, and the uppercase E to refer to points of ∂T.

Definition 1.8. Under the hypotheses of the Main Theorem, let $(\tilde{e}_1, \tilde{e}_2)$ be a pair of adjacent distinct oriented edges in $\tilde{T}$ with common initial vertex $\tilde{v}$. If $q(\tilde{e}_1) = q(\tilde{e}_2)$, we call this a *collapsing pair (of edges)* under q. Let $e = q(\tilde{e}_1)$ be the resulting oriented edge in T. For a vertex $w \in T$ (or end point $E \in \partial T$), we say the pair $(\tilde{e}_1, \tilde{e}_2)$ *faces* w (resp. E) if e points toward w (resp. E). This is the same as saying that the geodesic from $q(\tilde{v})$ to w (resp. E) passes through e.

The proof of the Main Theorem follows from two facts: Proposition 3.8 states that because q is not locally injective, all end points of T (with the possible exception of a single end point) are faced by a collapsing pair, while Proposition 3.4 states that local surjectivity of q forces any end point of T faced by a collapsing pair to lie outside $\Sigma^1(\rho)$.

The case where stabilizers on $\tilde{T}$ have type F_n. If we add the condition that the stabilizers under $\tilde{\rho}$ have type F_n, then we can prove that a point $E \in \partial T$ that is *not* faced by a collapsing pair lies in $\Sigma^n(\rho)$.

Theorem 1.9. *Assume the conditions of the Main Theorem. Also, suppose that G has type F_n and that for each point $\tilde{z}$ of $\tilde{T}$, the stabilizer $G_{\tilde{z}}$ has type F_n, for $n > 0$. Then $E \in \partial T$ lies in $\Sigma^n(\rho)$ if and only if there is no collapsing pair facing E.*

Corollary 1.10. *Let the group H have type F_n, and let $\varphi : H \to H$ be injective, so that $G = \langle H, t \mid a^t = \varphi(a)$ for all $a \in H\rangle$ is an ascending HNN-extension. If $\chi : G \twoheadrightarrow \mathbb{Z}$ maps $t \mapsto 1$ and $\langle\langle H\rangle\rangle \mapsto 0$, then χ represents a point in $\Sigma^n(G)$.*

This corollary is not new [Meinert 1996; 1997], but the approach is. For further discussion on this result, see [Bieri et al. 2010].

2. Controlled connectivity

In a CAT(0) space M, there is a notion of a *(visual) boundary* ∂M, which is obtained by taking equivalence classes of geodesic rays [Bridson and Haefliger 1999, Chapter II.8]. This boundary carries a topology, called the cone topology, induced by the topology on M. We call points of ∂M *end points*. CAT(0) spaces are contractible, and the boundary of a proper CAT(0) space is a compact space. Let τ be a geodesic ray in M. Following [Bieri and Geoghegan 2003a], we define the *Busemann function* $\beta_\tau : M \to \mathbb{R}$ by

$$\beta_\tau(p) = \lim_{t\to\infty}(t - d(\tau(t), p)).$$

For $r \in \mathbb{R}$, the set $HB_r(\tau) = {\beta_\tau}^{-1}([r, \infty))$ is called a *horoball* around E. Horoballs in CAT(0) spaces are contractible. We can view $HB_r(\tau)$ as the nested union of closed balls $\bigcup_{k\geq\max\{0,r\}} \overline{B_{k-r}(\tau(k))}$.

Definition 2.1. Fix $n \in \mathbb{N}$. Let G be a group having type F_n, and let M be a proper CAT(0) space admitting an isometric action $G \overset{\rho}{\curvearrowright} M$. Choose an n-dimensional $(n-1)$-connected CW-complex X^n on which X acts freely and cocompactly, and choose a continuous G-map $h : X^n \to M$. We call h a *control map*; one can be found because the action by G on X^n is free and M is contractible. Fix a geodesic ray τ representing $E \in \partial M$. For a horoball $HB_r(\tau)$ about E, denote the largest subcomplex of X^n contained in $h^{-1}(HB_r(\tau))$ by $X_{(\tau,r)}$. Finally, we need a notion of *lag function*: any $\lambda(r) > 0$ satisfying $r - \lambda(r) \to \infty$ as $r \to \infty$ is called a lag.

We say ρ is *controlled* $(n-1)$*-connected*, or CC^{n-1}, *over* E if for all $r \in \mathbb{R}$ and all $-1 \leq p \leq (n-1)$, there exists a lag λ such that every map $f : S^p \to X_{(\tau,r)}$ extends to a map $\tilde{f} : B^{p+1} \to X_{(\tau,r-\lambda(r))}$.[6]

Definition 2.2. The Bieri–Geoghegan invariant $\Sigma^n(\rho)$ is the subset of ∂M consisting of all end points over which ρ is controlled $(n-1)$-connected.

