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WILLMORE HYPERSURFACES WITH TWO DISTINCT PRINCIPAL CURVATURES IN $\mathbb{R}^{n+1}$

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We classify Willmore hypersurfaces of dimension $n \geq 3$ in $\mathbb{R}^{n+1}$ with two distinct principal curvatures under Möbius transformation group of $\mathbb{R}^{n+1}$. We also characterize conformally flat Willmore hypersurfaces in $\mathbb{R}^{n+1}$ for $n \geq 4$ in terms of the hyperelastic curves in two-dimensional real space forms.

1. Introduction

Let $f : M^n \to \mathbb{R}^{n+1}$ be an immersed hypersurface without umbilical point. The Willmore functional $W(f)$ is defined by

$$W(f) = \Big(\frac{n}{n-1}\Big)^{n/2} \int_{M^n} (|II|^2 - nH^2)^{n/2} dA,$$

where II and H denote, respectively, the second fundamental form and the mean curvature of f, and dA is the volume element of the induced metric (via f) on M^n. It was shown in [Chen 1974; Wang 1998] that the Willmore functional is invariant under the Möbius transformation group of $\mathbb{R}^{n+1}$. The critical points of the functional $W(f)$ are called Willmore hypersurfaces. Recently, the study of Willmore hypersurfaces has been a topic of increasing interest [Bryant 1984; Guo et al. 2001; Hertrich-Jeromin 2003; Li 2001; Li 2002; Palmer 1991]. The Euler–Lagrange equation for Willmore hypersurfaces has been computed in [Wang 1998] (or it can be found also in [Li 2001]). If $n = 2$, then we have the classical Willmore surfaces, and the functional $W(f)$ is equivalent to the classical Willmore functional $W_c(f) = \int_{M^2} H^2 dA$. Willmore himself [1982] conjectured that the minimum of the functional $W_c(f)$ for a topological torus is reached in the conformal class of the Clifford torus and is $2\pi^2$.

Let $M^2(c)$ denote a two-dimensional real space form of curvature c, and let $\gamma : I \to M^2(c)$ be an immersed curve. The one-dimensional version of $W_c(f)$ is

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defined by

$$F(\gamma)=\int_{\gamma}(\kappa^2+\lambda)ds,$$

where s and κ denote, respectively, the arclength parameter and the oriented curvature of $\gamma(s)$. The critical point of $F(\gamma)$ is called elastic curve. If the constant λ vanishes, the critical point of $F(\gamma)$ is called a free elastic curve.

Bernoulli introduced critical curves as a mathematical model for plane elastic curves, which were later classified by Euler. There have been extensive studies of elastic curves [Arroyo et al. 1999; 2003; Barros and Garay 1998; Bryant and Griffiths 1986; Langer and Singer 1984b; Langer and Singer 1984a]. Hertich-Jeromin [2003] has given the relationship between elastic curves and Willmore surfaces (or see [Bryant and Griffiths 1986; Langer and Singer 1984a]).

Theorem 1.1 [Hertrich-Jeromin 2003]. *A Willmore channel surface is Möbius equivalent to either*

(1) *a cylinder over a free elastic curve in a Euclidean* 2*-plane*,

(2) *a cone over a free elastic curve in a* 2*-sphere*, *or*

(3) *a surface of revolution over a free elastic curve in a hyperbolic* 2*-plane.*

A channel hypersurface is the envelope $f: I\times S^{n-1}\to\mathbb{R}^{n+1}$ of a 1-parameter family of hypersphere in $\mathbb{R}^{n+1}$. If it envelopes a sphere congruence $S: M^n\to S_1^{n+2}$ with $\mathrm{rank}(dS)\leq 1$, then f is called a branched channel hypersurface [Hertrich-Jeromin 2003].

