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QUANTUM AFFINE ALGEBRAS, CANONICAL BASES, AND q-DEFORMATION OF ARITHMETICAL FUNCTIONS

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We obtain affine analogs of the Gindikin–Karpelevich and Casselman–Shalika formulas as sums over Kashiwara and Lusztig's canonical bases. As suggested by these formulas, we define natural q-deformation of arithmetical functions such as (multi)partition functions and Ramanujan τ-functions, and prove various identities among them. In some examples we recover classical identities by taking limits. Additionally, we consider q-deformation of the Kostant function and study certain q-polynomials whose special values are weight multiplicities.

Introduction

This paper is a continuation of [Kim and Lee 2011]. The classical Gindikin–Karpelevich formula and the Casselman–Shalika formula express certain integrals of spherical functions over maximal unipotent subgroups of p-adic groups as products over all positive roots. In the previous paper, we expressed the products over positive roots as sums over Kashiwara and Lusztig's canonical bases. This idea first appeared in [Bump and Nakasuji 2010]. Let G be a split reductive p-adic group, χ an unramified character of T, the maximal torus, and f^0 the standard spherical vector corresponding to χ. Let z be the element of ${}^LT \subset {}^LG$, the L-group of G, corresponding to χ by the Satake isomorphism. Then

$$\int_{N_-(F)} f^0(n)\,dn = \prod_{\alpha\in\Delta^+} \frac{1-q^{-1}z^{\alpha}}{1-z^{\alpha}} = \sum_{b\in\boldsymbol{B}} (1-q^{-1})^{d(\phi_i(b))} z^{\mathrm{wt}(b)}, \tag{0-1}$$

$$\begin{aligned}\int_{N_-(F)} f^0(n)\psi_\lambda(n)\,dn &= \chi(V(\lambda)) \prod_{\alpha\in\Delta^+}(1-q^{-1}z^{\alpha}) \\ &= (-t)^M z^{2\rho}\,\chi(V(\lambda)) \prod_{\alpha\in\Delta^+}(1-t^{-1}z^{-\alpha}) \\ &= (-t)^M z^{\rho} \sum_{b'\otimes b\in\mathfrak{B}_\lambda\otimes\mathfrak{B}_\rho} G_\rho(b;q) z^{\mathrm{wt}(b'\otimes b)},\end{aligned} \tag{0-2}$$

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where Δ^+ is the set of positive roots, $\boldsymbol{B}$ is the canonical basis, $\mathfrak{B}_\lambda$ is the crystal basis with highest weight λ, and we set $M = |\Delta^+|$ and $t = q^{-1}$. Notice that in the Casselman–Shalika formula, we used crystal bases because they behave well with respect to the tensor product.

In the affine Kac–Moody groups, A. Braverman, D. Kazhdan, and M. Patnaik [Braverman et al. ≥ 2012] calculated the integral (0-1) and obtained a formula of the form

$$\int_{N_-(F)} f^0(n)\, dn = A \prod_{\alpha\in\Delta^+} \left(\frac{1-q^{-1}z^\alpha}{1-z^\alpha}\right)^{\operatorname{mult}\alpha}, \tag{0-3}$$

where A is a certain correction factor. When the underlying finite simple Lie algebra $\mathfrak{g}_{\mathrm{cl}}$ is simply laced of rank n, A is given by

$$\prod_{i=1}^{n}\prod_{j=1}^{\infty} \frac{1-q^{-d_i}z^{j\delta}}{1-q^{-d_i-1}z^{j\delta}},$$

where d_i's are the exponents of $\mathfrak{g}_{\mathrm{cl}}$, and δ is the minimal positive imaginary root.

In this paper, we use the explicit description of the canonical basis introduced by Beck, Chari, Pressley, and Nakajima [Beck et al. 1999; Beck and Nakajima 2004] to write the right-hand side of (0-3) as a sum over the canonical basis. Moreover, we obtain the generalization of (0-2). Namely, we prove the following (Theorem 1-16 and Corollary 2-12, respectively).

$$\prod_{\alpha\in\Delta^+} \left(\frac{1-q^{-1}z^\alpha}{1-z^\alpha}\right)^{\operatorname{mult}\alpha} = \sum_{b\in\boldsymbol{B}} (1-q^{-1})^{d(\phi(b))} z^{\operatorname{wt}(b)}, \tag{0-4}$$

$$\chi(V(\lambda))z^\rho \prod_{\alpha\in\Delta^+} (1-q^{-1}z^{-\alpha})^{\operatorname{mult}\alpha} = \sum_{b'\otimes b\in\mathfrak{B}_\lambda\otimes\mathfrak{B}_\rho} G_\rho(b;q)\, z^{\operatorname{wt}(b'\otimes b)}, \tag{0-5}$$

where $\boldsymbol{B}$ is the canonical basis of $\boldsymbol{U}^+$ (the positive part of the quantum affine algebra), and $\mathfrak{B}_\lambda$ is the crystal basis with highest weight λ. Here z is a formal variable. We also write the correction factor A as a sum over a canonical basis in the case when $\mathfrak{g}_{\mathrm{cl}}$ is simply laced.

We first prove (0-4) by induction, and deduce (0-5) from (0-4) and the Weyl–Kac character formula. In the course of the proof, we see that (0-5) can be considered as a q-deformation of the Weyl–Kac character formula. We also introduce $H_{\lambda+\rho}(\mu;q) \in \mathbb{Z}[q^{-1}]$ (Definition 2-2). It has many remarkable properties; its constant term is the multiplicity of the weight $\lambda-\mu$ in $V(\lambda)$, and the value at $q=-1$ is the multiplicity of the weight $\lambda+\rho-\mu$ in the tensor product $V(\lambda)\otimes V(\rho)$. It is also related to Kazhdan–Lusztig polynomials when $\mathfrak{g}$ is of finite type (Corollary 3-30).

When $q=-1$ and λ is a strictly dominant weight, the Casselman–Shalika formula (0-5) gives a formula for multiplicity of the weight ν in the tensor product

$V(\lambda-\rho)\otimes V(\rho)$ in terms of q-deformation of the Kostant partition function, generalizing the result of [Guillemin and Rassart 2004, Theorem 1] to affine Kac–Moody algebras; see (3-24). More precisely, we define $K_q^\infty(\mu)$ in a similar way as in [Guillemin and Rassart 2004], by

$$\sum_{\mu\in Q_+} K_q^\infty(\mu) z^\mu = \prod_{\alpha\in\Delta^+}\left(\frac{1-q^{-1}z^\alpha}{1-z^\alpha}\right)^{\operatorname{mult}\alpha}.$$

Note that when $q=\infty$, $K_q^\infty(\mu)$ is the classical Kostant partition function. Then we have

$$\dim(V(\lambda-\rho)\otimes V(\rho))_\nu = \sum_{w\in W}(-1)^{l(w)} K_{-1}^\infty(w\lambda-\nu).$$

Since the set of positive roots is infinite, the left-hand sides of (0-4) and (0-5) become infinite products. This leads to very interesting q-deformation of arithmetical functions such as multipartition functions and Fourier coefficients of modular forms. We indicate one example here.

We define $\epsilon_{q,n}(k)$ as

$$\prod_{k=1}^{\infty}(1-q^{-1}t^k)^n = \sum_{k=0}^{\infty}\epsilon_{q,n}(k)t^k.$$

Note that $\epsilon_{1,n}(k)$ is a classical arithmetic function related to modular forms. For example, we have $\epsilon_{1,24}(k)=\tau(k+1)$, where $\tau(k)$ is the Ramanujan τ-function. Thus the function $\epsilon_{q,n}(k)$ should be considered as a q-deformation of the function $\epsilon_{1,n}(k)$.

For a multipartition $\boldsymbol{p}=(\rho^{(1)},\dots,\rho^{(n)})\in\mathcal{P}(n)$, we define

$$p_{q,n}(k) = \sum_{\substack{\boldsymbol{p}\in\mathcal{P}(n)\\ |\boldsymbol{p}|=k}}(1-q^{-1})^{d(\boldsymbol{p})}, \qquad k\ge 1,$$

and set $p_{q,n}(0)=1$. Here $|\boldsymbol{p}|$ is the weight of the multipartition and the number $d(\boldsymbol{p})$ is defined in Section 1. Notice that if $q\to\infty$ and $k>0$, the function $p_{\infty,n}(k)$ is just the multipartition function with n-components. In particular, $p_{\infty,1}(k)=p(k)$, the usual partition function. Hence we can think of $p_{q,n}(k)$ as a q-deformation of the multipartition function.

It turns out that there are remarkable relations among these q-deformations. We prove (Theorem 3-8)

$$\epsilon_{q,n}(k)=\sum_{r=0}^{k}\epsilon_{1,n}(r)p_{q,n}(k-r),$$

which yields an infinite family of q-polynomial identities. We also obtain “classical” identities by taking limits.

When $n = 24$ and $q \to \infty$, the identity becomes a well-known recurrence formula for the Ramanujan τ-function:

$$0 = \sum_{r=0}^{k} \tau(r+1) p_{\infty,24}(k-r).$$

In fact, we prove another family of identities (Proposition 3-13) and obtain an intriguing characterization of the function $\epsilon_{q,n}(k)$. In Example 3-14, by taking $q = 1$, we write $\tau(k+1)$ as a sum of certain integers arising from the structure of the affine Lie algebra of type $A_4^{(1)}$.