Relationship to the BNSR invariant. If ρ fixes an end point E, then the pair (ρ, E) determines a homomorphism $\chi_{\rho,E} : G \to \mathbb{R}$, and E lies in $\Sigma^1(\rho)$ if and only if $\chi_{\rho,E}$ represents a point in $\Sigma^1(G)$ [Bieri and Geoghegan 2003a, §10.6]. In fact, we can obtain the classical BNSR invariant $\Sigma^n(G)$ as the special case where ρ is the action $G \curvearrowright G_{ab} \otimes \mathbb{R}$ [Bieri and Geoghegan 2003a, Chapter 10, Example A]. This is an action by translations on a finite-dimensional real vector space, so every end point is fixed, and $\partial(G_{ab} \otimes \mathbb{R}) \cong \mathrm{Hom}(G, \mathbb{R})$.

The question of finding a single technique for calculating Σ^1 for arbitrary group actions on trees seems out of reach currently. To see this, consider an action $G \overset{\rho}{\curvearrowright} T$ by translations, where T is a simplicial line. This corresponds to a homomorphism $\chi : G \twoheadrightarrow \mathbb{Z}$, and calculating $\Sigma^1(\rho)$ determines whether χ and $-\chi$ represent points of $\Sigma^1(G)$. However, it is known that $\ker \chi$ is finitely generated if and only if both do represent points of $\Sigma^1(G)$ [Bieri et al. 1987, Theorem B1]. Thus a method for calculating $\Sigma^1(\rho)$ even in the special case where the tree is a simplicial line would enable us to determine whether or not the kernel of an arbitrary homomorphism to $\mathbb{Z}$ is finitely generated.

3. Proof of the Main Theorem

An automorphism s of a tree T having no fixed point is said to be *hyperbolic*. For each such s, there is a unique line A_s, called the *axis* of s, stable under the action of the subgroup $\langle s \rangle$, that acts on A_s by translations. If e is an oriented edge of T, then s is said to *act coherently* on e if e and se are consistently oriented (that is, if they point in the same direction — neither toward each other nor away from each other). For an automorphism s, if $e \neq se$, then s acts coherently on e if and only if s is hyperbolic and both e and se lie on the axis of s [Serre 1980, Proposition 25].

Lemma 3.1. *Let T be a cocompact G-tree, and let $E \in \partial T$. Then for any geodesic ray τ representing E, any $r \in \mathbb{R}$, and any oriented edge e of T oriented toward E, there exists an element of the G-orbit of e that is oriented toward E and does not lie in $HB_r(\tau)$.*

Proof. The ray of oriented edges beginning at e and representing E, with all edges pointing toward E, contains infinitely many edges. Because the action is

[6]By convention, $S^{-1} = \varnothing$, and (-1)-connected means "nonempty".

cocompact, the pigeon-hole principle ensures that there must be edges e_1 and e_2 from this ray in the same G-orbit. Hence, there is an $h \in G$ with $he_1 = e_2$. Because e_1 and e_2 are consistently oriented, h is hyperbolic. Let v_1 be the terminus of e_1 (the vertex of e_1 where β_τ is maximized). By choosing $k \in \mathbb{Z}$ such that $|k| > \beta_\tau(v_1) - r$ and h^k moves e_1 away from E, we ensure that $h^k e_1$ is oriented toward E and does not lie in $HB_r(\tau)$. Thus $h^k e$ is the edge we seek. □

Observation 3.2. For trees $\tilde{T}$ and T, let $q : \tilde{T} \to T$ be locally surjective. If $\tau = (e_0, e_1, \dots)$ is a geodesic edge ray in T and $\tilde{e}_0$ is an edge of $\tilde{T}$ satisfying $q(\tilde{e}_0) = e_0$, then there exists a lift $\tilde{\tau}$ of τ to $\tilde{T}$ having initial edge $\tilde{e}_0$ and that is also a geodesic edge ray.

Observation 3.3. Given a nonempty connected G-graph Γ and minimal G-tree T, any G-morphism $h : \Gamma \to T$ is surjective.

