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We consider certain analogues of level-N Eisenstein series involving hyperbolic functions. By developing the method used in our previous work, we prove some relation formulas for these series at positive integers which include our previous results corresponding to the cases of level 1 and 2. Furthermore, using these results, we evaluate certain two-variable analogues of level-N Eisenstein series.

1. Introduction

In [Tsumura 2008], we considered an analogue of the Eisenstein series defined by

$$\mathcal{G}_k(i) = \sum_{m\in\mathbb{Z}\setminus\{0\}} \sum_{n\in\mathbb{Z}} \frac{(-1)^n}{\sinh(m\pi)(m+ni)^k} \tag{1-1}$$

$$= \sum_{m\in\mathbb{Z}\setminus\{0\}} \sum_{n\in\mathbb{Z}} \frac{1}{\sinh((m+ni)\pi)(m+ni)^k} \quad (k\in\mathbb{N}), \tag{1-2}$$

where $i=\sqrt{-1}$ and $\sinh x = (e^x - e^{-x})/2$. We evaluated $\mathcal{G}_{2p-1}(i)$ $(p\in\mathbb{N})$ in terms of π and the lemniscate constant ϖ defined by

$$\varpi = 2\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\Gamma(1/4)^2}{2\sqrt{2\pi}} = 2.6220575542921\ldots.$$

More precisely we gave

$$\mathcal{G}_{2p-1}(i) = \frac{2(-1)^p}{\pi} \sum_{j=1}^{p} (1-2^{1-2p+2j})\zeta(2p-2j)\big((-1)^j G_{2j}(i) - 2\zeta(2j)\big), \tag{1-3}$$

where $\zeta(s)$ is Riemann's zeta function and $G_{2j}(\tau)$ is the ordinary Eisenstein series defined by

$$G_{2j}(\tau) = \sum_{m\in\mathbb{Z}} \sum_{\substack{n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(m+n\tau)^{2j}} \tag{1-4}$$

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for $j \in \mathbb{N}$ and $\tau \in \mathbb{C}$ with $\operatorname{Im} \tau > 0$. Note that $G_2(\tau)$ is conditionally convergent with respect to the order of summation as above. We can view (1-3) as a double series analogue of the following formula given by Cauchy [1889] and Mellin [1902]:

$$\sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{(-1)^m}{\sinh(m\pi)m^{4k-1}} = \frac{2}{\pi}\sum_{j=0}^{2k}(1-2^{1-4k+2j})\zeta(4k-2j)(-1)^j(2^{1-2j}-1)\zeta(2j),$$

and similar formulas for the Dirichlet series involving hyperbolic functions; see, for example, [Berndt 1977; 1978; Meyer 2000].

As another type analogue of $G_{2j}(i)$, we considered

$$\mathfrak{C}_l^{\langle r\rangle}(i) = \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\sum_{n\in\mathbb{Z}} \frac{(\coth(m\pi))^r}{(m+ni)^l} \qquad (l \in \mathbb{N}_{\geq 3},\ r \in \mathbb{Z}_{\geq -1}), \tag{1-5}$$

where $\coth x = (e^x+e^{-x})/(e^x-e^{-x})$, and evaluated them in the case $l \equiv r \mod 2$; see [Tsumura 2009].

In [Komori et al. 2010], using a method completely different from the one in [Tsumura 2008; 2009], Komori, Matsumoto and the author evaluated

$$\mathcal{G}_k^{\langle r\rangle}(\tau) = \sum_{m\in\mathbb{Z}\setminus\{0\}}\sum_{n\in\mathbb{Z}} \frac{1}{\sinh((m+n\tau)\pi i/\tau)^r(m+n\tau)^k} \quad (r, k \in \mathbb{N}) \tag{1-6}$$

(and more generalized double series) for any $\tau \in \mathbb{C}$ with $\operatorname{Im} \tau > 0$.

In [Tsumura 2010] we considered analogues of level-2 Eisenstein series such as

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \frac{1}{\sinh((2m+1+(2n+1)i)\pi/2)(2m+1+(2n+1)i)^k}, \tag{1-7}$$

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \frac{1}{\cosh((2m+1+(2n+1)i)\pi/2)(2m+1+(2n+1)i)^k}, \tag{1-8}$$

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \frac{\tanh((2m+1+(2n+1)i)\pi/2)}{(2m+1+(2n+1)i)^l}, \tag{1-9}$$

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \frac{\coth((2m+1+(2n+1)i)\pi/2)}{(2m+1+(2n+1)i)^l} \tag{1-10}$$

for $k, l \in \mathbb{N}$ with $l \geq 3$, and evaluated them in terms of π and ϖ. Note that the level-N Eisenstein series is defined by

$$G_k(\tau; \underline{a} \bmod n) = \sum_{\substack{m\in\mathbb{Z}\\ m\equiv a_1 \bmod N}}\ \sum_{\substack{n\in\mathbb{Z}\\ n\equiv a_2 \bmod N\\ (m,n)\neq(0,0)}} \frac{1}{(m+n\tau)^k} \tag{1-11}$$

for $k \in \mathbb{N}_{\geq 2}$ and $\underline{a} = (a_1, a_2) \in \mathbb{Z}^2$ with $0 \leq a_1, a_2 < N$, which was studied by Hecke [Hecke 1937, Section 1] (see also, for example, [Koblitz 1993, Chapter III]).

In this paper, by developing the method used in [Tsumura 2008; 2009; 2010], we consider analogues of level-N Eisenstein series involving hyperbolic functions, namely

$$\mathfrak{C}_k^{\langle r\rangle}(\tau; \underline{a} \bmod n) = \sum_{\substack{m\in\mathbb{Z}\setminus\{0\} \\ m\equiv a_1 \bmod N}} \sum_{\substack{n\in\mathbb{Z} \\ n\equiv a_2 \bmod N}} \frac{\coth((m+n\tau)\pi i/N\tau)^r}{(m+n\tau)^k} \tag{1-12}$$

for $k \in \mathbb{N}_{\geq 2}$, $r \in \mathbb{Z}$ and $\underline{a} = (a_1, a_2) \in \mathbb{Z}^2$ with $0 \leq a_1, a_2 < N$. Note that (1-12) in the case $k = 2$ and $r = 2$ is conditionally convergent with respect to the order of summation as above. In fact, since $(\coth x)^2 = 1 + 1/(\sinh x)^2$, we have

$$\mathfrak{C}_2^{\langle 2\rangle}(\tau; \underline{a} \bmod n) = \sum_{\substack{m\in\mathbb{Z}\setminus\{0\} \\ m\equiv a_1 \bmod N}} \sum_{\substack{n\in\mathbb{Z} \\ n\equiv a_2 \bmod N}} \left(1 + \frac{1}{\sinh((m+n\tau)\pi i/N\tau)^2}\right) \frac{1}{(m+n\tau)^2}.$$

If we divide this double series into two parts, the first is conditionally convergent and the second is absolutely convergent. Considering $(\coth x)^{2\nu}$, we can inductively confirm that $\mathfrak{C}_2^{\langle 2\nu\rangle}(\tau; \underline{a} \bmod n)$ $(\nu \in \mathbb{N})$ is also conditionally convergent.

Outline of article. In Section 2, we state evaluation formulas for some quantities of the form (1-12) (see Theorem 2.1, whose proof is given in Section 3). We also evaluate (1-12) in terms of (1-11) and certain partial zeta values which will be defined by (2-4) (see Examples 2.5 and 2.6). This subsumes previous results on (1-5) corresponding to the case $(r, N) = (1, 1)$ [Tsumura 2009] and on (1-9) and (1-10) corresponding to the cases $(r, N) = (\pm 1, 2)$ [Tsumura 2010]. Here, for example, we give a new formula corresponding to the case $r = 2$:

$$\mathfrak{C}_4^{\langle 2\rangle}(i; (1, 1) \bmod 2) = -\frac{5\varpi^4 + 2\pi^3}{360}.$$

More generally, we give explicit formulas for level-N versions of these expressions (see Example 2.6). From these results, we evaluate the level-N version of (1-6), defined by

$$\mathcal{G}_k^{\langle r\rangle}(\tau; \underline{a} \bmod n) = \sum_{\substack{m\in\mathbb{Z}\setminus\{0\} \\ m\equiv a_1 \bmod N}} \sum_{\substack{n\in\mathbb{Z} \\ n\equiv a_2 \bmod N}} \frac{1}{\sinh((m+n\tau)\pi i/N\tau)^r (m+n\tau)^k} \tag{1-13}$$

(see Proposition 2.4; also Remark 3.9).

In Section 4, based on the results above, we evaluate a two-variable analogue of (1-11) defined by

$$\widetilde{G}_{j,k}(\tau;\ \underline{a} \bmod n) = \sum_{\substack{m\in\mathbb{Z}\\ m\equiv a_1 \bmod N}} \sum_{\substack{n\in\mathbb{Z}\\ n\equiv a_2 \bmod N\\ (m,n)\neq(0,0)}} \sum_{\substack{l\in\mathbb{Z}\\ l\equiv a_2 \bmod N\\ (m,l)\neq(0,0)}} \frac{1}{(m+l\tau)^j(m+n\tau)^k} \tag{1-14}$$

for $j,k\in\mathbb{N}_{\geq 2}$. Note that in the case $j=2$ or $k=2$, (1-14) is conditionally convergent with respect to the order of summation as above. We prove some relation formulas among $\widetilde{G}_{j,k}(\tau;\ \underline{a} \bmod n)$ and $\mathscr{G}_l^{\langle r\rangle}(\tau;\ \underline{a} \bmod n)$ (see Theorems 4.1 and 4.2), and evaluate $\widetilde{G}_{2p,2q}(i;\ \underline{a} \bmod n)$ (see Examples 4.3 and 4.4). For example, we obtain

$$\begin{aligned}\widetilde{G}_{4,4}(i;(1,1) \bmod 2) &= \sum_{\substack{m\in\mathbb{Z}\\ m\equiv 1 \bmod 2}} \sum_{\substack{n\in\mathbb{Z}\\ n\equiv 1 \bmod 2}} \sum_{\substack{l\in\mathbb{Z}\\ l\equiv 1 \bmod 2}} \frac{1}{(m+li)^4(m+ni)^4}\\ &= \frac{\varpi^8}{8960} - \frac{\varpi^4\pi^4}{17280} + \frac{\pi^7}{6048}.\end{aligned}$$

This paper contains a lot of examples of evaluation formulas. They were checked numerically using Mathematica 7.