In [Arroyo et al. 2003], the authors have defined free hyperelastic curves (also called free r-elastic curves), which are a generalization of the classical elastic curves. The hyperelastic curves are defined as the critical points of the functional

$$F^r(\gamma)=\int_{\gamma}\kappa^r ds.$$

They also computed the Euler–Lagrange equation for hyperelastic curves, and have studied the problem of the existence of closed hyperelastic curves.

In this paper, our purpose is to classify the Willmore hypersurfaces with two distinct principal curvatures. The main theorem of this paper is as follows.

Theorem 1.2. *Let $f: M^n\to\mathbb{R}^{n+1}$ for $n\geq 3$ be a Willmore hypersurface with two distinct principal curvatures. Then f is locally Möbius equivalent to one of the following hypersurfaces in $\mathbb{R}^{n+1}$:*

(1) *a cylinder over a free n-elastic curve in a Euclidean* 2*-plane*,

(2) *a cone over a free n-elastic curve in a* 2*-sphere*,

(3) *a rotational hypersurface over a free n-elastic curve in a hyperbolic* 2*-plane*,

(4) *the image of σ of the standard torus $S^k(\sqrt{(n-k)/n}) \times S^{n-k}(\sqrt{k/n})$ in S^{n+1} for $1 < k < n-1$. Here σ is the stereographic projection $\sigma : S^{n+1} \backslash \{(-1, \vec{0})\} \to \mathbb{R}^{n+1}$.*

It is a classical result that a hypersurface of dimension $n \geq 3$ in space forms has a principle curvature of multiplicity at least $n-1$ everywhere if and only if it is conformally flat. Cartan [1917] has given a local classification for conformally flat hypersurfaces in $\mathbb{R}^{n+1}$, and proved for $n \geq 4$ that $f : M^n \to \mathbb{R}^{n+1}$ is a conformally flat immersion if and only if f is a branched channel hypersurface. Thus the following corollary is a higher-dimensional version of Theorem 1.2.

Corollary 1.3. *For $n \geq 4$, let $f : M^n \to \mathbb{R}^{n+1}$ be a conformally flat Willmore hypersurface without umbilical point. Then f is locally Möbius equivalent to one of the following hypersurfaces in $\mathbb{R}^{n+1}$:*

(1) *a cylinder over a free n-elastic curve in a Euclidean 2-plane,*

(2) *a cone over a free n-elastic curve in a 2-sphere,*

(3) *a rotational hypersurface over a free n-elastic curve in a hyperbolic 2-plane.*

Remark 1.3.1. In fact, Theorem 1.2 gives a classification of Willmore hypersurfaces with two distinct principal curvatures in space forms. Since the Willmore functional is conformal invariant, the Willmore hypersurfaces in space forms are equivalent to each other by conformal diffeomorphisms σ^{-1} and τ; see [Liu et al. 2001]. Let H^{n+1} be the $(n+1)$-dimensional hyperbolic space defined by

$$H^{n+1} = \{(y_0, y_1, \dots, y_{n+1}) \mid -y_0^2 + y_1^2 + \cdots + y_{n+1}^2 = -1,\ y_0 > 0\}.$$

The conformal diffeomorphisms σ^{-1} and τ are defined by

$$\sigma^{-1} : \mathbb{R}^{n+1} \to S^{n+1} \setminus \{(-1, \vec{0})\}, \quad \sigma^{-1}(u) = \Big(\frac{1-|u|^2}{1+|u|^2}, \frac{2u}{1+|u|^2}\Big),$$

$$\tau : H^{n+1} \to S^{n+1}_+ \subset S^{n+1}, \qquad \tau(y) = \Big(\frac{1}{y_0}, \frac{\vec{y}}{y_0}\Big), \ y = (y_0, \vec{y}) \in H^{n+1},$$

where S^{n+1}_+ is the hemisphere in S^{n+1} whose the first coordinate is positive.

The paper is organized as follows. In Section 2, we review the elementary facts about Möbius geometry for hypersurfaces in $\mathbb{R}^{n+1}$. In Section 3, we present some examples of Willmore hypersurfaces in terms of the hyperelastic curves in two-dimensional space forms. In Section 4, we give the proof of Theorem 1.2 and Corollary 1.3.

