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## DESCENT FOR DIFFERENTIAL GALOIS THEORY OF DIFFERENCE EQUATIONS: CONFLUENCE AND $q$ -DEPENDENCE

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**The present paper contains two results that generalize and improve constructions of Hardouin and Singer. In the case of a derivation, we prove that the parametrized Galois theory for difference equations constructed by Hardouin and Singer can be descended from a differentially closed to an algebraically closed field. In the second part of the paper, we show that the theory can be applied to deformations of  $q$ -series to study the differential dependence with respect to  $x \frac{d}{dx}$  and  $q \frac{d}{dq}$ . We show that the parametrized difference Galois group (with respect to a convenient derivation defined in the text) of the Jacobi Theta function can be considered as the Galoisian counterpart of the heat equation.**

### Introduction

The present paper contains two results that generalize and improve constructions of the second author and M. Singer [HS 2008]. In the case of a derivation, we perform a descent of the constructions in that paper from a differentially closed to an algebraically closed field. The latter being much smaller, this is of good use in applications. In the second part of the paper, we show that the theory can be applied to deformations of  $q$ -series, which appear in many settings like those of quantum invariants and modular forms, to study the differential dependence with respect to  $x \frac{d}{dx}$  and  $q \frac{d}{dq}$ .

A specific Galois theory was constructed in [HS 2008] to study the differential relations among solutions of a difference linear system. This was done by attaching to a linear difference system a differential Picard–Vessiot ring, that is, a differential splitting ring, and, therefore, a differential algebraic group, in the sense of Kolchin, which we will call the Galois  $\Delta$ -group. Roughly, this is a matrix group defined

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as the zero set of algebraic differential equations. Both the differential Picard–Vessiot ring and the Galois  $\Delta$ -group were proved in that paper to be well defined under the assumption that the difference operator and the derivations commute with each other and that the field of constants for the difference operator is differentially closed. The differential closure of a differential field  $K$  is an enormous field that contains a solution of any system of algebraic differential equations with coefficients in  $K$  that has a solution in some differential extension of  $K$ . When one works with  $q$ -difference equations, the subfield of the field of meromorphic functions over  $\mathbb{C}^*$  of constants for the homothety  $x \mapsto qx$  is the field of elliptic functions: its differential closure is a very big field. The same happens for the shift  $x \mapsto x + 1$ , whose field of constants are periodic functions. In their applications, Hardouin and Singer prove that one can always descend the Galois  $\Delta$ -group with an *ad hoc* argument. Here we prove that, in the case of one derivation, we can actually suppose that the field of constants is algebraically closed and that the Galois  $\Delta$ -group descends from a differentially closed field to an algebraically closed one (see Proposition 1.20). In Section 1 we also obtain that the properties and the results used in the applications descend to an algebraically closed field, namely:

- The differential transcendence degree of an extension generated by a fundamental solution matrix of the difference equation is equal to the differential dimension of the Galois  $\Delta$ -group (Proposition 1.8).
- The sufficient and necessary condition for solutions of rank 1 difference equations to be differentially transcendental (Proposition 1.9).
- The sufficient and necessary condition for a difference system to admit a linear differential system totally integrable with the difference system (Corollary 1.26).

The proof of the descent (see Proposition 1.16) is based on an idea of M. Wibmer [2010], which he used in a parallel difference setting (see also [Cohn 1965]). The differential counterpart of his method is the differential prolongation of ideals, which goes back at least to E. Cartan, E. Vessiot, E. R. Kolchin and more recently to B. Malgrange in the nonlinear theory, but has never been exploited in this context. For a more general version of Proposition 1.16, see [Wibmer 2011].

Differential prolongations have been used in [Granier 2011] to adapt the ideas of B. Malgrange to build a Galois theory for nonlinear  $q$ -difference equations, and in [Di Vizio and Hardouin 2011a; 2011b; 2011c] to prove the comparison theorem between Granier’s groupoid, the intrinsic parameterized Galois group and the parametrized Galois group in [HS 2008] (see also [Di Vizio and Hardouin 2010]). So it appears that the differential prolongations are a common denominator of several differential and difference Galois theories.

In the second part of the paper, we show that the theory applies to investigate the differential relations with respect to the derivations  $\frac{d}{dx}$  and  $\frac{d}{dq}$  among solutions of

$q$ -difference equations over the field  $k(q, x)$  of rational functions with coefficients in a field  $k$  of zero characteristic. For instance, one can consider the Jacobi theta function

$$\theta_q(x) = \sum_{n \in \mathbb{Z}} q^{n(n-1)/2} x^n,$$

which is a solution of the  $q$ -difference equation  $y(qx) = qxy(x)$ . Let

$$\ell_q = x \frac{\theta'_q(x)}{\theta_q(x)} \quad \text{and} \quad \delta = \ell_q(x)x \frac{d}{dx} + q \frac{d}{dq}.$$

In this specific case, we show below that the Galois  $\delta$ -group can be considered as the Galoisian counterpart of the heat equation. In Theorem 2.6 we show that, for solutions of rank 1 equations  $y(qx) = a(x)y(x)$  with  $a(x) \in k(q, x)$ , the following facts are equivalent:

- Having  $a(x) = \mu x^r \frac{g(qx)}{g(x)}$ , for some  $r \in \mathbb{Z}$ ,  $g \in k(q, x)$  and  $\mu \in k(q)$ .
- Satisfying an algebraic  $\frac{d}{dx}$ -relation (over some field we are going to specify in the text below).
- Satisfying an algebraic  $\delta$ -relation.

In the higher rank case, we deduce from a general statement a necessary and sufficient condition on the Galois group so that a system  $Y(qx) = A(x)Y(x)$  can be completed in a compatible (that is integrable) way, by two linear differential systems in  $\frac{d}{dx}$  and  $x\ell_q \frac{d}{dx} + q \frac{d}{dq}$  (see Proposition 1.24 and Corollary 2.9). The condition consists in the property of the group of being, up to a conjugation, contained in the subgroup of constants of GL, that is the differential subgroup of GL whose points have coordinates that are annihilated by the derivations. A result of Cassidy [1972] says that, for a proper Zariski dense differential subgroup of a simple linear algebraic group, this is always the case.

***A breviary of difference-differential algebra.*** For the reader's convenience, we briefly recall some basic definitions of differential and difference algebra, that we will use in the text. See [Cohn 1965] and [Levin 2008] for the general theory.

By a  $(\sigma, \Delta)$ -field we will mean a field  $F$  of zero characteristic, equipped with an automorphism  $\sigma$  and a set of commuting derivations  $\Delta = \{\partial_1, \dots, \partial_n\}$ , such that  $\sigma \partial_i = \partial_i \sigma$  for any  $i = 1, \dots, n$ . We will use the terms  $(\sigma, \Delta)$ -ring,  $\Delta$ -ring,  $\Delta$ -field,  $\partial$ -ring,  $\partial$ -field for  $\partial \in \Delta$ ,  $\sigma$ -ring,  $\sigma$ -field,  $\dots$ , with the evident meaning, analogous to the definition of  $(\sigma, \Delta)$ -field. Notice that, if  $F$  is a  $(\sigma, \Delta)$ -field, the subfield  $K = F^\sigma$  of  $F$  of the invariant elements with respect to  $\sigma$  is a  $\Delta$ -field. We will say that a  $\Delta$ -field, and in particular  $K$ , is  $\Delta$ -closed if any system of algebraic differential equations in  $\partial_1, \dots, \partial_n$  with coefficients in  $K$  having a solution in an extension of  $K$  has a solution in  $K$  (see [McGrail 2000]).

A  $(\sigma, \Delta)$ -ideal of a  $(\sigma, \Delta)$ -ring is an ideal invariant under  $\sigma$  and the derivations in  $\Delta$ . A maximal  $(\sigma, \Delta)$ -ideal is an ideal which is maximal with respect to the property of being a  $(\sigma, \Delta)$ -ideal. Similar definitions can be given for  $\Delta$ -ideals.

A  $(\sigma, \Delta)$ -extension of  $F$  is a ring extension  $R$  of  $F$  equipped with an extension of  $\sigma$  and of the derivations of  $\Delta$ , such that the commutativity conditions are preserved.

A ring of  $\Delta$ -polynomials with coefficients in  $F$  is a ring of polynomials in infinitely many variables

$$F\{X_1, \dots, X_\nu\}_\Delta := F[X_i^{(\alpha)}; i = 1, \dots, \nu],$$

equipped with the differential structure such that

$$\partial_k(X_i^{(\alpha)}) = X_i^{(\alpha+e_k)},$$

for any indices  $i = 1, \dots, \nu$ ,  $k = 1, \dots, n$ , and  $\underline{\alpha} = (\alpha_1, \dots, \alpha_n)$  in  $\mathbb{Z}_{\geq 0}^n$ , with  $\underline{\alpha} + e_k = (\alpha_1, \dots, \alpha_k + 1, \dots, \alpha_n)$ .

We will need the ring of rational  $\Delta$ -functions over  $\mathrm{GL}_\nu(F)$ . This is a localization of a ring of  $\Delta$ -polynomials in the variables  $X_{i,j}$ , for  $i, j = 1, \dots, \nu$ :

$$F\left\{X_{i,j}, \frac{1}{\det(X_{i,j})}\right\}_\Delta := F\{X_{i,j}\}_\Delta \left[\frac{1}{\det(X_{i,j})}\right],$$

equipped with the induced differential structure. We write  $X$  for  $(X_{i,j})$ ,  $\partial^\alpha(X)$  for  $\partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}(X)$  and  $\det X$  for  $\det(X_{i,j})$ . If  $\Delta$  is empty, the ring  $F\{X, \det X^{-1}\}_\Delta$  is nothing else than the ring of rational functions  $F[X, \det X^{-1}]$ . By a  $\Delta$ -relation (over  $F$ ) satisfied by a given matrix  $U$  with coefficients in a  $(\sigma, \Delta)$ -extension  $R$  of  $F$ , we mean an element of  $F\{X, \det X^{-1}\}_\Delta$  that vanishes at  $U$ . The  $\Delta$ -relations satisfied by a chosen  $U$  form a  $\Delta$ -ideal.

## 1. Differential Galois theory for difference equations

**Introduction to differential Galois theory for difference equations.** In this section, we briefly recall some results from [HS 2008]. Let  $F$  be a field of zero characteristic equipped with an automorphism  $\sigma$ . We denote the subfield of  $F$  of all  $\sigma$ -invariant elements by  $K = F^\sigma$ .

**Definition 1.1.** A  $(\sigma)$ -difference module  $\mathcal{M} = (M, \Sigma)$  over  $F$  (also called  $\sigma$ -module over  $F$  or a  $F$ - $\sigma$ -module, for short) is a finite-dimensional  $F$ -vector space  $M$  together with a  $\sigma$ -semilinear bijection  $\Sigma : M \rightarrow M$ , that is, a bijection  $\Sigma$  such that  $\Sigma(\lambda m) = \sigma(\lambda)\Sigma(m)$  for all  $(\lambda, m) \in F \times M$ .