2. Relation formulas for $\mathfrak{C}_k^{\nu}(\alpha)$

From now on, we set $N\in\mathbb{N}$, $\underline{a}=(a_1,a_2)\in\mathbb{Z}^2$ with $0\leq a_1,a_2<N$ and $\tau\in\mathbb{C}$ with $\operatorname{Im}\tau>0$. For convenience, we set

$$\mathfrak{a} = \underline{a} \bmod N.$$

Theorem 2.1. *For $r\in\mathbb{Z}$ and $p\in\mathbb{N}$, we have*

$$\begin{aligned}&(N\tau)^{2p+1}\mathfrak{C}_{2p+1}^{\langle r+1\rangle}(\tau;\mathfrak{a})\\ &\quad = \frac{2i}{\pi}\sum_{\omega=1}^{p}\zeta(2p-2\omega)(N\tau)^{2\omega+2}\mathfrak{C}_{2\omega+2}^{\langle r\rangle}(\tau;\mathfrak{a}) + 2\zeta(2p)\frac{(N\tau)^3}{\pi^2}\mathfrak{C}_3^{\langle r-1\rangle}(\tau;\mathfrak{a})\end{aligned} \tag{2-1}$$

and

$$(N\tau)^{2p+2}\mathfrak{C}_{2p+2}^{\langle r+1\rangle}(\tau;\mathfrak{a}) = \frac{2i}{\pi}\sum_{\omega=0}^{p}\zeta(2p-2\omega)(N\tau)^{2\omega+3}\mathfrak{C}_{2\omega+3}^{\langle r\rangle}(\tau;\mathfrak{a}). \tag{2-2}$$

We will prove this theorem in the next section. Note that if we know the values $\mathfrak{C}_3^{\langle -1\rangle}(\tau;\mathfrak{a})$ and $\mathfrak{C}_4^{\langle 0\rangle}(\tau;\mathfrak{a})$, then we can inductively evaluate $\mathfrak{C}_k^{\langle r\rangle}(\tau;\mathfrak{a})$ for $k\in\mathbb{N}_{\geq 3}$ and $r\in\mathbb{Z}_{\geq -1}$ with $k\equiv r \bmod 2$, as follows.

By the definition (1-12), we can see that

$$\mathfrak{C}_{2k}^{\langle 0\rangle}(\tau;\mathfrak{a})=\begin{cases} G_{2k}(\tau;\mathfrak{a}) & \text{if } a_1\neq 0,\\ G_{2k}(\tau;\mathfrak{a})-\tau^{-2k}\displaystyle\sum_{\substack{n\in\mathbb{Z}\\ n\equiv a_2 \bmod N}}\frac{1}{n^{2k}} & \text{if } a_1=0 \text{ and } a_2\neq 0,\\ N^{-2k}\big(G_{2k}(\tau)-2\tau^{-2k}\zeta(2k)\big) & \text{if } a_1=a_2=0,\end{cases}\tag{2-3}$$

for $k\in\mathbb{N}_{\geq 2}$. For simplicity, we define a certain partial zeta value by

$$\tilde{\zeta}(l;a \bmod N):=\sum_{\substack{n\in\mathbb{Z}\setminus\{0\}\\ n\equiv a \bmod N}}\frac{1}{n^l}\qquad (l\in\mathbb{N}_{\geq 2}).\tag{2-4}$$

The proof of the next proposition will be given in Section 3 as well.

Proposition 2.2. *With the same notation*,

$$\mathfrak{C}_{3}^{\langle -1\rangle}(\tau;\mathfrak{a})=\begin{cases} \dfrac{i\pi}{N\tau}G_2(\tau;\mathfrak{a}) & \textit{if } a_1\neq 0,\\ \dfrac{i\pi}{N\tau}\big(G_2(\tau;\mathfrak{a})-\tau^{-2}\tilde{\zeta}(2;a_2 \bmod N)\big) & \textit{if } a_1=0 \text{ and } a_2\neq 0,\\ \dfrac{i\pi}{N^3\tau}\big(G_2(\tau)-2\tau^{-2}\zeta(2)\big) & \textit{if } a_1=a_2=0.\end{cases}$$

From Theorem 2.1 and Proposition 2.2, we derive:

Theorem 2.3. *For $r\in\mathbb{Z}_{\geq -1}$ and $k\in\mathbb{N}_{\geq 3}$ with $k\equiv r \bmod 2$,*

$$\tau^k\pi^r\mathfrak{C}_k^{\langle r\rangle}(\tau;\mathfrak{a})\in\mathbb{Q}\big[\tau,\ \pi,\ \big\{\tilde{\zeta}(2j;a_2 \bmod N),\ G_{2j}(\tau;\mathfrak{a})\big\}_{j\in\mathbb{N}}\big].\tag{2-5}$$

Proof. We prove (2-5) by induction on $r\geq -1$. First we assume $r=-1$. Since $k\equiv r \bmod 2$ with $k\geq 3$, we can write $k=2p+3$ $(p\geq 0)$. Hence we further use induction on p. When $p=0$, namely $k=3$, we immediately obtain the assertion from Proposition 2.2. Furthermore, by (2-2) with $r=-1$, we have

$$\begin{aligned}\frac{\pi}{i}(N\tau)^{2p+3}&\mathfrak{C}_{2p+3}^{\langle -1\rangle}(\tau;\mathfrak{a})\\ &=-(N\tau)^{2p+2}\mathfrak{C}_{2p+2}^{\langle 0\rangle}(\tau;\mathfrak{a})+\frac{2i}{\pi}\sum_{\omega=0}^{p-1}\zeta(2p-2\omega)(N\tau)^{2\omega+3}\mathfrak{C}_{2\omega+3}^{\langle -1\rangle}(\tau;\mathfrak{a}).\end{aligned}$$

Hence, by (2-3), we obtain the assertion by induction on p in the case $r=-1$.

Next we assume that the induction hypotheses hold for r. By multiplying the both sides of (2-1) and of (2-2) by π^{r+1}, we obtain the assertion in the case of $r+1$. Thus we complete the proof. □

As we noted in Section 1, using the relation $1/(\sinh x)^2=(\coth x)^2-1$ and the binomial theorem, we have the following relation between $\mathfrak{C}_k^{\langle r\rangle}(\tau;\mathfrak{a})$ and $\mathscr{G}_k^{\langle r\rangle}(\tau;\mathfrak{a})$ defined by (1-13).

Proposition 2.4. *For* $\nu \in \mathbb{N}$,

$$\mathscr{G}_k^{\langle 2\nu\rangle}(\tau;\mathfrak{a}) = \sum_{j=0}^{\nu} \binom{\nu}{j} (-1)^{\nu-j} \mathfrak{C}_k^{\langle 2j\rangle}(\tau;\mathfrak{a}). \tag{2-6}$$

Therefore, for $l \in \mathbb{N}$ *and* $\nu \in \mathbb{N}$,

$$\tau^{2l}\pi^{2\nu}\mathscr{G}_{2l}^{\langle 2\nu\rangle}(\tau;\mathfrak{a}) \in \mathbb{Q}\Big[\tau,\ \pi,\ \big\{\tilde{\zeta}(2k; a_2 \bmod N),\ G_{2k}(\tau;\mathfrak{a})\big\}_{k\in\mathbb{N}}\Big]. \tag{2-7}$$

Hence we can evaluate $\mathscr{G}_{2l}^{\langle 2\nu\rangle}(\tau;\mathfrak{a})$ by using the result on $\mathfrak{C}_{2l}^{\langle 2j\rangle}(\tau;\mathfrak{a})$ (see below). We will consider $\mathscr{G}_{2l+1}^{\langle 2\nu+1\rangle}(\tau;\mathfrak{a})$ in Remark 3.9.