These q-deformations of arithmetic functions essentially come from the observation that the Casselman–Shalika formula may be interpreted as a q-deformation of the Weyl–Kac character formula. In a forthcoming paper, we intend to study q-deformation of other arithmetical functions such as the divisor function, and obtain identities which become classical identities when $q = 1$ or $q \to \infty$.

1. Gindikin–Karpelevich formula

Let $\mathfrak{g}$ be an untwisted affine Kac–Moody algebra over $\mathbb{C}$. We denote by $I = \{0, 1, \dots, n\}$ the set of indices for simple roots. Let W be the Weyl group. We keep almost all the notations from [Beck and Nakajima 2004, Sections 2 and 3]. However, we use v for the parameter of a quantum group and reserve q for another parameter. Whenever there is a discrepancy in notations, we will make it clear.

We fix $\boldsymbol{h} = (\dots, i_{-1}, i_0, i_1, \dots)$ as in [Beck and Nakajima 2004, Section 3.1]. Then for any integers $m < k$, the product $s_{i_m} s_{i_{m+1}} \cdots s_{i_k} \in W$ is a reduced expression, as is the product $s_{i_k} s_{i_{k-1}} \cdots s_{i_m} \in W$. We set

$$\beta_k = \begin{cases} s_{i_0} s_{i_{-1}} \cdots s_{i_{k+1}}(\alpha_{i_k}) & \text{if } k \le 0, \\ s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_{i_k}) & \text{if } k > 0, \end{cases}$$

and define

$$\mathcal{R}(k) = \{\beta_0, \beta_{-1}, \dots, \beta_k\} \text{ for } k \le 0 \quad \text{and} \quad \mathcal{R}(k) = \{\beta_1, \beta_2, \dots, \beta_k\} \text{ for } k > 0.$$

Let $T_i = T''_{i,1}$ be the automorphism of $\boldsymbol{U}$ as in [Lusztig 1993, Section 37.1.3], and let

$$\boldsymbol{c}_+ = (c_0, c_{-1}, c_{-2}, \dots) \in \mathbb{N}^{\mathbb{Z}_{\le 0}} \quad \text{and} \quad \boldsymbol{c}_- = (c_1, c_2, \dots) \in \mathbb{N}^{\mathbb{Z}_{>0}}$$

be functions (or sequences) that are zero almost everywhere. We denote by $\mathscr{C}_>$ (respectively $\mathscr{C}_<$) the set of such functions $\boldsymbol{c}_+$ (respectively $\boldsymbol{c}_-$). Then we define

$$E_{\boldsymbol{c}_+} = E_{i_0}^{(c_0)} T_{i_0}^{-1}(E_{i_{-1}}^{(c_{-1})}) T_{i_0}^{-1} T_{i_{-1}}^{-1}(E_{i_{-2}}^{(c_{-2})}) \cdots$$

and

$$E_{\boldsymbol{c}_-} = \cdots T_{i_1} T_{i_2}(E_{i_3}^{(c_3)}) T_{i_1}(E_{i_2}^{(c_2)}) E_{i_1}^{(c_1)}.$$

We set

$$B(k)=\begin{cases}\{E_{\boldsymbol{c}_+} : c_m=0 \text{ for } m<k\} & \text{for } k\le 0,\\ \{E_{\boldsymbol{c}_-} : c_m=0 \text{ for } m>k\} & \text{for } k>0.\end{cases}$$

We denote by $\boldsymbol{B}$ the Kashiwara-Lusztig canonical basis for $\boldsymbol{U}^+$, the positive part of the quantum affine algebra.

Proposition 1-1 [Beck et al. 1999; Beck and Nakajima 2004]. *For each $E_{\boldsymbol{c}_+}\in B(k)$, $k\le 0$ (respectively $E_{\boldsymbol{c}_-}\in B(k)$, $k>0$), there exists a unique $b\in\boldsymbol{B}$ such that*

$$b\equiv E_{\boldsymbol{c}_+}\ (\textit{respectively } E_{\boldsymbol{c}_-})\quad \operatorname{mod} v^{-1}\mathbb{Z}[v^{-1}]. \tag{1-2}$$

We denote by $\boldsymbol{B}(k)$ the subset of $\boldsymbol{B}$ corresponding to $B(k)$ as in the above theorem. Then we define the map $\phi:\boldsymbol{B}(k)\to\mathscr{C}_>$ for $k\le 0$ (respectively $\mathscr{C}_<$ for $k>0$) to be $b\mapsto \boldsymbol{c}_+$ (respectively $\boldsymbol{c}_-$) such that the condition (1-2) holds. For an element $\boldsymbol{c}_+=(c_0,c_{-1},\dots)\in\mathscr{C}_>$ (respectively $\boldsymbol{c}_-=(c_1,c_2,\dots)\in\mathscr{C}_>$), we define $d(\boldsymbol{c}_+)$ (respectively $d(\boldsymbol{c}_-)$) to be the number of nonzero c_i's.

Proposition 1-3. *For each $k\in\mathbb{Z}$, we have*

$$\prod_{\alpha\in\mathscr{R}(k)}\frac{1-q^{-1}\boldsymbol{z}^\alpha}{1-\boldsymbol{z}^\alpha}=\sum_{b\in\boldsymbol{B}(k)}(1-q^{-1})^{d(\phi(b))}\boldsymbol{z}^{\mathrm{wt}(b)}. \tag{1-4}$$

Proof. First we assume $k>0$ and use induction on k. If $k=1$, then the identity (1-4) is easily verified. Now, using an induction argument, we obtain

$$\begin{aligned}\prod_{\alpha\in\mathscr{R}(k)}&\frac{1-q^{-1}\boldsymbol{z}^\alpha}{1-\boldsymbol{z}^\alpha}\\ &=\Bigl(\prod_{\alpha\in\mathscr{R}(k-1)}\frac{1-q^{-1}\boldsymbol{z}^\alpha}{1-\boldsymbol{z}^\alpha}\Bigr)\frac{1-q^{-1}\boldsymbol{z}^{\beta_k}}{1-\boldsymbol{z}^{\beta_k}}\\ &=\Bigl(\sum_{b\in\boldsymbol{B}(k-1)}(1-q^{-1})^{d(\phi(b))}\boldsymbol{z}^{\mathrm{wt}(b)}\Bigr)\Bigl(1+\sum_{j\ge1}(1-q^{-1})\boldsymbol{z}^{j\beta_k}\Bigr)\\ &=\sum_{b\in\boldsymbol{B}(k-1)}(1-q^{-1})^{d(\phi(b))}\boldsymbol{z}^{\mathrm{wt}(b)}+\sum_{j\ge1}\sum_{b\in\boldsymbol{B}(k-1)}(1-q^{-1})^{d(\phi(b))+1}\boldsymbol{z}^{\mathrm{wt}(b)+j\beta_k}.\end{aligned}$$

On the other hand, since $b'\in\boldsymbol{B}(k)$ satisfies

$$b'\equiv bT_{i_1}T_{i_2}\cdots T_{i_k}(E_k^{(j)})\quad \operatorname{mod} v^{-1}\mathbb{Z}[v^{-1}]$$

for unique $b\in\boldsymbol{B}(k-1)$ and $j\ge 0$, we can write $\boldsymbol{B}(k)$ as a disjoint union

$$\boldsymbol{B}(k)=\bigcup_{j\ge0}\{b'\in\boldsymbol{B}(k):\phi(b')=(c_1,\dots,c_{k-1},j,0,0,\dots),\ c_i\in\mathbb{N}\}.$$

Now it is clear that

$$\sum_{b\in \boldsymbol{B}(k)} (1-q^{-1})^{d(\phi(b))} z^{\mathrm{wt}(b)} = \sum_{b\in \boldsymbol{B}(k-1)} (1-q^{-1})^{d(\phi(b))} z^{\mathrm{wt}(b)} + \sum_{j\geq 1}\sum_{b\in \boldsymbol{B}(k-1)} (1-q^{-1})^{d(\phi(b))+1} z^{\mathrm{wt}(b)+j\beta_k}.$$

This completes the proof of the case $k > 0$. The case $k \leq 0$ can be proved in a similar way through a downward induction. □

We set

$$\mathcal{R}_> = \bigcup_{k\leq 0} \mathcal{R}(k) \quad \text{and} \quad \mathcal{R}_< = \bigcup_{k>0} \mathcal{R}(k).$$

Similarly, we set

$$\boldsymbol{B}_> = \bigcup_{k\leq 0} \boldsymbol{B}(k) \quad \text{and} \quad \boldsymbol{B}_< = \bigcup_{k>0} \boldsymbol{B}(k).$$

Corollary 1-5. *We have*

$$\prod_{\alpha\in\mathcal{R}_>} \frac{1-q^{-1}z^{\alpha}}{1-z^{\alpha}} = \sum_{b\in \boldsymbol{B}_>} (1-q^{-1})^{d(\phi(b))} z^{\mathrm{wt}(b)}. \tag{1-6}$$

The same identity is true if $\mathcal{R}_>$ and $\boldsymbol{B}_>$ are replaced with $\mathcal{R}_<$ and $\boldsymbol{B}_<$, respectively.