One can attach a  $\sigma$ -difference module  $\mathcal{M}_A := (F^\nu, \Sigma_A)$ , with  $\Sigma_A : F^\nu \rightarrow F^\nu$ ,  $Y \mapsto A^{-1}\sigma(Y)$ , to a  $\sigma$ -difference system

$$(1-1) \quad \sigma(Y) = AY, \quad \text{with } A \in \mathrm{GL}_\nu(F) \text{ for some } \nu \in \mathbb{Z}_{>0},$$

so that the horizontal (that is invariant) vectors with respect to  $\Sigma_A$  correspond to the solutions of  $\sigma(Y) = AY$ . Conversely, the choice of an  $F$ -basis  $\underline{e}$  of a difference module  $\mathcal{M}$  leads to a  $\sigma$ -difference system  $\sigma(Y) = AY$ , with  $A \in \mathrm{GL}_v(F)$ , which corresponds to the equation for the horizontal vectors of  $\mathcal{M}$  with respect to  $\Sigma_A$  in the chosen basis.

We define a *morphism of ( $\sigma$ -)difference modules over  $F$*  to be an  $F$ -linear map between the underlying  $F$ -vector spaces, commuting with the  $\sigma$ -semilinear operators. As defined above, the ( $\sigma$ -)difference modules over  $F$  form a Tannakian category (see [Deligne 1990]), that is, a category equivalent over the algebraic closure  $\overline{K}$  of  $K$  to the category of finite-dimensional representations of an affine group scheme. The affine group scheme corresponding to the sub-Tannakian category generated by the ( $\sigma$ -)difference module  $\mathcal{M}_A$ , whose non-Tannakian construction we are going to sketch below, is called Picard–Vessiot group of (1-1). Its structure measures the algebraic relations satisfied by the solutions of (1-1).

Let  $\Delta := \{\partial_1, \dots, \partial_n\}$  be a set of commuting derivations of  $F$  such that, for all  $i = 1, \dots, n$ , we have  $\sigma \circ \partial_i = \partial_i \circ \sigma$ . In [HS 2008] it was proved, among other things, that the category of  $\sigma$ -modules carries also a  $\Delta$ -structure, that is, it is a differential Tannakian category as defined by Ovchinnikov [2009]. The latter is equivalent to a category of finite-dimensional representations of a differential group scheme (see [Kolchin 1973]), whose structure measures the differential relations satisfied by the solutions of the  $\sigma$ -difference modules. In the next section, we describe the Picard–Vessiot approach to the theory in [HS 2008] (in opposition to the differential Tannakian approach), that is, the construction of minimal rings containing the solutions of  $\sigma(Y) = AY$  and their derivatives with respect to  $\Delta$ , whose automorphism group is a concrete incarnation of the differential group scheme defined by the differential Tannakian equivalence. We will implicitly consider the usual Galois theory of ( $\sigma$ -)difference equations by allowing  $\Delta$  to be the empty set (see for instance [van der Put and Singer 1997]): we will informally refer to this theory and the objects considered in it as classical.

*$\Delta$ -Picard–Vessiot rings.* Let  $F$  be a  $(\sigma, \Delta)$ -field as above, with  $K = F^\sigma$ . Let us consider a  $\sigma$ -difference system

$$(1-2) \quad \sigma(Y) = AY,$$

with  $A \in \mathrm{GL}_v(F)$ , as in (1-1).

**Definition 1.2** [HS 2008, Definition 6.10]. A  $(\sigma, \Delta)$ -extension  $\mathcal{R}$  of  $F$  is a  *$\Delta$ -Picard–Vessiot extension for (1-2)* if

- (1)  $\mathcal{R}$  is a simple  $(\sigma, \Delta)$ -ring, that is, it has no nontrivial ideal stable under both  $\sigma$  and  $\Delta$ ,

- (2)  $\mathcal{R}$  is generated as a  $\Delta$ -ring by  $Z \in \mathrm{GL}_v(\mathcal{R})$  and  $1/\det(Z)$ , where  $Z$  is a fundamental solution matrix of (1-2).

One can formally construct such an object as follows. We consider the ring of rational  $\Delta$ -functions  $F\{X, \det X^{-1}\}_\Delta$ . We want to equip it with a  $(\sigma, \Delta)$ -algebra structure, respecting the commutativity conditions for  $\sigma$  and the  $\partial_i$ . Therefore, we set  $\sigma(X) = AX$  and

$$(1-3) \quad \begin{aligned} \sigma(X^\alpha) &= \sigma(\partial^\alpha X) = \partial^\alpha(\sigma(X)) = \partial^\alpha(AX) \\ &= \sum_{\underline{i}+\underline{j}=\alpha} \binom{\alpha_1}{i_1} \cdots \binom{\alpha_n}{i_n} \partial^i(A) X^j, \end{aligned}$$

for each multi-index  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ . The quotient  $\mathcal{R}$  of  $F\{X, \det X^{-1}\}_\Delta$  by a maximal  $(\sigma, \Delta)$ -ideal obviously satisfies the conditions of the definition above and hence is a  $\Delta$ -Picard–Vessiot ring.

**Proposition 1.3** [HS 2008, Propositions 6.14 and 6.16]. *Suppose the field  $K$  is  $\Delta$ -closed.*

- (1) *The ring of constants  $\mathcal{R}^\sigma$  of a  $\Delta$ -Picard–Vessiot ring  $\mathcal{R}$  for (1-2) is equal to  $K$ , that is, there are no new constants with respect to  $\sigma$ , compared to  $F$ .*
- (2) *Two  $\Delta$ -Picard–Vessiot rings for (1-2) are isomorphic as  $(\sigma, \Delta)$ -rings.*

*$\Delta$ -Picard–Vessiot groups.* From here through Corollary 1.12 (page 86), we will assume that  $K$  is a  $\Delta$ -closed field.

Let  $\mathcal{R}$  be a  $\Delta$ -Picard–Vessiot ring for (1-2). Notice that, as in classical Galois theory for  $(\sigma)$ -difference equations, the ring  $\mathcal{R}$  does not need to be a domain. One can show that it is in fact the direct sum of a finite number of copies of an integral domain, therefore one can consider the ring  $L$  of total fractions of  $\mathcal{R}$  which is isomorphic to the product of a finite number of copies of one field (see [HS 2008]).

**Definition 1.4.** The group  $\mathrm{Gal}^\Delta(\mathcal{M}_A)$  (also denoted  $\mathrm{Aut}^{\sigma, \Delta}(L/F)$ ) of the automorphisms of  $L$  that fix  $F$  and commute with  $\sigma$  and  $\Delta$  is called the  $\Delta$ -Picard–Vessiot group of (1-2). We will also call it the Galois  $\Delta$ -group of (1-2).

**Remark 1.5.** The group  $\mathrm{Gal}^\Delta(\mathcal{M}_A)$  consists in the  $K$ -points of a linear algebraic  $\Delta$ -subgroup of  $\mathrm{GL}_v(K)$  in the sense of Kolchin. That is a subgroup of  $\mathrm{GL}_v(K)$  defined by a  $\Delta$ -ideal of  $K\{X, \det X^{-1}\}_\Delta$ .

Below, we recall some fundamental properties of the Galois  $\Delta$ -group, which are the starting point for proving the Galois correspondence:

**Proposition 1.6** [HS 2008, Lemma 6.19]. (1) *The ring  $L^{\mathrm{Gal}^\Delta(\mathcal{M}_A)}$  of elements of  $L$  fixed by  $\mathrm{Gal}^\Delta(\mathcal{M}_A)$  is  $F$ .*

(2) Let  $H$  be an algebraic  $\Delta$ -subgroup of  $\text{Gal}^\Delta(\mathcal{M}_A)$ . If  $L^H = F$ , then  $H$  is equal to  $\text{Gal}^\Delta(\mathcal{M}_A)$ .

As we have already pointed out, the Galois  $\emptyset$ -group is an algebraic group defined over  $K$  and corresponds to the classical Picard–Vessiot group attached to the  $\sigma$ -difference system (1-2) (see [van der Put and Singer 1997; Sauloy 2004]):

**Proposition 1.7** [HS 2008, Proposition 6.21]. *The algebraic  $\Delta$ -group  $\text{Gal}^\Delta(\mathcal{M}_A)$  is a Zariski dense subset of  $\text{Gal}^\emptyset(\mathcal{M}_A)$ .*

*Differential dependence and total integrability.* The  $\Delta$ -Picard–Vessiot ring  $\mathcal{R}$  of (1-2) is a  $\text{Gal}^\Delta(\mathcal{M}_A)$ -torsor in the sense of Kolchin. This implies in particular that the  $\Delta$ -relations satisfied by a fundamental solution of the  $\sigma$ -difference system (1-2) are entirely determined by  $\text{Gal}^\Delta(\mathcal{M}_A)$ :

**Proposition 1.8** [HS 2008, Proposition 6.29]. *The  $\Delta$ -transcendence degree of  $\mathcal{R}$  over  $F$  is equal to the  $\Delta$ -dimension of  $\text{Gal}(\mathcal{M}_A)$ .*

Since the  $\Delta$ -subgroups of  $\mathbb{G}_a^n$  coincide with the zero set of a homogeneous linear  $\Delta$ -polynomial  $L(Y_1, \dots, Y_n)$  (see [Cassidy 1972]), we have:

**Proposition 1.9.** *Let  $a_1, \dots, a_n \in F$  and let  $S$  be a  $(\sigma, \Delta)$ -extension of  $F$  such that  $S^\sigma = K$ . If  $z_1, \dots, z_n \in S$  satisfy  $\sigma(z_i) - z_i = a_i$  for  $i = 1, \dots, n$ , then  $z_1, \dots, z_n \in S$  satisfy a nontrivial  $\Delta$ -relation over  $F$  if and only if there exists a nonzero homogeneous linear differential polynomial  $L(Y_1, \dots, Y_n)$  with coefficients in  $K$  and an element  $f \in F$  such that  $L(a_1, \dots, a_n) = \sigma(f) - f$ .*

*Proof.* If  $\Delta = \{\partial_1\}$ , the proposition coincides with Proposition 3.1 in [HS 2008]. The proof in the case of many derivations is a straightforward generalization of their argument.  $\square$

The following proposition relates the structural properties of the Galois  $\Delta$ -group (see the remark immediately below for the definition of constant  $\Delta$ -group) with the holonomy of a  $\sigma$ -difference system:

**Proposition 1.10.** *The following statements are equivalent:*

- (1) *The  $\Delta$ -Galois group  $\text{Gal}^\Delta(\mathcal{M}_A)$  is conjugate over  $K$  to a constant  $\Delta$ -group.*
- (2) *For all  $i = 1, \dots, n$ , there exists a  $B_i \in M_n F$  such that the set of linear systems*

$$\begin{aligned} \sigma(Y) &= AY \\ \partial_1 Y &= B_1 Y \\ &\dots \\ \partial_n Y &= B_n Y \end{aligned}$$



is integrable, that is, the matrices  $B_i$  and  $A$  satisfy the functional equations deduced from the commutativity of the operators:

$$\partial_i(B_j) + B_j B_i = \partial_j(B_i) + B_i B_j \quad \text{and} \quad \sigma(B_j)A = \partial_j(A) + AB_j,$$

for any  $i, j = 1, \dots, n$ .