Example 2.5. In the case $N = 1$, we simply denote (1-12) by $\mathfrak{C}_k^{\langle r\rangle}(\tau)$. Then, combining Theorem 2.1, Proposition 2.2 and (2-3), we obtain

$$\begin{aligned}
\mathfrak{C}_3^{\langle -1\rangle}(\tau) &= i(-\pi^3 + 3\pi\tau^2 G_2(\tau))/(3\tau^3),\\
\mathfrak{C}_5^{\langle -1\rangle}(\tau) &= i(-2\pi^5 + 5\pi^3\tau^2 G_2(\tau) + 15\pi\tau^4 G_4(\tau))/(15\tau^5),\\
\mathfrak{C}_3^{\langle 1\rangle}(\tau) &= i(-4\pi^4 + 15\tau^2 G_2(\tau)\pi^2 - 45\tau^4 G_4(\tau))/(45\tau^3\pi),\\
\mathfrak{C}_5^{\langle 1\rangle}(\tau) &= i(-4\pi^6 + 7\tau^2 G_2(\tau)\pi^4 + 105\tau^4 G_4(\tau)\pi^2 - 315\tau^6 G_6(\tau))/(315\tau^5\pi),\\
\mathfrak{C}_4^{\langle 2\rangle}(\tau) &= (16\pi^6 - 84\tau^2 G_2(\tau)\pi^4 + 630\tau^4 G_4(\tau)\pi^2 - 945\tau^6 G_6(\tau))/(945\tau^4\pi^2),\\
\mathfrak{C}_6^{\langle 2\rangle}(\tau) &= (64\pi^8 - 180\tau^2 G_2(\tau)\pi^6 - 945\tau^4 G_4(\tau)\pi^4 + 9450\tau^6 G_6(\tau)\pi^2\\
&\qquad - 14175\tau^8 G_8(\tau))/(14175\tau^6\pi^2),\\
\mathfrak{C}_3^{\langle 3\rangle}(\tau) &= i(-44\pi^6 + 189\tau^2 G_2(\tau)\pi^4 - 945\tau^4 G_4(\tau)\pi^2 + 945\tau^6 G_6(\tau))/(945\tau^3\pi^3),\\
\mathfrak{C}_5^{\langle 3\rangle}(\tau) &= i(-4\pi^8 - 45\tau^2 G_2(\tau)\pi^6 + 1260\tau^4 G_4(\tau)\pi^4 - 4725\tau^6 G_6(\tau)\pi^2\\
&\qquad + 4725\tau^8 G_8(\tau))/(4725\tau^5\pi^3),\\
\mathfrak{C}_4^{\langle 4\rangle}(\tau) &= (208\pi^8 - 1080\tau^2 G_2(\tau)\pi^6 + 8505\tau^4 G_4(\tau)\pi^4 - 18900\tau^6 G_6(\tau)\pi^2\\
&\qquad + 14175\tau^8 G_8(\tau))/(14175\tau^4\pi^4),\\
\mathfrak{C}_6^{\langle 4\rangle}(\tau) &= (1024\pi^{10} - 2376\tau^2 G_2(\tau)\pi^8 - 30690\tau^4 G_4(\tau)\pi^6 + 270270\tau^6 G_6(\tau)\pi^4\\
&\qquad - 623700\tau^8 G_8(\tau)\pi^2 + 467775\tau^{10} G_{10}(\tau))/(467775\tau^6\pi^4).
\end{aligned}$$

The case $\tau = i$ was studied in [Tsumura 2009], and we recover the results found there. For example,

$$\begin{aligned}
\mathfrak{C}_4^{\langle 2\rangle}(i) &= \frac{42\varpi^4 + 16\pi^4 - 84\pi^3}{945},\\
\mathfrak{C}_4^{\langle 4\rangle}(i) &= \frac{27\varpi^8 + 567\varpi^4\pi^4 + 208\pi^8 - 1080\pi^7}{14175\pi^4}.
\end{aligned}$$

By Proposition 2.4, we can inductively evaluate $\mathscr{G}_{2\nu}^{\langle 2l\rangle}(\tau)$ in terms of $G_{2j}(\tau)$ and $\zeta(2k)$. This fact was already given in [Komori et al. 2010] by a totally different method. Here we recover, for example,

$$\mathscr{G}_4^{\langle 2\rangle}(i) = \frac{-21\varpi^4 + 37\pi^4 - 84\pi^3}{945}, \tag{2-8}$$

$$\mathscr{G}_4^{\langle 4\rangle}(i) = \frac{27\varpi^8 + 252\varpi^4\pi^4 - 587\pi^8 + 1440\pi^7}{14175\pi^4}. \tag{2-9}$$

Next we consider the case $\tau = \rho = e^{2\pi i/3}$. We recall the properties of $G_{2k}(\rho)$. For the details, see [Koblitz 1993; Nesterenko and Philippon 2001; Serre 1970; Waldschmidt 1999]; also [Komori et al. 2010]. Let

$$\widetilde{\varpi} = \frac{\Gamma(1/3)^3}{2^{4/3}\pi} = 2.42865064788758\cdots \tag{2-10}$$

which is an analogue of the lemniscate constant ϖ. Then we obtain $G_2(\rho) = 2\pi\rho/\sqrt{3}$,

$$G_6(\rho) = \frac{\widetilde{\varpi}^6}{35}, \quad G_{12}(\rho) = \frac{\widetilde{\varpi}^{12}}{7007}, \quad G_{18}(\rho) = \frac{\widetilde{\varpi}^{18}}{1440257}, \ldots \tag{2-11}$$

and $G_k(\rho) = 0$ for $k \ge 3$ with $6 \nmid k$. Using these results, we can evaluate $\mathfrak{C}_k^{\langle r\rangle}(\rho)$, similarly to the case $\tau = i$, for example,

$$\mathfrak{C}_4^{\langle 2\rangle}(\rho) = \frac{-27\widetilde{\varpi}^6 + 16\pi^6 - 56\sqrt{3}\pi^5}{945\rho\pi^2},$$

$$\mathfrak{C}_4^{\langle 4\rangle}(\rho) = \frac{-18900\widetilde{\varpi}^6\pi^2 + 7280\pi^8 - 25200\sqrt{3}\pi^7}{496125\rho\pi^4}.$$

From these results, we recover these formulas from [Komori et al. 2010]:

$$\mathscr{G}_4^{\langle 2\rangle}(\rho) = \frac{-27\widetilde{\varpi}^6 + 37\pi^6 - 56\sqrt{3}\pi^5}{945\rho\pi^2}, \tag{2-12}$$

$$\mathscr{G}_4^{\langle 4\rangle}(\rho) = \frac{270\widetilde{\varpi}^6 - 587\pi^6 + 960\sqrt{3}\pi^5}{14175\rho\pi^2}. \tag{2-13}$$

Example 2.6. We consider the case $N > 1$, $a_1 \neq 0$ and $a_2 \neq 0$. We simply denote the level-N Eisenstein series by $G_{2j}^{\frac{a}{}}(\tau)$ instead of $G_{2j}(\tau; \mathfrak{a})$. Then we have the following formulas which are explicit examples of the main result in this paper:

$$\mathfrak{C}_3^{\langle -1\rangle}(\tau; \mathfrak{a}) = iG_2^{\frac{a}{}}(\tau)\pi/(N\tau),$$

$$\mathfrak{C}_5^{\langle -1\rangle}(\tau; \mathfrak{a}) = i(G_2^{\frac{a}{}}(\tau)\pi^3 + 3N^2\tau^2 G_4^{\frac{a}{}}(\tau)\pi)/(3N^3\tau^3),$$

$$\mathfrak{C}_3^{\langle 1\rangle}(\tau; \mathfrak{a}) = i(G_2^{\frac{a}{}}(\tau)\pi^2 - 3N^2\tau^2 G_4^{\frac{a}{}}(\tau))/(3N\tau\pi),$$

$$\mathfrak{C}_5^{\langle 1\rangle}(\tau;\mathfrak{a}) = i(G_2^{\underline{a}}(\tau)\pi^4 + 15iN^2\tau^2G_4^{\underline{a}}(\tau)\pi^2 - 45N^4\tau^4G_6^{\underline{a}}(\tau))/(45N^3\tau^3\pi),$$
$$\mathfrak{C}_4^{\langle 2\rangle}(\tau;\mathfrak{a}) = (-4G_2^{\underline{a}}(\tau)\pi^4 + 30N^2\tau^2G_4^{\underline{a}}(\tau)\pi^2 - 45N^4\tau^4G_6^{\underline{a}}(\tau))/(45N^2\tau^2\pi^2),$$
$$\mathfrak{C}_6^{\langle 2\rangle}(\tau;\mathfrak{a}) = (-4G_2^{\underline{a}}(\tau)\pi^6 - 21N^2\tau^2G_4^{\underline{a}}(\tau)\pi^4 + 210N^4\tau^4G_6^{\underline{a}}(\tau)\pi^2 - 315N^6\tau^6G_8^{\underline{a}}(\tau))/(315N^4\tau^4\pi^2),$$
$$\mathfrak{C}_3^{\langle 3\rangle}(\tau;\mathfrak{a}) = i(9G_2^{\underline{a}}(\tau)\pi^4 - 45N^2\tau^2G_4^{\underline{a}}(\tau)\pi^2 + 45N^4\tau^4G_6^{\underline{a}}(\tau))/(45N\tau\pi^3),$$
$$\mathfrak{C}_5^{\langle 3\rangle}(\tau;\mathfrak{a}) = i(-G_2^{\underline{a}}(\tau)\pi^6 + 28N^2\tau^2G_4^{\underline{a}}(\tau)\pi^4 - 105N^4\tau^4G_6^{\underline{a}}(\tau)\pi^2 + 105N^6\tau^6G_8^{\underline{a}}(\tau))/(105N^3\tau^3\pi^3),$$
$$\mathfrak{C}_4^{\langle 4\rangle}(\tau;\mathfrak{a}) = (-8G_2^{\underline{a}}(\tau)\pi^6 + 63N^2\tau^2G_4^{\underline{a}}(\tau)\pi^4 - 140N^4\tau^4G_6^{\underline{a}}(\tau)\pi^2 + 105N^6\tau^6G_8^{\underline{a}}(\tau))/(105N^2\tau^2\pi^4),$$
$$\mathfrak{C}_6^{\langle 4\rangle}(\tau;\mathfrak{a}) = (-24G_2^{\underline{a}}(\tau)\pi^8 - 310N^2\tau^2G_4^{\underline{a}}(\tau)\pi^6 + 2730N^4\tau^4G_6^{\underline{a}}(\tau)\pi^4 - 6300N^6\tau^6G_8^{\underline{a}}(\tau)\pi^2 + 4725N^8\tau^8G_{10}^{\underline{a}}(\tau))/(4725N^4\tau^4\pi^4).$$