2. Möbius invariants of hypersurfaces in $\mathbb{R}^{n+1}$

Wang [1998] defined Möbius invariants of submanifolds in S^{n+1} and gave a congruence theorem of hypersurfaces in S^{n+1}. Through the inverse stereographic projection σ^{-1}, we can regard hypersurfaces in $\mathbb{R}^{n+1}$ as hypersurfaces in S^{n+1}. In this section we define Möbius invariants and give a congruence theorem of hypersurfaces in $\mathbb{R}^{n+1}$ in a way similar to [Wang 1998]. See also [Liu et al. 2001].

Let $\mathbb{R}^{n+3}_1$ be the Lorentz space, that is, $\mathbb{R}^{n+3}$ with the inner product $\langle\cdot,\cdot\rangle$ defined by

$$\langle x, y\rangle = -x_0y_0 + x_1y_1 + \cdots + x_{n+2}y_{n+2}$$

for $x = (x_0, x_1, \dots, x_{n+2})$ and $y = (y_0, y_1, \dots, y_{n+2}) \in \mathbb{R}^{n+3}$.

Let $f : M^n \to \mathbb{R}^{n+1}$ be a hypersurface without umbilical point, and let II and H denote the second fundamental form and the mean curvature of f, respectively. The Möbius position vector $Y : M^n \to \mathbb{R}^{n+3}_1$ of f is

$$Y = \rho\Big(\frac{1+|f|^2}{2}, \frac{1-|f|^2}{2}, f\Big), \quad \text{where } \rho^2 = \frac{n}{n-1}(|II|^2 - nH^2).$$

Theorem 2.1 [Wang 1998]. *Two hypersurfaces $f, \tilde{f} : M^n \to \mathbb{R}^{n+1}$ are Möbius equivalent if and only if there exists $T \in O(n+2, 1)$ such that $\tilde{Y} = YT$.*

It follows immediately from Theorem 2.1 that

$$g = \langle dY, dY\rangle = \rho^2 df \cdot df$$

is a Möbius invariant, called the Möbius metric of f; see [Wang 1998].

Let Δ be the Laplacian with respect to g, we define

$$N = -\frac{1}{n}\Delta Y - \frac{1}{2n^2}\langle\Delta Y, \Delta Y\rangle Y.$$

Then we have

$$\langle Y, Y\rangle = 0, \quad \langle N, Y\rangle = 1, \quad \langle N, N\rangle = 0.$$

Let $\{E_1, \dots, E_n\}$ be a local orthonormal basis for g with dual basis $\{\omega_1, \dots, \omega_n\}$, If we write $Y_i = E_i(Y)$, then we have

$$\langle Y_i, Y\rangle = \langle Y_i, N\rangle = 0 \quad \text{and} \quad \langle Y_i, Y_j\rangle = \delta_{ij} \quad \text{for } 1 \le i, j \le n.$$

Let ξ be the mean curvature sphere of f given by

$$\xi = \Big(\frac{1+|f|^2}{2}H + f\cdot e_{n+1}, \frac{1-|f|^2}{2}H - f\cdot e_{n+1}, Hf + e_{n+1}\Big),$$

where e_{n+1} is the unit normal vector field of f in $\mathbb{R}^{n+1}$. By direct computations, we have

$$\langle \xi, Y\rangle = \langle \xi, N\rangle = \langle \xi, Y_i\rangle = 0 \quad \text{and} \quad \langle \xi, \xi\rangle = 1.$$

Then $\{Y, N, Y_1, \dots, Y_n, \xi\}$ forms a moving frame in $\mathbb{R}^{n+3}_1$ along M^n. In this section we will use the range of indices $1 \le i, j, k \le n$. The structure equations are given by

$$\begin{aligned} dY &= Y_i\omega_i, \\ dN &= A_{ij}\omega_i Y_j + C_i\omega_i\xi, \\ dY_i &= -A_{ij}\omega_j Y - \omega_i N + \omega_{ij}Y_j + B_{ij}\omega_j\xi, \\ d\xi &= -C_i\omega_i Y - \omega_i B_{ij}Y_j, \end{aligned}$$

where ω_{ij} is the connection form of the Möbius metric g and is defined by the structure equations $d\omega_i = \omega_{ij} \wedge \omega_j$ and $\omega_{ij} + \omega_{ji} = 0$. Here and henceforth we use Einstein summation on repeated indices.