Let $\boldsymbol{c}_0 = (\rho^{(1)}, \rho^{(2)}, \dots, \rho^{(n)})$ be a multipartition with n components, that is, each component $\rho^{(i)}$ is a partition. We denote by $\mathcal{P}(n)$ the set of all multipartitions with n components. Let $S_{\boldsymbol{c}_0}$ be defined as in [Beck and Nakajima 2004, p. 352] and set

$$B_0 = \{S_{\boldsymbol{c}_0} : \boldsymbol{c}_0 \in \mathcal{P}(n)\}.$$

Proposition 1-7 [Beck et al. 1999; Beck and Nakajima 2004]. *For each $S_{\boldsymbol{c}_0} \in B_0$, there exists a unique $b \in \boldsymbol{B}$ such that*

$$b \equiv S_{\boldsymbol{c}_0} \mod v^{-1}\mathbb{Z}[v^{-1}]. \tag{1-8}$$

We denote by $\boldsymbol{B}_0$ the subset of $\boldsymbol{B}$ corresponding to B_0. Using the same notation ϕ as we used for $\boldsymbol{B}(k)$, we define a function $\phi : \boldsymbol{B}_0 \to \mathcal{P}(n)$, $b \mapsto \boldsymbol{c}_0$, such that the condition (1-8) is satisfied.

For a partition $\boldsymbol{p} = (1^{m_1}2^{m_2}\cdots r^{m_r}\cdots)$, we define

$$d(\boldsymbol{p}) = \#\{r : m_r \neq 0\} \quad \text{and} \quad |\boldsymbol{p}| = m_1 + 2m_2 + 3m_3 + \cdots.$$

Then for a multipartition $\boldsymbol{c}_0 = (\rho^{(1)}, \rho^{(2)}, \dots, \rho^{(n)}) \in \mathcal{P}(n)$, we set

$$d(\boldsymbol{c}_0) = d(\rho^{(1)}) + d(\rho^{(2)}) + \cdots + d(\rho^{(n)}).$$

We obtain from the definition of $S_{\boldsymbol{c}_0}$ that if $\phi(b) = \boldsymbol{c}_0$ then

$$\mathrm{wt}(b) = |\boldsymbol{c}_0|\delta,$$

where $|\boldsymbol{c}_0| = |\rho^{(1)}| + \cdots + |\rho^{(n)}|$ is the weight of the multipartition $\boldsymbol{c}_0$.

Proposition 1-9. *We have*

$$\text{(1-10)}\quad \prod_{\alpha\in\Delta_{\mathrm{im}}^{+}} \left(\frac{1-q^{-1}z^{\alpha}}{1-z^{\alpha}}\right)^{\mathrm{mult}\,\alpha} = \prod_{k=1}^{\infty}\left(\frac{1-q^{-1}z^{k\delta}}{1-z^{k\delta}}\right)^{n} = \sum_{b\in\boldsymbol{B}_0}(1-q^{-1})^{d(\phi(b))}z^{\mathrm{wt}(b)},$$

where Δ_{im}^{+} is the set of positive imaginary roots of $\mathfrak{g}$.

Proof. The first equality follows from the fact that $\Delta_{\mathrm{im}}^{+} = \{\delta, 2\delta, 3\delta, \dots\}$ and $\mathrm{mult}(k\delta) = n$ for all $k = 1, 2, \dots$. Now we consider the second equality and assume $n = 1$. Then we have

$$\text{(1-11)}\qquad \prod_{k=1}^{\infty}\left(\frac{1-q^{-1}z^{k\delta}}{1-z^{k\delta}}\right) = \prod_{k=1}^{\infty}\left(1+\sum_{j=1}^{\infty}(1-q^{-1})z^{jk\delta}\right).$$

We consider the generating function of the partition function $p(m)$:

$$\text{(1-12)}\qquad \sum_{m=0}^{\infty} p(m)z^{m\delta} = \prod_{k=1}^{\infty}\left(1+\sum_{j=1}^{\infty}z^{jk\delta}\right) = \sum_{\rho^{(1)}\in\mathscr{P}(1)} z^{|\rho^{(1)}|\delta} = \sum_{b\in\boldsymbol{B}_0} z^{\mathrm{wt}(b)}.$$

Comparing (1-11) and (1-12), we see that if we expand the product in the right-hand side of (1-11) into a sum, the coefficient of $z^{|\rho^{(1)}|\delta}$ will be a power of $(1-q^{-1})$ and the exponent of $(1-q^{-1})$ is exactly the number $d(\rho^{(1)})$. Therefore, we obtain

$$\prod_{k=1}^{\infty}\left(\frac{1-q^{-1}z^{k\delta}}{1-z^{k\delta}}\right) = \sum_{\rho^{(1)}\in\mathscr{P}(1)} (1-q^{-1})^{d(\rho^{(1)})}z^{|\rho^{(1)}|\delta} = \sum_{b\in\boldsymbol{B}_0}(1-q^{-1})^{d(b)}z^{\mathrm{wt}(b)}.$$

Next we assume that $n = 2$. Then we have

$$\begin{aligned}
\prod_{k=1}^{\infty}&\left(\frac{1-q^{-1}z^{k\delta}}{1-z^{k\delta}}\right)^{2} \\
&= \left(\sum_{\rho^{(1)}\in\mathscr{P}(1)} (1-q^{-1})^{d(\rho^{(1)})}z^{|\rho^{(1)}|\delta}\right)\left(\sum_{\rho^{(2)}\in\mathscr{P}(1)} (1-q^{-1})^{d(\rho^{(2)})}z^{|\rho^{(2)}|\delta}\right) \\
&= \sum_{(\rho^{(1)},\rho^{(2)})\in\mathscr{P}(2)} (1-q^{-1})^{d(\rho^{(1)})+d(\rho^{(2)})}z^{(|\rho^{(1)}|+|\rho^{(2)}|)\delta} \\
&= \sum_{b\in\boldsymbol{B}_0}(1-q^{-1})^{d(b)}z^{\mathrm{wt}(b)}.
\end{aligned}$$

It is now clear that this argument naturally generalizes to the case $n > 2$. $\square$

Let us consider the correction factor A in (0-3). We will make a modification of the formula (1-10) to write A as a sum over $\boldsymbol{B}_0$ in the case when the underlying classical Lie algebra $\mathfrak{g}_{\mathrm{cl}}$ is simply laced. For a partition $\boldsymbol{p} = (1^{m_1}2^{m_2}\cdots)$ and $d_i \in \mathbb{N}$, we define

$$Q_{d_i}(\boldsymbol{p}, j) = \begin{cases} (1-q)q^{-(d_i+1)m_j} & \text{if } m_j \neq 0, \\ 1 & \text{if } m_j = 0, \end{cases} \quad \text{and} \quad Q_{d_i}(\boldsymbol{p}) = \prod_{j=1}^{\infty} Q_{d_i}(\boldsymbol{p}, j).$$

For a multipartition $\boldsymbol{p} = (\rho^{(1)}, \dots, \rho^{(n)})$ and $d_i \in \mathbb{N}$, $i = 1, \dots, n$, we define

$$Q_{d_1,\dots,d_n}(\boldsymbol{p}) = \prod_{i=1}^{n} Q_{d_i}(\rho^{(i)}).$$

Then we obtain:

Corollary 1-13. *Assume that $\mathfrak{g}_{\mathrm{cl}}$ is simply laced. Then we have*

$$A = \prod_{i=1}^{n}\prod_{j=1}^{\infty} \frac{1-q^{-d_i}z^{j\delta}}{1-q^{-d_i-1}z^{j\delta}} = \sum_{b\in\boldsymbol{B}_0} Q(\phi(b))z^{\mathrm{wt}(b)},$$

where the d_i's are the exponents of $\mathfrak{g}_{\mathrm{cl}}$ and we write $Q(\boldsymbol{p}) = Q_{d_1,\dots,d_n}(\boldsymbol{p})$.

Proof. The first equality is a result in [Braverman et al. $\geq$ 2012] and the second can be obtained using a similar argument as in the proof of Proposition 1-9. □

Let $\mathscr{C} = \mathscr{C}_{>} \times \mathscr{P}(n) \times \mathscr{C}_{<}$ as in [Beck and Nakajima 2004].

Theorem 1-14 [Beck et al. 1999; Beck and Nakajima 2004]. *There is a bijection between the sets $\boldsymbol{B}$ and $\mathscr{C}$ such that for each $\boldsymbol{c} = (\boldsymbol{c}_+, \boldsymbol{c}_0, \boldsymbol{c}_-) \in \mathscr{C}$, there exists a unique $b \in \boldsymbol{B}$ such that*

$$b \equiv E_{\boldsymbol{c}_+} S_{\boldsymbol{c}_0} E_{\boldsymbol{c}_-} \mod v^{-1}\mathbb{Z}[v^{-1}]. \tag{1-15}$$

Then we naturally extend the function ϕ to a bijection of $\boldsymbol{B}$ onto $\mathscr{C}$ and the number $d(\boldsymbol{c})$ is also defined by $d(\boldsymbol{c}) = d(\boldsymbol{c}_+) + d(\boldsymbol{c}_0) + d(\boldsymbol{c}_-)$ for each $\boldsymbol{c} \in \mathscr{C}$.