In [Tsumura 2010], we studied the case when $(N, a_1, a_2, \tau) = (2, 1, 1, i)$ and $r = \pm 1$, based on [Katayama 1978]. In this case, as mentioned in both of these papers, we see $G_2^{(1,1)}(i) = -\pi/4$, $G_{4k+2}^{(1,1)}(i) = 0$ and $G_{4k}^{(1,1)}(i) \in \mathbb{Q}\cdot\varpi^{4k}$ $(k \in \mathbb{N})$, which can be concretely calculated; for example,

$$G_4^{(1,1)}(i) = -\frac{\varpi^4}{48}, \quad G_8^{(1,1)}(i) = \frac{\varpi^8}{8960}, \quad G_{12}^{(1,1)}(i) = -\frac{\varpi^{12}}{1689600}.$$

Hence, by the formulas above, we can explicitly evaluate $\mathfrak{C}_k^{\langle r\rangle}(\tau; (1, 1) \bmod 2)$ when $k \equiv r \bmod 2$. In particular, when $r = \pm 1$, these coincide with the results given in [Tsumura 2010]. As examples in the cases $r = 2, 4$, we give

$$\mathfrak{C}_4^{\langle 2\rangle}(i; (1, 1) \bmod 2) = -\frac{5\varpi^4 + 2\pi^3}{360},$$
$$\mathfrak{C}_4^{\langle 4\rangle}(i; (1, 1) \bmod 2) = \frac{3\varpi^8 - 21\varpi^4\pi^4 - 8\pi^7}{1680\pi^4},$$

and

$$\mathscr{G}_4^{\langle 2\rangle}(i; (1, 1) \bmod 2) = \frac{5\varpi^8 - 4\pi^3}{720}, \tag{2-14}$$

$$\mathscr{G}_4^{\langle 4\rangle}(i; (1, 1) \bmod 2) = \frac{9\varpi^8 - 28\varpi^4\pi^4 + 32\pi^7}{5040\pi^4}. \tag{2-15}$$

3. Proofs of Theorem 2.1 and Proposition 2.2

For $\underline{a} = (a_1, a_2) \in \mathbb{Z}^2$ with $0 \le a_1, a_2 < N$, we set $\beta = (a_1 + a_2\tau)/N$ for simplicity. We fix a small $\varepsilon > 0$. For $u \in [1, 1+\varepsilon]$, $r \in \mathbb{Z}$ and $k \in \mathbb{N}$, we define

$$\widehat{\mathfrak{D}}_k^{\langle r\rangle}(\tau;\beta;u)=\sideset{}{^*}\sum_{m\in\mathbb{Z}}\sum_{n=1}^{\infty}\frac{u^{-n}\coth((m+\beta+n\tau)\pi i/\tau)^r}{\sinh((m+\beta+n\tau)\pi i/\tau)(m+\beta+n\tau)^k}+\sideset{}{^*}\sum_{m\in\mathbb{Z}}\sum_{n=1}^{\infty}\frac{u^{-n}\coth((m+\beta-n\tau)\pi i/\tau)^r}{\sinh((m+\beta-n\tau)\pi i/\tau)(m+\beta-n\tau)^k}+\sideset{}{^*}\sum_{m\in\mathbb{Z}}\frac{\coth((m+\beta)\pi i/\tau)^r}{\sinh((m+\beta)\pi i/\tau)(m+\beta)^k},\tag{3-1}$$

where $\sum_{m\in\mathbb{Z}}^{*}$ stands for the sum over $m\in\mathbb{Z}\setminus\{0\}$ if $a_1=0$ and over $m\in\mathbb{Z}$ if $a_1\neq 0$.

When $u>1$, we define $\widehat{\mathfrak{D}}_{-k}^{\langle r\rangle}(\tau;\beta;u)$ for $k\in\mathbb{Z}_{\geq 0}$ by (3-1). This is well-defined in the following sense. Since $\sinh(x)=0$ implies $x\in\pi i\mathbb{Z}$, the equality

$$\sinh\Bigl((m+\beta+n\tau)\frac{\pi i}{\tau}\Bigr)=\sinh\Bigl(\bigl(Nm+a_1+(Nn+a_2)\,\tau\bigr)\frac{\pi i}{N\tau}\Bigr)=0\quad(m,n\in\mathbb{Z})$$

implies $(a_1,a_2)=(0,0)$ and $m=0$. Similarly, $\cosh(x)=0$ implies $x\in\pi i/2+\pi i\mathbb{Z}$, so the equality

$$\cosh\Bigl((m+\beta+n\tau)\,\frac{\pi i}{\tau}\Bigr)=\cosh\Bigl(\bigl(Nm+a_1+(Nn+a_2)\,\tau\bigr)\frac{\pi i}{N\tau}\Bigr)=0\quad(m,n\in\mathbb{Z})$$

implies $(a_1,a_2)=(0,N/2)$ and $m=0$. Hence, by the definition of $\sum^*$ a few lines above, we see that (3-1) is absolutely convergent under the conditions above, that is, well-defined.

Since $\cosh(n\pi i)=(-1)^n$ and $\sinh(n\pi i)=0$, we can rewrite (3-1) as

$$\widehat{\mathfrak{D}}_k^{\langle r\rangle}(\tau;\beta;u)=\sideset{}{^*}\sum_{m\in\mathbb{Z}}\frac{\coth((m+\beta)\pi i/\tau)^r}{\sinh((m+\beta)\pi i/\tau)}\times\Biggl(\sum_{n=1}^{\infty}(-u)^{-n}\biggl(\frac{1}{(m+\beta+n\tau)^k}+\frac{1}{(m+\beta-n\tau)^k}\biggr)+\frac{1}{(m+\beta)^k}\Biggr).\tag{3-2}$$

When $k\geq 2$, we see that $\widehat{\mathfrak{D}}_k^{\langle r\rangle}(\tau;\beta;u)$ converges absolutely and uniformly for u in $[1,1+\varepsilon]$. Furthermore, when $k=1$, we have

$$\widehat{\mathfrak{D}}_1^{\langle r\rangle}(\tau;\beta;u)=\sideset{}{^*}\sum_{m\in\mathbb{Z}}\frac{2(m+\beta)\coth((m+\beta)\pi i/\tau)^r}{\sinh((m+\beta)\pi i/\tau)}\sum_{n=1}^{\infty}\frac{(-u)^{-n}}{(m+\beta)^2-n^2\tau^2}+\sideset{}{^*}\sum_{m\in\mathbb{Z}}\frac{\coth((m+\beta)\pi i/\tau)^r}{\sinh((m+\beta)\pi i/\tau)(m+\beta)},\tag{3-3}$$

which converges absolutely and uniformly for u in $[1,1+\varepsilon]$. Hence, for any $k\in\mathbb{N}$, we have

$$\lim_{u\to 1}\widehat{\mathfrak{D}}_k^{\langle r\rangle}(\tau;\beta;u)=\widehat{\mathfrak{D}}_k^{\langle r\rangle}(\tau;\beta;1)$$
$$=\sideset{}{^*}\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{N}}\frac{\coth((m+\beta+n\tau)\pi i/\tau)^r}{\sinh((m+\beta+n\tau)\pi i/\tau)(m+\beta+n\tau)^k}. \tag{3-4}$$

Now we let

$$\mathcal{S}_r(\theta;\tau;\beta)=\sideset{}{^*}\sum_{m\in\mathbb{Z}}\frac{\coth((m+\beta)\pi i/\tau)^r e^{(m+\beta)i\theta/\tau}}{\sinh((m+\beta)\pi i/\tau)}. \tag{3-5}$$

Set $A=\mathrm{Re}(i/\tau)$ and $B=\mathrm{Im}(i/\tau)$. Then $A>0$ because $\mathrm{Im}\,\tau>0$. We further let $D(R):=\{\theta\in\mathbb{C}:|\theta|<R\}$ be the closed disk of radius R, where $R>0$.

Lemma 3.1. *$\mathcal{S}_r(\theta;\tau;\beta)$ converges absolutely for $\theta\in D\,(A\pi/(A+|B|))$.*

Proof. Let $\theta\in D(A\pi/(A+|B|))$ and set $(a,b)=(\mathrm{Re}\,\theta,\mathrm{Im}\,\theta)$. Then

$$|a|,\ |b|<\frac{A\pi}{A+|B|}. \tag{3-6}$$

Here we consider the order of $\mathcal{S}_r(\theta;\tau;\beta)$, namely

$$\mathcal{S}_r(\theta;\tau;\beta)=O\left(e^{|m|\,\mathrm{Re}((\pm\theta-\pi)i/\tau)}\right)\qquad(|m|\to\infty),$$

which implies the maximum of two cases corresponding to $\pm\theta$. By (3-6), we have

$$\begin{aligned}\mathrm{Re}((\pm\theta-\pi)i/\tau)&=\mathrm{Re}((\pm a-\pi\pm bi)(A+Bi))=(\pm a-\pi)A\mp bB\\&\le(|a|-\pi)A+|b||B|<\left(\frac{A\pi}{A+|B|}-\pi\right)A+\frac{A|B|\pi}{A+|B|}=0.\end{aligned}$$

Therefore we have the assertion. □

As in [Tsumura 2008, § 2], we set

$$\mathfrak{H}(\theta;u):=-\frac{1}{2}\left(\frac{e^{\theta}}{e^{\theta}+u}+\frac{e^{-\theta}}{e^{-\theta}+u}\right) \tag{3-7}$$

for $\theta\in\mathbb{C}$ and $u\in[1,1+\varepsilon]$. This function is holomorphic for $\theta\in D(\pi)$, and satisfies

$$\lim_{u\to 1}\mathfrak{H}(\theta;u)=-\tfrac{1}{2}\qquad(\theta\in D(\pi)). \tag{3-8}$$

We also set

$$J_r(\theta;\tau;\beta;u):=\mathcal{S}_r(\theta;\tau;\beta)(2\mathfrak{H}(i\theta;u)+1). \tag{3-9}$$

