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MOSHE ADRIAN

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ON THE LOCAL LANGLANDS CORRESPONDENCES OF DEBACKER–REEDER AND REEDER FOR GL(ℓ, F), WHERE ℓ IS PRIME

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We prove that the conjectural depth-zero local Langlands correspondence of DeBacker and Reeder agrees with the known depth-zero local Langlands correspondence for the group GL(ℓ, F), where ℓ is prime and F is a nonarchimedean local field of characteristic 0. We also prove that if one assumes a certain compatibility condition between Adler's and Howe's constructions of supercuspidal representations, then the conjectural positive-depth local Langlands correspondence of Reeder also agrees with the known positive-depth local Langlands correspondence for GL(ℓ, F).

1. Introduction

Let F be a nonarchimedean local field of characteristic zero. Let $\mathbf{G}$ be a connected reductive group defined over F. The local Langlands correspondence asserts that there is a finite to one map from the set of admissible representations of $\mathbf{G}(F)$ to the set of Langlands parameters of $\mathbf{G}(F)$, satisfying various conditions. Until recently, this has only been proven for special cases of groups such as GL(n, F), Sp(4, F), and $U(3)$. The local Langlands correspondence for GL(n, F) was proven by Harris and Taylor, and independently by Henniart.

More recently, DeBacker and Reeder, in two papers that will be cited throughout the text, described conjectural local Langlands correspondences for a more general class of groups and certain classes of Langlands parameters. These correspondences are still conjectural, despite satisfying several requirements that the Langlands correspondence should have. One would therefore like to know whether they agree at least with the proven correspondences in the known cases.

We prove that the correspondence introduced in [DeBacker and Reeder 2009] (henceforth [DB-R]) agrees with the known correspondence for GL(ℓ, F), while the one in [Reeder 2008] (henceforth [R]) agrees with the known correspondence

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for $\mathrm{GL}(\ell, F)$ if one assumes a certain compatibility condition, which we describe later.

For $\mathrm{GL}(n, F)$, the constructions of Harris–Taylor, Henniart, and DeBacker–Reeder (and Reeder) use different methods. We first recall the classical construction of the tame local Langlands correspondence for $\mathrm{GL}(\ell, F)$ as in [Moy 1986]. We note that a tame local Langlands correspondence for $\mathrm{GL}(n, F)$ was conjectured there for general n. In view of [Bushnell and Henniart 2005], Moy's correspondence is indeed correct for $\mathrm{GL}(\ell, F)$, ℓ a prime.

Definition 1.1. Let E/F be an extension of degree ℓ, ℓ relatively prime to the residual characteristic of F, and let χ be a character of E^*. The pair $(E/F, \chi)$ is called *admissible* if χ does not factor through the norm from a proper subfield of E containing F.

We write $\mathbb{P}_\ell(F)$ for the set of F-isomorphism classes of admissible pairs $(E/F, \chi)$ where E/F is a degree-ℓ extension (for more information about admissible pairs, see [Moy 1986]). Let $\mathbb{A}^0_\ell(F)$ denote the set of supercuspidal representations of $\mathrm{GL}(\ell, F)$. Howe [1977] constructs a map

$$\mathbb{P}_\ell(F) \to \mathbb{A}^0_\ell(F), \quad (E/F, \chi) \mapsto \pi_\chi.$$

This map is a bijection [Moy 1986]. Let $\mathbb{G}^0_\ell(F)$ denote the set of irreducible ℓ-dimensional representations of W_F, where W_F is the Weil group of F. We then have a bijection [Moy 1986]

$$\mathbb{P}_\ell(F) \to \mathbb{G}^0_\ell(F), \quad (E/F, \chi) \mapsto \mathrm{Ind}_{W_E}^{W_F}(\chi) =: \phi(\chi).$$

The local Langlands correspondence is then given by

$$\phi(\chi) \mapsto \pi_{\chi\Delta_\chi}$$

for some subtle finite order character Δ_χ of E^* [Bushnell and Henniart 2005]. In the case of depth-zero supercuspidal representations, there is only one extension E/F to deal with, namely, the unramified extension of F of degree ℓ.

On the other hand, the constructions of [DB-R] and [R] extensively use Bruhat–Tits theory. To a certain class of Langlands parameters for an unramified connected reductive group $\mathbf{G}$, they associate a character of a torus, to which they attach a collection of supercuspidal representations on the pure inner forms of $\mathbf{G}(F)$, a conjectural L-packet. They are also able to isolate the part of this packet corresponding to a particular pure inner form, and prove that their correspondences satisfy various natural conditions, such as stability.

Specifically, we prove the following. Let E/F be the unramified degree-ℓ extension, ℓ a prime. To any tame, regular, semisimple, elliptic, Langlands parameter (TRSELP) for $\mathrm{GL}(\ell, F)$, we show that DeBacker–Reeder theory attaches the character $\chi\Delta_\chi$ of E^*, to which is attached the representation $\pi_{\chi\Delta_\chi}$. This will prove that their correspondence agrees with the correspondence of [Moy 1986] for GL(ℓ, F).

We then prove the same for Reeder's construction, if one assumes a certain compatibility condition, which we describe now. The construction in [R] begins by canonically attaching a certain admissible pair $(L/F, \Omega)$ to a Langlands parameter for GL(ℓ, F). His construction then inputs this admissible pair into the theory of [Adler 1998] in order to construct a supercuspidal representation $\pi(L, \Omega)$ of GL(ℓ, F). The compatibility condition that we will need to assume is that $\pi(L, \Omega)$ is the same supercuspidal representation that is attached to $(L/F, \Omega)$ via the construction in [Howe 1977]. We remark that this compatibility condition does not seem to be known to the experts.

Although Moy's correspondence agrees with DeBacker and Reeder's (and also with Reeder's, assuming the above compatibility), some important details are different. One interesting and subtle difference lies in the passage from a Langlands parameter to a character of a torus. To illustrate it, we rewrite both correspondences to include their factorization through characters of elliptic tori as $\{$Langlands parameters from [DB-R] or [R] for GL(ℓ, F)$\} \to \mathbb{P}_\ell(F) \to \mathbb{A}^0_\ell(F)$.

Then, the correspondence of Moy is given by

$$\phi(\chi) = \mathrm{Ind}_{W_E}^{W_F}(\chi) \mapsto (E/F, \chi) \mapsto \pi_{\chi\Delta_\chi},$$

whereas the correspondences of DeBacker–Reeder (and Reeder, assuming the compatibility) are given by

$$\phi(\chi) = \mathrm{Ind}_{W_E}^{W_F}(\chi) \mapsto (E/F, \chi\Delta_\chi) \mapsto \pi_{\chi\Delta_\chi}.$$

We now briefly present an outline of the paper. In Section 2, we introduce some notation that we will need throughout. In Section 3, we briefly recall some of the key components to the construction from [DB-R]. In Section 4, we recall the tame local Langlands correspondence for GL(ℓ, F) as explained in [Moy 1986]. In Sections 5 and 6, we work out the DeBacker–Reeder theory for GL(ℓ, F), and we show that the correspondences of DeBacker–Reeder and Moy agree for GL(ℓ, F). Finally, in Section 7, we work out the theory of [R] for GL(ℓ, F), where ℓ is prime, and we show that under the compatibility condition, the correspondences of Reeder and Moy agree for GL(ℓ, F).

2. Notation

Let F denote a nonarchimedean local field of characteristic zero. We let $\mathfrak{o}_F$ denote the ring of integers of F, $\mathfrak{p}_F$ its maximal ideal, $\mathfrak{f}$ the residue field of F, q the order of $\mathfrak{f}$, and p the characteristic of $\mathfrak{f}$. Let $\mathfrak{f}_\mathfrak{m}$ denote the degree-m extension of $\mathfrak{f}$. We let ϖ denote a uniformizer of F. Let F^u denote the maximal unramified extension of F. We have the canonical projection

$$\Pi : \mathfrak{o}_F^* \to \mathfrak{o}_F^*/(1+\mathfrak{p}_F) \cong \mathfrak{f}^*$$

We denote by W_F the Weil group of F, I_F the inertia subgroup of W_F, I_F^+ the wild inertia subgroup of W_F, and W_F^{ab} the abelianization of W_F. We denote by W_F' the Weil–Deligne group, we set $W_t := W_F/I_F^+$, and we set $I_t := I_F/I_F^+$. We fix an element $\Phi \in \mathrm{Gal}(\overline{F}/F)$ whose inverse induces the map $x \mapsto x^q$ on $\mathfrak{F} := \bar{\mathfrak{f}}$, and if E/F is the unramified extension of degree ℓ, we fix an element $\Phi_E \in \mathrm{Gal}(\overline{E}/E)$ whose inverse induces the map $x \mapsto x^{q^\ell}$ on $\mathfrak{F} := \bar{\mathfrak{f}}$. Let $\mathbf{G}$ be an unramified connected reductive group over F, and set $G = \mathbf{G}(F^u)$. We fix $\mathbf{T} \subset \mathbf{G}$, an F^u-split maximal torus which is defined over F and maximally F-split, and set $T = \mathbf{T}(F^u)$. We write $X := X_*(\mathbf{T})$, W_o for the finite Weyl group $N_G(T)/T$, and set $N := N_G(T)$. Recall that the extended affine Weyl group is defined by $W := X \rtimes W_o$, and that the affine Weyl group is defined by $W^o := \Psi \rtimes W_o$, where Ψ is the coroot lattice in X. We let $\mathscr{A} := \mathscr{A}(T)$ be the apartment of T. We denote by θ the automorphism of X, W induced by Φ. If E/F is a finite Galois extension, then we denote by $\aleph_{E/F}$ the local class field theory character of F^* with respect to the extension E/F. If $\chi \in \widehat{E^*}$ satisfies $\chi|_{1+\mathfrak{p}_E} \equiv 1$, then $\chi|_{\mathfrak{o}_E^*}$ factors to a character, denoted χ_o, of the multiplicative group of the residue field of E, given by $\chi_o(x) := \chi(u)$ for any $u \in \mathfrak{o}_E^*$ such that $\Pi(u) = x$. If E/F is the degree-ℓ unramified extension, where ℓ is prime, we once and for all fix a generator ξ of $\mathrm{Gal}(E/F)$. We also fix a generator of $\mathrm{Gal}(\mathfrak{f}_\ell/\mathfrak{f})$, which, abusing notation, we also denote by ξ. If χ is a character of E^* or $\mathfrak{f}_\ell^*$, we let χ^ξ denote the character given by $\chi^\xi(x) := \chi(\xi(x))$. If L/K is a Galois quadratic extension, we let the map $x \mapsto \overline{x}$ denote the nontrivial Galois automorphism of L/K. If A is a group and B is a normal subgroup of A, we denote the image of $a \in A$ in A/B by $[a]$. If $\phi : C \to D$ is a group homomorphism and ϕ is trivial on a normal subgroup $M \lhd C$, then we will abuse notation and write $\phi|_{C/M}$ for the factorization of ϕ to a map $C/M \to D$. For example, the Langlands parameters in [DB-R] are trivial on the wild inertia subgroup I_F^+ of the inertia group I_F. Therefore, if ϕ is such a Langlands parameter and $I_t := I_F/I_F^+$, we will write $\phi|_{I_t}$ to denote the factorization of $\phi|_{I_F}$ to the quotient I_t.