Theorem 1-16. *We have*

$$\prod_{\alpha\in\Delta^+} \left(\frac{1-q^{-1}z^{\alpha}}{1-z^{\alpha}}\right)^{\mathrm{mult}\,\alpha} = \sum_{b\in\boldsymbol{B}} (1-q^{-1})^{d(\phi(b))} z^{\mathrm{wt}(b)}. \tag{1-17}$$

Proof. Recall that $\Delta^+ = \Delta^+_{\mathrm{re}} \cup \Delta^+_{\mathrm{im}}$, $\Delta^+_{\mathrm{re}} = \mathscr{R}_{>} \cup \mathscr{R}_{<}$, and $\mathrm{mult}\,\alpha = 1$ for $\alpha \in \Delta^+_{\mathrm{re}}$. Then the identity of the theorem follows from Corollary 1-5, Proposition 1-9, and Theorem 1-14. □

2. Casselman–Shalika formula

For the functions $\boldsymbol{c}_+ = (c_0, c_{-1}, c_{-2}, \dots) \in \mathscr{C}_>$ and $\boldsymbol{c}_- = (c_1, c_2, \dots) \in \mathscr{C}_<$, we define

$$|\boldsymbol{c}_+| = c_0 + c_{-1} + c_{-2} + \cdots \quad \text{and} \quad |\boldsymbol{c}_-| = c_1 + c_2 + \cdots .$$

For a multipartition $\boldsymbol{c}_0 = (\rho^{(1)}, \rho^{(2)}, \dots, \rho^{(n)}) \in \mathscr{P}(n)$, set $|\boldsymbol{c}_0| = |\rho^{(1)}| + \cdots + |\rho^{(n)}|$ as in Section 1.

Using similar arguments as in Section 1, we obtain the following identities.

Proposition 2-1. (1) *For each $k \in \mathbb{Z}$,*

$$\prod_{\alpha \in \mathscr{R}(k)} (1 - q^{-1} z^{\alpha})^{-1} = \sum_{b \in \boldsymbol{B}(k)} q^{-|\phi(b)|} z^{\mathrm{wt}(b)}.$$

(2) *The following identity is still true if $\mathscr{R}_>$ and $\boldsymbol{B}_>$ are replaced with $\mathscr{R}_<$ and $\boldsymbol{B}_<$, respectively.*

$$\prod_{\alpha \in \mathscr{R}_>} (1 - q^{-1} z^{\alpha})^{-1} = \sum_{b \in \boldsymbol{B}_>} q^{-|\phi(b)|} z^{\mathrm{wt}(b)}.$$

(3) $\displaystyle\prod_{\alpha \in \Delta^+_{\mathrm{im}}} \left(1 - q^{-1} z^{\alpha}\right)^{-\operatorname{mult}\alpha} = \prod_{k=1}^{\infty} \left(1 - q^{-1} z^{k\delta}\right)^{-n} = \sum_{b \in \boldsymbol{B}_0} q^{-|\phi(b)|} z^{\mathrm{wt}(b)}.$

(4) $\displaystyle\prod_{\alpha \in \Delta^+} (1 - q^{-1} z^{\alpha})^{-\operatorname{mult}\alpha} = \sum_{b \in \boldsymbol{B}} q^{-|\phi(b)|} z^{\mathrm{wt}(b)}.$

Let $P_+ = \{\lambda \in P : \langle h_i, \lambda\rangle \geq 0 \text{ for all } i \in I\}$. Recall that the irreducible $\mathfrak{g}$-module $V(\lambda)$ is integrable if and only if $\lambda \in P_+$ [Kac 1990, Lemma 10.1].

Definition 2-2. Let $\lambda \in P_+$. We define $H_\lambda(\,\cdot\,; q) : Q_+ \to \mathbb{Z}[q^{-1}]$ using the generating series

$$\begin{aligned}\sum_{\mu \in Q_+} H_\lambda(\mu; q) z^{\lambda - \mu} &= \sum_{w \in W} (-1)^{\ell(w)} \sum_{b \in \boldsymbol{B}} (1 - q^{-1})^{d(\phi(b))} z^{w\lambda - \mathrm{wt}(b)} \\ &= \Bigl(\sum_{w \in W} (-1)^{\ell(w)} z^{w\lambda}\Bigr)\Bigl(\sum_{b \in \boldsymbol{B}} (1 - q^{-1})^{d(\phi(b))} z^{-\mathrm{wt}(b)}\Bigr),\end{aligned}$$

and we write

$$\chi_q(V(\lambda)) = \sum_{\mu \in Q_+} H_\lambda(\mu; q) z^{\lambda - \mu}.$$

We denote by $\chi(V(\lambda))$ the usual character of $V(\lambda)$. We have the element $d \in \mathfrak{h}$ such that $\alpha_0(d) = 1$ and $\alpha_j(d) = 0$, $j \in I \setminus \{0\}$. We define $\rho \in \mathfrak{h}^*$ as in [Kac 1990, Chapter 6] by $\rho(h_j)=1$, $j \in I$ and $\rho(d)=0$. By the Weyl–Kac character formula,

$$\frac{\sum_{w\in W}(-1)^{\ell(w)}z^{w(\lambda+\rho)-\rho}}{\prod_{\alpha\in\Delta^+}(1-z^{-\alpha})^{\operatorname{mult}\alpha}}=\chi(V(\lambda)).$$

In particular, if $\lambda=0$, then

$$\sum_{w\in W}(-1)^{\ell(w)}z^{w\rho}=z^{\rho}\prod_{\alpha\in\Delta^+}(1-z^{-\alpha})^{\operatorname{mult}\alpha}.$$

By Theorem 1-16,

$$\sum_{b\in\boldsymbol{B}}(1-q^{-1})^{d(\phi(b))}z^{-\mathrm{wt}(b)}=\prod_{\alpha\in\Delta^+}\left(\frac{1-q^{-1}z^{-\alpha}}{1-z^{-\alpha}}\right)^{\operatorname{mult}\alpha}.$$

Thus we obtain

$$\begin{aligned}\chi_q(V(\rho))&=\left(\sum_{w\in W}(-1)^{\ell(w)}z^{w\rho}\right)\left(\sum_{b\in\boldsymbol{B}}(1-q^{-1})^{d(\phi(b))}z^{-\mathrm{wt}(b)}\right)\\&=z^{\rho}\prod_{\alpha\in\Delta^+}(1-z^{-\alpha})^{\operatorname{mult}\alpha}\prod_{\alpha\in\Delta^+}\left(\frac{1-q^{-1}z^{-\alpha}}{1-z^{-\alpha}}\right)^{\operatorname{mult}\alpha}\\&=z^{\rho}\prod_{\alpha\in\Delta^+}(1-q^{-1}z^{-\alpha})^{\operatorname{mult}\alpha}.\end{aligned}$$

Therefore we have proved that

$$\chi_q(V(\rho))=z^{\rho}\prod_{\alpha\in\Delta^+}(1-q^{-1}z^{-\alpha})^{\operatorname{mult}\alpha}. \tag{2-3}$$

When $q=-1$ in (2-3), we have the following identity from [Kac 1990, Exercise 10.1].

Lemma 2-4. $\chi_{-1}(V(\rho))=z^{\rho}\prod_{\alpha\in\Delta^+}(1+z^{-\alpha})^{\operatorname{mult}\alpha}=\chi(V(\rho)).$

Remark 2-5. By Definition 2-2,

$$\chi_{-1}(V(\rho))=\sum_{\mu\in Q_+}H_{\rho}(\mu;-1)z^{\rho-\mu}=z^{\rho}\prod_{\alpha\in\Delta^+}(1+z^{-\alpha})^{\operatorname{mult}\alpha}.$$

Therefore, if $H_\rho(\mu;-1)\neq 0$, $\rho-\mu$ must be a weight of $V(\rho)$ and $H_\rho(\mu;-1)$ is the multiplicity of $\rho-\mu$ in $V(\rho)$.

Now we have the following affine analog of the Casselman–Shalika formula.

Corollary 2-6.

$$\chi_q(V(\lambda+\rho))=\chi(V(\lambda))\chi_q(V(\rho)). \tag{2-7}$$

Proof. By Definition 2-2 and Theorem 1-16,

$$\chi_q(V(\lambda+\rho)) = \left(\sum_{w\in W}(-1)^{\ell(w)} z^{w(\lambda+\rho)}\right)\prod_{\alpha\in\Delta^+}\left(\frac{1-q^{-1}z^{-\alpha}}{1-z^{-\alpha}}\right)^{\operatorname{mult}\alpha}.$$

By the Weyl–Kac character formula and (2-3), the right-hand side is

$$\chi(V(\lambda))\chi_q(V(\rho)).$$

□

Remark 2-8. When $q=1$, we see that $\chi_1(V(\lambda+\rho))z^{-\rho}$ is the numerator of the Weyl–Kac character formula. Hence we can think of (2-7) as a q-deformation of Weyl–Kac character formula. Since $\chi_\infty(V(\rho)) = z^\rho$, by setting $q=\infty$, we have

$$\chi_\infty(V(\lambda+\rho)) = z^\rho\chi(V(\lambda)).$$

Hence we may consider $\chi_q(V(\lambda+\rho))z^{-\rho}$ as a q-deformation of $\chi(V(\lambda))$. Moreover, by Definition 2-2,

$$\sum_{\mu\in Q_+} H_{\lambda+\rho}(\mu;\infty)z^{\lambda-\mu} = \chi(V(\lambda)).$$

Therefore, $H_{\lambda+\rho}(\mu;\infty)$ is the multiplicity of the weight $\lambda-\mu$ in $V(\lambda)$.