Proposition 3.4. *Let T be a cocompact G-tree, and let $\tilde{T}$ be a minimal G-tree. Suppose $q : \tilde{T} \to T$ is a G-morphism that is locally surjective. If $E \in \partial T$ is such that there exists a collapsing pair facing E, then E does not lie in $\Sigma^1(\rho)$.*

Proof. Let Γ be a free cocompact G-graph, and choose any G-morphism $h : \Gamma \to \tilde{T}$. Then the composition $q \circ h$ is a suitable control map for determining controlled connectivity over E.

Let $\tau : [0, \infty) \to T$ be a geodesic edge ray representing E. We show that for any lag $\lambda > 0$, there exist points in the subgraph $\Gamma_{(\tau,0)}$ that cannot be connected via a path in $\Gamma_{(\tau,-\lambda)}$.

By Lemma 3.1, we can choose a collapsing pair $(\tilde{e}_1, \tilde{e}_2)$ facing E but whose image in T does not lie in $HB_{-\lambda}(\tau)$. Let $\tilde{v}$ be the vertex shared by $\tilde{e}_1$ and $\tilde{e}_2$, and let v be its image in T. Let γ be the geodesic ray representing E and emanating from v. By Observation 3.2, there exist two distinct lifts $\tilde{\gamma}_i$ $(i = 1, 2)$ of γ to $\tilde{T}$, with $\tilde{\gamma}_i$ having initial edge $\tilde{e}_i$. Because γ and τ both represent E, they eventually merge, so that γ intersects $HB_r(\tau)$ nontrivially for all $r \in \mathbb{R}$. Hence, both $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ intersect $q^{-1}(HB_r(\tau))$ for all r.

By design, $\tilde{\gamma}_1 \cap \tilde{\gamma}_2 = \tilde{v}$, and $\tilde{\gamma}_1 \cup \tilde{\gamma}_2$ is a line. By Observation 3.3, h is onto, so that $\tilde{\gamma}_1 \cup \tilde{\gamma}_2$ lies in the image of h. Choose a vertex $\tilde{y}_i \in \tilde{\gamma}_i \cap q^{-1}(HB_0(\tau))$, and choose $x_i \in h^{-1}(\tilde{y}_i)$. Then both x_i lie in $\Gamma_{(\tau,0)}$, but any path through $\Gamma_{(\tau,-\lambda)}$ joining x_1 to x_2 would be mapped to a path in $q^{-1}(HB_{-\lambda}(\tau))$ joining $\tilde{y}_1$ to $\tilde{y}_2$. Since $\tilde{T}$ is a tree, no such path exists. □

Lemma 3.5. *Let T be a minimal G-tree and let $\mathscr{E}$ be a nonempty G-invariant set of oriented edges. Then there is no vertex v in T such that all edges of $\mathscr{E}$ are oriented away from v.*

Proof. The full subtree of T on the vertex subset

$$\{v \mid \text{each edge of } \mathscr{E} \text{ is oriented away from } v\}$$

is a proper G-invariant subtree. By minimality, this set must be empty. □

Corollary 3.6. *Let T be a cocompact G-tree and let $\tilde{T}$ be a minimal G-tree. Let $q : \tilde{T} \to T$ be a G-morphism that is surjective but not locally injective. Then every vertex of T is faced by a collapsing pair.*

Proof. Let $\tilde{\mathscr{E}}$ be the set of oriented edges of $\tilde{T}$ that are part of a collapsing pair. This is a G-invariant set, and it is nonempty because q is not locally injective. By Lemma 3.5, each vertex $\tilde{v}$ of $\tilde{T}$ must therefore have an edge $\tilde{e}$ in $\tilde{\mathscr{E}}$ oriented toward $\tilde{v}$. Set $v = q(\tilde{v})$. Then if $q(\tilde{e})$ is not oriented toward v, the image of the path from $\tilde{e}$ to $\tilde{v}$ must contain points of backtracking. The point of backtracking closest to v gives rise to a collapsing pair facing v. Because q is surjective, all vertices of T are of this form. □

Observation 3.7. If a cocompact G-tree T has a nonempty G-invariant subtree T', then T is a Hausdorff neighborhood of T'. Hence, T and T' have the same set of end points.