Since $A\pi/(A+|B|)\le\pi$, it follows from Lemma 3.1 that $J_r(\theta;\tau;\beta;u)$ is holomorphic for $\theta\in D(A\pi/(A+|B|))$. Hence, for each $u\in[1,1+\varepsilon]$, we can expand $J_r(\theta;\tau;\beta;u)$ as

$$J_r(\theta;\tau;\beta;u)=\sum_{n=0}^{\infty}\Lambda_n^{\langle r\rangle}(\tau;\beta;u)\frac{\theta^n}{n!}\quad (\theta\in D(A\pi/(A+|B|))). \tag{3-10}$$

By Cauchy's integral theorem, for any $\gamma\in\mathbb{R}$ with $0<\gamma<A\pi/(A+|B|)$, we have

$$\frac{|\Lambda_n^{\langle r\rangle}(\tau;\beta;u)|}{n!}\le\frac{1}{2\pi}\int_{C_\gamma}|J_r(\theta;\tau;\beta;u)||z|^{-n-1}|dz|\le\frac{M_\gamma}{\gamma^n}\quad (n\in\mathbb{Z}_{\ge 0}), \tag{3-11}$$

where $C_\gamma: z=\gamma e^{it}$ $(0\le t\le 2\pi)$ and

$$M_\gamma:=\max_{(z,u)\in C_\gamma\times[1,1+\varepsilon]}|J_r(z;\tau;\beta;u)|.$$

Hence the right-hand side of (3-10) is uniformly convergent in $u\in[1,1+\varepsilon]$ if $\theta\in D(A\pi/(A+|B|))$. By (3-8) and (3-9), we have $J_r(\theta;\tau;\beta;u)\to 0$ as $u\to 1$. Therefore we see that

$$\Lambda_n^{\langle r\rangle}(\tau;\beta;u)\to 0\quad (u\to 1;\ n\in\mathbb{Z}_{\ge 0}). \tag{3-12}$$

Lemma 3.2. *For $u\in(1,1+\varepsilon]$ and $\theta\in D(A\pi/(A+|B|))$,*

$$J_r(\theta;\tau;\beta;u)=\sum_{j=0}^{\infty}\widehat{\mathfrak{D}}_{-j}^{\langle r\rangle}(\tau;\beta;u)\frac{\theta^j}{j!}, \tag{3-13}$$

that is, $\widehat{\mathfrak{D}}_{-j}^{\langle r\rangle}(\tau;\beta;u)=\Lambda_j^{\langle r\rangle}(\tau;\beta;u)$, for $j\in\mathbb{Z}_{\ge 0}$.

Proof. When $u>1$, from (3-7), we have (see [Tsumura 2008, Lemma 2.1])

$$\mathfrak{H}(i\theta;u)=\sum_{n=1}^{\infty}(-u)^{-n}\cos(n\theta).$$

Therefore, from (3-5) and (3-9), we have

$$\begin{aligned}J_r(\theta;\tau;\beta;u)&=\sum_{m\in\mathbb{Z}}{}^{*}\sum_{n=1}^{\infty}(-u)^{-n}\frac{\coth((m+\beta)\pi i/\tau)^r e^{(m+\beta+n\tau)i\theta/\tau}}{\sinh((m+\beta)\pi i/\tau)}\\&\quad+\sum_{m\in\mathbb{Z}}{}^{*}\sum_{n=1}^{\infty}(-u)^{-n}\frac{\coth((m+\beta)\pi i/\tau)^r e^{(m+\beta-n\tau)i\theta/\tau}}{\sinh((m+\beta)\pi i/\tau)}\\&\quad+\sum_{m\in\mathbb{Z}}{}^{*}\frac{\coth((m+\beta)\pi i/\tau)^r e^{(m+\beta)i\theta/\tau}}{\sinh((m+\beta)\pi i/\tau)}.\end{aligned} \tag{3-14}$$

Using the Maclaurin expansion of e^x and the definition of $\widehat{\mathfrak{D}}_{-k}^{\langle r\rangle}(\tau;\beta;u)$ in (3-1), namely in (3-2), we complete the proof. □

Lemma 3.3. *For $r \in \mathbb{Z}$ and $k \in \mathbb{N}$,*

$$N^{k+2}\mathfrak{C}_{k+2}^{\langle r\rangle}(\tau;\mathfrak{a}) = \sideset{}{^*}\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \frac{\coth((m+\beta+n\tau)\pi i/\tau)^r}{(m+\beta+n\tau)^{k+2}} = \sum_{j=0}^{[k/2]} \widehat{\mathfrak{D}}_{k+1-2j}^{\langle r\rangle}(\tau;\beta;1)\frac{(i\pi/\tau)^{2j+1}}{(2j+1)!}. \tag{3-15}$$

Proof. The first equality comes from the definition (1-12) and $\beta = (a_1 + a_2\tau)/N$. We prove the second equality. We first assume $k \in \mathbb{Z}_{\geq 0}$. For $u \in [1, 1+\varepsilon]$, we set

$$\begin{aligned}\Phi_r(\theta;k;\tau;\beta;u) = &\sideset{}{^*}\sum_{m\in\mathbb{Z}}\sum_{n=1}^{\infty}(-u)^{-n}\frac{\coth((m+\beta)\pi i/\tau)^r e^{(m+\beta+n\tau)i\theta/\tau}}{\sinh((m+\beta)\pi i/\tau)(m+\beta+n\tau)^{k+2}}\\ &+\sideset{}{^*}\sum_{m\in\mathbb{Z}}\sum_{n=1}^{\infty}(-u)^{-n}\frac{\coth((m+\beta)\pi i/\tau)^r e^{(m+\beta-n\tau)i\theta/\tau}}{\sinh((m+\beta)\pi i/\tau)(m+\beta-n\tau)^{k+2}}\\ &+\sideset{}{^*}\sum_{m\in\mathbb{Z}}\frac{\coth((m+\beta)\pi i/\tau)^r e^{(m+\beta)i\theta/\tau}}{\sinh((m+\beta)\pi i/\tau)(m+\beta)^{k+2}},\end{aligned} \tag{3-16}$$

which converges absolutely and uniformly in $u \in [1, 1+\varepsilon]$ if $\theta \in D(A\pi/(A+|B|))$. If $u > 1$, it follows from Lemma 3.2 that

$$\begin{aligned}\Phi_r(\theta;k;\tau;\beta;u) &= \sum_{j=0}^{\infty}\widehat{\mathfrak{D}}_{k+2-j}^{\langle r\rangle}(\tau;\beta;u)\frac{(i\theta/\tau)^j}{j!}\\ &= \sum_{j=0}^{k+1}\widehat{\mathfrak{D}}_{k+2-j}^{\langle r\rangle}(\tau;\beta;u)\frac{(i\theta/\tau)^j}{j!} + \sum_{j=k+2}^{\infty}\Lambda_{j-k-2}^{\langle r\rangle}(\tau;\beta;u)\frac{(i\theta/\tau)^j}{j!}.\end{aligned} \tag{3-17}$$

By considering

$$\lim_{u\to 1}\tfrac{1}{2}\left\{\Phi_r(\theta;k;\tau;\beta;u) - \Phi_r(-\theta;k;\tau;\beta;u)\right\},$$

and using (3-4) and (3-11), we can let $u \to 1$ on the both sides of (3-17) if θ lies in $D(A\pi/(A+|B|))$. By (3-12), we have

$$\frac{1}{2}\sideset{}{^*}\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\frac{(-1)^n\coth((m+\beta)\pi i/\tau)^r\left(e^{(m+\beta+n\tau)i\theta/\tau} - e^{-(m+\beta+n\tau)i\theta/\tau}\right)}{\sinh((m+\beta)\pi i/\tau)(m+\beta+n\tau)^{k+2}} = \sum_{\nu=0}^{[k/2]}\widehat{\mathfrak{D}}_{k+1-2\nu}^{\langle r\rangle}(\tau;\beta;1)\frac{(i\theta/\tau)^{2\nu+1}}{(2\nu+1)!} \tag{3-18}$$

for $\theta \in D(A\pi/(A+|B|))$. Moreover, we claim that the left-hand side of (3-18) is absolutely convergent on the region $\Omega(\tau) := \bigcup_{n\geq 1} \mathfrak{X}_n(\tau)$, where

$$\mathfrak{X}_n(\tau) = \left\{ \theta \in \mathbb{C} : \left| \theta - \left(1 - \frac{1}{n}\right)\pi \right| < \frac{A\pi}{(A+|B|)n} \right\}.$$

Actually we know that the left-hand side of (3-18) is

$$O\left(e^{|m|\mathrm{Re}((\pm\theta-\pi)i/\tau)}|m+\beta+n\tau|^{-k-2}\right) \qquad (|m|, |n| \to \infty).$$

Hence we aim to prove $\mathrm{Re}((\pm\theta - \pi)i/\tau) < 0$ for any $\theta \in \Omega(\tau)$. In fact, for any n and any $\theta \in \mathfrak{X}_n$, we set $(a, b) = (\mathrm{Re}\,\theta, \mathrm{Im}\,\theta)$. Then

$$|a| < \left(1 - \frac{1}{n}\right)\pi + \frac{A\pi}{(A+|B|)n} \quad \text{and} \quad |b| < \frac{A\pi}{(A+|B|)n}.$$

Hence, by recalling that $A = \mathrm{Re}(i/\tau)$ and $B = \mathrm{Im}(i/\tau)$, we obtain the claim, since

$$\begin{aligned}\mathrm{Re}((\pm\theta-\pi)i/\tau) &= \mathrm{Re}((\pm a-\pi\pm bi)(A+Bi)) = (\pm a-\pi)A \mp Bb \\ &\leq (|a|-\pi)A + |B|\,|b| < -\frac{A\pi}{n} + \frac{A^2\pi}{(A+|B|)n} + \frac{A|B|\pi}{(A+|B|)n} = 0.\end{aligned}$$

On the other hand, it is clear that the right-hand side of (3-18) is holomorphic for $\theta \in \Omega(\tau)$, so (3-18) holds for $\theta \in \Omega(\tau)$.