By setting $q=-1$ in (2-7), and by Lemma 2-4 we get the following.

Lemma 2-9.
$$\begin{aligned}\chi_{-1}(V(\lambda+\rho)) &= \sum_{\mu\in Q_+} H_{\lambda+\rho}(\mu;-1)z^{\lambda+\rho-\mu}\\ &= \chi(V(\lambda))\chi(V(\rho)) = \chi(V(\lambda)\otimes V(\rho)).\end{aligned}$$

Hence, $H_{\lambda+\rho}(\mu;-1)$ is the multiplicity of the weight $\lambda+\rho-\mu$ in the tensor product $V(\lambda)\otimes V(\rho)$.

Before we investigate further the implication of the Casselman–Shalika formula (2-7), we need the following lemma.

Lemma 2-10. *Assume that $\lambda_1,\lambda_2\in P_+$. Then the set of weights of $V(\lambda_1)\otimes V(\lambda_2)$ is the same as that of $V(\lambda_1+\lambda_2)$.*

Proof. Suppose that $\lambda_1,\lambda_2\in P_+$. Let $V(\lambda_1)$ and $V(\lambda_2)$ be the integrable highest weight modules with highest weights λ_1 and λ_2, respectively. By [Kac 1990, p. 211], $V(\lambda_1+\lambda_2)$ occurs in $V(\lambda_1)\otimes V(\lambda_2)$ with multiplicity one. Hence it is enough to prove that any weight of $V(\lambda_1)\otimes V(\lambda_2)$ is a weight of $V(\lambda_1+\lambda_2)$.

If V_1 and V_2 are modules in the category $\mathcal{O}$, the weight space of $(V_1\otimes V_2)_\mu$ for $\mu\in\mathfrak{h}^*$, is given by

$$(V_1\otimes V_2)_\mu = \sum_{\nu\in\mathfrak{h}^*}(V_1)_\nu\otimes(V_2)_{\mu-\nu}.$$

Hence weights of $V(\lambda_1) \otimes V(\lambda_2)$ are of the form $\mu_1 + \mu_2$, where μ_1 and μ_2 are weights of $V(\lambda_1)$ and $V(\lambda_2)$, respectively. Furthermore, since $V(\lambda_1) \otimes V(\lambda_2)$ is completely reducible, a weight $\mu_1 + \mu_2$ of $V(\lambda_1) \otimes V(\lambda_2)$ is a weight of the module $V(\lambda)$ for some $\lambda \in P_+$ that appears in the decomposition of $V(\lambda_1) \otimes V(\lambda_2)$.

It follows from [Kac 1990, Corollary 10.1] that we can choose $w \in W$ such that $w(\mu_1 + \mu_2) \in P_+$. Then, by [Kac 1990, Proposition 11.2], we need only show that $w(\mu_1 + \mu_2)$ is nondegenerate with respect to $\lambda_1 + \lambda_2$. By [Kac 1990, Lemma 11.2], $w\mu_1$ and $w\mu_2$ are nondegenerate with respect to λ_1 and λ_2, respectively. Now, from the definition of nondegeneracy [Kac 1990, p. 190], we see that $w\mu_1 + w\mu_2$ is nondegenerate with respect to $\lambda_1 + \lambda_2$. $\square$

Now we use crystal bases, namely, bases at $v = 0$, since they behave nicely under tensor products. Let $\mathfrak{B}_\lambda$ be the crystal basis associated to a dominant integral weight $\lambda \in P_+$. We choose $G_\rho(\,\cdot\,; q) : \mathfrak{B}_\rho \to \mathbb{Z}[q^{-1}]$ by assigning any element of $\mathbb{Z}[q^{-1}]$ to each $b \in \mathfrak{B}_\rho$ so that

$$H_\rho(\mu; q) = \sum_{\substack{b \in \mathfrak{B}_\rho \\ \mathrm{wt}(b) = \rho - \mu}} G_\rho(b; q). \tag{2-11}$$

By Remark 2-5, it is enough to consider $\mu \in Q_+$ such that $\rho - \mu$ is a weight of $b \in \mathfrak{B}_\rho$.

Using the function $G_\rho(\,\cdot\,; q)$, we can rewrite the Casselman–Shalika formula in Corollary 2-6 in a familiar form:

Corollary 2-12.

$$\begin{aligned} \sum_{\mu \in Q_+} H_{\lambda+\rho}(\mu; q) z^{\lambda+\rho-\mu} &= \chi(V(\lambda)) z^\rho \prod_{\alpha \in \Delta^+} (1 - q^{-1} z^{-\alpha})^{\operatorname{mult} \alpha} \\ &= \sum_{b' \otimes b \in \mathfrak{B}_\lambda \otimes \mathfrak{B}_\rho} G_\rho(b; q) z^{\mathrm{wt}(b' \otimes b)}. \end{aligned} \tag{2-13}$$

Proof. The first equality is obvious from (2-3) and Corollary 2-6. For the second equality, we obtain

$$\begin{aligned} &\chi(V(\lambda)) z^\rho \prod_{\alpha \in \Delta^+} (1 - q^{-1} z^{-\alpha})^{\operatorname{mult} \alpha} \\ &\quad = \chi(V(\lambda)) \chi_q(V(\rho)) = \Bigl(\sum_{b' \in \mathfrak{B}_\lambda} z^{\mathrm{wt}(b')} \Bigr) \Bigl(\sum_{\mu \in Q_+} H_\rho(\mu; q)\, z^{\rho-\mu} \Bigr) \\ &\quad = \Bigl(\sum_{b' \in \mathfrak{B}_\lambda} z^{\mathrm{wt}(b')} \Bigr) \Bigl(\sum_{b \in \mathfrak{B}_\rho} G_\rho(b; q) z^{\mathrm{wt}(b)} \Bigr) = \sum_{b' \otimes b \in \mathfrak{B}_\lambda \otimes \mathfrak{B}_\rho} G_\rho(b; q) z^{\mathrm{wt}(b' \otimes b)}. \qquad \square \end{aligned}$$

The following proposition provides useful information on $H_{\lambda+\rho}(\mu; q) \in \mathbb{Z}[q^{-1}]$.

Proposition 2-14. *Assume that $\lambda \in P_+$. We then have that $H_{\lambda+\rho}(\mu; q)$ is a nonzero polynomial if and only if $\lambda+\rho-\mu$ is a weight of $V(\lambda+\rho)$.*

Proof. We obtain from (2-13) that if $H_{\lambda+\rho}(\mu; q) \neq 0$, then $\lambda+\rho-\mu$ is a weight of $V(\lambda)\otimes V(\rho)$. Then $\lambda+\rho-\mu$ is a weight of $V(\lambda+\rho)$ by Lemma 2-10. Conversely, assuming that $\lambda+\rho-\mu$ is a weight of $V(\lambda+\rho)$, it is also a weight of $V(\lambda)\otimes V(\rho)$. By Lemma 2-9,

$$\sum_{\mu' \in Q_+} H_{\lambda+\rho}(\mu'; -1) z^{\lambda+\rho-\mu'} = \chi(V(\lambda)\otimes V(\rho)).$$

Since $\lambda+\rho-\mu$ is a weight of $V(\lambda)\otimes V(\rho)$, the coefficient $H_{\lambda+\rho}(\mu; -1) \neq 0$. Then $H_{\lambda+\rho}(\mu; q)$ is a nonzero polynomial. □

3. Applications

We give several applications of our formulas to q-deformation of (multi)partition functions and modular forms, and the Kostant function and the multiplicity formula. We also obtain formulas for $H_\lambda(\mu; q)$.

3.1. ***Multipartition functions and modular forms.*** We will write $\mathcal{P} = \mathcal{P}(1)$. For a partition $\boldsymbol{p} = (1^{m_1} 2^{m_2} \cdots r^{m_r} \cdots) \in \mathcal{P}$, we define

$$\kappa_q(\boldsymbol{p}) = \begin{cases} (-q^{-1})^{\sum m_r} & \text{if } m_r = 0 \text{ or } 1 \text{ for all } r, \\ 0 & \text{otherwise.} \end{cases}$$

We define for $k \geq 1$

$$\epsilon_q(k) = \sum_{\substack{\boldsymbol{p}\in\mathcal{P} \\ |\boldsymbol{p}|=k}} \kappa_q(\boldsymbol{p})$$

and set $\epsilon_q(0) = 1$. For example, $\epsilon_q(5) = 2q^{-2} - q^{-1}$ and $\epsilon_q(6) = -q^{-3} + 2q^{-2} - q^{-1}$.