*Proof.* The proof is a straightforward generalization of Proposition 2.9 in [HS 2008] to the case of several derivations.  $\square$

**Remark 1.11.** Let  $K$  be a  $\Delta$ -field and  $C$  its subfield of  $\Delta$ -constants. A linear  $\Delta$ -group  $G \subset \mathrm{GL}_\nu$  defined over  $K$  is said to be a constant  $\Delta$ -group (or  $\Delta$ -constant, for short) if one of the following equivalent statements hold:

- The set of differential polynomials  $\partial_h(X_{i,j})$ , for  $h = 1, \dots, n$  and  $i, j = 1, \dots, \nu$ , belong to the defining ideal of  $G$  in the differential Hopf algebra  $K\{X_{i,j}, \frac{1}{\det(X_{i,j})}\}_\Delta$  of  $\mathrm{GL}_\nu$  over  $K$ .
- The differential Hopf algebra of  $G$  over  $K$  is an extension of scalars of a finitely generated Hopf algebra over  $C$ .
- The points of  $G$  in  $K$  (which is  $\Delta$ -closed!) coincide with the  $C$ -points of an algebraic group defined over  $C$ .

For instance, let  $\mathbb{G}_m$  be the multiplicative group defined over  $K$ . Its differential Hopf algebra is  $K\{x, \frac{1}{x}\}_\Delta$ , that is, the  $\Delta$ -ring generated by  $x$  and  $\frac{1}{x}$ . The constant  $\Delta$ -group  $\mathbb{G}_m(C)$  corresponds to the differential Hopf algebra

$$\frac{K\{x, \frac{1}{x}\}_\Delta}{(\partial_h(x) ; h = 1, \dots, n)} \cong C\left[x, \frac{1}{x}\right] \otimes_C K,$$

where  $C\left[x, \frac{1}{x}\right]$  is a  $\Delta$ -ring with the trivial action of the derivations in  $\Delta$ .

According to [Cassidy 1972], if  $H$  is a Zariski dense  $\Delta$ -subgroup of a simple linear algebraic group  $G \subset \mathrm{GL}_\nu(K)$ , defined over a differentially closed field  $K$ , then either  $H = G$  or there exists  $P \in \mathrm{GL}_\nu(K)$  such that  $PHP^{-1}$  is a constant  $\Delta$ -subgroup of  $PGP^{-1}$ . Therefore:

**Corollary 1.12.** *If  $\mathrm{Gal}^\partial(\mathcal{M}_A)$  is simple, we are either in the situation of the proposition above or there are no  $\Delta$ -relations among the solutions of  $\sigma(Y) = AY$ .*

**Descent over an algebraically closed field.** For the remainder of this section, we consider a  $(\sigma, \Delta)$ -field  $F$ , where  $\sigma$  is an automorphism of  $F$  and  $\Delta = \{\partial\}$  is a set containing only one derivation. Moreover we suppose that  $\sigma$  commutes with  $\partial$ .

We have recalled above the theory developed in [HS 2008], where it was assumed that the field of constants  $K$  is differentially closed. Although in most applications in that paper an *ad hoc* descent argument proves that one can consider

smaller, nondifferentially closed, fields of constants, the assumption that  $K$  is  $\Delta$ -closed is quite restrictive. We show here that if the  $\sigma$ -constants  $K$  of  $F$  form an algebraically closed field, we can construct a  $\partial$ -Picard–Vessiot ring, whose ring of  $\sigma$ -constants coincides with  $K$ , which allows us to descend the group introduced in the previous section from a  $\partial$ -closed field to an algebraically closed field. This kind of results were first tackled in [Peón Nieto 2011] using model theoretic arguments. Here we use an idea of M. Wibmer of developing a differential analogue of [Wibmer 2010, Lemma 2.16]. Wibmer [2011] has given a more general version of Proposition 1.16. A Tannakian approach to the descent of parametrized Galois groups can be found in [Gillet et al. 2011].

For now, we do not make any assumption on  $K = F^\sigma$ . Let  $\Theta$  be the semi-group generated by  $\partial$  and, for all  $k \in \mathbb{Z}_{\geq 0}$ , let  $\Theta_{\leq k}$  be the set of elements of  $\Theta$  of order less or equal to  $k$ . We endow the differential rational function ring  $\mathcal{S} := K\{X, 1/\det(X)\}_{\partial}$ , where  $X = (X_{i,j})$ , with the grading associated to the usual ranking, that is, we consider for all  $k \in \mathbb{Z}_{\geq 0}$  the rational function ring

$$\mathcal{S}_k := K[\beta(X), 1/\det(X); \quad \text{for all } \beta \in \Theta_{\leq k}].$$

Of course, we have  $\partial(\mathcal{S}_k) \subset \mathcal{S}_{k+1}$ . It is convenient to set  $\mathcal{S}_{-1} = K$ .

**Definition 1.13** [Lando 1970]. The prolongation of an ideal  $\mathfrak{J}_k$  of  $\mathcal{S}_k$  is the ideal  $\pi_1(\mathfrak{J}_k)$  of  $\mathcal{S}_{k+1}$  generated by  $\Theta_{\leq 1}(\mathfrak{J}_k)$ . We say that a prime ideal  $\mathfrak{J}_k$  of  $\mathcal{S}_k$ , for  $k \geq 0$ , is a differential kernel of length  $k$  if the prime ideal  $\mathfrak{J}_{k-1} := \mathfrak{J}_k \cap \mathcal{S}_{k-1}$  of  $\mathcal{S}_{k-1}$  is such that  $\pi_1(\mathfrak{J}_{k-1}) \subset \mathfrak{J}_k$ .

Notice that by this definition, any prime ideal  $\mathfrak{J}_0$  of  $\mathcal{S}_0$  is a differential kernel.

**Remark 1.14.** Lando [1970] defines a differential kernel as a finitely generated field extension  $F(A_0, A_1, \dots, A_k)/F$ , together with an extension of  $\partial$  to a derivation of  $F(A_0, A_1, \dots, A_{k-1})$  into  $F(A_0, A_1, \dots, A_k)$  such that  $\partial(A_i) = A_{i+1}$  for  $i = 0, \dots, k-1$ . We can recover  $\mathfrak{J}_{k-1}$  as the kernel of the  $F$ -morphism

$$\mathcal{S}_k = F[X, \dots, \partial^k(X)] \rightarrow F(A_0, A_1, \dots, A_k), \quad \partial^i(X) \mapsto A_i.$$

We will not use Lando’s point of view here.

Notice that we have used a prolongation  $\pi_1$  due to Malgrange [2005] to rewrite Lando’s definition. One should pay attention to the fact that Malgrange also considers a weak prolongation  $\tilde{\pi}_1$  [Chapter V], which we won’t consider here.

**Proposition 1.15** [Lando 1970, Proposition 1]. *Let  $k \geq 0$  be an integer and let  $\mathfrak{J}_k$  be a differential kernel of  $\mathcal{S}_k$ . There exists a differential kernel  $\mathfrak{J}_{k+1}$  of  $\mathcal{S}_{k+1}$  such that  $\mathfrak{J}_k = \mathfrak{J}_{k+1} \cap \mathcal{S}_k$ .*

Let  $\sigma(Y) = AY$  be a  $\sigma$ -difference system with coefficient in  $F$  as (1-2),  $\mathcal{R}$  a  $\partial$ -Picard–Vessiot ring for  $\sigma(Y) = AY$ , constructed as on pages 83–84 under the

assumption that  $K$  is  $\partial$ -closed, and  $\mathfrak{J}$  be the defining ideal of  $\mathcal{R}$ , that is, the ideal such that

$$\mathcal{R} \cong F\{X, 1/\det(X)\}_{\partial}/\mathfrak{J}.$$

Then Proposition 6.21 in [HS 2008] says that  $\mathfrak{J}_k := \mathfrak{J} \cap \mathcal{S}_k$  is a maximal  $\sigma$ -ideal of  $\mathcal{S}_k$  endowed with the  $\sigma$ -structure induced by  $\sigma(X) = AX$ , which implies that  $\mathfrak{J}$  itself is a  $\sigma$ -maximal ideal of  $\mathcal{S}$ . In order to prove the descent of the  $\partial$ -Picard–Vessiot ring  $\mathcal{R}$ , we are going to proceed somehow in the opposite way. Without any assumption on  $K$ , we will construct a sequence  $(\mathfrak{J}_k)_{k \in \mathbb{N}}$  of  $\sigma$ -maximal ideals of  $\mathcal{S}_k$  such that  $\bigcup_{k \in \mathbb{N}} \mathfrak{J}_k$  is a  $\sigma$ -maximal ideal of  $\mathcal{S}$  stable by  $\partial$ . Such an ideal will provide us with a  $\partial$ -Picard–Vessiot ring  $\mathcal{R}^{\#}$ , which will be a simple  $\sigma$ -ring. If, moreover,  $K$  is algebraically closed, we will be able to compare its group of automorphisms and the Galois  $\partial$ -group of the previous section.

**Proposition 1.16.** *Let  $A \in \mathrm{GL}_v(F)$ . Then there exists a  $(\sigma, \partial)$ -extension  $\mathcal{R}^{\#}$  of  $F$  such that:*

- (1)  $\mathcal{R}^{\#}$  is generated over  $F$  as a  $\partial$ -ring by  $Z \in \mathrm{GL}_v(\mathcal{R}^{\#})$  and  $1/\det(Z)$  for some matrix  $Z$  satisfying  $\sigma(Z) = AZ$ .
- (2)  $\mathcal{R}^{\#}$  is a simple  $\sigma$ -ring, that is, it has no nontrivial ideals stable under  $\sigma$ .

**Remark 1.17.** Of course, a simple  $\sigma$ -ring carrying a  $\partial$ -ring structure is a simple  $(\sigma, \partial)$ -ring and thus  $\mathcal{R}^{\#}$  is a  $\partial$ -Picard–Vessiot ring in the sense of Definition 1.2.