3. Review of construction of DeBacker and Reeder

We first review some of the basic theory from [DB-R]. We first fix a pinning $(\hat{T}, \hat{B}, \{x_\alpha\})$ for the dual group $\hat{G}$. The operator $\hat{\theta}$ dual to θ extends to an automorphism of $\hat{T}$. There is a unique extension of $\hat{\theta}$ to an automorphism of $\hat{G}$, satisfying $\hat{\theta}(x_\alpha) = x_{\theta\cdot\alpha}$ (see [DB-R, Section 3.2]). Following [DB-R], we may form the semidirect product ${}^L G := \langle\hat{\theta}\rangle \ltimes \hat{G}$.

Definition 3.1. Let W'_F denote the Weil–Deligne group. A Langlands parameter $\phi : W'_F \to {}^L G$ is called a *tame regular semisimple elliptic Langlands parameter* (abbreviated TRSELP) if

(1) ϕ is trivial on I_F^+;

(2) the centralizer of $\phi(I_F)$ in $\hat{G}$ is a torus;

(3) $C_{\hat{G}}(\phi)^o = (\hat{Z}^{\hat{\theta}})^o$, where $\hat{Z}$ denotes the center of $\hat{G}$.

Condition (2) forces ϕ to be trivial on SL(2, $\mathbb{C}$). Let $\hat{N} = N_{\hat{G}}(\hat{T})$. After conjugating by $\hat{G}$, we may assume that $\phi(I_F) \subset \hat{T}$ and $\phi(\Phi) = \hat{\theta} f$, where $f \in \hat{N}$. Let $\hat{w}$ be the image of f in $\hat{W}_o$, and let w be the element of W_o corresponding to $\hat{w}$.

Let ϕ be a TRSELP with associated w and set $\sigma = w\theta$. σ is an automorphism of X. Let $\hat{\sigma}$ be the automorphism dual to σ, and let n be the order of σ. We set $\hat{G}_{ab} := \hat{G}/\hat{G}'$, where $\hat{G}'$ denotes the derived group of $\hat{G}$. Let

$$^L T_\sigma := \langle \hat{\sigma} \rangle \ltimes \hat{T}.$$

Associated to ϕ, DeBacker and Reeder [DB-R, Chapter 4] define a $\hat{T}$-conjugacy class of Langlands parameters

$$\phi_T : W_F \to {}^L T_\sigma \tag{1}$$

as follows. Set $\phi_T := \phi$ on I_F, and $\phi_T(\Phi) := \hat{\sigma} \ltimes \tau$ where $\tau \in \hat{T}$ is any element whose class in $\hat{T}/(1 - \hat{\sigma})\hat{T}$ corresponds to the image of f in $\hat{G}_{ab}/(1 - \hat{\theta})\hat{G}_{ab}$ under the bijection

$$\hat{T}/(1 - \hat{\sigma})\hat{T} \xrightarrow{\sim} \hat{G}_{ab}/(1 - \hat{\theta})\hat{G}_{ab} \tag{2}$$

Chapter 4 of [DB-R] gives a canonical bijection between $\hat{T}$-conjugacy classes of admissible homomorphisms $\phi : W_t \to {}^L T_\sigma$ and depth-zero characters of T^{Φ_σ} where $\Phi_\sigma := \sigma \otimes \Phi^{-1}$. We briefly summarize this construction. Let $\mathbb{T} := X \otimes \mathfrak{F}^*$. Given automorphisms α, β of abelian groups A, B, respectively, let $\mathrm{Hom}_{\alpha,\beta}(A, B)$ denote the set of homomorphisms $f : A \to B$ such that $f \circ \alpha = \beta \circ f$. The twisted norm map

$$N_\sigma : \mathbb{T}^{\Phi_\sigma^n} \to \mathbb{T}^{\Phi_\sigma}$$

given by $N_\sigma(t) = t\,\Phi_\sigma(t)\,\Phi_\sigma^2(t) \ldots \Phi_\sigma^{n-1}(t)$ induces isomorphisms

$$\mathrm{Hom}(\mathbb{T}^{\Phi_\sigma}, \mathbb{C}^*) \xrightarrow{\sim} \mathrm{Hom}_{\Phi_\sigma,\mathrm{Id}}(\mathbb{T}^{\Phi_\sigma^n}, \mathbb{C}^*) \xrightarrow{\sim} \mathrm{Hom}_{\Phi_\sigma,\mathrm{Id}}(X \otimes \mathfrak{f}_n^*, \mathbb{C}^*)$$

Moreover, the map $s \mapsto \chi_s$ gives an isomorphism

$$\mathrm{Hom}_{\Phi,\hat{\sigma}}(\mathfrak{f}_n^*, \hat{T}) \xrightarrow{\sim} \mathrm{Hom}_{\Phi_\sigma,\mathrm{Id}}(X \otimes \mathfrak{f}_n^*, \mathbb{C}^*),$$

where $\chi_s(\lambda \otimes a) := \lambda(s(a))$. The canonical projection $I_t \to \mathfrak{f}_m^*$ induces an isomorphism as Φ-modules

$$I_t/(1 - \mathrm{Ad}(\Phi)^m)\, I_t \xrightarrow{\sim} \mathfrak{f}_m^*,$$

where Ad denotes the adjoint action. Since $\hat{\sigma}$ has order n, we have

$$\mathrm{Hom}_{\Phi,\hat{\sigma}}(\mathfrak{f}_n^*, \hat{T}) \cong \mathrm{Hom}_{\mathrm{Ad}(\Phi),\hat{\sigma}}(I_t, \hat{T}).$$

Therefore, the map $s \mapsto \chi_s$ is a canonical bijection

$$\mathrm{Hom}_{\mathrm{Ad}(\Phi),\hat{\sigma}}(I_t, \hat{T}) \xrightarrow{\sim} \mathrm{Hom}(\mathbb{T}^{\Phi_\sigma}, \mathbb{C}^*)$$

Moreover, we have an isomorphism

$${}^0T^{\Phi_\sigma} \times X^\sigma \xrightarrow{\sim} T^{\Phi_\sigma}, \quad (\gamma, \lambda) \mapsto \gamma\lambda(\varpi),$$

where ${}^0T$ is the group of $\mathfrak{o}_{F^u}$-points of $\mathbf{T}$.

Finally, note that $\hat{T}/(1-\hat{\sigma})\hat{T}$ is the character group of X^σ, whereby

$$\tau \in \hat{T}/(1-\hat{\sigma})\hat{T}$$

corresponds to $\chi_\tau \in \mathrm{Hom}(X^\sigma, \mathbb{C}^*)$, where $\chi_\tau(\lambda) := \lambda(\tau)$. Therefore, we have a canonical bijection between $\hat{T}$-conjugacy classes of admissible homomorphisms $\phi : W_t \to {}^L T_\sigma$ and depth-zero characters

$$\chi_\phi := \chi_s \otimes \chi_\tau \in \mathrm{Irr}(T^{\Phi_\sigma}), \tag{3}$$

where $s := \phi|_{I_t}$, $\phi(\Phi) = \hat{\sigma} \ltimes \tau$, and where we have inflated χ_s to ${}^0T^{\Phi_\sigma}$.