From the definitions, we have

$$\prod_{k=1}^{\infty} (1 - q^{-1} t^k) = 1 + \sum_{\boldsymbol{p}\in\mathcal{P}} \kappa_q(\boldsymbol{p}) t^{|\boldsymbol{p}|} = 1 + \sum_{k=1}^{\infty} \epsilon_q(k) t^k.$$

Then it follows from Euler's pentagonal number theorem that when $q = 1$, we have

$$\epsilon_1(k) = \begin{cases} (-1)^m & \text{if } k = \tfrac{1}{2} m(3m \pm 1), \\ 0 & \text{otherwise.} \end{cases} \tag{3-1}$$

We also define for $k \geq 1$

$$p_q(k) = \sum_{\substack{\boldsymbol{p}\in\mathcal{P} \\ |\boldsymbol{p}|=k}} (1 - q^{-1})^{d(\boldsymbol{p})},$$

where $d(\boldsymbol{p})$ is the same as in the previous sections, and we set $p_q(0) = 1$. Note that if $k > 0$, $p_\infty(k) = p(k)$. Hence we can think of $p_q(k)$ as a q-deformation of the partition function.

Proposition 3-2. *If* $k > 0$, *then*

$$\epsilon_q(k) - p_q(k) = \sum_{m=1}^{\infty} (-1)^m \{ p_q(k - \tfrac{1}{2}m(3m-1)) + p_q(k - \tfrac{1}{2}m(3m+1)) \}, \tag{3-3}$$

where we define $p_q(M) = 0$ *for all negative integers* M.

Proof. We put $n = 1$ in Proposition 1-9 and obtain

$$\prod_{k=1}^{\infty} (1 - q^{-1} z^{k\delta}) = \left(\sum_{\boldsymbol{p} \in \mathcal{P}} (1 - q^{-1})^{d(\boldsymbol{p})} z^{|\boldsymbol{p}|\delta} \right) \prod_{k=1}^{\infty} (1 - z^{k\delta}).$$

After the change of variables $z^\delta = t$, we obtain

$$\begin{aligned} 1 + \sum_{k=1}^{\infty} \epsilon_q(k) t^k &= \prod_{k=1}^{\infty} (1 - q^{-1} t^k) \\ &= \left(\sum_{\boldsymbol{p} \in \mathcal{P}} (1 - q^{-1})^{d(\boldsymbol{p})} t^{|\boldsymbol{p}|} \right) \prod_{k=1}^{\infty} (1 - t^k) \\ &= \left(1 + \sum_{k=1}^{\infty} p_q(k) t^k \right) \left(1 + \sum_{m=1}^{\infty} (-1)^m \left\{ t^{\frac{1}{2}m(3m-1)} + t^{\frac{1}{2}m(3m+1)} \right\} \right), \end{aligned}$$

where we use the definition of $p_q(k)$ and (3-1) in the last equality. We obtain the identity (3-3) by expanding the product and equating the coefficient of t^k with $\epsilon_q(k)$. □

As a corollary of the proof of Proposition 3-2, we obtain:

Corollary 3-4. *Let* $(a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k)$. *Then*

$$\sum_{n=0}^{\infty} \frac{(q^{-1}; t)_n}{(t; t)_n} t^n = \sum_{k=0}^{\infty} p_q(k) t^k.$$

Proof. By the q-binomial theorem,

$$\prod_{k=1}^{\infty} (1 - q^{-1} t^k) = \left(\sum_{n=0}^{\infty} \frac{(q^{-1}; t)_n}{(t; t)_n} t^n \right) \prod_{k=1}^{\infty} (1 - t^k).$$

Comparing this with the identity in the proof of Proposition 3-2, we obtain the result. □

Remark 3-5. When $q \to \infty$, we have

$$\sum_{n=0}^{\infty} \frac{t^n}{(t;t)_n} = \sum_{\boldsymbol{p}\in\mathcal{P}} t^{|\boldsymbol{p}|} = \sum_{n=0}^{\infty} p(n)t^n.$$

This is a special case of [Andrews 1976, Corollary 2.2].

We generalize Proposition 3-2 to the case of multipartitions. For a multipartition $\boldsymbol{p} = (\rho^{(1)}, \dots, \rho^{(n)}) \in \mathcal{P}(n)$, we define

$$\kappa_q(\boldsymbol{p}) = \prod_{i=1}^{n} \kappa_q(\rho^{(i)}),$$

and for $k \geq 1$,

$$\epsilon_{q,n}(k) = \sum_{\substack{\boldsymbol{p}\in\mathcal{P}(n)\\ |\boldsymbol{p}|=k}} \kappa_q(\boldsymbol{p}), \tag{3-6}$$

and set $\epsilon_{q,n}(0) = 1$. From the definitions, we have

$$\prod_{k=1}^{\infty}(1-q^{-1}t^k)^n = 1 + \sum_{\boldsymbol{p}\in\mathcal{P}(n)} \kappa_q(\boldsymbol{p})t^{|\boldsymbol{p}|} = \sum_{k=0}^{\infty} \epsilon_{q,n}(k)t^k.$$

One can see that if $k > 0$, we have $\epsilon_{\infty,n}(k) = 0$.

Remark 3-7. Note that $\epsilon_{1,n}(k)$ is a classical arithmetic function related to modular forms. For example, we have $\epsilon_{1,24}(k) = \tau(k+1)$, where $\tau(k)$ is the Ramanujan τ-function. Thus the function $\epsilon_{q,n}(k)$ should be considered as a q-deformation of the function $\epsilon_{1,n}(k)$.

We also define for $k \geq 1$

$$p_{q,n}(k) = \sum_{\substack{\boldsymbol{p}\in\mathcal{P}(n)\\ |\boldsymbol{p}|=k}} (1-q^{-1})^{d(\boldsymbol{p})},$$

and set $p_{q,n}(0) = 1$. Notice that if $k > 0$, the function $p_{\infty,n}(k)$ is nothing but the multipartition function with n-components. Hence we can think of $p_{q,n}(k)$ as a q-deformation of the multipartition function.

Theorem 3-8. *If* $k > 0$, *then*

$$\epsilon_{q,n}(k) = \sum_{r=0}^{k} \epsilon_{1,n}(r)p_{q,n}(k-r). \tag{3-9}$$

Proof. From Proposition 1-9 we obtain

$$\prod_{k=1}^{\infty}(1-q^{-1}z^{k\delta})^n = \Bigl(\sum_{\boldsymbol{p}\in\mathcal{P}(n)} (1-q^{-1})^{d(\boldsymbol{p})}z^{|\boldsymbol{p}|\delta}\Bigr) \prod_{k=1}^{\infty}(1-z^{k\delta})^n.$$

After the change of variables $z^\delta = t$, we obtain from the definitions

$$\begin{aligned}\sum_{k=0}^{\infty} \epsilon_{q,n}(k)t^k &= \Bigl(\sum_{\boldsymbol{p}\in\mathcal{P}(n)} (1-q^{-1})^{d(\boldsymbol{p})} t^{|\boldsymbol{p}|}\Bigr) \prod_{k=1}^{\infty}(1-t^k)^n \\ &= \Bigl(\sum_{r=0}^{\infty} p_{q,n}(r)t^r\Bigr)\Bigl(\sum_{s=0}^{\infty} \epsilon_{1,n}(s)t^s\Bigr). \qquad \square\end{aligned}$$

By taking $q \to \infty$, we obtain the identity

$$0 = \sum_{r=0}^{k} \epsilon_{1,n}(r)\, p_{\infty,n}(k-r),$$

where $p_{\infty,n}(k)$ is the multipartition function with n-components. This is an easy consequence of the identities

$$\prod_{k=1}^{\infty}(1-t^k)^n = \sum_{k=0}^{\infty} \epsilon_{1,n}(k)t^k \quad \text{and} \quad \prod_{k=1}^{\infty}(1-t^k)^{-n} = \sum_{k=0}^{\infty} p_{\infty,n}(k)t^k.$$

Example 3-10. When the affine Kac–Moody algebra $\mathfrak{g}$ is of type $X_{24}^{(1)}$, with $X = A, B, C$, or D, we have

$$\epsilon_{q,24}(k) = \sum_{r=0}^{k} \tau(r+1)\, p_{q,24}(k-r) \quad \text{and} \quad 0 = \sum_{r=0}^{k} \tau(r+1)\, p_{\infty,24}(k-r),$$

where $\tau(k)$ is the Ramanujan τ-function. If $k = 2$, the first identity becomes

$$\epsilon_{q,24}(2) = \tau(1)p_{q,24}(2) + \tau(2)p_{q,24}(1) + \tau(3)p_{q,24}(0).$$