*Proof.* Let  $\mathcal{S} = F\{X, \frac{1}{\det(X)}\}_{\partial}$  be the differential rational function  $\sigma$ -ring in the variables  $X = (X_{i,j})$  and let the ideals  $\mathcal{S}_k$ ,  $k \in \mathbb{N}$  be as above. We define a  $\sigma$ -ring structure on  $\mathcal{S}$  as in (1-3), so that, in particular,  $\sigma(X) = AX$ . We will prove by induction on  $k \geq 0$  that there exists a maximal  $\sigma$ -ideal  $\mathfrak{J}_k$  of  $\mathcal{S}_k$  such that  $\mathfrak{J}_k$  is a differential kernel of length  $k$  and  $\mathfrak{J}_{k-1} = \mathfrak{J}_k \cap \mathcal{S}_{k-1}$ . For  $k = 0$ , we can take  $\mathfrak{J}_0$  to be any  $\sigma$ -maximal ideal of  $\mathcal{S}_0$ . Then the  $\sigma$ -ring  $\mathcal{S}_0/\mathfrak{J}_0$  is a classical Picard–Vessiot ring for  $\sigma(Y) = AY$  in the sense of [van der Put and Singer 1997]. It follows from Proposition 1.15 that there exists a differential kernel  $\mathfrak{J}_1$  of  $\mathcal{S}_1$  such that  $\mathfrak{J}_1 \cap \mathcal{S}_0 = \mathfrak{J}_0$  and  $\pi_1(\mathfrak{J}_0) \subset \mathfrak{J}_1$ .

Now, let us construct  $\mathfrak{J}_{k+1}$  starting from  $\mathfrak{J}_{k-1}$  and  $\mathfrak{J}_k$ . By Corollary 1.16 in [van der Put and Singer 1997], both  $\mathfrak{J}_{k-1}$  and  $\mathfrak{J}_k$  can be written as intersections

$$\mathfrak{J}_{k-1} = \bigcap_{i=0}^{t_{k-1}} \mathfrak{J}_i^{(k-1)} \quad \text{and} \quad \mathfrak{J}_k = \bigcap_{i=0}^{t_k} \mathfrak{J}_i^{(k)},$$

where the  $\mathfrak{J}_i^{(k-1)}$  and  $\mathfrak{J}_i^{(k)}$  are prime ideals of  $\mathcal{S}_{k-1}$  and  $\mathcal{S}_k$ , respectively.<sup>1</sup> We shall assume that these representations are minimal and, so, unique. Then the prime

<sup>1</sup>We point out, although the remark plays no role in the proof, that the Ritt–Raudenbush theorem on the  $\partial$ -Noetherianity of  $\mathcal{S}$  implies that the sequence of integers  $(t_k)_{k \in \mathbb{N}}$  becomes stationary.

ideals  $\mathfrak{J}_i^{(k-1)}$  (respectively  $\mathfrak{J}_i^{(k)}$ ) are permuted by  $\sigma$ , and for any  $i = 1, \dots, t_k$ , there exists  $j \in \{1, \dots, t_{k-1}\}$  such that  $\mathfrak{J}_i^{(k)} \cap \mathcal{S}_{k-1} = \mathfrak{J}_j^{(k-1)}$ .

The last assertion means that, for all  $i = 0, \dots, t_k$ , the prime ideal  $\mathfrak{J}_i^{(k)}$  is a differential kernel of  $\mathcal{S}_k$ . Proposition 1.15 implies that  $\pi_1(\mathfrak{J}_i^{(k)})$ , and hence  $\bigcap_{i=0}^{t_k} \pi_1(\mathfrak{J}_i^{(k)})$ , is a proper  $\sigma$ -ideal of  $\mathcal{S}_{k+1}$ . Therefore there exists a  $\sigma$ -maximal ideal  $\mathfrak{I}_{k+1}$  of  $\mathcal{S}_{k+1}$  containing  $\bigcap_{i=0}^{t_k} \pi_1(\mathfrak{J}_i^{(k)})$ . Moreover, the  $\sigma$ -maximality of  $\mathfrak{I}_k$  and the inclusion  $\mathfrak{I}_k \subset \mathfrak{I}_{k+1} \cap \mathcal{S}_k$  imply that  $\mathfrak{I}_k = \mathfrak{I}_{k+1} \cap \mathcal{S}_k$ , which ends the recursive argument. The ideal  $\mathfrak{I} = \bigcup_{k \in \mathbb{N}} \mathfrak{I}_k$  is clearly  $\sigma$ -maximal in  $\mathcal{S}$  and  $\partial$ -stable. Then  $\mathcal{R}^\# := \mathcal{S}/\mathfrak{I}$  satisfies the requirements.  $\square$

**Remark 1.18.** By Lemma 6.8 in [HS 2008], where the assumption that the field  $K$  is  $\partial$ -closed plays no role, we have that there exists a set of idempotents  $e_0, \dots, e_r \in \mathcal{R}^\#$  such that  $\mathcal{R}^\# = \mathcal{R}_0 \oplus \dots \oplus \mathcal{R}_r$ , where  $\mathcal{R}_i = e_i \mathcal{R}^\#$  is an integral domain and  $\sigma$  permutes the set  $\{\mathcal{R}_0, \dots, \mathcal{R}_r\}$ . Let  $L_i$  be the fraction field of  $\mathcal{R}_i$ . The total field of fraction  $L^\#$  of  $\mathcal{R}^\#$  is equal to  $L_0 \oplus \dots \oplus L_r$ .

If  $\tilde{K}$  denotes a  $\partial$ -closure of  $K$  and  $\tilde{F}$  is the fraction field of  $\tilde{K} \otimes_K F$  equipped with an extension of  $\sigma$  that acts as the identity on  $\tilde{K}$ , then the construction of  $\mathcal{R}^\#$  over  $\tilde{F}$  gives a ring which is isomorphic to a  $\partial$ -Picard–Vessiot extension of the  $\sigma$ -difference system  $\sigma(Y) = AY$  viewed as a  $\sigma$ -difference system with coefficients in  $\tilde{F}$ , in the sense of Definition 1.2.

Notice that two Picard–Vessiot rings as in the proposition above may require a finitely generated extension of  $K$  to become isomorphic (see [Wibmer 2011]).

**Corollary 1.19.** *If  $K = F^\sigma$  is an algebraically closed field, the set of  $\sigma$ -constants of  $\mathcal{R}^\#$  is equal to  $K$ .*

*Proof.* Let  $c \in \mathcal{R}^\#$  be a  $\sigma$ -constant. In the notation of the previous proof, there exists  $k \in \mathbb{N}$  such that  $c \in \mathcal{R}_k := \mathcal{S}_k/\mathfrak{I}_k$ . By construction,  $\mathcal{R}_k$  is a simple  $\sigma$ -ring and a finitely generated  $F$ -algebra. By Lemma 1.8 in [van der Put and Singer 1997], the  $\sigma$ -constants of  $\mathcal{R}_k$  coincide with  $K$ .  $\square$

**Proposition 1.20.** *Let  $\sigma(Y) = AY$  be a linear  $\sigma$ -difference system with coefficients in  $F$  and let  $\mathcal{R}^\#$  be the  $\partial$ -Picard–Vessiot ring constructed in Proposition 1.16. If  $K = F^\sigma$  is algebraically closed, the functor*

$$(1-4) \quad \begin{array}{ccc} \text{Aut}^{\sigma, \partial} : K\text{-}\partial\text{-algebras} & \longrightarrow & \text{Groups} \\ S & \longmapsto & \text{Aut}^{\sigma, \partial}(\mathcal{R}^\# \otimes_K S/F \otimes_K S) \end{array}$$

*is representable by a linear algebraic  $\partial$ -group scheme  $G_A$  defined over  $K$ . Moreover,  $G_A$  becomes isomorphic to  $\text{Gal}^\partial(\mathcal{M}_A)$  over a differential closure of  $K$  (see Definition 1.4 above).*

**Remark 1.21.** Without getting into too many details, the representability of the functor (1-4) is precisely the definition of a linear algebraic  $\partial$ -group scheme  $G_A$  defined over  $K$ .

*Proof.* The first assertion is proved exactly as in [HS 2008, p. 368], where the authors only use the fact that the constants of the  $\partial$ -Picard–Vessiot ring do not increase with respect to the base field  $F$ , the assumption that  $K$  is  $\partial$ -closed being used to prove this property of  $\partial$ -Picard–Vessiot rings. The second assertion is just a consequence of the theory of differential Tannakian categories (see [Gillet et al. 2011]), which asserts that two differential fiber functors become isomorphic on a common  $\partial$ -closure of their fields of definition.  $\square$

**Definition 1.22.** We say that  $G_A$  is the  $\partial$ -group scheme attached to  $\sigma(Y) = AY$ .

Since we are working with schemes and not with the points of linear algebraic  $\partial$ -groups in a  $\partial$ -closure of  $K$ , we need to consider functorial definitions of  $\partial$ -subgroup schemes and invariants (see [Maurischat 2010] in the case of iterative differential equations). A  $\partial$ -subgroup functor  $H$  of the functor  $G_A$  is a  $\partial$ -group functor

$$H : K\text{-}\partial\text{-algebras} \rightarrow \text{Groups}$$

such that for all  $K$ - $\partial$ -algebra  $S$ , the group  $H(S)$  is a subgroup of  $G_A(S)$ . So, let  $L^\#$  be the total ring of fractions of  $\mathcal{R}^\#$  and let  $H$  be a  $\partial$ -subgroup functor of  $G_A$ . We say that  $r = \frac{a}{b} \in L^\#$ , with  $a, b \in \mathcal{R}^\#$ ,  $b$  not a zero divisor, is an invariant of  $H$  if for all  $K$ - $\partial$ -algebra  $S$  and all  $h \in H(S)$ , we have

$$h(a \otimes 1).(b \otimes 1) = (a \otimes 1).h(b \otimes 1).$$

We denote the ring of invariant of  $L^\#$  under the action of  $H$  by  $(L^\#)^H$ .

The Galois correspondence is proved by classical arguments starting from the following theorem.

**Theorem 1.23.** *Let  $H$  be a  $\partial$ -subgroup functor  $H$  of  $G_A$ . Then  $(L^\#)^H = K$  if and only if  $H = G_A$ .*

*Proof.* The proof relies on the same arguments as Theorem 11.4 in [Maurischat 2010] and Lemma 6.19 in [HS 2008].  $\square$

*Descent of the criteria for hypertranscendence.* The main consequence of Proposition 1.20 and Theorem 1.23 is that Propositions 1.8 and 1.9 remain valid if one only assumes that the  $\sigma$ -constants  $K$  of the base field  $F$  form an algebraically closed field and replaces the differential Picard–Vessiot group  $\text{Gal}^\Delta(\mathcal{M}_A)$  defined over the differential closure of  $K$  by the  $\Delta$ -group scheme  $G_A$  defined over  $K$ .