To get the depth-zero L-packet associated to ϕ, one implements the component group

$$\mathrm{Irr}(C_\phi) \cong [X/(1-w\theta)\,X]_{\mathrm{tor}}$$

as follows. We set X_w to be the preimage of $[X/(1-w\theta)\,X]_{\mathrm{tor}}$ in X. To $\lambda \in X_w$, DeBacker and Reeder associate a 1-cocycle u_λ, hence a twisted Frobenius $\Phi_\lambda = \mathrm{Ad}(u_\lambda) \circ \Phi$. Moreover, to λ, they associate a facet J_λ, and hence a parahoric subgroup G_λ associated to J_λ. Let $\mathbb{G}_\lambda := G_\lambda/G_\lambda^+$. Let W_λ be the subgroup of W^o generated by reflections in the hyperplanes containing J_λ. Then to λ, DeBacker and Reeder associate an element $w_\lambda \in W_\lambda$. Fix once and for all a lift $\dot{w}$ of w to N. Using this lift, DeBacker and Reeder also associate a lift $\dot{w}_\lambda$ of w_λ to N. By Lang's theorem, there exists $p_\lambda \in G_\lambda$ such that $p_\lambda^{-1}\Phi_\lambda(p_\lambda) = \dot{w}_\lambda$. We then define $T_\lambda := \mathrm{Ad}(p_\lambda)\,T$, and set $\chi_\lambda := \chi_\phi \circ \mathrm{Ad}(p_\lambda)^{-1}$. Since χ_λ is depth-zero, its restriction to ${}^0T_\lambda^{\Phi_\lambda}$ factors through a character χ_λ^0 of $\mathbb{T}_\lambda^{\Phi_\lambda}$, where $\mathbb{T}_\lambda^{\Phi_\lambda}$ is the projection of ${}^0T^{\Phi_\lambda}$ in $\mathbb{G}_\lambda$. Therefore, χ_λ^0 gives rise to an irreducible cuspidal Deligne–Lusztig representation κ_λ^0 of $\mathbb{G}_\lambda^{\Phi_\lambda}$. Inflate κ_λ^0 to a representation of $G_\lambda^{\Phi_\lambda}$, and define an extension to $Z^{\Phi_\lambda} G_\lambda^{\Phi_\lambda}$ by

$$\kappa_\lambda := \chi_\lambda \otimes \kappa_\lambda^0,$$

where Z denotes the center of G. This makes sense since $(Z \cap G_\lambda)^{\Phi_\lambda}$ acts on κ_λ^0 via the restriction of χ_λ^0. Finally, form the representation

$$\pi_\lambda := \operatorname{Ind}_{Z^{\Phi_\lambda} G_\lambda^{\Phi_\lambda}}^{G^{\Phi_\lambda}} \kappa_\lambda,$$

where Ind denotes smooth induction. Then DeBacker and Reeder construct a packet $\Pi(\phi)$ of representations on the pure inner forms of G, parametrized by $\operatorname{Irr}(C_\phi)$, using the above construction, where C_ϕ is the component group of ϕ.

4. Existing description of the tame local Langlands correspondence for GL(ℓ, F)

In this section, we describe the construction of the tame local Langlands correspondence for GL(ℓ, F) as explained in [Moy 1986], where ℓ is a prime.

4A. ***Depth-zero supercuspidal representations of* GL(ℓ, F).** Let $(E/F, \chi)$ be an admissible pair, where χ has level 0 and E/F has degree ℓ. By definition of admissible pair, this implies that E/F is unramified, and the residue field of E is $\mathfrak{f}_\ell$. We have $\chi|_{1+\mathfrak{p}_E} = 1$, so $\chi|_{\mathfrak{o}_E^*}$ is the inflation of the character χ_o of $\mathfrak{f}_\ell^*$. By the theory of finite groups of Lie type, this character gives rise to an irreducible cuspidal representation λ' of GL(ℓ, $\mathfrak{f}$), which is the irreducible cuspidal Deligne–Lusztig representation corresponding to the elliptic torus $\mathfrak{f}_\ell^* \subset \mathrm{GL}(\ell, \mathfrak{f})$ and the character χ_o of $\mathfrak{f}_\ell^*$. Let λ be the inflation of λ' to GL(ℓ, $\mathfrak{o}_F$). We may extend λ to a representation Λ of $K(F) := F^*\mathrm{GL}(\ell, \mathfrak{o}_F)$ by setting $\Lambda|_{F^*} = \chi|_{F^*}$, and then induce the resulting representation to $G(F) = \mathrm{GL}(\ell, F)$. Set

$$\pi_\chi = \operatorname{cInd}_{K(F)}^{G(F)} \Lambda,$$

where cInd denotes compact induction. Let $\mathbb{P}_\ell(F)_0$ be the subset of admissible pairs $(E/F, \chi)$ such that χ has level zero and $\mathbb{A}_\ell^0(F)_0$ be the subset of depth-zero supercuspidal representations of GL(ℓ, F).

Proposition 4.1. *Suppose that $p \neq \ell$. The map $(E/F, \chi) \mapsto \pi_\chi$ induces a bijection*

$$\mathbb{P}_\ell(F)_0 \to \mathbb{A}_\ell^0(F)_0$$

Proof. See [Moy 1986]. □

4B. ***Positive depth supercuspidal representations of* GL(ℓ, F), *ℓ a prime.*** In this section we recall the parametrization of the positive depth supercuspidal representations via admissible pairs, following [Moy 1986]. Let $\mathbb{A}_\ell^0(F)^+$ denote the set of all positive depth irreducible supercuspidal representations of GL(ℓ, F), and let

$\mathbb{P}_\ell(F)^+$ denote the set of all admissible pairs $(E/F, \chi) \in \mathbb{P}_\ell(F)$ such that χ has positive level.

Proposition 4.2. *Suppose that $p \neq \ell$. There is a map $(E/F, \chi) \mapsto \pi_\chi$ that induces a bijection*

$$\mathbb{P}_\ell(F)^+ \to \mathbb{A}_\ell^0(F)^+$$

Proof. See [Moy 1986]. □

4C. *Langlands parameters.* Let $\mathbb{G}_\ell^0(F)$ be the set of equivalence classes of irreducible smooth ℓ-dimensional representations of W_F. Recall that there is a local Artin reciprocity isomorphism given by $W_E^{ab} \cong E^*$. Then, if $(E/F, \chi) \in \mathbb{P}_\ell(F)$, χ gives rise to a character of W_E^{ab}, which we can pullback to a character, also denoted χ, of W_E. We can then form the induced representation $\phi(\chi) := \mathrm{Ind}_{W_E}^{W_F} \chi$ of W_F.

Theorem 4.3. *Suppose $p \neq \ell$. If $(E/F, \chi) \in \mathbb{P}_\ell(F)$, the representation $\phi(\chi)$ of W_F is irreducible. The map $(E/F, \chi) \mapsto \phi(\chi)$ induces a bijection*

$$\mathbb{P}_\ell(F) \to \mathbb{G}_\ell^0(F)$$

Proof. See [Moy 1986]. □

For the next theorem, we will need to associate to any admissible pair $(E/F, \chi)$ in $\mathbb{P}_\ell(F)$ a specific character Δ_χ of E^*. We will not define Δ_χ in general, but only for the cases that we need in this paper. For the general definition of Δ_χ associated to any admissible pair $(E/F, \chi) \in \mathbb{P}_\ell(F)$, see [Moy 1986].

Definition 4.4. If $(E/F, \chi)$ is an admissible pair in which E/F is quadratic and unramified, define Δ_χ to be the unique quadratic unramified character of E^*. If $(E/F, \chi)$ is an admissible pair in which E/F is of degree ℓ and unramified, where ℓ is an odd prime, then define Δ_χ to be the trivial character of E^*.

Theorem 4.5 (Tame local Langlands correspondence [Moy 1986]). *Suppose $p \neq \ell$. For $\phi \in \mathbb{G}_\ell^0(F)$, define $\pi(\phi) = \pi_{\chi\Delta_\chi}$ in the notation of Propositions 4.1 and 4.2, for any $(E/F, \chi) \in \mathbb{P}_\ell(F)$ such that $\phi \cong \phi(\chi)$. The map*

$$\pi : \mathbb{G}_\ell^0(F) \to \mathbb{A}_\ell^0(F)$$

is the local Langlands correspondence for supercuspidal representations of $\mathrm{GL}(\ell, F)$.

5. The case of GL(ℓ, F)

For Sections 5 and 6, we consider the group $\mathbf{G}(F) = \mathrm{GL}(\ell, F)$, where ℓ is prime. We will show that the conjectural correspondence of [DB-R] agrees with the local Langlands correspondence for $\mathrm{GL}(\ell, F)$ given in Section 4.

Let $\phi : W_F \to {}^LG$ be a TRSELP for $\mathbf{G}(F) = \mathrm{GL}(\ell, F)$. This is equivalent to an irreducible admissible $\phi : W_F \to \mathrm{GL}(\ell, \mathbb{C})$ that is trivial on the wild inertia group.

By Section 4C, we have $\phi = \mathrm{Ind}_{W_E}^{W_F}(\chi)$ for some admissible pair $(E/F, \chi)$, where χ has level zero and E/F is of degree ℓ and unramified. We will need the relative Weil group [Tate 1979, Chapter 1]

$$W_{E/F} := W_F/[W_E, W_E]^c,$$

where c denotes closure and $[W_E, W_E]$ denotes the commutator subgroup of W_E. The representation $\phi = \mathrm{Ind}_{W_E}^{W_F}(\chi)$ factors through $W_{E/F}$, since

$$\phi|_{W_E} = \chi \oplus \chi^{\xi} \oplus \cdots \oplus \chi^{\xi^{\ell-1}}.$$

We begin by calculating the character χ_ϕ from (3). Note that

$${}^L G = \langle \hat{\theta} \rangle \times \mathrm{GL}(\ell, \mathbb{C}).$$

$\hat{T}$ is the diagonal maximal torus in $\hat{G} = \mathrm{GL}(\ell, \mathbb{C})$, and after conjugation, we may assume $\phi(I_F) \subset \hat{T}$. Moreover, $\phi(\Phi) = \hat{\theta} f$ for some $f \in \hat{N}$ such that $\hat{w}$ is a cycle of length ℓ in the Weyl group S_ℓ, the symmetric group on ℓ letters. The reason for this requirement on the Weyl group element is that ϕ is TRSELP and hence elliptic. In particular, ellipticity is equivalent to requiring that the image of ϕ is not contained in any proper Levi subgroup of ${}^L G$ [DB-R, Section 3.4]. After conjugating the TRSELP by a permutation matrix in $N_{\hat{G}}(\hat{T})$, we may assume without loss of generality that $\hat{w} = (1\ 2\ 3\ \cdots\ \ell) \in S_\ell$ since all cycles of length ℓ are conjugate in S_ℓ. Note that this choice implies that $w = (1\ 2\ 3\ \cdots\ \ell) \in S_\ell$. The arguments in the remainder of the paper are the same for all other allowable choices of $\hat{w}$.