Through some computations, we obtain

$$\epsilon_{q,24}(2) = 276q^{-2} - 24q^{-1}.$$

On the other hand, we have

$$\begin{aligned}\tau(1)&p_{q,24}(2) + \tau(2)p_{q,24}(1) + \tau(3)p_{q,24}(0) \\ &= p_{q,24}(2) - 24p_{q,24}(1) + 252 \\ &= \{276(1-q^{-1})^2 + 48(1-q^{-1})\} - 24\cdot 24(1-q^{-1}) + 252 \\ &= 276(1-q^{-1})^2 - 528(1-q^{-1}) + 252 \\ &= 276q^{-2} - 24q^{-1} = \epsilon_{q,24}(2).\end{aligned}$$

We also see that

$$\tau(1)p_{\infty,24}(2) + \tau(2)p_{\infty,24}(1) + \tau(3)p_{\infty,24}(0) = 324 - 24\cdot 24 + 252 = 0.$$

Now we consider the whole set of positive roots, not just the set of imaginary positive roots, and obtain interesting identities. We begin with the identity (2-3). Recalling the description of the set of positive roots, we obtain

$$\begin{aligned}(3\text{-}11)\quad \sum_{\mu\in Q_+} H_\rho(\mu;q)z^{-\mu} &= z^{-\rho}\chi_q(V(\rho)) = \prod_{\alpha\in\Delta_+}(1-q^{-1}z^{-\alpha})^{\operatorname{mult}\alpha}\\ &= \biggl(\prod_{k=1}^{\infty}(1-q^{-1}z^{-k\delta})^n \prod_{\alpha\in\Delta_{\mathrm{cl}}}(1-q^{-1}z^{\alpha-k\delta})\biggr)\prod_{\alpha\in\Delta_{\mathrm{cl}}^+}(1-q^{-1}z^{-\alpha}),\end{aligned}$$

where Δ_{cl} is the set of classical roots.

Let

$$\mathfrak{X} = \Bigl\{\sum_{\alpha\in Q_+} c_\alpha z^{-\alpha} : c_\alpha\in\mathbb{C}\Bigr\}$$

be the set of (infinite) formal sums. Recall that we have the element $d\in\mathfrak{h}$ such that $\alpha_0(d)=1$ and $\alpha_j(d)=0$, $j\in I\setminus\{0\}$. Let $\mathfrak{h}_{\mathbb{Z}}$ be the $\mathbb{Z}$-span of $\{h_0,h_1,\dots,h_n,d\}$. We then define the evaluation map $\mathrm{EV}_t : \mathfrak{X}\times\mathfrak{h}_{\mathbb{Z}}\to\mathbb{C}[[t]]$ by

$$\mathrm{EV}_t\Bigl(\sum_\alpha c_\alpha z^{-\alpha}, s\Bigr) = \sum_\alpha c_\alpha t^{\alpha(s)},\qquad s\in\mathfrak{h}_{\mathbb{Z}}.$$

Then we see that $\mathrm{EV}_t(\,\cdot\,,d)$ is the same as the *basic specialization* in [Kac 1990, p. 219] with q replaced by t. We apply $\mathrm{EV}_t(\,\cdot\,,d)$ to (3-11) and obtain

$$(3\text{-}12)\quad (1-q^{-1})^{|\Delta_{\mathrm{cl}}^+|}\prod_{k=1}^{\infty}(1-q^{-1}t^k)^{\dim\mathfrak{g}_{\mathrm{cl}}} = \sum_{k=0}^{\infty}\Bigl(\sum_{\mu\in Q_{+,\mathrm{cl}}} H_\rho(k\alpha_0+\mu;q)\Bigr)t^k,$$

where $\mathfrak{g}_{\mathrm{cl}}$ is the finite-dimensional simple Lie algebra corresponding to $\mathfrak{g}$, and $Q_{+,\mathrm{cl}}$ is the $\mathbb{Z}_{\geq 0}$-span of $\{\alpha_1,\dots,\alpha_n\}$. We write $|\Delta_{\mathrm{cl}}^+|=r$ and $\dim\mathfrak{g}_{\mathrm{cl}}=N$ so that $N=2r+n$. By comparing (3-12) with the identity

$$\prod_{k=1}^{\infty}(1-q^{-1}t^k)^n = \sum_{k=0}^{\infty}\epsilon_{q,n}(k)t^k,$$

we obtain:

Proposition 3-13. $$\epsilon_{q,N}(k) = \sum_{\mu\in Q_{+,\mathrm{cl}}}\frac{H_\rho(k\alpha_0+\mu;q)}{(1-q^{-1})^r}.$$

By Definition 2-2, $\epsilon_{q,N}(k)$ is a power series in q^{-1} in the above formula. However, one can see from (3-6) that $\epsilon_{q,N}(k)$ is actually a polynomial in q^{-1}.

Example 3-14. We take $\mathfrak{g}$ to be of type $A_4^{(1)}$. Then the classical Lie algebra $\mathfrak{g}_{\mathrm{cl}}$ is of type A_4, and $r = |\Delta_{\mathrm{cl}}^+| = 10$ and $N = \dim \mathfrak{g}_{\mathrm{cl}} = 24$. Taking the limit $q \to 1$, we obtain

$$\tau(k+1) = \lim_{q\to 1} \sum_{\mu\in Q_{+,\mathrm{cl}}} \frac{H_\rho(k\alpha_0+\mu; q)}{(1-q^{-1})^{10}}.$$

Therefore the sum $\sum_{\mu\in Q_{+,\mathrm{cl}}} H_\rho(k\alpha_0+\mu; q)$ is always divisible by $(1-q^{-1})^{10}$. But Lehmer's conjecture predicts that the sum is never divisible by $(1-q^{-1})^{11}$.

3.2. ***The Kostant function and $H_\lambda(\mu; q)$.*** In this section, let $\mathfrak{g}$ be an untwisted affine Kac–Moody algebra (affine type) or a finite-dimensional simple Lie algebra (finite type).

Definition 3-15. We define the functions $K_q^\infty(\mu)$ and $K_q^1(\mu)$ by

$$\sum_{\mu\in Q_+} K_q^\infty(\mu) z^\mu = \prod_{\alpha\in\Delta_+} \left(\frac{1-q^{-1}z^\alpha}{1-z^\alpha} \right)^{\mathrm{mult}\,\alpha} = \sum_{b\in \boldsymbol{G}} (1-q^{-1})^{d(\phi(b))} z^{\mathrm{wt}(b)}$$

and

$$\sum_{\mu\in Q_+} K_q^1(\mu) z^\mu = \prod_{\alpha\in\Delta_+} (1-q^{-1}z^\alpha)^{-\mathrm{mult}\,\alpha} = \sum_{b\in \boldsymbol{G}} q^{-|\phi(b)|} z^{\mathrm{wt}(b)}.$$

We set $K_q^\infty(\mu) = K_q^1(\mu) = 0$ if $\mu \notin Q_+$.

Remark 3-16. (1) Note that both $K_\infty^\infty(\mu)$ with $q=\infty$ and $K_1^1(\mu)$ with $q=1$ are equal to the classical Kostant partition function $K(\mu)$. Hence both of them can be considered as q-deformations of the Kostant function.

(2) The function $K_q^1(\mu)$ was introduced by Lusztig [1983] for finite types; see also [Kato 1982]. On the other hand, the function $K_q^\infty(\mu)$ for finite types can be found in the work of Guillemin and Rassart [2004].

We obtain from the Casselman–Shalika formula (Corollary 2-6)

$$\begin{aligned} z^{-\lambda}\chi(V(\lambda)) &= \sum_{\beta\in Q_+} (\dim V(\lambda)_{\lambda-\beta}) z^{-\beta} \\ &= z^{-\lambda-\rho}\chi_q(V(\lambda+\rho)) \prod_{\alpha\in\Delta_+} (1-q^{-1}z^{-\alpha})^{-\mathrm{mult}\,\alpha} \\ &= \left(\sum_{\mu\in Q_+} H_{\lambda+\rho}(\mu; q) z^{-\mu} \right) \left(\sum_{\nu\in Q_+} K_q^1(\nu) z^{-\nu} \right). \end{aligned}$$

Therefore, we have a q-deformation of the Kostant multiplicity formula:

Proposition 3-17. $\dim V(\lambda)_{\lambda-\beta} = \sum_{\mu\in Q_+} H_{\lambda+\rho}(\mu; q) K_q^1(\beta-\mu).$

In order to see that this is indeed a q-deformation of the Kostant multiplicity formula, we need to determine the value of $H_{\lambda+\rho}(\mu; 1)$.

Lemma 3-18. *We have*

$$H_{\lambda+\rho}(\mu; 1) = \begin{cases} (-1)^{\ell(w)} & \text{if } w \circ \lambda = -\mu \text{ for some } w \in W, \\ 0 & \text{otherwise}, \end{cases}$$

where we define $w \circ \lambda = w(\lambda+\rho) - \lambda - \rho$ for $w \in W$ and $\lambda \in P_+$.

Note that such a $w \in W$ is unique if it exists, so there is no ambiguity in the assertion.