Gorchinskiy and Ovchinnikov [2012] exhibit a linear differential algebraic group defined over  $K$  which can be conjugated to a constant  $\partial$ -group only over a transcendental extension of  $K$ . Thus, one has to be careful when trying to prove a

descended version of Proposition 1.10. We go back to the multiple derivation case to state the following proposition, which is a generalization of Corollary 3.12 in [HS 2008]:

**Proposition 1.24.** *We suppose that  $F$  is a  $(\sigma, \Delta)$ -field,<sup>2</sup> with  $\Delta = \{\partial_1, \dots, \partial_n\}$ , such that  $K = F^\sigma$  is an algebraically closed field and  $F = K(x_1, \dots, x_n)$  a purely transcendental extension of transcendence basis  $x_1, \dots, x_n$ . We suppose also that  $\sigma$  induces an automorphisms of  $K(x_i)$  for any  $i = 1, \dots, n$ .*

*Let  $\tilde{K}$  be a  $\Delta$ -closure of  $K$  and  $\tilde{F}$  be the fraction field of  $\tilde{K} \otimes_K F$  equipped with an extension of  $\sigma$  acting as the identity on  $\tilde{K}$ , so that one can consider  $\text{Gal}^\Delta(\mathcal{M}_A)$  as in Definition 1.4.*

*We consider a  $\sigma$ -difference system  $\sigma(Y) = AY$  with coefficients in  $F$ . The following statements are equivalent:*

- (1)  *$\text{Gal}^\Delta(\mathcal{M}_A)$  is conjugate over  $\tilde{K}$  to a constant  $\Delta$ -group.*
- (2) *There exist  $B_1, \dots, B_n \in M_\nu F$  such that the system*

$$\begin{cases} \sigma(Y) = AY \\ \partial_i Y = B_i Y, \quad i = 1, \dots, n \end{cases}$$

*is integrable.*

**Remark 1.25.** Since  $\sigma$  induces an automorphism of  $K(x_i)$  for any  $i = 1, \dots, n$ , it acts on  $x_i$  through a Möbius transformation. By choosing another transcendence basis of  $F/K$ , we can suppose that either  $\sigma(x_i) = q_i x$  for some  $q_i \in K$  or that  $\sigma(x_i) = x_i + h_i$  for some  $h_i \in K$ . We are therefore in the most classical situation.

*Proof.* First of all we prove that the field  $\tilde{F}$  is a purely transcendental extension of  $\tilde{K}$ . Suppose that  $\sigma$  acts periodically on  $x_1$ , so that  $r$  is the minimal positive integer such that  $\sigma^r(x_1) = x_1$ . Then the polynomial

$$(T - x_1)(T - \sigma(x_1)) \cdots (T - \sigma^{r-1}(x_1))$$

has coefficients in  $K$  and vanishes at  $x_1$ . This is impossible since  $x_1$  is transcendental over  $K$ . Therefore we deduce that  $\sigma$  does not acts periodically on  $x_1$ , or on any element of the transcendence basis.

Suppose now that  $x_1, \dots, x_n$ , considered as elements of  $\tilde{F}$ , satisfy an algebraic relation over  $\tilde{K}$ . This means that there exists a nonzero polynomial  $P$  in  $\tilde{K}[T_1, \dots, T_n]$  such that  $P(x_1, \dots, x_n) = 0$ . We can suppose that there exists  $i_0 = 1, \dots, n$  such that  $\sigma(x_i) = q_i x$  for  $i \leq i_0$  and  $\sigma(x_i) = x_i + h_i$  for  $i > i_0$  (see remark above) and choose  $P$  so that the number of monomials appearing in its expression is minimal. Let  $T_1^{\alpha_1} \cdots T_n^{\alpha_n}$  be a monomial of maximal degree appearing in  $P$ . We can of course suppose that its coefficient is 1. Then the polynomial

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<sup>2</sup>We recall that we assume that the derivations of  $\Delta$  and  $\sigma$  commute with each other.

$P(q_1 T_1, \dots, q_{i_0} T_{i_0}, T_{i_0+1} + h_{i_0}, \dots, T_n + h_n)$  also annihilates at  $x_1, \dots, x_n$ . It contains a monomial of higher degree of the form  $q_1^{\alpha_1} \dots q_{i_0}^{\alpha_{i_0}} T_1^{\alpha_1} \dots T_n^{\alpha_n}$ . Therefore

$$P(T_1, \dots, T_n) - q_1^{-\alpha_1} \dots q_{i_0}^{-\alpha_{i_0}} P(q_1 T_1, \dots, q_{i_0} T_{i_0}, T_{i_0+1} + h_{i_0}, \dots, T_n + h_n)$$

also vanishes at  $x_1, \dots, x_n$  and contains less terms than  $P$ , against our assumptions on  $P$ . This proves that  $x_1, \dots, x_n$  are algebraically independent over  $\tilde{K}$  and hence that  $\tilde{F} = \tilde{K}(x_1, \dots, x_n)$ .

Now let us prove the proposition. The group  $\text{Gal}^\Delta(\mathcal{M}_A)$  is the  $\Delta$ -Galois group of  $\sigma(Y) = AY$  considered as a  $\sigma$ -difference system with coefficients in  $\tilde{F}$ . Then by Proposition 1.10, there exists  $\tilde{B}_i \in M_v \tilde{F}$  such that

$$(1-5) \quad \begin{aligned} \sigma(\tilde{B}_i) &= A \tilde{B}_i A^{-1} + \partial_i(A) A^{-1} \\ \partial_i(\tilde{B}_j) + \tilde{B}_j \tilde{B}_i &= \partial_j(\tilde{B}_i) + \tilde{B}_i \tilde{B}_j. \end{aligned}$$

Now, we can replace the coefficients in  $\tilde{B}_i$  before the monomials in  $\underline{x}$  by indeterminates. From (1-5), we obtain the system

$$\begin{aligned} \sigma(\bar{B}_i) &= A \bar{B}_i A^{-1} + \partial_i(A) A^{-1} \\ \partial_i(\bar{B}_j) + \bar{B}_j \bar{B}_i &= \partial_j(\bar{B}_i) + \bar{B}_i \bar{B}_j \end{aligned}$$

with indeterminate coefficients, which possess a specialization in  $\tilde{K}$ . By clearing the denominators and identifying the coefficients of the monomials in the transcendence basis  $\underline{x}$ , we see that this system is equivalent to a finite set of polynomial equations with coefficient in  $K$ . Since  $K$  is algebraically closed and this set of equations has a solution in  $\tilde{K}$ , it must have a solution in  $K$ . Thus, there exists  $B_i \in M_v(F)$  with the required properties.

On the other hand, if there exists  $B_i \in M_v F$  with the required commutativity properties, then it follows from Proposition 1.10 that  $\text{Gal}^\Delta(\mathcal{M}_A)$  is conjugate over  $\tilde{K}$  to a constant  $\Delta$ -group.  $\square$

Going back to the one-derivative situation, we have:

**Corollary 1.26.** *We suppose that  $F/K$  is a purely transcendental extension as in the previous proposition. Let  $\sigma(Y) = AY$  be a  $\sigma$ -difference system with coefficients in  $F$  and let  $G_A$  be its  $\partial$ -group scheme over  $K$ . The following statements are equivalent:*

- (1)  $G_A$  is conjugate over  $\tilde{K}$  to a constant  $\partial$ -group.
- (2) There exists  $B \in M_v(F)$  such that  $\sigma(B) = ABA^{-1} + \partial(A)A^{-1}$ , that is, such that the following system is integrable:

$$\begin{cases} \sigma(Y) = AY \\ \partial Y = BY. \end{cases}$$

*Proof.* Let us assume that  $G_A$  is conjugate over  $\tilde{K}$  to a constant  $\partial$ -group. Since  $G_A$  becomes isomorphic to  $\text{Gal}^\partial(\mathcal{M}_A)$  over  $\tilde{K}$ , the latter is conjugate over  $\tilde{K}$  to a constant  $\partial$ -group and we can conclude by the proposition above. On the other hand, if there exists  $B \in M_\nu(F)$  such that  $\sigma(B) = ABA^{-1} + \partial(A)A^{-1}$ , then it follows from Proposition 1.10 that  $\text{Gal}^\partial(\mathcal{M}_A)$  is conjugate over  $\tilde{K}$  to a constant  $\partial$ -group. Therefore the same holds for  $G_A$ . So we can apply the previous proposition.  $\square$

## 2. Confluence and $q$ -dependence

**Differential Galois theory for  $q$ -dependence.** Let  $k$  be a characteristic zero field,  $k(q)$  the field of rational functions in  $q$  with coefficients in  $k$  and  $K$  a finite extension of  $k(q)$ . We fix an extension  $|\cdot|$  to  $K$  of the  $q^{-1}$ -adic valuation on  $k(q)$ . This means that  $|\cdot|$  is defined on  $k[q]$  in the following way: there exists  $d \in \mathbb{R}$ ,  $d > 1$ , such that  $|f(q)| = d^{\deg_q(f)}$  for any  $f \in k[q]$ . It extends by multiplicativity to  $k(q)$ . By definition, we have  $|q| > 1$  and therefore it makes sense to consider elliptic functions with respect to  $|\cdot|$ . So let  $(C, |\cdot|)$  be the smallest valuated extension of  $(K, |\cdot|)$  which is both complete and algebraically closed,  $\text{Mer}(C^*)$  the field of meromorphic functions over  $C^* := C \setminus \{0\}$  with respect to  $|\cdot|$ , that is, the field of fractions of the analytic functions over  $C^*$ , and  $C_E$  the field of elliptic functions on the torus  $C^*/q^{\mathbb{Z}}$ , that is, the subfield of  $\text{Mer}(C^*)$  invariant with respect to the  $q$ -difference operators  $\sigma_q : f(x) \mapsto f(qx)$ .

Since the derivation  $\delta_q = q \frac{d}{dq}$  is continuous on  $k(q)$  with respect to  $|\cdot|$ , it naturally acts on the completion of  $K$  with respect to  $|\cdot|$ , and therefore on the completion of its algebraic closure, which coincides with  $C$  (see Chapter 3 in [Robert 2000]). It extends to  $\text{Mer}(C^*)$  by setting  $\delta_q x = 0$ . The fact that  $\delta_x = x \frac{d}{dx}$  acts on  $\text{Mer}(C^*)$  is straightforward. We notice that

$$\begin{aligned} \delta_x \circ \sigma_q &= \sigma_q \circ \delta_x, \\ \delta_q \circ \sigma_q &= \sigma_q \circ (\delta_x + \delta_q). \end{aligned}$$

This choice of the derivations is not optimal, in the sense that we would like to have two derivations commuting with each other and, more important, commuting with  $\sigma_q$ . We are going to reduce to this assumption in two steps. First of all, we consider the logarithmic derivative  $\ell_q(x) = \delta_x(\theta_q)/\theta_q$  of the Jacobi Theta function

$$\theta_q(x) = \sum_{n \in \mathbb{Z}} q^{-n(n-1)/2} x^n.$$

We recall that, if  $|q| > 1$ , the formal series  $\theta_q$  naturally defines a meromorphic function on  $C^*$  and satisfies the  $q$ -difference equation

$$\theta_q(qx) = qx\theta_q(x),$$



so that  $\ell_q(qx) = \ell_q(x) + 1$ . This implies that  $\sigma_q \delta_x (\ell_q) = \delta_x (\ell_q)$  and hence that  $\delta_x (\ell_q)$  is an elliptic function.