Let us first calculate χ_s, where $s := \phi|_{I_t}$ (recall again that $\phi|_{I_F^+} \equiv 1$, so $\phi|_{I_F}$ factors to I_t).

Proposition 5.1. *Let $\phi = \mathrm{Ind}_{W_E}^{W_F}(\chi)$ and set $s = \phi|_{I_t}$, where $(E/F, \chi)$ is an admissible pair as above. Then, the isomorphism*

$$\mathrm{Hom}_{\mathrm{Ad}(\Phi),\hat{\sigma}}(I_t, \hat{T}) \xrightarrow{\sim} \mathrm{Hom}_{\Phi,\hat{\sigma}}(\mathfrak{f}_\ell^*, \hat{T})$$

sends s to $\tilde{\beta}_s$, where

$$\tilde{\beta}_s(x) = \begin{pmatrix} \chi_o(x) & 0 & 0 & \dots & 0 \\ 0 & \chi_o^{\xi}(x) & 0 & \dots & 0 \\ 0 & 0 & \chi_o^{\xi^2}(x) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \chi_o^{\xi^{\ell-1}}(x) \end{pmatrix}$$

Proof. Since $\hat{\sigma}$ has order ℓ, $s \in \mathrm{Hom}_{\mathrm{Ad}(\Phi),\hat{\sigma}}(I_t, \hat{T})$ is trivial on $(1 - \mathrm{Ad}(\Phi)^\ell)\, I_t$, so factors to $I_t/(1 - \mathrm{Ad}(\Phi)^\ell)\, I_t$. We first note that the isomorphisms

$$I_t \cong \varprojlim_m \mathfrak{f}_m^*, \qquad I_t/(1 - \mathrm{Ad}(\Phi)^\ell)\, I_t \xrightarrow{\sim} \mathfrak{f}_\ell^*$$

are induced by local Artin reciprocity [R, Chapter 5]. Moreover, the map

$$\mathrm{Hom}_{\mathrm{Ad}(\Phi),\hat{\sigma}}(I_t, \hat{T}) \to \mathrm{Hom}_{\Phi,\hat{\sigma}}(\mathfrak{f}_\ell^*, \hat{T})$$

comes from the diagram

$$\begin{tikzcd} & I_t \arrow[r, "s"] \arrow[d] & \hat{T} \\ \mathfrak{f}_\ell^* \arrow[r, "\sim"] & I_t/(1 - \mathrm{Ad}(\Phi)^\ell)\, I_t \arrow[ur] & \end{tikzcd}$$

Recall that ϕ factors through $W_{E/F}$. Hence, we also have the commutative diagram

$$\begin{tikzcd} W_F \arrow[r, "\phi"] \arrow[d] & \mathrm{GL}(\ell, \mathbb{C}) \\ W_{E/F} \arrow[ur, "\beta"] & \end{tikzcd}$$

It is a fact that $W_{E/F}$ is an extension of $\mathrm{Gal}(E/F)$ by E^*, and can be described by generators and relations as follows. The generators are $\{z \in E^*\}$ and an element j where $j \in W_{E/F}$ satisfies $j^\ell = \varpi$ and $jzj^{-1} = \xi(z)$. Then the map $W_F \to W_{E/F}$ sends I_F to $\mathfrak{o}_E^*$ and Φ to j.

Let us calculate the map β. Consider the canonical sequence

$$1 \to W_E/[W_E, W_E]^c \to W_F/[W_E, W_E]^c \to W_F/W_E \cong \mathrm{Gal}(E/F) \to 1$$

Recall that ϕ is trivial on $[W_E, W_E]^c$. To calculate $\beta|_{E^*}$, it suffices to calculate $\phi|_{W_E}$ since $W_E/[W_E, W_E]^c \cong E^*$ by Artin reciprocity. But

$$\phi|_{W_E} = \chi \oplus \chi^\xi \oplus \cdots \oplus \chi^{\xi^{\ell-1}}.$$

Therefore,

$$\beta(t) = \begin{pmatrix} \chi(t) & 0 & 0 & \dots & 0 \\ 0 & \chi^\xi(t) & 0 & \dots & 0 \\ 0 & 0 & \chi^{\xi^2}(t) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \chi^{\xi^{\ell-1}}(t) \end{pmatrix}$$

Moreover, since ϕ is irreducible, we have that $\beta(j) \in N_{\mathrm{GL}(\ell,\mathbb{C})}(\hat{T})$ represents $\hat{w}$.

Since $\phi|_{I_F^+} \equiv 1$, we have that $\beta|_{1+\mathfrak{p}_E} \equiv 1$, so $\beta|_{\mathfrak{o}_E^*}$ factors to a map

$$\tilde{\beta}_s : \mathfrak{f}_\ell^* \to \mathrm{GL}(\ell, \mathbb{C}),$$

given by

$$\tilde{\beta}_s(x) = \begin{pmatrix} \chi_o(x) & 0 & 0 & \dots & 0 \\ 0 & \chi_o^{\xi}(x) & 0 & \dots & 0 \\ 0 & 0 & \chi_o^{\xi^2}(x) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \chi_o^{\xi^{\ell-1}}(x) \end{pmatrix} \quad \text{for all } x \in \mathfrak{f}_\ell^*. \qquad \square$$

Proposition 5.2. *Let $\phi = \mathrm{Ind}_{W_E}^{W_F}(\chi)$ and set $s = \phi|_{I_t}$ as above. Then the composite isomorphism*

$$\mathrm{Hom}_{\mathrm{Ad}(\Phi),\hat{\sigma}}(I_t, \hat{T}) \xrightarrow{\sim} \mathrm{Hom}_{\Phi,\hat{\sigma}}(\mathfrak{f}_\ell^*, \hat{T}) \xrightarrow{\sim} \mathrm{Hom}_{\Phi_\sigma,\mathrm{Id}}(X \otimes \mathfrak{f}_\ell^*, \mathbb{C}^*) \xrightarrow{\sim} \mathrm{Hom}_{\Phi_\sigma,\mathrm{Id}}(\mathbb{T}^{\Phi_\sigma^\ell}, \mathbb{C}^*)$$

sends s to ${}^\ell\chi_o^E$, where $s = \phi|_{I_t}$ and ${}^\ell\chi_o^E(x_1, x_2, \dots, x_\ell) := \chi_o(x_1)\chi_o^{\xi}(x_2)\dots\chi_o^{\xi^{\ell-1}}(x_\ell)$.

Proof. The composite isomorphism

$$\mathrm{Hom}_{\Phi,\hat{\sigma}}(\mathfrak{f}_\ell^*, \hat{T}) \xrightarrow{\sim} \mathrm{Hom}_{\Phi_\sigma,\mathrm{Id}}(X \otimes \mathfrak{f}_\ell^*, \mathbb{C}^*) \xrightarrow{\sim} \mathrm{Hom}_{\Phi_\sigma,\mathrm{Id}}(\mathbb{T}^{\Phi_\sigma^\ell}, \mathbb{C}^*)$$

is given by $\tilde{\alpha} \longmapsto \{\lambda(x) \mapsto \lambda(\tilde{\alpha}(x))\}$, where $x \in \mathfrak{f}_\ell^*$ and $\lambda \in X = X_*(T) = X^*(\hat{T})$. Note that $\mathbb{T}$ splits over $\mathfrak{f}_\ell$ and $\mathbb{T}^{\Phi_\sigma^\ell} \cong \mathfrak{f}_\ell^* \times \cdots \times \mathfrak{f}_\ell^*$. Then, it is easy to see that under this composite isomorphism, $\tilde{\beta}_s$ (where $\tilde{\beta}_s$ is as in Proposition 5.1) maps to the homomorphism

$$(x_1, x_2, \dots, x_\ell) \mapsto \chi_o(x_1)\, \chi_o^{\xi}(x_2) \cdots \chi_o^{\xi^{\ell-1}}(x_\ell) \quad \text{for all } x_1, x_2, \dots, x_\ell \in \mathfrak{f}_\ell^*$$

by considering the standard basis of cocharacters of X. $\square$

Proposition 5.3. *The isomorphism*

$$\mathrm{Hom}_{\Phi_\sigma,\mathrm{Id}}(\mathbb{T}^{\Phi_\sigma^\ell}, \mathbb{C}^*) \xrightarrow{\sim} \mathrm{Hom}(\mathbb{T}^{\Phi_\sigma}, \mathbb{C}^*) \tag{4}$$

is given by $\Lambda \mapsto \Lambda'$, where $\Lambda'(a) := \Lambda((x_1, x_2, \dots, x_\ell))$ whenever $a \in \mathfrak{f}_\ell^*$ and $(x_1, x_2, \dots, x_\ell) \in \mathfrak{f}_\ell^* \times \mathfrak{f}_\ell^* \times \cdots \times \mathfrak{f}_\ell^*$ satisfies $x_1 x_2^{q^{\ell-1}} x_3^{q^{\ell-2}} \cdots x_\ell^{q} = a$.

Proof. Recall that the isomorphism (4) is abstractly given by $\Lambda \mapsto \Lambda'$, where $\Lambda'(a) := \Lambda((x_1, x_2, \dots, x_\ell))$ for any $(x_1, x_2, \dots, x_\ell) \in \mathfrak{f}_\ell^* \times \mathfrak{f}_\ell^* \times \cdots \times \mathfrak{f}_\ell^*$ such that $N_\sigma((x_1, x_2, \dots, x_\ell)) = a$.