Proof. From Definition 2-2, we obtain

$$\sum_{\mu \in Q_+} H_{\lambda+\rho}(\mu; 1) z^{\lambda+\rho-\mu} = \sum_{w \in W} (-1)^{\ell(w)} z^{w(\lambda+\rho)}.$$

The condition $\lambda+\rho-\mu = w(\lambda+\rho)$ is equivalent to $w \circ \lambda = -\mu$. □

Now we take $q = 1$ in Proposition 3-17 and use Lemma 3-18 to obtain the classical Kostant multiplicity formula

$$\dim V(\lambda)_{\lambda-\beta} = \sum_{w \in W} (-1)^{\ell(w)} K(w \circ \lambda + \beta).$$

Note that the sum is actually a finite sum. Indeed, we have $w \circ \lambda < 0$ for each $w \in W$ and $w \circ \lambda + \beta \geq 0$ only for finitely many $w \in W$ for fixed $\lambda \in P_+$ and $\beta \in Q_+$. For the same reason, the sum in (3-23) below is also a finite sum.

Remark 3-19. In Section 2 we obtained (Remark 2-8 and Lemma 2-9)

$$H_{\lambda+\rho}(\mu; \infty) = \dim V(\lambda)_{\lambda-\mu}, \tag{3-20}$$

$$H_{\lambda+\rho}(\mu; -1) = \dim(V(\lambda) \otimes V(\rho))_{\lambda+\rho-\mu}. \tag{3-21}$$

When $\mathfrak{g}$ is of finite type, we define $H_\lambda(\mu; q)$ as in Definition 2-2, and we can prove the analogous results. See [Kim and Lee 2011] for details.

We next derive a formula for $H_{\lambda+\rho}(\mu; q)$:

Proposition 3-22.

$$H_{\lambda+\rho}(\mu; q) = \sum_{w \in W} (-1)^{\ell(w)} K_q^\infty(w \circ \lambda + \mu). \tag{3-23}$$

Proof. From the definitions we have

$$\begin{aligned} \chi_q(V(\lambda+\rho)) &= \sum_{\mu \in Q_+} H_{\lambda+\rho}(\mu; q) z^{\lambda+\rho-\mu} \\ &= \Big(\sum_{w \in W} (-1)^{\ell(w)} z^{w(\lambda+\rho)}\Big)\Big(\sum_{\nu \in Q_+} K_q^\infty(\nu) z^{-\nu}\Big). \end{aligned}$$

The identity comes from expanding the product and comparing the coefficients. □

If we take the limit $q \to \infty$ in (3-23), we have, from (3-20),

$$\dim V(\lambda)_{\lambda-\mu} = \sum_{w\in W} (-1)^{\ell(w)} K(w\circ\lambda+\mu),$$

which is again the classical Kostant multiplicity formula.

If we take $q=-1$ in (3-23), we obtain, from (3-21),

$$\dim(V(\lambda)\otimes V(\rho))_{\lambda+\rho-\mu} = \sum_{w\in W} (-1)^{\ell(w)} K_{-1}^{\infty}(w\circ\lambda+\mu). \tag{3-24}$$

This is a generalization of the formula in [Guillemin and Rassart 2004, Theorem 1] to the affine case.

Example 3-25. Assume that $\mathfrak{g}$ is of type $A_1^{(1)}$. We write

$$\mu = m\alpha_0 + n\alpha_1 = (m,n) \in Q_+$$

and set $\lambda=0$ in (3-23). Through standard computation, we obtain

$$\{w\rho+\mu-\rho : w\in W\} = \left\{ \left(m - \frac{k(k+1)}{2}, n - \frac{k(k-1)}{2} \right) \,\middle|\, k\in\mathbb{Z} \right\}.$$

Thus we have

$$H_\rho(m,n;q) = \sum_{k\in\mathbb{Z}} (-1)^k K_q^{\infty}\left(m - \frac{k(k+1)}{2}, n - \frac{k(k-1)}{2} \right).$$

By taking the limit as $q\to\infty$, we obtain, for $(m,n)\neq(0,0)$,

$$0 = \sum_{k\in\mathbb{Z}} (-1)^k K\left(m - \frac{k(k+1)}{2}, n - \frac{k(k-1)}{2} \right).$$

In this case, $K(m,n)$ counts the number of vector partitions of (m,n) into parts of the forms (a,a), $(a-1,a)$, or $(a,a-1)$. Then we have obtained (3-9) [Carlitz 1965, p. 148].

We further investigate properties of the function $H_\lambda(\mu;q)$. From the definitions of $K_q^{\infty}(\mu)$ and $K_q^{1}(\mu)$, we have

$$\begin{aligned}\left(\sum_{\mu\in Q_+} K_q^{\infty}(\mu) z^{\mu} \right) \left(\sum_{\nu\in Q_+} K_q^{1}(\nu) z^{\nu} \right) \\ &= \prod_{\alpha\in\Delta_+} \left(\frac{1-q^{-1}z^{\alpha}}{1-z^{\alpha}} \right)^{\operatorname{mult}\alpha} \prod_{\alpha\in\Delta_+} (1-q^{-1}z^{\alpha})^{-\operatorname{mult}\alpha} \\ &= \prod_{\alpha\in\Delta_+} (1-z^{\alpha})^{-\operatorname{mult}\alpha} = \sum_{\beta\in Q_+} K(\beta) z^{\beta},\end{aligned}$$

where $K(\beta)$ is the classical Kostant function. Thus we have

$$\sum_{\mu \in Q_+} K_q^{\infty}(\mu) K_q^1(\beta - \mu) = K(\beta), \tag{3-26}$$

and we obtain, for $\beta > 0$,

$$K_q^{\infty}(\beta) = K(\beta) - K_q^1(\beta) - \sum_{0<\nu<\beta} K_q^{\infty}(\nu) K_q^1(\beta - \nu), \tag{3-27}$$

and $K_q^{\infty}(0) = K_q^1(0) = K(0) = 1$.

Then we obtain from Proposition 3-22

$$\begin{aligned} H_{\lambda+\rho}(\mu; q) = H_{\lambda+\rho}(\mu; 1) + \sum_{w \in W} (-1)^{\ell(w)} K(w \circ \lambda + \mu) - \sum_{w \in W} (-1)^{\ell(w)} K_q^1(w \circ \lambda + \mu) \\ - \sum_{\substack{w \in W \\ w \circ \lambda + \mu > 0}} (-1)^{\ell(w)} \sum_{0 < \nu < w \circ \lambda + \mu} K_q^{\infty}(\nu) K_q^1(w \circ \lambda + \mu - \nu), \end{aligned}$$

where $H_{\lambda+\rho}(\mu; 1)$ plays the role of a correction term for the case $w \circ \lambda + \mu = 0$. See Lemma 3-18 for the value of $H_{\lambda+\rho}(\mu; 1)$. We also used the fact that

$$K(\beta) = K_q^1(\beta) = K_q^{\infty}(\beta) = 0$$

unless $\beta \geq 0$.

Now we apply the classical Kostant formula and get:

Proposition 3-28. *Assume that $\lambda \in P_+$ and $\mu \in Q_+$. Then we have*

$$\begin{aligned} H_{\lambda+\rho}(\mu; q) = H_{\lambda+\rho}(\mu; 1) + \dim V(\lambda)_{\lambda-\mu} - \sum_{w \in W} (-1)^{\ell(w)} K_q^1(w \circ \lambda + \mu) \\ - \sum_{\substack{w \in W \\ w \circ \lambda + \mu > 0}} (-1)^{\ell(w)} \sum_{0 < \nu < w \circ \lambda + \mu} K_q^{\infty}(\nu) K_q^1(w \circ \lambda + \mu - \nu). \end{aligned}$$

For the rest of this section, we assume that $\mathfrak{g}$ is of finite type. We denote by $\rho^{\vee}$ the element of $\mathfrak{h}$ defined by $\langle \alpha_i, \rho^{\vee} \rangle = 1$ for all the simple roots α_i. The following identity was conjectured by Lusztig [1983] and proved by S. Kato [1982].

Proposition 3-29. *For $\lambda \in P_+$ and $\mu \in Q_+$, we have*

$$\sum_{w \in W} (-1)^{\ell(w)} K_q^1(w \circ \lambda + \mu) = q^{-\langle \mu, \rho^{\vee} \rangle} P_{w_{\lambda-\mu}, w_{\lambda}}(q),$$

where w_{ν} is the element in the affine Weyl group $\widehat{W}$ corresponding to $\nu \in P_+$, and $P_{w_{\lambda-\mu}, w_{\lambda}}(q)$ is the Kazhdan–Lusztig polynomial.

Hence, from Proposition 3-28, we obtain:

Corollary 3-30. $H_{\lambda+\rho}(\mu; q) = H_{\lambda+\rho}(\mu; 1) + \dim V(\lambda)_{\lambda-\mu} - q^{-\langle\mu,\rho^\vee\rangle} P_{w_{\lambda-\mu},w_\lambda}(q)$

$$- \sum_{\substack{w\in W\\ w\circ\lambda+\mu>0}} (-1)^{\ell(w)} \sum_{0<\nu<w\circ\lambda+\mu} K_q^\infty(\nu) K_q^1(w\circ\lambda+\mu-\nu).$$

Setting $q = 1$, and noting that $K_1^\infty(\beta) = 0$ if $\beta > 0$, we obtain the famous property of the Kazhdan–Lusztig polynomial:

Corollary 3-31. $$\dim V(\lambda)_{\lambda-\mu} = P_{w_{\lambda-\mu},w_\lambda}(1).$$

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