**Lemma 2.1.** *The derivations*

$$\delta_x \quad \text{and} \quad \delta = \ell_q(x)\delta_x + \delta_q$$

of  $\text{Mer}(C^*)$  commute with  $\sigma_q$ .

*Proof.* For  $\delta_x$ , it is clear. For  $\delta$ , we have

$$\begin{aligned} \delta \circ \sigma_q (f(q, x)) &= [\ell_q(x)\delta_x + \delta_q] \circ \sigma_q (f(q, x)) \\ &= \sigma_q \circ [(\ell_q(q^{-1}x) + 1)\delta_x + \delta_q] f(q, x) \\ &= \sigma_q \circ [\ell_q(x)\delta_x + \delta_q] f(q, x) \\ &= \sigma_q \circ \delta (f(q, x)). \end{aligned} \quad \square$$

**Corollary 2.2.** (1) *The derivations  $\delta_x, \delta$  of  $\text{Mer}(C^*)$  stabilize  $C_E$  in  $\text{Mer}(C^*)$ .*

(2) *The field of constants  $\text{Mer}(C^*)^{\delta_x, \delta}$  of  $\text{Mer}(C^*)$  with respect to  $\delta_x, \delta$  is equal to the algebraic closure  $\bar{k}$  of  $k$  in  $C$ .*

*Proof.* The first part of the proof immediately follows from the lemma above. The constants of  $\text{Mer}(C^*)$  with respect to  $\delta_x$  coincide with  $C$ . As far as the constants of  $C$  with respect to  $\delta$  is concerned, we are reduced to determining the constants  $C^{\delta_q}$  of  $C$  with respect to  $\delta_q$ . Since the topology induced by  $|\cdot|$  on  $k$  is trivial, one concludes that  $C^{\delta_q}$  is the algebraic closure of  $k$  in  $C$ .  $\square$

Since

$$[\delta_x, \delta] = \delta_x \circ \delta - \delta \circ \delta_x = \delta_x(\ell_q(x))\delta_x,$$

we can consider a  $(\delta_x, \delta)$ -closure  $\tilde{C}_E$  of  $C_E$  (see [Yaffe 2001; Pierce 2003; Singer 2007]). We extend  $\sigma_q$  to the identity of  $\tilde{C}_E$ . The  $(\delta_x, \delta)$ -field of  $\sigma_q$ -constants  $\tilde{C}_E$  almost satisfies the hypothesis of [HS 2008], apart from the fact that  $\delta_x, \delta$  do not commute with each other:

**Lemma 2.3.** *There exists  $h \in \tilde{C}_E$  satisfying the differential equation*

$$\delta(h) = \delta_x(\ell_q(x))h,$$

such that the derivations

$$\partial_1 = h\delta_x \quad \text{and} \quad \partial_2 = \delta = \ell_q(x)\delta_x + \delta_q$$

commute with each other and with  $\sigma_q$ .

*Proof.* Since  $\ell_q(qx) = \ell_q(x) + 1$ , we have  $\delta_x(\ell_q(x)) \in C_E$ . Therefore, we are looking for a solution  $h$  of a linear differential equation of order 1, with coefficients in  $C_E$ . Let us suppose that  $h \in \tilde{C}_E$  exists. Then since  $\sigma_q h = h$ , the identity

$\sigma_q \circ \partial_i = \partial_i \circ \sigma_q$  follows from Lemma 2.1 for  $i = 1, 2$ . The verification of the fact that  $\partial_1 \circ \partial_2 = \partial_2 \circ \partial_1$  is straightforward and, therefore, left to the reader.

We now prove the existence of  $h$ . Consider the differential rational function ring  $\mathcal{S} := C_E\{y, \frac{1}{y}\}_{\delta_x}$ . We endow  $\mathcal{S}$  with an extension of  $\delta$  by

$$\delta(\delta_x^n(y)) = \delta_x^{n+1}(\ell_q)y$$

for all  $n \geq 0$ . Since  $\delta_x \circ \delta = \delta_x(\ell_q(x))\delta_x + \delta \circ \delta_x$ , the definition of  $\delta$  over  $\mathcal{S}$  is consistent and the commutativity relation between  $\delta_x$  and  $\delta$  extends from  $C_E$  to  $\mathcal{S}$ . Now let  $\mathfrak{M}$  be a maximal  $(\delta_x, \delta)$ -maximal ideal of  $\mathcal{S}$ . Then, the ring  $\mathcal{S}/\mathfrak{M}$  is a simple  $(\delta_x, \delta)$ - $C_E$ -algebra. By van der Put and Singer [2003, Lemma 1.17], it is also an integral domain. Let  $L$  be the quotient field of  $\mathcal{S}/\mathfrak{M}$ . The field  $L$  is a  $(\delta_x, \delta)$ -field extension of  $C_E$  which contains a solution of the equation  $\delta(y) = \delta_x(\ell_q(x))y$ . Since  $\tilde{C}_E$  is the  $(\delta_x, \delta)$ -closure of  $C_E$ , there exists  $h \in \tilde{C}_E$  satisfying the differential equation  $\delta(h) = \delta_x(\ell_q(x))h$ .  $\square$

Let  $\Delta = \{\partial_1, \partial_2\}$ . Notice that, since  $\ell_q(qx) = \ell_q(x) + 1$  and  $\sigma_q$  commutes with  $\Delta$ , we have

$$\sigma_q \partial_1(\ell_q(x)) = \partial_1(\ell_q(x)) \quad \text{and} \quad \sigma_q(\partial_2(\ell_q(x))) = \partial_2(\ell_q(x)),$$

and thus  $\partial_i(\ell_q(x)) \in C_E$  for  $i = 1, 2$ . We conclude that the subfield  $C_E(x, \ell_q(x))$  of  $\text{Mer}(C^*)$  is actually a  $(\sigma_q, \Delta)$ -field. Moreover, extending the action of  $\sigma_q$  trivially to  $\tilde{C}_E$ , we can consider the  $(\sigma_q, \Delta)$ -field  $\tilde{C}_E(x, \ell_q(x))$ . Since the fields  $C_E(x, \ell_q)$  and  $\tilde{C}_E$  are linearly disjoint over  $C_E$  (see Lemma 6.11 in [HS 2008]),  $\tilde{C}_E(x, \ell_q(x))$  has a  $\Delta$ -closed field of constants, which coincide with  $\tilde{C}_E$ .

**Galois  $\Delta$ -group and  $q$ -dependence.** The previous subsection showed that one can attach two linear differential algebraic groups to a  $q$ -difference system  $\sigma_q(Y) = A(x)Y$  with  $A \in \text{GL}_n(C(x))$ :

(1) The group  $\text{Gal}^\Delta(\mathcal{M}_A)$  obtained by applying Definition 1.4 to the  $(\sigma_q, \Delta)$ -field  $\tilde{C}_E(x, \ell_q)$ . This group is defined over  $\tilde{C}_E$  and measures all differential relations satisfied by the solutions of the  $q$ -difference equation with respect to  $\delta_x$  and  $\delta_q$ . However its computation may be a little difficult. Indeed, since the derivations of  $\Delta$  are themselves defined above  $\tilde{C}_E$ , there is no hope of a general descent argument. Nonetheless, in some special cases, one can use the linear disjunction of the field  $\tilde{C}_E^\Delta$  of  $\Delta$ -constants and  $C_E(x, \ell_q)$  above  $C_E$  to simplify the computations.

(2) The Galois  $\partial_2$ -group  $\text{Gal}^{\partial_2}(\mathcal{M}_A)$  which corresponds to Definition 1.4 applied to the  $(\sigma_q, \partial_2)$ -field  $\tilde{C}_E(x, \ell_q)$ .

Let us consider the  $q$ -difference system

$$(2-1) \quad Y(qx) = A(x)Y(x), \quad \text{with } A \in \text{GL}_v(C_E(x, \ell_q)).$$

In view of Proposition 1.20, the Galois  $\partial_2$ -group  $\text{Gal}^{\partial_2}(\mathcal{M}_A)$  attached to (2-1) is defined above the algebraic closure  $\overline{C}_E$  of  $C_E$ . We will prove below that, in fact, it descends to  $C_E$  and thus reduce all the computations to calculus over the field of elliptic functions.

The field  $C_E(x, \ell_q)$  is a subfield of the field of meromorphic functions over  $C^*$ , therefore (2-1) has a fundamental solution matrix  $U \in \text{GL}_v(\text{Mer}(C^*))$ . In fact, the existence of such a fundamental solution  $U$  is actually equivalent to the triviality of the pullback on  $C^*$  of vector bundles over the torus  $C^*/q^{\mathbb{Z}}$  (see [Praagman 1986] for an analytic argument). The  $(\sigma_q, \partial_2)$ -ring

$$\mathcal{R}_{\text{Mer}} := C_E(x, \ell_q)\{U, \det U^{-1}\}_{\partial_2} \subset \text{Mer}(C^*)$$

is generated as a  $(\sigma_q, \partial_2)$ -ring by a fundamental solution  $U$  of (2-1) and by  $\det U^{-1}$  and has the property that  $\mathcal{R}_{\text{Mer}}^{\sigma_q} = C_E$ , in fact

$$C_E \subset \mathcal{R}^{\sigma_q} \subset \text{Mer}(C^*)^{\sigma_q} = C_E.$$

Notice that  $\mathcal{R}_{\text{Mer}}$  does not need to be a simple  $(\sigma_q, \partial_2)$ -ring. For this reason we call it a weak  $\partial_2$ -Picard–Vessiot ring. We have:

**Proposition 2.4.**  *$\text{Aut}^{\sigma_q, \partial_2}(\mathcal{R}_{\text{Mer}}/C_E(x, \ell_q))$  consists of the  $C_E$ -points of a linear algebraic  $\partial_2$ -group  $G_{C_E}$  defined over  $C_E$  such that  $G_{C_E} \otimes_{C_E} \overline{C}_E \cong \text{Gal}^{\partial_2}(\mathcal{M}_A)$ .*

*Proof.* See the proof of Theorem 3.5 in [Di Vizio and Hardouin 2011c], which gives an analogous statement for the derivation  $x \frac{d}{dx}$ .  $\square$