We need some preliminaries. First note that

$$\begin{aligned}\Phi_\sigma((x_1, x_2, \dots, x_\ell)) &= w\, \Phi^{-1}((x_1, x_2, \dots, x_\ell)) \\ &= w(x_1^q, x_2^q, \dots, x_\ell^q) = (x_\ell^q, x_1^q, x_2^q, \dots, x_{\ell-1}^q).\end{aligned}$$

If we make the identification of $\mathbb{T}^{\Phi_\sigma^\ell}$ with tuples $(x_1, x_2, \dots, x_\ell) \in \mathfrak{f}_\ell^* \times \mathfrak{f}_\ell^* \times \cdots \times \mathfrak{f}_\ell^*$, then we have that since we made our choice of $w = (1\ 2\ 3\ \cdots\ \ell) \in S_\ell$, we get

$$\begin{aligned}\mathbb{T}^{\Phi_\sigma} &= \big\{(x_1, \dots, x_\ell) \in \mathfrak{f}_\ell^* \times \mathfrak{f}_\ell^* \times \cdots \times \mathfrak{f}_\ell^* : (x_\ell^q, x_1^q, \dots, x_{\ell-1}^q) = (x_1, x_2, \dots, x_\ell)\big\} \\ &= \big\{(x_1, x_1^q, x_1^{q^2}, \dots, x_1^{q^{\ell-1}}) : x_1 \in \mathfrak{f}_\ell\big\}.\end{aligned}$$

If $(x_1, x_2, \dots, x_\ell) \in \mathfrak{f}_\ell^* \times \mathfrak{f}_\ell^* \times \cdots \times \mathfrak{f}_\ell^* = \mathbb{T}^{\Phi_\sigma^\ell}$, then

$$\begin{aligned}&N_\sigma((x_1, x_2, \dots, x_\ell)) \\ &\quad= (x_1, x_2, \dots, x_\ell)\, \Phi_\sigma((x_1, x_2, \dots, x_\ell)) \cdots \Phi_\sigma^{\ell-1}((x_1, x_2, \dots, x_\ell)) \\ &\quad= (x_1, x_2, \dots, x_\ell)\,(x_\ell^q, x_1^q, x_2^q, \dots, x_{\ell-1}^q) x_{\ell-2}^{q^2}) \dots (x_2^{q^{\ell-1}}, x_3^{q^{\ell-1}}, \dots, x_\ell^{q^{\ell-1}}, x_1^{q^{\ell-1}}) \\ &\quad= (x_1 x_2^{q^{\ell-1}} x_3^{q^{\ell-2}} \cdots x_\ell^q,\ x_2 x_3^{q^{\ell-1}} x_4^{q^{\ell-2}} \cdots x_1^q,\ \dots,\ x_\ell x_1^{q^{\ell-1}} x_2^{q^{\ell-2}} \cdots x_{\ell-1}^q)\end{aligned}$$

Therefore, $N_\sigma : \mathbb{T}^{\Phi_\sigma^\ell} \to \mathbb{T}^{\Phi_\sigma}$ is the map

$$(x_1, x_2, \dots, x_\ell) \mapsto x_1 x_2^{q^{\ell-1}} x_3^{q^{\ell-2}} \cdots x_\ell^q$$

for all $(x_1, x_2, \dots, x_\ell) \in \mathfrak{f}_\ell^* \times \mathfrak{f}_\ell^* \times \cdots \times \mathfrak{f}_\ell^*$. □

We now need to obtain a character of ${}^0T^{\Phi_\sigma}$ from a character of $\mathbb{T}^{\Phi_\sigma}$. In our case, ${}^0T^{\Phi_\sigma} = \mathfrak{o}_E^*$, which has a canonical projection map $\mathfrak{o}_E^* = {}^0T^{\Phi_\sigma} \xrightarrow{\eta} \mathbb{T}^{\Phi_\sigma} = \mathfrak{f}_\ell^*$. Then, given $\zeta \in \mathrm{Hom}(\mathbb{T}^{\Phi_\sigma}, \mathbb{C}^*)$, we obtain a character μ of ${}^0T^{\Phi_\sigma} = \mathfrak{o}_E^*$ given by $\mu(z) := \zeta(\eta(z))$, $z \in \mathfrak{o}_E^*$. Let us more explicitly calculate such a μ, given some $\Lambda' \in \mathrm{Hom}(\mathbb{T}^{\Phi_\sigma}, \mathbb{C}^*)$ that comes from $\Lambda \in \mathrm{Hom}_{\Phi_\sigma,\mathrm{Id}}(\mathbb{T}^{\Phi_\sigma^\ell}, \mathbb{C}^*)$ as in (4). Let $z \in \mathfrak{o}_E^*$. Then $\mu(z) = \Lambda'(\eta(z)) = \Lambda((\eta(z), 1, 1, \dots, 1))$, by Proposition 5.3.

We may now calculate the character χ_s that arises from ϕ, where $s = \phi|_{I_t}$ and $\phi = \mathrm{Ind}_{W_E}^{W_F}(\chi)$. The above analysis and Proposition 5.2 shows that

$$\chi_s(z) = {}^\ell\chi_o^E((\eta(z), 1, 1, \dots, 1)) = \chi_o(\eta(z)) = \chi(z),$$

where $z \in \mathfrak{o}_E^*$. It remains to compute χ_τ. First note that if we make the identification $X = \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$, then $X^\sigma = \{(k, k, \dots, k) : k \in \mathbb{Z}\}$. Let $\lambda_{(k,k,\dots,k)} \in X^\sigma$ denote the character of $\hat{T}$ corresponding to $(k, k, \dots, k) \in \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ via this identification.

Proposition 5.4. *Let $\ell = 2$. The character $\chi_\tau : X^\sigma \to \mathbb{C}^*$ is given by*

$$\lambda_{(k,k)} \mapsto (-\chi(\varpi))^k.$$

Proof. Note that $\hat{\theta} = 1$ and $\hat{G}' = \mathrm{SL}(2, \mathbb{C})$, so τ is any element whose class in $\hat{T}/(1-\hat{\sigma})\hat{T}$ corresponds to the image of f in $\mathrm{GL}(2, \mathbb{C})/\mathrm{SL}(2, \mathbb{C})$ under the bijection

$$\hat{T}/(1-\hat{\sigma})\hat{T} \xrightarrow{\sim} \mathrm{GL}(2, \mathbb{C})/\mathrm{SL}(2, \mathbb{C})$$

as in (2). We thus need to compute f first.

Recall that $\phi(\Phi) = \beta(j)$, where β is as in the proof of Proposition 5.1. Recall that since ϕ is irreducible, then

$$\beta(j) = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$$

for some $a, b \in \mathbb{C}^*$. After conjugation by $\hat{G}$, we may assume that $b = 1$. But since $j^2 = \varpi$, we have

$$\begin{pmatrix} \chi(\varpi) & 0 \\ 0 & \chi^{\xi}(\varpi) \end{pmatrix} = \beta(\varpi) = \beta(j^2) = \beta(j)^2 = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

Therefore, $a = \chi(\varpi)$ and so

$$\beta(j) = \begin{pmatrix} 0 & \chi(\varpi) \\ 1 & 0 \end{pmatrix},$$

and we may take

$$f = \begin{pmatrix} 0 & \chi(\varpi) \\ 1 & 0 \end{pmatrix}.$$

We now note that the bijection

$$\hat{T}/(1-\hat{\sigma})\hat{T} \xrightarrow{\sim} \hat{G}_{ab}/(1-\hat{\theta})\hat{G}_{ab}$$

is induced by the inclusion $\hat{T} \hookrightarrow \hat{G}$ [DB-R, Section 4.3]. Now, we have that

$$\left[\begin{pmatrix} -\chi(\varpi) & 0 \\ 0 & 1 \end{pmatrix}\right] = \left[\begin{pmatrix} 0 & \chi(\varpi) \\ 1 & 0 \end{pmatrix}\right] \in \mathrm{GL}(2, \mathbb{C})/\mathrm{SL}(2, \mathbb{C})$$

since

$$\begin{pmatrix} 0 & \chi(\varpi) \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} -\chi(\varpi) & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C})$$

Therefore, since

$$\begin{pmatrix} -\chi(\varpi) & 0 \\ 0 & 1 \end{pmatrix} \in \hat{T},$$

we may set

$$\tau = \begin{pmatrix} -\chi(\varpi) & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $\chi_\tau : X^\sigma \to \mathbb{C}^*$ is given by

$$\chi_\tau(\lambda_{(k,k)}) = \lambda_{(k,k)}(\tau) = \lambda_{(k,k)}\begin{pmatrix} -\chi(\varpi) & 0 \\ 0 & 1 \end{pmatrix} = (-\chi(\varpi))^k. \qquad \square$$

Proposition 5.5. *Let ℓ be an odd prime. The character $\chi_\tau : X^\sigma \to \mathbb{C}^*$ is given by*

$$\lambda_{(k,k,\dots,k)} \mapsto \chi(\varpi)^k.$$

Proof. Note that $\hat{\theta} = 1$ and $\hat{G}' = \mathrm{SL}(\ell, \mathbb{C})$, so τ is any element whose class in $\hat{T}/(1-\hat{\sigma})\hat{T}$ corresponds to the image of f in $\mathrm{GL}(\ell, \mathbb{C})/\mathrm{SL}(\ell, \mathbb{C})$ under the bijection

$$\hat{T}/(1-\hat{\sigma})\hat{T} \xrightarrow{\sim} \mathrm{GL}(\ell, \mathbb{C})/\mathrm{SL}(\ell, \mathbb{C})$$

as in (2). We thus need to compute f first.