Proposition 3.8. *Let T be a cocompact G-tree and let $\tilde{T}$ be a minimal G-tree. Suppose $q : \tilde{T} \to T$ is a G-morphism that is not locally injective. Then there exists at most one point $E_0 \in \partial T$ such that no collapsing pairs face E_0.*

Proof. By Observation 3.7, the ends of T and the ends of $q(\tilde{T})$ are the same, so we may assume q is surjective. By Corollary 3.6, each vertex of T is faced by a collapsing pair in $\tilde{T}$. If two points of ∂T were not faced by a collapsing pair, then no vertex on the line between them would be faced by a collapsing pair. Hence, there can be at most one point of ∂T not faced by a collapsing pair. □

This proposition has an interesting consequence. If such an end E_0 exists, it must clearly be fixed by ρ. Yet points of the boundary that are fixed by ρ correspond to homomorphisms $G \to \mathbb{R}$, and such an end point lies in $\Sigma^n(\rho)$ if and only if the corresponding homomorphism lies in the BNSR invariant $\Sigma^n(G)$, as discussed in Section 2. Since we only consider simplicial trees, such points in fact correspond to homomorphisms $G \twoheadrightarrow \mathbb{Z}$.

Corollary 3.9. *Under the conditions of Proposition 3.8, if an end point $E_0 \in \partial T$ is faced by no collapsing pair in $\tilde{T}$, then there exists a canonically associated discrete character $\chi : G \to \mathbb{Z}$ such that $E_0 \in \Sigma^n(\rho)$ if and only if $[\chi] \in \Sigma^n(G)$, the BNSR invariant.*

Proof of the Main Theorem. Because q is not locally injective, Proposition 3.8 ensures that there is at most one end point faced by a collapsing pair. Because q is locally surjective, Proposition 3.4 ensures that every end point faced by a collapsing pair lies outside $\Sigma^1(\rho)$. □

The case where stabilizers under $\tilde{\rho}$ have type F_n. Recall the topological construction of the Bass–Serre tree, discussed in [Geoghegan 2008, §6.2; Scott and Wall 1979]: the action $\tilde{\rho}$ corresponds to a graph of groups decomposition of G. From this we can build a $K(G, 1)$ X admitting the quotient $G\backslash\tilde{T}$ as a retract. Let $p : \tilde{X} \twoheadrightarrow X$ be the universal covering projection. There is a natural G-map $h : \tilde{X} \twoheadrightarrow \tilde{T}$, and it is clear from the construction of h that $h^{-1}(A) \subseteq \tilde{X}$ is contractible for any connected subset $A \subseteq \tilde{T}$. If for an integer $n \geq 1$ all point stabilizers under $\tilde{\rho}$ have type F_n, then we can take X to have compact n-skeleton. Hence, letting Γ be the n-skeleton of $\tilde{X}$, the composition $\bar{h} = q \circ h|_\Gamma : \Gamma \to T$ is an appropriate control map for ρ.

Definition 3.10. While the map q does not induce a map $\partial\tilde{T} \to \partial T$, each geodesic ray in T can be lifted to one or more geodesic rays in $\tilde{T}$ (see Observation 3.2) as long as q is locally surjective. Hence, given $E \in \partial T$, we can consider the set $q^{-1}(E) \subseteq \partial\tilde{T}$ of end points represented by lifts of rays representing E.

Lemma 3.11. *If q is locally surjective, then $q^{-1}(E)$ is a singleton if and only if there are no collapsing pairs facing E.*

Proof. Suppose that $q^{-1}(E)$ is not a singleton. Then for τ representing E, there exist two distinct lifts $\tilde{\tau}_1$ and $\tilde{\tau}_2$, representing distinct points $\tilde{E}_1$ and $\tilde{E}_2$ of $\partial\tilde{T}$. If these lifts are not disjoint, then where they split (as they must, eventually) there is a collapsing pair facing E. If they are disjoint, consider the geodesic path P through $\tilde{T}$ connecting them. The image of P in T is a finite subtree of T. Choose any vertex $v \neq \tau(0)$ that is a leaf of this subtree. This leaf and the corresponding edge lie under a collapsing pair of edges of P facing E.

Now suppose there is a collapsing pair $(\tilde{e}_1, \tilde{e}_2)$ of edges of $\tilde{T}$ facing E. Let e be their common image in T, and let ζ be the geodesic ray in T representing E and beginning with the edge e. Then there are distinct lifts $\tilde{\zeta}_1$ and $\tilde{\zeta}_2$ of ζ, each representing a distinct end point of $\tilde{T}$. Hence $q^{-1}(E)$ is not a singleton. □

Proof of Theorem 1.9. If there is a collapsing pair facing E, then by Proposition 3.4, $E \notin \Sigma^1(\rho)$.

If there is no collapsing pair facing E, we take the control map $\bar{h}$ described above. By construction of $\bar{h}$, we need only show that $q^{-1}(HB_r(\tau))$ is connected for any horoball $HB_r(\tau)$ about E.