Therefore, as in Proposition 1.9, one can prove:

**Corollary 2.5.** *Let  $a_1, \dots, a_n \in C_E(x, \ell_q)$  and let  $S \subset \text{Mer}(C^*)$  be a  $(\sigma_q, \partial_2)$ -extension of  $C_E(x, \ell_q)$  such that  $S^{\sigma_q} = C_E$ . If  $z_1, \dots, z_n \in S$  satisfy  $\sigma(z_i) - z_i = a_i$  for  $i = 1, \dots, n$ , then  $z_1, \dots, z_n \in S$  satisfy a nontrivial  $\partial_2$ -relation over  $C_E(x, \ell_q)$  if and only if there exists a nonzero homogeneous linear differential polynomial  $L(Y_1, \dots, Y_n)$  with coefficients in  $C_E$  and an element  $f \in C_E(x, \ell_q)$  such that  $L(a_1, \dots, a_n) = \sigma(f) - f$ .*

*Proof.* The proof strictly follows Proposition 3.1 in [HS 2008], so we only sketch it. First of all, if such an  $L$  exists,  $L(z_1, \dots, z_n) - f$  is  $\sigma_q$ -invariant and therefore belongs to  $C_E$ , which gives a nontrivial  $\partial_2$ -relation among the  $z_1, \dots, z_n$ . On the other hand, we can suppose that  $S$  coincides with the ring  $\mathcal{R}_{\text{Mer}}$  introduced above for the system  $\sigma_q Y = AY$ , where  $A$  is the diagonal block matrix

$$A = \text{diag} \left( \begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & a_n \\ 0 & 1 \end{pmatrix} \right).$$

In fact, a solution matrix of  $\sigma_q Y = AY$  is given by

$$U = \text{diag} \left( \begin{pmatrix} 1 & z_1 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & z_n \\ 0 & 1 \end{pmatrix} \right) \in \text{GL}_{2n}(\text{Mer}(C^*)).$$

This implies that  $\text{Aut}^{\sigma_q, \partial_2}(\mathcal{R}_{\text{Mer}}/C_E(x, \ell_q))$  is a  $\partial_2$ -subgroup of  $(C_E, +)^n$ , so there exists a nonzero homogeneous linear differential polynomial  $L(Y_1, \dots, Y_n)$  with coefficients in  $C_E$  such that  $\text{Aut}^{\sigma_q, \partial_2}(\mathcal{R}_{\text{Mer}}/C_E(x, \ell_q))$  is contained in the set of zeros of  $L$  in  $(C_E, +)^n$ . We set  $f = L(z_1, \dots, z_n)$ . A Galoisian argument shows that  $f \in C_E(x, \ell_q)$  and that  $L(a_1, \dots, a_n) = \sigma_q(f) - f$ .  $\square$

**Galoisian approach to heat equation.** We want to show how the computation of the Galois  $\partial_2$ -group of the  $q$ -difference equation  $y(qx) = qxy(x)$  leads to the heat equation. We recall that the Jacobi theta function satisfies  $\theta_q(qx) = qx\theta_q(x)$ . Corollary 2.5 applied to this equation becomes: The function  $\theta_q$  satisfies a  $\partial_2$ -relation with coefficients in  $C_E(x, \ell_q)$  if and only if there exist  $a_1, \dots, a_m \in C_E$  and  $f \in C_E(x, \ell_q)$  such that

$$\sum_{i=0}^m a_i \partial_2^i \left( \frac{\partial_2(qx)}{qx} \right) = \sigma_q(f) - f.$$

A simple computation leads to

$$\frac{\partial_2(qx)}{qx} = \ell_q + 1 = \sigma_q \left( \frac{1}{2}(\ell_q^2 + \ell_q) \right) - \left( \frac{1}{2}(\ell_q^2 + \ell_q) \right)$$

and therefore to

$$\sigma_q \left( 2 \frac{\partial_2(\theta_q)}{\theta_q} - (\ell_q^2 + \ell_q) \right) = 2 \frac{\partial_2(\theta_q)}{\theta_q} - (\ell_q^2 + \ell_q).$$

The last identity is equivalent to the fact that

$$(2-2) \quad 2 \frac{\partial_2(\theta_q)}{\theta_q} - (\ell_q^2 + \ell_q) = 2 \frac{\delta_q(\theta_q)}{\theta_q} + \ell_q^2 - \ell_q$$

is an elliptic function and implies that

$$(2-3) \quad \text{Gal}^{\partial_2}(\mathcal{M}_{q,x}) \subset \left\{ \frac{\partial_2(c)}{c} = 0 \right\}.$$

The heat equation<sup>3</sup>

$$(2-4) \quad 2\delta_q \theta_q = -\delta_x^2 \theta_q + \delta_x \theta_q$$

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<sup>3</sup>One deduces from (2-4) that  $\tilde{\theta}(q, x) := \theta_q(q^{1/2}x)$  satisfies the equation  $2\delta_q \tilde{\theta} = \delta_x^2 \tilde{\theta}$ . To recover the classical form of the heat equation, it is enough make the change of variables  $q = \exp(-2i\pi\tau)$  and  $x = \exp(2i\pi z)$ .

can be rewritten as

$$2 \frac{\delta_q \theta_q}{\theta_q} + \ell_q^2 - \ell_q = -\delta_x(\ell_q).$$

Since  $\delta_x(\ell_q)$  is an elliptic function, taking into account (2-2), we see that the Galois  $\partial_2$ -group of  $\theta_q$ , as described in (2-3), can somehow be viewed as a Galoisian counterpart of the heat equation.

**On the  $q$ -hypertranscendence of  $q$ -difference equations of rank 1.** We want to study the  $q$ -dependence of the solutions of a  $q$ -difference equation of the form  $y(qx) = a(x)y(x)$ , where  $a(x) \in k(q, x)$ ,  $a(x) \neq 0$ .

**Theorem 2.6.** *Let  $u$  be a nonzero meromorphic solution of  $y(qx) = a(x)y(x)$ ,  $a(x) \in k(q, x)$ , in the sense of the previous subsections. The following statements are equivalent:*

- (1)  $a(x) = \mu x^r g(qx)/g(x)$  for some  $r \in \mathbb{Z}$ ,  $g \in k(q, x)$  and  $\mu \in k(q)$ .
- (2)  $u$  is a solution of a nontrivial algebraic  $\delta_x$ -relation over  $C_E(x, \ell_q)$  (and therefore over  $C(x)$ ).
- (3)  $u$  is a solution of a nontrivial algebraic  $\partial_2$ -relation over  $C_E(x, \ell_q)$ .

First of all, we remark that the equivalence between (1) and (2) follows from Theorem 1.1 in [Hardouin 2008] (replacing  $\mathbb{C}$  by  $k(q)$  is not an obstacle in the proof). Moreover it is equivalent to be  $\delta_x$ -algebraic over  $C_E(x, \ell_q)$  or over  $C(x)$ , since  $C_E(\ell_q)$  is  $\delta_x$ -algebraic over  $C(x)$ . The equivalence between (1) and (3) is proved in the proposition below:

**Proposition 2.7.** *Let  $u$  be a nonzero meromorphic solution of  $y(qx) = a(x)y(x)$ , with  $a(x) \in k(q, x)$ . Then  $u$  satisfies an algebraic differential equation with respect to  $\partial_2$  with coefficients in  $C_E(x, \ell_q)$  if and only if  $a(x) = \mu x^r g(qx)/g(x)$  for some  $r \in \mathbb{Z}$ ,  $g \in k(q, x)$  and  $\mu \in k(q)$ .*

*Proof.* By Lemma 3.3 in [Hardouin 2008], there exists  $f(x) \in k(q, x)$  such that  $a(x) = \tilde{a}(x)f(qx)/f(x)$  and  $\tilde{a}(x)$  has the property that if  $\alpha$  is a zero (respectively a pole) of  $\tilde{a}(x)$ , then  $q^n \alpha$  is neither a zero nor a pole of  $\tilde{a}(x)$  for any  $n \in \mathbb{Z} \setminus \{0\}$ . Replacing  $u$  by  $u/f(x)$ , we can suppose that  $a(x) = \tilde{a}(x)$  and we can write  $a(x)$  in the form

$$a(x) = \mu x^r \prod_{i=1}^s (x - \alpha_i)^{l_i},$$

where  $\mu \in k(q)$ ,  $r \in \mathbb{Z}$ ,  $l_1, \dots, l_s \in \mathbb{Z}$  and, for all  $i = 1, \dots, s$ , the  $\alpha_i$  are nonzero elements of a fixed algebraic closure of  $k(q)$ , such that  $q^{\mathbb{Z}} \alpha_i \cap q^{\mathbb{Z}} \alpha_j = \emptyset$  if  $i \neq j$ . By Corollary 2.5, the solutions of  $y(qx) = a(x)y(x)$  will satisfy a nontrivial algebraic

differential equation in  $\partial_2$  if and only if there exists  $f \in C_E(x, \ell_q)$ ,  $a_1, \dots, a_m \in C_E$  such that

$$\sum_{i=0}^m a_i \partial_2^i \left( \frac{\partial_2(a(x))}{a(x)} \right) = f(qx) - f(x).$$

We can suppose that  $a_m = 1$ . Notice that

$$\frac{\partial_2(a(x))}{a(x)} = \frac{\partial_2(x^r)}{x^r} + \frac{\delta_q(\mu)}{\mu} + \sum_{i=1}^s \frac{\partial_2(x - \alpha_i)^{l_i}}{(x - \alpha_i)^{l_i}},$$

where

$$\frac{\partial_2(x^r)}{x^r} = r\ell_q(x) = \left(\frac{1}{2}r(\ell_q^2 - \ell_q)\right)(qx) - \left(\frac{1}{2}r(\ell_q^2 - \ell_q)\right)(x),$$

with  $\ell_q^2 - \ell_q \in C_E(x, \ell_q)$ , and

$$\frac{\delta_q(\mu)}{\mu} = \left( \frac{\delta_q(\mu)}{2\mu} \ell_q \right) (qx) - \left( \frac{\delta_q(\mu)}{2\mu} \ell_q \right) (x), \quad \text{with } \frac{\delta_q(\mu)}{2\mu} \ell_q \in C_E(x, \ell_q).$$

It remains to show that a solution of  $y(qx) = a(x)y(x)$  satisfies a nontrivial differential equation in  $\partial_2$  if and only if there exists  $h \in C_E(x, \ell_q)$  such that

$$(2-5) \quad \sum_{j=0}^m a_j \partial_2^j \left( \sum_{i=1}^s \frac{l_i(x\ell_q(x) - \delta_q(\alpha_i))}{(x - \alpha_i)} \right) = h(qx) - h(x).$$

If we prove that (2-5) never holds, we can conclude that a solution of the equation  $y(qx) = a(x)y(x)$  satisfies a nontrivial algebraic differential equation in  $\partial_2$  with coefficients in  $C_E(x, \ell_q)$  if and only if  $a(x) = \mu x^r$  (modulo the reduction done at the beginning of the proof). For all  $i = 1, \dots, s$  and  $j = 0, \dots, m$ , the fact that  $\partial_2 \ell_q(x) \in C_E$  allows to prove inductively that

$$(2-6) \quad \partial_2^j \left( \frac{l_i(x\ell_q(x) - \delta_q(\alpha_i))}{(x - \alpha_i)} \right) = \frac{l_i(-1)^j j! (x\ell_q(x) - \delta_q(\alpha_i))^{j+1}}{(x - \alpha_i)^{j+1}} + \frac{h_{i,j}}{(x - \alpha_i)^j},$$

for some  $h_{i,j} \in C_E[\ell_q]$ .