Finally we claim that $\Omega(\tau) \supset [(1-1/L)\pi, \pi)$, where $L = \max(1, |B|/2A)$. In order to prove this, we only have to prove $\mathfrak{X}_n(\tau) \cap \mathfrak{X}_{n+1}(\tau) \neq \varnothing$ for all $n \geq L$, because any $\mathfrak{X}_n(\tau)$ is the disk whose center is on the real axis. More precisely, we have to prove

$$\left(1 - \frac{1}{n}\right)\pi + \frac{A\pi}{(A+|B|)n} \geq \left(1 - \frac{1}{n+1}\right)\pi - \frac{A\pi}{(A+|B|)(n+1)},$$

if $n \geq L$. In fact, this can be easily verified. Hence we obtain the claim. Therefore (3-18) holds for $\theta \in [(1-1/L)\pi, \pi)$. If we set $\theta = \pi$ on the left-hand side of (3-18), we have

$$\sum_{m\in\mathbb{Z}}^{*} \sum_{n\in\mathbb{Z}} \frac{\coth((m+\beta+n\tau)\pi i/\tau)^r}{(m+\beta+n\tau)^{k+2}},$$

which is absolutely convergent if $k \geq 1$. Hence, by Abel's theorem, (3-18) holds for $\theta = \pi$, which implies (3-15). Thus we complete the proof. □

Remark 3.4. As stated in the proof, (3-18) holds for $k = 0$ if $\theta \in [(1-1/L)\pi, \pi)$, because the left-hand side of (3-18) converges absolutely even if $k = 0$ and θ is in $[(1-1/L)\pi, \pi)$. We claim that (3-18) holds for $\theta = \pi$ when $(k, r) = (0, 0)$.

In fact, by setting $(k, r, \theta) = (0, 0, \pi)$ on the left-hand side of (3-18), we have

$$\sum_{m\in\mathbb{Z}}^{*}\sum_{n\in\mathbb{Z}}\frac{1}{(m+\beta+n\tau)^2}=N^2G_2(\tau;\mathfrak{a})-\frac{\delta_{a_1,0}N^2}{\tau^2}\times\begin{cases}N^2\tilde{\zeta}(2;a_2 \bmod N) & \text{if } a_2\neq 0,\\ 2\zeta(2) & \text{if } a_2=0,\end{cases}$$

where $\delta_{p,q}$ is the Kronecker delta. Therefore it follows from Abel's theorem that (3-18) holds for $k = 0$ and $\theta = \pi$. Hence we obtain

$$\widehat{\mathfrak{D}}_1^{\langle 0\rangle}(\tau;\beta;1)=\frac{N^2\tau}{i\pi}G_2(\tau;\mathfrak{a})-\frac{\delta_{a_1,0}}{i\pi\tau}\begin{cases}N^2\tilde{\zeta}(2;a_2 \bmod N) & \text{if } a_2\neq 0,\\ 2\zeta(2) & \text{if } a_2=0.\end{cases}\tag{3-19}$$

For $k \in \mathbb{N}$, we differentiate (3-18) in $\theta \in [(1-1/L)\pi, \pi)$. Then

$$\frac{1}{2}\sum_{m\in\mathbb{Z}}^{*}\sum_{n\in\mathbb{Z}}\frac{(-1)^n\coth((m+\beta)\pi i/\tau)^r\left(e^{(m+\beta+n\tau)i\theta/\tau}+e^{-(m+\beta+n\tau)i\theta/\tau}\right)}{\sinh((m+\beta)\pi i/\tau)(m+\beta+n\tau)^{k+1}}=\sum_{\nu=0}^{[k/2]}\widehat{\mathfrak{D}}_{k+1-2\nu}^{\langle r\rangle}(\tau;\beta;1)\frac{(i\theta/\tau)^{2\nu}}{(2\nu)!}.\tag{3-20}$$

If $k \geq 2$, both sides on (3-20) converge absolutely and uniformly in $[(1-1/L)\pi, \pi]$. Hence, by letting $\theta \to \pi$, we have:

Lemma 3.5. *For $r \in \mathbb{Z}$ and $k \in \mathbb{N}$ with $k \geq 2$,*

$$N^{k+1}\mathfrak{C}_{k+1}^{\langle r+1\rangle}(\tau;\mathfrak{a})=\sum_{j=0}^{[k/2]}\widehat{\mathfrak{D}}_{k+1-2j}^{\langle r\rangle}(\tau;\beta;1)\frac{(i\pi/\tau)^{2j}}{(2j)!}.\tag{3-21}$$

Letting $k = 2p+\mu$ for $p \in \mathbb{N}$ and $\mu \in \{0, 1\}$ in (3-15) and (3-21), we have

$$N^{2p+2+\mu}\mathfrak{C}_{2p+2+\mu}^{\langle r\rangle}(\tau;\mathfrak{a})=\sum_{j=0}^{p}\widehat{\mathfrak{D}}_{2p+1+\mu-2j}^{\langle r\rangle}(\tau;\beta;1)\frac{(i\pi/\tau)^{2j+1}}{(2j+1)!},\tag{3-22}$$

$$N^{2p+1+\mu}\mathfrak{C}_{2p+1+\mu}^{\langle r+1\rangle}(\tau;\mathfrak{a})=\sum_{j=0}^{p}\widehat{\mathfrak{D}}_{2p+1+\mu-2j}^{\langle r\rangle}(\tau;\beta;1)\frac{(i\pi/\tau)^{2j}}{(2j)!}.\tag{3-23}$$

Note that (3-22) also holds for $p = 0$ if $\mu = 1$, because (3-15) holds for $k = 1$. Here we use the following result given in our previous work.

Lemma 3.6 [Tsumura 2007, Lemma 4.4]. *Let $\{P_{2h}\}$, $\{Q_{2h}\}$, $\{R_{2h}\}$ be sequences satisfying*

$$P_{2h}=\sum_{j=0}^{h}R_{2h-2j}\frac{(i\pi)^{2j}}{(2j)!},\quad Q_{2h}=\sum_{j=0}^{h}R_{2h-2j}\frac{(i\pi)^{2j}}{(2j+1)!}\qquad(h\in\mathbb{Z}_{\geq 0}).\tag{3-24}$$

Then

$$P_{2h} = -2\sum_{\omega=0}^{h} \zeta(2h-2\omega) Q_{2\omega} \qquad (h \in \mathbb{Z}_{\geq 0}). \tag{3-25}$$

Multiply the both sides of (3-22) and (3-23) by $\tau^{2p+2+\mu}$ and $\tau^{2p+1+\mu}$, respectively. Then apply Lemma 3.5 with $P_0 = Q_0 = R_0 = \tau^{1+\mu}\widehat{\mathfrak{D}}^{\langle r\rangle}_{1+\mu}(\tau;\beta;1)$ and

$$\begin{aligned} P_{2h} &= (N\tau)^{2p+1+\mu}\mathfrak{C}^{\langle r+1\rangle}_{2p+1+\mu}(\tau;\mathfrak{a}),\\ Q_{2h} &= \frac{1}{i\pi}(N\tau)^{2p+2+\mu}\mathfrak{C}^{\langle r\rangle}_{2p+2+\mu}(\tau;\mathfrak{a}),\\ R_{2h} &= \tau^{2h+1+\mu}\widehat{\mathfrak{D}}^{\langle r\rangle}_{2h+1+\mu}(\tau;\beta;1) \end{aligned}$$

for $h \in \mathbb{N}$. Then it follows from (3-25) that

$$\begin{aligned} (N\tau)^{2p+1+\mu}&\mathfrak{C}^{\langle r+1\rangle}_{2p+1+\mu}(\tau;\mathfrak{a}) \\ &= -2\sum_{\omega=1}^{p}\zeta(2p-2\omega)\frac{1}{i\pi}(N\tau)^{2\omega+2+\mu}\mathfrak{C}^{\langle r\rangle}_{2\omega+2+\mu}(\tau;\mathfrak{a}) \\ &\qquad - 2\zeta(2p)\tau^{1+\mu}\widehat{\mathfrak{D}}^{\langle r\rangle}_{1+\mu}(\tau;\beta;1) \end{aligned} \tag{3-26}$$

for $p \in \mathbb{N}$. In order to complete the proof of Theorem 2.1, we have to determine $\widehat{\mathfrak{D}}^{\langle r\rangle}_{1+\mu}(\tau;\beta;1)$ for $\mu = 0, 1$. As noted above, (3-22) holds for $p = 0$ when $\mu = 1$, namely

$$N^3\mathfrak{C}^{\langle r\rangle}_3(\tau;\mathfrak{a}) = \widehat{\mathfrak{D}}^{\langle r\rangle}_2(\tau;\beta;1)\frac{i\pi}{\tau}. \tag{3-27}$$

Moreover, we obtain the following.

Lemma 3.7. *For $r \in \mathbb{Z}$,*

$$\frac{i\pi}{\tau}\widehat{\mathfrak{D}}^{\langle r\rangle}_1(\tau;\beta;1) = \widehat{\mathfrak{D}}^{\langle r-1\rangle}_2(\tau;\beta;1) = \frac{N^3\tau}{i\pi}\mathfrak{C}^{\langle r-1\rangle}_3(\tau;\mathfrak{a}). \tag{3-28}$$

Proof. The second equality comes from (3-27) by replacing r with $r-1$. So we will prove the first equality.