Recall that ϕ factors through $W_{E/F}$, and we have the commutative diagram

$$\begin{tikzcd} W_F \arrow[r, "\phi"] \arrow[d] & \mathrm{GL}(\ell, \mathbb{C}) \\ W_{E/F} \arrow[ur, "\beta"] & \end{tikzcd}$$

From Proposition 5.1, we have $\phi(\Phi) = \beta(j)$. To compute $\beta(j)$, recall that because of our choice of $\hat{w}$, we have

$$\beta(j) = \begin{pmatrix} 0 & 0 & 0 & \dots & a_1 \\ a_2 & 0 & 0 & \dots & 0 \\ 0 & a_3 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & a_\ell & 0 \end{pmatrix}$$

for some $a_i \in \mathbb{C}^*$. After conjugation the Langlands parameter by an element in $\hat{G}$ of the form

$$\begin{pmatrix} 0 & x_2 & 0 & 0 & \dots & 0 \\ 0 & 0 & x_3 & 0 & \dots & 0 \\ 0 & 0 & 0 & x_4 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \dots & x_\ell \\ x_1 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

we may assume that $a_2 = a_3 = \cdots = a_\ell = 1$. Therefore,

$$\begin{pmatrix} \chi(\varpi) & 0 & 0 & \dots & 0 \\ 0 & \chi^{\xi}(\varpi) & 0 & \dots & 0 \\ 0 & 0 & \chi^{\xi^2}(\varpi) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \chi^{\xi^{\ell-1}}(\varpi) \end{pmatrix} = \beta(\varpi) = \beta(j^{\ell}) = \beta(j)^{\ell} = \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_1 & 0 & \dots & 0 \\ 0 & 0 & a_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_1 \end{pmatrix}$$

Hence, $a_1 = \chi(\varpi)$, so we may take $f = \begin{pmatrix} 0 & 0 & 0 & \dots & \chi(\varpi) \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$ Now, we have

$$\left[\begin{pmatrix} \chi(\varpi) & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}\right] = \left[\begin{pmatrix} 0 & 0 & 0 & \dots & \chi(\varpi) \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}\right] \in \mathrm{GL}(\ell, \mathbb{C})/\mathrm{SL}(\ell, \mathbb{C})$$

since

$$\begin{pmatrix} 0 & 0 & 0 & \dots & \chi(\varpi) \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \chi(\varpi) & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix},$$

which is an element of SL(ℓ, $\mathbb{C}$). Therefore, we may set

$$\tau = \begin{pmatrix} \chi(\varpi) & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

Then $\chi_\tau : X^{\sigma} \to \mathbb{C}^*$ is given by

$$\chi_\tau(\lambda_{(k,k,\dots,k)}) = \lambda_{(k,k,\dots,k)}(\tau) = (\chi(\varpi))^k. \qquad \square$$

Recall that we have computed χ_ϕ on $\mathfrak{o}_E^*$. It remains to compute $\chi_\phi(\varpi)$. Because of the isomorphism

$$^0T^{\Phi_\sigma} \times X^\sigma \xrightarrow{\sim} T^{\Phi_\sigma}, \quad (\gamma, \lambda) \mapsto \gamma\lambda(\varpi),$$

we need to compute $\chi_\phi(1, \lambda_{(1,1,\ldots,1)})$.

Proposition 5.6. *Let $\ell = 2$. Then $\chi_\phi = \chi\,\Delta_\chi$, where $\phi = \mathrm{Ind}_{W_E}^{W_F}(\chi)$.*

Proof. We have that $\chi_\phi(1, \lambda_{(1,1)}) = \chi_s(1)\chi_\tau(\lambda_{(1,1)}) = -\chi(\varpi)$. Therefore, $\chi_\phi(\varpi) = -\chi(\varpi)$. Recall that we have shown that $\chi_\phi|_{\mathfrak{o}_E^*} = \chi|_{\mathfrak{o}_E^*}$. Since $\ell = 2$, Δ_χ is the unique quadratic unramified character of E^*. Therefore, we have that $\Delta_\chi(\varpi) = -1$ and $\Delta_\chi|_{\mathfrak{o}_E^*} \equiv 1$, so $\chi_\phi = \chi\,\Delta_\chi$. □

Proposition 5.7. *Let ℓ be an odd prime. $\chi_\phi = \chi\,\Delta_\chi$.*

Proof. We have that $\chi_\phi(1, \lambda_{(1,1,\ldots,1)}) = \chi_s(1)\,\chi_\tau(\lambda_{(1,1,\ldots,1)}) = \chi(\varpi)$. Therefore, $\chi_\phi(\varpi) = \chi(\varpi)$. Recall that we have shown that $\chi_\phi|_{\mathfrak{o}_E^*} = \chi|_{\mathfrak{o}_E^*}$. Therefore, $\chi_\phi = \chi$. But recall that Δ_χ is trivial since ℓ is an odd prime, so we have $\chi_\phi = \chi\,\Delta_\chi$. □

6. From a character of a torus to a representation for GL(ℓ, F)

In this section we determine the representation that DeBacker and Reeder assign to a TRSELP for GL(ℓ, F), using the results from Section 5. Note that

$$[X/(1 - w\theta)\,X]_{\mathrm{tor}} = 0,$$

so we may let $\lambda = 0$ (recall that $\lambda \in X_w$). The proof of [DB-R, Lemma 2.7.2] implies that we may take $u_\lambda = 1$, and therefore $\Phi_\lambda = \Phi$. It is also easy to see that we may take $w_\lambda = w$ ([DB-R, Section 2.7]) and $\dot{w}_\lambda = \dot{w}$, where $\dot{w}$ is a fixed choice of lift of w. Since the theory of [DB-R] is independent of any choices, we are free to choose a specific lift $\dot{w}$, which we do now.

Let $f(x)$ be a monic irreducible polynomial of degree ℓ over $\mathfrak{f}$. Let $\tilde{f}(x)$ be a monic lift of $f(x)$ to $F[x]$. We may write $E = F(\delta)$, where δ is a root of $\tilde{f}(x)$. First set

$$\tilde{w} := \begin{pmatrix} 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}.$$

Recall that we need to find $p_\lambda \in G_\lambda$ such that $p_\lambda^{-1}\Phi(p_\lambda) = \dot{w}_\lambda$. By choosing the basis $1, \delta, \delta^2, \ldots, \delta^{\ell-1}$ for E over F, we may embed E^* into GL(ℓ, F) in the standard way. Denote this embedding by $\varphi : E^* \hookrightarrow \mathrm{GL}(\ell, F)$.

Lemma 6.1. *There exists an $A \in G_\lambda$ such that*

$$A \begin{pmatrix} t & 0 & 0 & 0 & \dots & 0 \\ 0 & \xi(t) & 0 & 0 & \dots & 0 \\ 0 & 0 & \xi^2(t) & 0 & \dots & 0 \\ 0 & 0 & 0 & \xi^3(t) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \xi^{\ell-1}(t) \end{pmatrix} A^{-1} = \varphi(t)$$

for all $t = a_0 + a_1\delta + a_2\delta^2 + \cdots + a_{\ell-1}\delta^{\ell-1} \in E^*$, $a_i \in F$.

Proof. Suppose $R(x)$ is a polynomial of degree ℓ in $F[x]$ that splits over E. Then we get an isomorphism

$$\begin{aligned} E[x]/(R(x)) &\xrightarrow{\sim} \bigoplus_{i=1}^{\ell} E, \\ p(x)0 &\mapsto (p(a_1), p(a_2), \dots, p(a_\ell)), \end{aligned}$$

where the a_i are the roots of $R(x)$. Setting $R(x)$ to now be the minimal polynomial of δ, and considering the basis $1, x, x^2, \dots, x^{\ell-1}$ of $E[x]/(R(x))$ over E, we get an isomorphism

$$\begin{aligned} E[x]/(R(x)) &\xrightarrow{G} E \oplus E \oplus \cdots \oplus E \\ 1 &\mapsto (1, 1, 1, \dots, 1) \\ x &\mapsto \big(\delta, \Phi(\delta), \Phi^2(\delta), \dots, \Phi^{\ell-1}(\delta)\big) \\ x^2 &\mapsto \big(\delta^2, \Phi(\delta)^2, \Phi^2(\delta)^2, \dots, \Phi^{\ell-1}(\delta)^2\big) \\ &\dots \\ x^{\ell-1} &\mapsto \big(\delta^{\ell-1}, \Phi(\delta)^{\ell-1}, \Phi^2(\delta)^{\ell-1}, \dots, \Phi^{\ell-1}(\delta)^{\ell-1}\big) \end{aligned}$$

This transformation yields the matrix

$$V := \begin{pmatrix} 1 & \delta & \delta^2 & \delta^3 & \dots & \delta^{\ell-1} \\ 1 & \Phi(\delta) & \Phi(\delta)^2 & \Phi(\delta)^3 & \dots & \Phi(\delta)^{\ell-1} \\ 1 & \Phi^2(\delta) & \Phi^2(\delta)^2 & \Phi^2(\delta)^3 & \dots & \Phi^2(\delta)^{\ell-1} \\ 1 & \Phi^3(\delta) & \Phi^3(\delta)^2 & \Phi^3(\delta)^3 & \dots & \Phi^3(\delta)^{\ell-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \Phi^{\ell-1}(\delta) & \Phi^{\ell-1}(\delta)^2 & \Phi^{\ell-1}(\delta)^3 & \dots & \Phi^{\ell-1}(\delta)^{\ell-1} \end{pmatrix}.$$

We then set $A := V^{-1}$. Note that what we have really done here is the following. We first have taken the standard E-basis $e'_1, e'_2, \dots, e'_\ell$ of $E \oplus E \oplus \cdots \oplus E$ and pulled it back by G to get a basis $e_1, e_2, \dots, e_\ell$ of $E[x]/(R(x))$. We have then shown that the standard embedding of an element $w \in E^*$ in GL(ℓ, F) (by considering its action on the basis $1, \delta, \delta^2, \ldots, \delta^{\ell-1}$) can be diagonalized over E with respect to the basis $e_1, e_2, \ldots, e_\ell$.