Since  $x$  is transcendental over  $C_E(\ell_q)$ , we can consider

$$f(x) = \sum_{j=0}^m a_j \partial_2^j \left( \sum_{i=1}^s \frac{l_i(x\ell_q(x) - \delta_q(\alpha_i))}{(x - \alpha_i)} \right)$$

as a rational function in  $x$  with coefficients in  $C_E(\ell_q)$ . In the partial fraction decomposition of  $f(x)$ , the polar term in  $\frac{1}{(x - \alpha_i)}$  of highest order is

$$\frac{l_i(-1)^m (m!) (x\ell_q(x) - \delta_q(\alpha_i))^{m+1}}{(x - \alpha_i)^{m+1}}.$$

By the partial fraction decomposition theorem, identities (2-5) and (2-6) imply that this last term appears either in the decomposition of  $h(x)$  or in the decomposition of  $h(qx)$ . In both cases, there exists  $s \in \mathbb{Z}^*$  such that the term  $1/(q^s x - \alpha_i)^{m+1}$  appears in the partial fraction decomposition of  $h(qx) - h(x)$ . This is in contradiction with the assumption that the poles  $\alpha_i$  of  $a(x)$  satisfy  $q^{\mathbb{Z}}\alpha_i \cap q^{\mathbb{Z}}\alpha_j = \emptyset$  if  $i \neq j$ .  $\square$

**Remark 2.8.** The Jacobi theta function is an illustration of the theorem above. Notice that a meromorphic solution of the  $q$ -difference equations of the form  $y(qx) = a(x)y(x)$  with  $a(x) = \mu x^r g(qx)/g(x) \in k(q, x)$  is given by

$$y(x) = \theta_q(\mu x/q^r)/\theta_q(x)^{r-1} g(x) \in \text{Mer}(\mathbb{C}^*).$$

**Integrability in  $x$  and  $q$ .** Let  $Y(qx) = AY(x)$  be a  $q$ -difference system with  $A \in \text{GL}_v(\mathbb{C}(x, \ell_q))$  and  $\Delta = \{\partial_1, \partial_2\}$  as in Lemma 2.3. We deduce the following from Proposition 1.24.

**Corollary 2.9.** *The Galois  $\Delta$ -group  $\text{Gal}^\Delta(\mathcal{M}_A)$  of  $Y(qx) = AY(x)$  is  $\Delta$ -constant (see Remark 1.11) if and only if there exist square matrices  $B_1, B_2 \in M_v(\overline{\mathbb{C}}_E(x, \ell_q))$  such that the system*

$$\begin{aligned}\sigma_q(Y) &= AY(x) \\ \delta_x Y &= B_1 Y \\ \partial_2 Y &= B_2 Y\end{aligned}$$

is integrable, in the sense that these matrices satisfy

$$(2-7) \quad \sigma_q(B_1) = AB_1A^{-1} + \delta_x(A).A^{-1},$$

$$(2-8) \quad \sigma_q(B_2) = AB_2A^{-1} + \partial_2(A).A^{-1},$$

$$(2-9) \quad \partial_2(B_1) + B_1B_2 + \delta_x(\ell_q)B_1 = \delta_x(B_2) + B_2B_1.$$

*Proof.* It follows from Proposition 1.10 that there exist  $B_1, B_2 \in M_v(\tilde{\mathbb{C}}_E(x, \ell_q))$  such that the system

$$\begin{aligned}\sigma_q(Y) &= AY(x) \\ \partial_1 Y &= hB_1 Y \\ \partial_2 Y &= B_2 Y\end{aligned}$$

is integrable. This is easily seen to be equivalent to (2-7), (2-8) and (2-9). Notice that, as in Proposition 1.24, we have:

- $x$  and  $\ell_q$  are transcendental and algebraically independent over  $\overline{\mathbb{C}}_E$  (see for instance Theorem 3.6 in [Hardouin 2008]).
- $\sigma_q(x) = qx$  and  $\sigma_q(\ell_q) = \ell_q + 1$ .

It follows that  $\overline{\mathbb{C}}_E(x, \ell_q) \otimes_{\overline{\mathbb{C}}_E} \tilde{\mathbb{C}}_E = \tilde{\mathbb{C}}_E(x, \ell_q)$  and that, therefore, the same argument as in the proof of Proposition 1.24 allows us to conclude.  $\square$

**Example 2.10.** We want to study the  $q$ -dependence of the  $q$ -difference system

$$(2-10) \quad Y(qx) = \begin{pmatrix} \lambda & \eta \\ 0 & \lambda \end{pmatrix} Y(x),$$

where  $\lambda, \eta \in k(q, x)$ . First of all, the solutions of the equation  $y(qx) = \lambda y(x)$  will admit a  $\Delta$ -relation if and only if  $\lambda = \mu x^r g(qx)/g(x)$  for some  $r \in \mathbb{Z}$ ,  $g \in k(q, x)$  and  $\mu \in k(q)$  (see Theorem 2.6). To simplify the exposition we can suppose that  $\lambda = \mu x^r$ . Then clearly the Galois  $\Delta$ -group of  $y(qx) = \lambda y(x)$  is  $\Delta$ -constant if and only if  $\mu \in k(q)$  is a power of  $q$ , that is, if and only if  $\mu \in q^{\mathbb{Z}}$ . Let  $\mu = q^s$  for  $s \in \mathbb{Z}$ . A solution of  $y(qx) = \lambda y(x)$  is given by

$$y(x) = x^{s-r} \theta_q(x)^r.$$

To obtain an integrable system

$$y(qx) = \mu x^r y(x), \quad \delta_x y = b_1 y \quad \text{and} \quad \partial_2 y = b_2 y,$$

satisfying (2-7), (2-8) and (2-9), it is enough to take

$$b_1 = \frac{\delta_x(y)}{y} \quad \text{and} \quad b_2 = \frac{\partial_2(y)}{y}.$$

If  $\mu = q^s \in q^{\mathbb{Z}}$  then  $b_1 = r \ell_q(x) + s - r \in C_E(x, \ell_q(x))$ . The same hypothesis on  $\mu$  implies that  $b_2 \in C_E(x, \ell_q(x))$ , in fact we have

$$\sigma_q \left( \frac{\partial_2(\theta_q(x))}{\theta_q(x)} \right) = (\ell_q(x) + 1) + \frac{\partial_2(\theta_q(x))}{\theta_q(x)},$$

which implies that

$$\frac{\partial_2(\theta_q(x))}{\theta_q(x)} = \frac{\ell_q(x)(\ell_q(x) + 1)}{2} + e(x), \quad \text{for some } e(x) \in C_E.$$

To go back to the initial system, we have to find

$$B_1 = \begin{pmatrix} b_1 & \alpha(x) \\ 0 & b_1 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} b_2 & \beta(x) \\ 0 & b_2 \end{pmatrix} \quad \text{in } M_2(\overline{C}_E(x, \ell_q)),$$

satisfying (2-7), (2-8) and (2-9). We find that the  $q$ -difference system (2-10) has constant Galois  $\Delta$ -group if and only if there exist  $\alpha(x), \beta(x) \in \overline{C}_E(x, \ell_q)$  such that

- $\delta_x(\eta) = r\eta + (\alpha(qx) - \alpha(x))\mu x^r$ ,
- $\delta_q(\eta) = \frac{\delta_q \mu}{\mu} \eta + (\beta(qx) - \beta(x) + \ell_q(\alpha(qx) - \alpha(x))) \mu x^r$ ,
- $\partial_2(\alpha(x)) + \delta_x(\ell_q)\alpha(x) = \delta_x(\beta(x))$ .



**Remark 2.11.** Pulita [2008] shows that given a  $p$ -adic differential equation of the form  $dY/dx = GY$  with coefficients in some classical algebras of functions, or a  $q$ -difference equation  $\sigma_q(Y) = AY$ , one can always complete it into an integrable system:

$$\begin{aligned}\sigma_q(Y) &= AY \\ \frac{dY}{dx} &= GY \\ \frac{dY}{dq} &= 0.\end{aligned}$$

This is not the case in the complex framework, even after extending the coefficients from  $C_E(\ell_q, x)$  to  $\text{Mer}(C^*)$ .

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# PACIFIC JOURNAL OF MATHEMATICS

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On slim double Lie groupoids	1
NICOLAS ANDRUSKIEWITSCH, JESUS OCHOA ARANGO and ALEJANDRO TIRABOSCHI	
Topological classification of quasitoric manifolds with second Betti number 2	19
SUYOUNG CHOI, SEONJEONG PARK and DONG YOUP SUH	
Refined Kato inequalities for harmonic fields on Kähler manifolds	51
DANIEL CIBOTARU and PENG ZHU	
Deformation retracts to the fat diagonal and applications to the existence of peak solutions of nonlinear elliptic equations	67
E. NORMAN DANCER, JONATHAN HILLMAN and ANGELA PISTOIA	
Descent for differential Galois theory of difference equations: confluence and $q$ -dependence	79
LUCIA DI VIZIO and CHARLOTTE HARDOUIN	
Modulation and natural valued quiver of an algebra	105
FANG LI	
Willmore hypersurfaces with two distinct principal curvatures in $\mathbb{R}^{n+1}$	129
TONGZHU LI	
Variational inequality for conditional pressure on a Borel subset	151
YUAN LI, ERCAI CHEN and WEN-CHIAO CHENG	
New homotopy 4-spheres	165
DANIEL NASH	
Combinatorial constructions of three-dimensional small covers	177
YASUZO NISHIMURA	
On a theorem of Paul Yang on negatively pinched bisectional curvature	201
AERYEONG SEO	
Orders of elements in finite quotients of Kleinian groups	211
PETER B. SHALEN	
A new algorithm for finding an l.c.r. set in certain two-sided cells	235
JIAN-YI SHI	
Addendum to the article Superconnections and parallel transport	253
FLORIN DUMITRESCU	



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