For $i = 1, 2$, let $\tilde{z}_i$ be a point in $q^{-1}(HB_r(\tau))$, and let z_i be its image in T. We find a path between $\tilde{z}_1$ and $\tilde{z}_2$ lying in $q^{-1}(HB_r(\tau))$.

There is a unique geodesic ray ζ_i in T that emanates from z_i and represents E. Let $\tilde{\zeta}_i$ be the lift of ζ_i to $\tilde{T}$ emanating from $\tilde{z}_i$. Since ζ_i lies in $HB_r(\tau)$, $\tilde{\zeta}_i$ lies in $q^{-1}(HB_r(\tau))$. Also, since $q^{-1}(E)$ is a singleton, $\tilde{\zeta}_1(\infty) = \tilde{\zeta}_2(\infty)$. Hence, $\tilde{\zeta}_1$ and $\tilde{\zeta}_2$ must eventually merge. The closure of $(\operatorname{im}\tilde{\zeta}_1 \cup \operatorname{im}\tilde{\zeta}_2) - (\operatorname{im}\tilde{\zeta}_1 \cap \operatorname{im}\tilde{\zeta}_2)$ is the geodesic connecting $\tilde{z}_1$ to $\tilde{z}_2$. □

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References

[Bass 1993] H. Bass, "Covering theory for graphs of groups", *J. Pure Appl. Algebra* **89**:1-2 (1993), 3–47. MR 94j:20028 Zbl 0805.57001

[Bieri and Geoghegan 2003a] R. Bieri and R. Geoghegan, "Connectivity properties of group actions on non-positively curved spaces", *Mem. Amer. Math. Soc.* **161**:765 (2003), xiv+83. MR 2004m:57001 Zbl 1109.20035

[Bieri and Geoghegan 2003b] R. Bieri and R. Geoghegan, "Topological properties of SL_2 actions on the hyperbolic plane", *Geom. Dedicata* **99** (2003), 137–166. MR 2004e:20068 Zbl 1039.20020

[Bieri et al. 1987] R. Bieri, W. D. Neumann, and R. Strebel, "A geometric invariant of discrete groups", *Invent. Math.* **90**:3 (1987), 451–477. MR 89b:20108 Zbl 0642.57002

[Bieri et al. 2010] R. Bieri, R. Geoghegan, and D. H. Kochloukova, "The sigma invariants of Thompson's group F", *Groups Geom. Dyn.* **4**:2 (2010), 263–273. MR 2595092 (2011h:20113 Zbl 1214.20048

[Bridson and Haefliger 1999] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften **319**, Springer, Berlin, 1999. MR 2000k:53038 Zbl 0988.53001

[Geoghegan 2008] R. Geoghegan, *Topological methods in group theory*, Graduate Texts in Mathematics **243**, Springer, New York, 2008. MR 2008j:57002 Zbl 1141.57001

[Lehnert 2009] R. Lehnert, *Kontrollierter zusammenhang von gruppenoperationen auf bäumen*, Diploma thesis, Goethe Universität Frankfurt am Main, 2009.

[Meinert 1996] H. Meinert, "The homological invariants for metabelian groups of finite Prüfer rank: a proof of the Σ^m-conjecture", *Proc. London Math. Soc.* (3) **72**:2 (1996), 385–424. MR 98b:20082 Zbl 0852.20042

[Meinert 1997] H. Meinert, "Actions on 2-complexes and the homotopical invariant Σ^2 of a group", *J. Pure Appl. Algebra* **119**:3 (1997), 297–317. MR 98g:20084 Zbl 0879.57010

[Ontaneda 2005] P. Ontaneda, "Cocompact CAT(0) spaces are almost geodesically complete", *Topology* **44**:1 (2005), 47–62. MR 2005m:57002 Zbl 1068.53026

[Rehn 2007] W. H. Rehn, *Kontrollierter Zusammenhang über symmetrischen Räumen*, Ph.D. thesis, Goethe Universität Frankfurt am Main, 2007, Available at http://publikationen.ub.uni-frankfurt.de/frontdoor/index/index/docId/360.

[Scott and Wall 1979] P. Scott and C. T. C. Wall, "Topological methods in group theory", pp. 137–203 in *Homological group theory*, edited by C. T. C. Wall, London Math. Soc. Lecture Note Ser. **36**, Cambridge Univ. Press, 1979. MR 81m:57002 Zbl 0423.20023

[Serre 1980] J.-P. Serre, *Trees*, Springer, Berlin, 1980. MR 82c:20083 Zbl 0548.20018

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Keith Jones
Department of Mathematics
Trinity College
300 Summit Street
Hartford, CT 06106
United States
keith.jones@trincoll.edu

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