Note that V is a Vandermonde matrix. Therefore, its determinant is

$$\prod_{0 \leq i < j \leq \ell-1} (\Phi^j(\delta) - \Phi^i(\delta)),$$

which has valuation zero. Since we also have that the entries of V are contained in $\mathfrak{o}_E^*$, we conclude that V, and hence A, is contained in G_λ. □

Set $\tilde{s} = \tilde{w}^{-1} A^{-1} \Phi(A)$. We now fix our choice of lift $\dot{w}$ of w by setting $\dot{w} := \tilde{s}^{-1} \tilde{w}\, \tilde{s}\, \Phi(\tilde{s})$, which we shall show is a legitimate lift. We claim first that we may set $p_\lambda = A\tilde{s}$, and that $\tilde{s} \in G_\lambda \cap T$. Since we will show that $\tilde{s} \in G_\lambda \cap T$, this shows that $p_\lambda \in G_\lambda$, which is required. To prove all of this, consider the adjoint action of $A^{-1}\Phi(A)$ on T. First, for $s \in T^{\Phi_w}$, we have

$$\begin{aligned}(A^{-1}\Phi(A)) \cdot \Phi(s) &= A^{-1}\Phi(A)\Phi(s)\Phi(A)^{-1}A \\ &= A^{-1}\Phi(AsA^{-1})A = A^{-1}AsA^{-1}A = s\end{aligned}$$

since Lemma 6.1 implies that AsA^{-1} is fixed by Φ.

We therefore have that $(A^{-1}\Phi(A)) \cdot \Phi(s) = w \cdot \Phi(s)$ for all $s \in T^{\Phi_w}$. Since T^{Φ_w} is dense in T in the Zariski topology, we have that $(A^{-1}\Phi(A)) \cdot \Phi(s) = w \cdot \Phi(s)$ for all $s \in T$. This implies that $(\tilde{w}^{-1}A^{-1}\Phi(A)) \cdot s = s$ for all $s \in T$ since $\tilde{w}$ is clearly a lift of w, which means that

$$\tilde{w}^{-1}A^{-1}\Phi(A)\, s\, (\tilde{w}^{-1}A^{-1}\Phi(A))^{-1} = s \quad \text{for all } s \in T.$$

This means that $\tilde{w}^{-1}A^{-1}\Phi(A) \in C_G(T) = T$, so in particular $\tilde{w}^{-1}A^{-1}\Phi(A) = \tilde{s} \in T$. But $A, \tilde{w} \in G_\lambda$ implies that $\tilde{w}^{-1}A^{-1}\Phi(A) \in G_\lambda$, which implies that $\tilde{s} \in G_\lambda \cap T$. This shows that $p_\lambda \in G_\lambda$, which is required. Moreover,

$$p_\lambda^{-1}\Phi(p_\lambda) = (A\tilde{s})^{-1}\Phi(A\tilde{s}) = \tilde{s}^{-1}A^{-1}\Phi(A)\,\Phi(\tilde{s}) = \tilde{s}^{-1}\tilde{w}\,\tilde{s}\,\Phi(\tilde{s}) = \dot{w}.$$

Finally, $\dot{w}$ is a lift of w since $\tilde{w}$ is, and since $\tilde{s} \in T$, proving the claim.

Thus, we have a p_λ such that $p_\lambda^{-1}\Phi_\lambda(p_\lambda) = \dot{w}$, and $\dot{w}$ is indeed a lift of w. Then if we define $T_\lambda := \mathrm{Ad}(p_\lambda)T$, we get that $T_\lambda^{\Phi_\lambda}$ is the image of E^* under φ. This is crucial, since the depth-zero supercuspidals of $\mathrm{GL}(\ell, F)$ are constructed in Section 4A by first fixing an the embedding of E^* into $\mathrm{GL}(\ell, F)$. The overall construction does not depend on the choice of embedding. We have fixed the embedding φ. DeBacker and Reeder are attaching a depth-zero supercuspidal representation of $\mathrm{GL}(\ell, F)$ to a Langlands parameter, and we need to show that their depth-zero supercuspidal matches the depth-zero supercuspidal attached in Theorem 4.5 (the latter of which, again, uses the construction in Section 4A, which assumes a fixed embedding, which we are assuming without loss of generality is φ).

Note that we have a simple description for the map $\mathrm{Ad}(p_\lambda)^{-1} : T_\lambda^{\Phi_\lambda} \to T^{\Phi_w}$, namely,

$$\phi(t) \mapsto \mathrm{diag}\big(t, \xi(t), \xi^2(t), \dots, \xi^{\ell-1}(t)\big),$$

where $\mathrm{diag}(d_1, d_2, \dots, d_\ell)$ denotes the diagonal $\ell \times \ell$ matrix with $d_1, d_2, \dots, d_\ell$ on the diagonal, and where $t = a_0 + a_1\delta + a_2\delta^2 + \cdots + a_{\ell-1}\delta^{\ell-1}$. Note that

$$T^{\Phi_w} = \big\{\mathrm{diag}\big(a_0, \xi(a_0), \xi^2(a_0), \dots, \xi^{\ell-1}(a_0)\big) : a_0 \in E^*\big\}$$

Finally, because of Propositions 5.6 and 5.7 and the definition of χ_λ, we have:

Proposition 6.2. *χ_λ is given by*

$$\chi_\lambda(\varphi(t)) = \chi(t)\,\Delta_\chi(t)$$

for all $t = a_0 + a_1\delta + a_2\delta^2 + \cdots + a_{\ell-1}\delta^{\ell-1} \in E^*$.

Let us sum up the data that we have obtained so far. Given a TRSELP for GL(ℓ, F), we have obtained a torus T^{Φ_w}. Given $\lambda = 0 \in X_w$, we have constructed $T_\lambda^{\Phi_\lambda}$ and p_λ. We have $T_\lambda^{\Phi_\lambda} \cong E^*$. From ϕ we have constructed a character χ_ϕ of T^{Φ_w}. Via $\mathrm{Ad}(p_\lambda)$, we transported χ_ϕ to a character χ_λ of $T_\lambda^{\Phi_\lambda}$. We have shown that $\chi_\lambda = \chi\,\Delta_\chi$. note that the restriction of χ_λ to ${}^0T_\lambda^{\Phi_\lambda}$ factors through a character χ_λ^0 of $\mathbb{T}_\lambda^{\Phi_\lambda}$. Then, the packet of representations that DeBacker–Reeder construct in [DB-R] from the data that we have obtained thus far is the single representation

$$\mathrm{Ind}_{F^*\mathrm{GL}(\ell,\mathfrak{o}_F)}^{\mathrm{GL}(\ell,F)}(\chi_\lambda \otimes \kappa_\lambda^0) = \pi_{\chi\Delta_\chi}$$

Recall that in Section 4C, the local Langlands correspondence for GL(ℓ, F), where ℓ is prime, was given as

$$\mathrm{Ind}_{W_E}^{W_F}(\chi) \mapsto \pi_{\chi\Delta_\chi}$$

We have therefore shown that the correspondence of DeBacker–Reeder coincides with the local Langlands correspondence.

7. The positive-depth correspondence of Reeder for GL(ℓ, F)

In this section, we prove that the correspondence of [R] agrees with the local Langlands correspondence of [Moy 1986] for GL(ℓ, F), where ℓ is an arbitrary prime, if one assumes a certain compatibility condition, which we describe now. Reeder's construction begins by canonically attaching a certain admissible pair $(L/F, \Omega)$ to a Langlands parameter for GL(ℓ, F). His construction then inputs this admissible pair into the theory of [Adler 1998] in order to construct a supercuspidal representation $\pi(L, \Omega)$ of GL(ℓ, F). The compatibility condition that we will need to assume is that $\pi(L, \Omega)$ is the same supercuspidal representation that is attached to $(L/F, \Omega)$ via the construction in [Howe 1977].

Most of the arguments and setup are the same as in the depth-zero case, so there is not much to prove here. We first very briefly review the construction of Reeder and refer to [R] for definitions and notions that are not explained here.

7A. *Review.* Let $\mathbf{G}$ be an F-quasisplit and F^u-split connected reductive group. Let $\mathbf{B} \subset \mathbf{G}$ be a Borel subgroup defined over F, and $\mathbf{T}$ a maximal torus of $\mathbf{B}$.

The Langlands parameters considered in [R] are the maps

$$\phi : W_F \to {}^L G = \langle \hat{\theta} \rangle \ltimes \hat{G}$$

such that:

(1) ϕ is trivial on $I^{(r+1)}$ and nontrivial on $I^{(r)}$ for some integer $r > 0$. Here, $\{I^{(k)}\}_{k \geq 0}$ is a filtration on I_F defined in [R, Section 5.2].

(2) The centralizer of $\phi(I^{(r)})$ in $\hat{G}$ is a maximal torus of $\hat{G}$. This is the *regularity* condition.

(3) $\phi(\Phi) \in \hat{\theta} \ltimes \hat{G}$, and the centralizer of $\phi(W_F)$ in $\hat{G}$ is finite, modulo $\hat{Z}^{\hat{\theta}}$. This is the *ellipticity* condition.

We may conjugate ϕ by an element of $\hat{G}$ so that $\phi(I_F) \subset \hat{T}$, and $\phi(\Phi) = \hat{\theta} f$, where $f \in \hat{N}$. Let $\hat{w}$ be the image of f in $\hat{W}_o$, and let w be the element of W_o dual to $\hat{w}$. We say that the element w is *associated* to ϕ.

Set $\sigma = w\theta$ and suppose its action on X has order n. From an above such Langlands parameter, Reeder defines a $\hat{T}$-conjugacy class of Langlands parameters

$$\phi_T : W_F \to {}^L T_\sigma$$

in the exact same way as in the depth-zero case. In particular, the element τ is defined in the same way.