As we stated in Remark 3.4, (3-18) holds for $k = 0$ if $\theta \in [(1-1/L)\pi, \pi)$. Hence we see that

$$\begin{aligned} \frac{1}{2}\sum_{m\in\mathbb{Z}}{}^{*}\sum_{n\in\mathbb{Z}}&\frac{(-1)^n\coth((m+\beta)\pi i/\tau)^r\left(e^{(m+\beta+n\tau)i\theta/\tau}-e^{-(m+\beta+n\tau)i\theta/\tau}\right)}{\sinh((m+\beta)\pi i/\tau)(m+\beta+n\tau)^2} \\ &= \widehat{\mathfrak{D}}^{\langle r\rangle}_1(\tau;\beta;1)\frac{i\theta}{\tau} \end{aligned} \tag{3-29}$$

holds for $\theta \in [(1-1/L)\pi, \pi)$. On the other hand, (3-20) with r replaced by $r-1$ and k by 1 becomes

$$\frac{1}{2}\sum_{m\in\mathbb{Z}}^{*}\sum_{n\in\mathbb{Z}}\frac{(-1)^n \coth((m+\beta)\pi i/\tau)^{r-1}\left(e^{(m+\beta+n\tau)i\theta/\tau}+e^{-(m+\beta+n\tau)i\theta/\tau}\right)}{\sinh((m+\beta)\pi i/\tau)(m+\beta+n\tau)^2} = \widehat{\mathfrak{D}}_2^{\langle r-1\rangle}(\tau;\beta;1), \tag{3-30}$$

which also holds for $\theta \in [(1-1/L)\pi, \pi)$. Now we subtract (3-30) from (3-29) of each side. Then we have

$$\frac{1}{2}\sum_{m\in\mathbb{Z}}^{*}\sum_{n\in\mathbb{Z}}\frac{(-1)^n \coth((m+\beta)\pi i/\tau)^{r-1}\Delta(\theta)}{\sinh((m+\beta)\pi i/\tau)(m+\beta+n\tau)^2} = \widehat{\mathfrak{D}}_1^{\langle r\rangle}(\tau;\beta;1)\frac{i\theta}{\tau} - \widehat{\mathfrak{D}}_2^{\langle r-1\rangle}(\tau;\beta;1), \tag{3-31}$$

where $\Delta(\theta)$ is equal to

$$\begin{aligned}
&\coth\Bigl((m+\beta)\frac{i\pi}{\tau}\Bigr)\bigl(e^{(m+\beta+n\tau)\frac{i\theta}{\tau}} - e^{-(m+\beta+n\tau)\frac{i\theta}{\tau}}\bigr) - \bigl(e^{(m+\beta+n\tau)\frac{i\theta}{\tau}} + e^{-(m+\beta+n\tau)\frac{i\theta}{\tau}}\bigr)\\
&= \frac{1}{2\sinh\Bigl((m+\beta)\frac{i\pi}{\tau}\Bigr)}\Bigl(\bigl(e^{(m+\beta)\frac{i\theta}{\tau}}+e^{-(m+\beta)\frac{i\theta}{\tau}}\bigr)\bigl(e^{(m+\beta)\frac{i\theta}{\tau}}e^{ni\theta}-e^{-(m+\beta)\frac{i\theta}{\tau}}e^{ni\theta}\bigr)\\
&\qquad -\bigl(e^{(m+\beta)\frac{i\theta}{\tau}} - e^{-(m+\beta)\frac{i\theta}{\tau}}\bigr)\bigl(e^{(m+\beta)\frac{i\theta}{\tau}}e^{ni\theta} + e^{-(m+\beta)\frac{i\theta}{\tau}}e^{ni\theta}\bigr)\Bigr)\\
&= \frac{i\sin(n\theta)}{\sinh((m+\beta)\pi i/\tau)}.
\end{aligned}$$

Therefore the left-hand side of (3-31) is absolutely and uniformly convergent in $\theta \in [(1-1/L)\pi, \pi]$. Hence, letting $\theta \to \pi$ on the both sides of (3-31) and noting $\sin(n\pi) = 0$, we have

$$0 = \widehat{\mathfrak{D}}_1^{\langle r\rangle}(\tau;\beta;1)\frac{i\pi}{\tau} - \widehat{\mathfrak{D}}_2^{\langle r-1\rangle}(\tau;\beta;1). \qquad \square$$

Proofs of Theorem 2.1 and Proposition 2.2. Combining (3-26) and (3-28), we obtain the proof of Theorem 2.1. Combining (3-19) and (3-28), we obtain the proof of Proposition 2.2. $\square$

Remark 3.8. The left-hand side of (3-29) in the case $\theta = \pi$ and $r = 2\nu$ ($\nu \in \mathbb{Z}_{\geq 0}$) coincides with $\mathfrak{C}_2^{\langle 2\nu\rangle}(\tau;\mathfrak{a})$, which is conditionally convergent as we noted in Section 1. Therefore, by Abel's theorem, we can let $\theta \to \pi$ on the both sides of (3-29). Hence we have

$$N^2\mathfrak{C}_2^{\langle 2\nu\rangle}(\tau;\mathfrak{a}) = \widehat{\mathfrak{D}}_1^{\langle 2\nu\rangle}(\tau;\beta;1)\frac{i\pi}{\tau} \qquad (\nu \in \mathbb{Z}_{\geq 0}). \tag{3-32}$$

Therefore, by (3-28), we have

$$\mathfrak{C}_2^{\langle 2\nu\rangle}(\tau;\mathfrak{a}) = \frac{i\pi}{N\tau}\mathfrak{C}_3^{\langle 2\nu-1\rangle}(\tau;\mathfrak{a}). \tag{3-33}$$

Remark 3.9. Combining Lemmas 3.3 and 3.7, and using Examples 2.5 and 2.6, we can inductively evaluate

$$\begin{aligned}\widehat{\mathfrak{D}}_{2p+1}^{\langle 2\nu\rangle}(\tau;\beta;1) &= \sideset{}{^*}\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{N}}\frac{\coth((m+\beta+n\tau)\pi i/\tau)^{2\nu}}{\sinh((m+\beta+n\tau)\pi i/\tau)(m+\beta+n\tau)^{2p+1}}\\ &= N^{2p+1}\sum_{\substack{j\in\mathbb{Z}\setminus\{0\}\\ j\equiv a_1 \bmod N}}\sum_{\substack{l\in\mathbb{Z}\\ l\equiv a_2 \bmod N}}\frac{(\coth((j+l\tau)\pi i/N\tau))^{2\nu}}{\sinh((j+l\tau)\pi i/N\tau)(j+l\tau)^{2p+1}}\end{aligned}$$

in terms of $G_{2k}(\tau;\mathfrak{a})$ and $\tilde{\zeta}(2d;a_2 \bmod N)$ $(k,d\in\mathbb{N})$. Therefore, by using the relation $1/(\sinh x)^2 = (\coth x)^2 - 1$ repeatedly, we see that

$$\mathcal{G}_{2j+1}^{\langle 2\nu+1\rangle}(\tau;\mathfrak{a}) = \sum_{\mu=0}^{\nu}\binom{\nu}{\mu}(-1)^{\nu-\mu}N^{-2j-1}\widehat{\mathfrak{D}}_{2j+1}^{\langle 2\nu\rangle}(\tau;\beta;1),$$

which can be evaluated in terms of $G_{2k}(\tau;\mathfrak{a})$ and $\tilde{\zeta}(2d;a_2 \bmod N)$.

4. Two-variable analogues of level-N Eisenstein series

In this section, we aim to evaluate two-variable analogues of level-N Eisenstein series $\widetilde{G}_{j,k}(\tau;\ \mathfrak{a})$ $(j,k\in\mathbb{N}_{\geq 2})$ defined by (1-14).

As well as in the previous section, we set $\beta = (a_1+a_2\tau)/N$ $(0\leq a_1,a_2<N)$. Since $\operatorname{Im}\tau>0$, namely $\operatorname{Re}(i/\tau)>0$, it follows from the binomial theorem that

$$\begin{aligned}\frac{1}{(\sinh((m+\beta)\pi i/\tau))^{2\nu}} &= 2^{2\nu}\frac{e^{-2\nu(m+\beta)\pi i/\tau}}{(1-e^{-2(m+\beta)\pi i/\tau})^{2\nu}}\\ &= 2^{2\nu}e^{-2\nu(m+\beta)\pi i/\tau}\sum_{j=0}^{\infty}\binom{j+2\nu-1}{2\nu-1}e^{-2j(m+\beta)\pi i/\tau},\end{aligned}$$

if $m>0$. By putting $\mu = j+\nu$, we conclude that this equals

$$\begin{aligned}&\frac{2^{2\nu}e^{-2\pi i\nu(m+\beta)/\tau}}{(2\nu-1)!}\sum_{\mu=\nu}^{\infty}(\mu+\nu-1)\cdots(\mu+1)\mu(\mu-1)\cdots(\mu-\nu+1)e^{-2\pi i(\mu-\nu)(m+\beta)/\tau}\\ &\qquad = \frac{2^{2\nu}}{(2\nu-1)!}\sum_{\mu=1}^{\infty}\mu\prod_{l=1}^{\nu-1}(\mu-l)(\mu+l)e^{-2\pi i\mu(m+\beta)/\tau}.\end{aligned} \tag{4-1}$$

Recall the Stirling numbers of the first kind, $\{c(n,k)\}$, defined by

$$F_n(X) = X(X-1)(X-2)\cdots(X-n+1) = \sum_{k=0}^{n} c(n,k)X^k$$

(see, for example, [Stanley 1997]). Using these numbers, we define $\{\alpha(n,k)\}$ by

$$\widetilde{\mathscr{F}}_n(X) = X\prod_{l=1}^{n-1}(X-l)(X+l) \left(= \frac{(-1)^n F_n(X)F_n(-X)}{X} \right) = \sum_{k=0}^{2n-1} \alpha(n,k)X^k. \tag{4-2}$$

Hence we have

$$\alpha(n,j) = (-1)^n \sum_{\omega=0}^{j+1} (-1)^\omega c(n, j+1-\omega)c(n,\omega)$$

for $0 \le j \le 2n-1$. Since $\widetilde{\mathscr{F}}_n(-X) = -\widetilde{\mathscr{F}}_n(X)$, we have $\alpha(n,2j) = 0$ for $0 \le j < n$. By (4-1), we have

$$\frac{1}{(\sinh((m+\beta)\pi i/\tau))^{2\nu}} = \frac{2^{2\mu}}{(2\nu-1)!} \sum_{j=1}^{\nu} \alpha(\nu, 2j-1) \sum_{\mu=1}^{\infty} \mu^{2j-1} e^{-2\pi i\mu(m+\beta)/\tau}, \tag{4-3}$$

when $m > 0$. Here we recall the summation formula from [Lipschitz 1889]:

$$\sum_{l\in\mathbb{Z}} \frac{1}{(z+l)^k} = (-1)^k \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} e^{2\pi i n z} \tag{4-4}$$

for $k \in \mathbb{N}$ with $k \ge 2$ and $z \in \mathbb{C}$ with $\operatorname{Im} z > 0$. This formula also holds for $k = 1$ as follows:

$$\lim_{L\to\infty} \sum_{l=-L}^{L} \frac{1}{z+l} = -\pi i - 2\pi i \sum_{n=1}^{\infty} e^{2\pi i n z} \tag{4-5}$$

for $z \in \mathbb{C}$ with $\operatorname{Im} z > 0$ (see [Pribitkin 2002, Section 5]).