As in the depth-zero case, a bijection is later given between $\hat{T}$-conjugacy classes of continuous homomorphisms

$$\phi : W_F / I^{(r+1)} \to {}^L T_\sigma$$

for which $\phi(\Phi) \in \hat{\sigma} \ltimes \hat{T}$ and characters of T^{Φ_σ} that are trivial on $T_{r+1}^{\Phi_\sigma}$, where $\{T_k\}_{k \geq 0}$ is the canonical filtration on T [R, Section 5.3]. This is done as follows. We have a composite isomorphism [R, Section 5.3]

$$\begin{aligned} \text{(5)} \qquad \mathrm{Hom}_{\mathrm{Ad}(\Phi),\hat{\sigma}}\big(I_F/I^{(r+1)}, \hat{T}\big) &\cong \mathrm{Hom}_{\mathrm{Ad}(\Phi),\hat{\sigma}}\big(I_F/I_n^{(r+1)}, \hat{T}\big) \\ &= \mathrm{Hom}_{\Phi,\hat{\sigma}}\big(\mathfrak{o}_n^*/(1+\mathfrak{p}_n^{r+1}), \hat{T}\big) \\ &= \mathrm{Hom}_{\Phi_\sigma,\mathrm{Id}}\big(X \otimes (\mathfrak{o}_n^*/(1+\mathfrak{p}_n^{r+1})), \mathbb{C}^*\big) \\ &= \mathrm{Hom}_{\Phi_\sigma,\mathrm{Id}}\big({}^0T^{\Phi_\sigma^n}/T_{r+1}^{\Phi_\sigma^n}, \mathbb{C}^*\big) \\ &= \mathrm{Hom}\big({}^0T^{\Phi_\sigma}/T_{r+1}^{\Phi_\sigma}, \mathbb{C}^*\big). \end{aligned}$$

Under this composite isomorphism, $s := \phi|_{I_F}$ maps to a character

$$\chi_s \in \mathrm{Hom}({}^0T^{\Phi_\sigma}/T^{\Phi_\sigma}_{r+1}, \mathbb{C}^*).$$

Then, if $\phi(\Phi) = \hat{\sigma} \ltimes \tau$, we get that τ gives rise to a character of X^σ given by $\chi_\tau(\lambda) := \lambda(\tau)$ for $\lambda \in X^\sigma$, just as in the depth-zero case. Recalling that $T^{\Phi_\sigma} = {}^0T^{\Phi_\sigma} \times X^\sigma$, we define a character χ_ϕ of T^{Φ_σ} by $\chi_\phi := \chi_s \otimes \chi_\tau$, which is our desired character of T^{Φ_σ} constructed from the Langlands parameter ϕ.

As in the depth-zero case, we have the set X_w. To $\lambda \in X_w$, Reeder associates a 1-cocycle u_λ, hence a twisted Frobenius $\Phi_\lambda = \mathrm{Ad}(u_\lambda) \circ \Phi$. Moreover, to λ is associated an affine Weyl group element w_λ, a parahoric subgroup G_{x_λ}, and an element $p_\lambda \in G_{x_\lambda}$ such that $p_\lambda^{-1}\Phi_\lambda(p_\lambda)$ is a lift of w_λ. We then define $T_\lambda := \mathrm{Ad}(p_\lambda)\,T$ and set $\chi_\lambda := \chi_\phi \circ \mathrm{Ad}(p_\lambda)^{-1}$. To the torus T_λ and the character χ_λ, we apply the construction of [Adler 1998] to obtain a supercuspidal representation. Then, Reeder constructs a packet $\Pi(\phi)$ of representations on the pure inner forms of G, parametrized by $\mathrm{Irr}(C_\phi)$, using the above construction.

7B. *The case of* GL(ℓ, F). We now consider the group $\mathbf{G}(F) = \mathrm{GL}(\ell, F)$, for ℓ prime. Let $\phi : W_F \to {}^LG$ be one of the Langlands parameters for $\mathbf{G}(F) = \mathrm{GL}(\ell, F)$ that is considered in Section 7A.

Lemma 7.1. *$\phi = \mathrm{Ind}_{W_E}^{W_F}(\chi)$ for some admissible pair $(E/F, \chi)$, where χ has positive level and E/F is of degree ℓ and unramified.*

Proof. The proof is similar as in the GL(2, F) case, but we include it for completeness purposes. As in the depth-zero case in Section 5, we may conjugate ϕ by an element of $\hat{G}$ so that the Weyl group element w that is associated to ϕ is the Weyl group element $(1\ 2\ 3\ \cdots\ \ell)$ in the symmetric group on ℓ letters. We know that ϕ is an irreducible admissible $\phi : W_F \to \mathrm{GL}(\ell, \mathbb{C})$ that is trivial on $I^{(r+1)}$ and nontrivial on $I^{(r)}$ for some integer $r > 0$. Let E be the degree ℓ unramified extension of F. Again, any representation $\mathrm{Ind}_{W_E}^{W_F}(\Omega)$ where $(E/F, \Omega)$ is an admissible pair is equivalent to the representation $\kappa : W_F \to \mathrm{GL}(\ell, \mathbb{C})$ satisfying:

(1) $\kappa|_{W_E}$ is given by $\Omega \in \widehat{E^*}$ by the local Langlands correspondence for tori.

(2)
$$\kappa(\Phi) = \begin{pmatrix} 0 & 0 & 0 & \dots & \Omega(\varpi) \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

We want to show that ϕ satisfies the two conditions above, for some admissible pair $(E/F, \chi)$. Let's restrict ϕ to W_E. By the composite isomorphism (5), $\phi|_{I_E}$ gives rise to a character $\ddot{\chi}$ of $\mathfrak{o}_E^*$. Then, by following the composite isomorphism (5) backwards, one sees that

$$\phi(x) = \begin{pmatrix} \ddot{\chi}(r_\ell(x)) & 0 & 0 & \dots & 0 \\ 0 & \ddot{\chi}^{\xi}(r_\ell(x)) & 0 & \dots & 0 \\ 0 & 0 & \ddot{\chi}^{\xi^2}(r_\ell(x)) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \ddot{\chi}^{\xi^{\ell-1}}(r_\ell(x)) \end{pmatrix}$$

as in the depth-zero case. Now, as in Propositions 5.4 and 5.5, we know that

$$\phi(\Phi) = \begin{pmatrix} 0 & 0 & 0 & \dots & a \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

for some $a \in \mathbb{C}^*$, because of the ellipticity condition on ϕ. Therefore, we have that

$$\phi(\Phi_E) = \phi(\Phi^\ell) = \phi(\Phi)^\ell = \begin{pmatrix} a & 0 & 0 & \dots & 0 \\ 0 & a & 0 & \dots & 0 \\ 0 & 0 & a & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & a \end{pmatrix}.$$

Then $\ddot{\chi}$ extends to a character, denoted χ, of E^*, by setting

$$\chi(\varpi) := a \quad \text{and} \quad \chi|_{\mathfrak{o}_E^*} := \ddot{\chi}|_{\mathfrak{o}_E^*}.$$

One can now see that $\phi = \mathrm{Ind}_{W_E}^{W_F}(\chi)$. By the regularity condition on ϕ, we get that $\ddot{\chi} \neq \ddot{\chi}^{\xi}$, and thus $(E/F, \chi)$ is an admissible pair. Finally, χ has positive level since $r > 0$. □

Proposition 7.2. *Let $\ell = 2$. Then $\chi_\phi = \chi\Delta_\chi$.*

Proof. The analogous arguments as in the depth-zero case show that $\chi_\phi|_{\mathfrak{o}_E^*} = \chi|_{\mathfrak{o}_E^*}$. In particular, let $z \in \mathfrak{o}_E^*$. Let $x \in I_F$ be any element such that $r_2(x) = z$ (where r_2 is as in [R, Section 5.1]), and let Γ be the cocharacter $t \mapsto \left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then

$$N_\sigma\big(\Gamma \otimes r_2(x)\big) = \begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix}.$$

Moreover, by the same arguments as in Proposition 5.1, we get

$$\phi(x) = \begin{pmatrix} \chi\ (r_2(x)) & 0 \\ 0 & \chi\big(\overline{r_2(x)}\big) \end{pmatrix},$$

so that $\Gamma(\phi(x)) = \chi(z)$, where here we are viewing Γ as a character of $\hat{T}$. Finally, as we may take τ to be the same element as in the depth-zero case, we have that $\chi_\phi(\varpi) = -\chi(\varpi)$, so that $\chi_\phi = \chi\Delta_\chi$. □

Proposition 7.3. *Let ℓ be an odd prime. Then*

$$\chi_\phi = \chi\Delta_\chi.$$

Proof. A reasoning analogous to that of Proposition 7.2 and the depth-zero case works here. □

Note that $[X/(1-w\theta)X]_{\mathrm{tor}} = 0$, so we may let $\lambda = 0$ (recall that $\lambda \in X_w$). It is easy to see that we may again take $u_\lambda = 1$, and therefore $\Phi_\lambda = \Phi$. It is also easy to see that we may take $w_\lambda = w$ (see [R, Section 6.4]), and we may also take the same p_λ as in the depth-zero case in Section 6. So we have the same T_λ as in Section 6 and the analogous χ_λ.

We have therefore shown that if we assume the compatibility condition in the beginning of Section 7, then by Proposition 7.3, the Reeder construction attaches the representation $\pi_{\chi\Delta_\chi}$ to the Langlands parameter $\phi = \mathrm{Ind}_{W_E}^{W_F}(\chi)$. This shows that as long as we assume this compatibility condition, the correspondences of [R] and [Moy 1986] agree for GL(ℓ, F), where ℓ is an odd prime.

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MOSHE ADRIAN
MATHEMATICS DEPARTMENT
UNIVERSITY OF UTAH
155 S 1400 E, ROOM 233
SALT LAKE CITY, UT 84112-0090
UNITED STATES
madrian@math.utah.edu

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