We can set $z = -(m+\beta)/\tau$ in (4-4), because we have $\operatorname{Im}(-(m+\beta)/\tau) > 0$. Then we see that (4-3) is equal to

$$\begin{aligned} &\frac{1}{(\sinh((m+\beta)\pi i/\tau))^{2\nu}} \\ &\quad = \frac{2^{2\mu}}{(2\nu-1)!} \sum_{j=1}^{\nu} \alpha(\nu, 2j-1) \frac{(2j-1)!}{(2\pi i)^{2j}} \sum_{l\in\mathbb{Z}} \frac{1}{(-(m+\beta)/\tau+l)^{2j}} \\ &\quad = \frac{2^{2\mu}}{(2\nu-1)!} \sum_{j=1}^{\nu} \alpha(\nu, 2j-1) \frac{(2j-1)!}{(2\pi i/\tau)^{2j}} \sum_{l\in\mathbb{Z}} \frac{1}{(m+\beta+l\tau)^{2j}}, \end{aligned} \tag{4-6}$$

by replacing l by $-l$. This holds for $m>0$. When $m<0$, by replacing (m,l,β) by $(-m,-l,-\beta)$ in (4-6), we have

$$\begin{aligned}\frac{1}{(\sinh((-m-\beta)\pi i/\tau))^{2\nu}}\\ &= \frac{2^{2\mu}}{(2\nu-1)!}\sum_{j=1}^{\nu}\alpha(\nu,2j-1)\frac{(2j-1)!}{(2\pi i/\tau)^{2j}}\sum_{l\in\mathbb{Z}}\frac{1}{(-m-\beta-l\tau)^{2j}},\end{aligned}$$

which coincides with (4-6). This implies that (4-6) also holds for $m<0$.

On the other hand, by (1-13), we have

$$\begin{aligned}\mathscr{G}_{2q}^{\langle 2p\rangle}(\tau;\mathfrak{a}) &= \sum_{\substack{k\in\mathbb{Z}\setminus\{0\}\\ k\equiv a_1 \bmod N}}\sum_{\substack{l\in\mathbb{Z}\\ l\equiv a_2 \bmod N}}\frac{1}{\sinh((k+l\tau)\pi i/N\tau)^{2p}(k+l\tau)^{2q}}\\ &= N^{-2q}\sum_{m\in\mathbb{Z}}{}^{*}\sum_{n\in\mathbb{Z}}\frac{1}{\sinh((m+\beta+n\tau)\pi i/\tau)^{2p}(m+\beta+n\tau)^{2q}}\\ &= N^{-2q}\sum_{m\in\mathbb{Z}}{}^{*}\sum_{n\in\mathbb{Z}}\frac{1}{\sinh((m+\beta)\pi i/\tau)^{2p}(m+\beta+n\tau)^{2q}}\end{aligned}$$

for $p,q\in\mathbb{N}$. Therefore, by (4-6) for any $m\in\mathbb{Z}\setminus\{0\}$, we have

$$\begin{aligned}\mathscr{G}_{2q}^{\langle 2p\rangle}(\tau;\mathfrak{a}) = \frac{2^{2p}N^{-2q}}{(2p-1)!}\times\sum_{j=1}^{p}\alpha(p,2j-1)\frac{(2j-1)!}{(2\pi i/\tau)^{2j}}\\ \sum_{m\in\mathbb{Z}}{}^{*}\sum_{n\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}\frac{1}{(m+\beta+l\tau)^{2j}(m+\beta+n\tau)^{2q}}.\end{aligned}$$

By (1-14) and $\beta=(a_1+a_2\tau)/N$, we have

$$\begin{aligned}\sum_{m\in\mathbb{Z}}{}^{*}\sum_{n\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}\frac{1}{(m+\beta+l\tau)^{2j}(m+\beta+n\tau)^{2q}}\\ = N^{2j+2q}\Big(\widetilde{G}_{2j,2q}(\tau;\ \mathfrak{a}) - \frac{\delta_{a_1,0}}{\tau^{2j+2q}}\tilde{\zeta}(2j;a_2 \bmod N)\tilde{\zeta}(2q;a_2 \bmod N)\Big),\end{aligned}$$

where $\tilde{\zeta}$ is defined by (2-4). Combining these relations, we obtain:

Theorem 4.1. *For* $p,q\in\mathbb{N}$,

$$\begin{aligned}\mathscr{G}_{2q}^{\langle 2p\rangle}(\tau;\mathfrak{a}) = \frac{2^{2p}}{(2p-1)!}\sum_{j=1}^{p}\alpha(p,2j-1)\frac{(2j-1)!}{(2\pi i/N\tau)^{2j}}\\ \times\Big(\widetilde{G}_{2j,2q}(\tau;\ \mathfrak{a}) - \frac{\delta_{a_1,0}}{\tau^{2j+2q}}\tilde{\zeta}(2j;a_2 \bmod N)\tilde{\zeta}(2q;a_2 \bmod N)\Big).\end{aligned}\tag{4-7}$$

By multiplying the both sides of (4-7) by $\tau^{2q}\pi^{2p}$, we can inductively obtain the following theorem by Proposition 2.4 and the fact $G_{2k}(\tau)\in\mathbb{Q}[G_4(\tau),G_6(\tau)]$ for $k\in\mathbb{N}_{\geq 2}$ (see [Koblitz 1993, Chapter III, § 2]).

Theorem 4.2. *For $p,q\in\mathbb{N}$,*

$$\tau^{2(p+q)}\widetilde{G}_{2p,2q}(\tau;\mathfrak{a})\in\mathbb{Q}\big[\tau,\ \pi,\ \big\{\tilde{\zeta}(2k;a_2 \bmod N),\ G_{2k}(\tau;\mathfrak{a})\big\}_{k\in\mathbb{N}}\big].$$

In particular when $N=1$, put $\widetilde{G}_{2p,2q}(\tau)=\widetilde{G}_{2p,2q}(\tau;(0,0) \bmod 1)$. Then

$$\tau^{2(p+q)}\widetilde{G}_{2p,2q}(\tau)\in\mathbb{Q}\big[\tau,\ \pi,\ G_2(\tau),\ G_4(\tau),\ G_6(\tau)\big].$$

Actually, combining (4-7) and the results given in Section 2, we can concretely evaluate $\widetilde{G}_{2p,2q}(\tau;\ \mathfrak{a})$ as follows.

Example 4.3. We set $N=1$, $(a_1,a_2)=(0,0)$, $p=1,2$, $q=2$ and $\tau=i$. By (4-2), we see that $\alpha(1,1)=1$, $\alpha(2,1)=-1$ and $\alpha(2,3)=1$. By substituting (2-8) and (2-9) into (4-7), we obtain

$$\begin{aligned}\widetilde{G}_{2,4}(i)&=-\frac{\varpi^4\pi^2}{45}+\frac{2}{63}\pi^6-\frac{4}{45}\pi^5,\\ \widetilde{G}_{4,4}(i)&=\frac{1}{525}\varpi^8+\frac{2}{675}\varpi^4\pi^4-\frac{2}{135}\pi^8+\frac{8}{189}\pi^7.\end{aligned}$$

Set $\tau=\rho$. Then, by substituting (2-12) and (2-13) into (4-7), we obtain

$$\widetilde{G}_{2,4}(\rho)=\frac{\widetilde{\varpi}^6}{35}-\frac{2}{63}\pi^6+\frac{8\sqrt{3}}{135}\pi^5,\quad \widetilde{G}_{4,4}(\rho)=\rho\Big(-\frac{2}{135}\pi^8+\frac{16\sqrt{3}}{567}\pi^7\Big).$$

Example 4.4. We set $N=2$, $(a_1,a_2)=(1,1)$, $p=1,2$, $q=2$ and $\tau=i$. By substituting (2-14) and (2-15) into (4-7), we obtain

$$\begin{aligned}\widetilde{G}_{2,4}(i;(1,1) \bmod 2)&=\frac{\varpi^4\pi^2}{576}-\frac{\pi^5}{720},\\ \widetilde{G}_{4,4}(i;(1,1) \bmod 2)&=\frac{\varpi^8}{8960}-\frac{\varpi^4\pi^4}{17280}+\frac{\pi^7}{6048}.\end{aligned}$$

Remark 4.5. Pasles and Pribitkin [2001] studied two-variable Lipschitz summation formulas. At present, it is unclear whether or not the results stated above can be obtained from their formula.

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