The tensors

$$\boldsymbol{A} = A_{ij}\omega_i \otimes \omega_j, \quad \Phi = C_i\omega_i, \quad \boldsymbol{B} = B_{ij}\omega_i \otimes \omega_j$$

are called the Blaschke tensor, the Möbius form and the Möbius second fundamental form of f, respectively. The covariant derivatives of C_i, A_{ij} and B_{ij} are defined by

$$C_{i,j}\omega_j = dC_i + C_j\omega_{ji}, \tag{1}$$

$$A_{ij,k}\omega_k = dA_{ij} + A_{ik}\omega_{kj} + A_{kj}\omega_{ki}, \tag{2}$$

$$B_{ij,k}\omega_k = dB_{ij} + B_{ik}\omega_{kj} + B_{kj}\omega_{ki}. \tag{3}$$

The integrability conditions for the structure equations are given by

$$A_{ij,k} - A_{ik,j} = B_{ik}C_j - B_{ij}C_k, \tag{4}$$

$$C_{i,j} - C_{j,i} = (B_{ik}A_{kj} - B_{jk}A_{ki}), \quad B_{ij,k} - B_{ik,j} = \delta_{ij}C_k - \delta_{ik}C_j, \tag{5}$$

$$R_{ijkl} = B_{ik}B_{jl} - B_{il}B_{jk} + \delta_{ik}A_{jl} + \delta_{jl}A_{ik} - \delta_{il}A_{jk} - \delta_{jk}A_{il}, \tag{6}$$

$$R_{ij} := R_{ikjk} = -B_{ik}B_{kj} + (\operatorname{tr}\boldsymbol{A})\delta_{ij} + (n-2)A_{ij}, \tag{7}$$

$$B_{ii} = 0, \quad (B_{ij})^2 = \frac{n-1}{n}, \quad \operatorname{tr}\boldsymbol{A} = A_{ii} = \frac{1}{2n}(1 + n^2\kappa), \tag{8}$$

where R_{ijkl} denote the components of the curvature tensor of g and

$$\kappa = \frac{1}{n(n-1)}R_{ijij}$$

is its normalized Möbius scalar curvature. When $n \ge 3$, we know that all coefficients in the structure equations are determined by $\{g, \boldsymbol{B}\}$ and we have this:

Theorem 2.2 [Wang 1998]. *For $n \ge 3$, two hypersurfaces $f : M^n \to \mathbb{R}^{n+1}$ and $\tilde{f} : M^n \to \mathbb{R}^{n+1}$ are Möbius equivalent if and only if there exists a diffeomorphism*

$\varphi : M^n \to M^n$ that preserves the Möbius metric and the Möbius second fundamental form.

The Möbius invariants and Euclidean invariants are related [Liu et al. 2001]:

$$\begin{aligned}
&B_{ij} = \rho^{-1}(h_{ij} - H\delta_{ij}),\\
&C_i = -\rho^{-2}\big(e_i(H) + (h_{ij} - H\delta_{ij})e_j(\log\rho)\big),\\
&A_{ij} = -\rho^{-2}\big(\text{hess}_{ij}(\log\rho) - e_i(\log\rho)e_j(\log\rho) - Hh_{ij}\big)\\
&\qquad\qquad - \tfrac{1}{2}\rho^{-2}(|\nabla\log\rho|^2 + H^2)\delta_{ij},
\end{aligned} \tag{9}$$

where hess_{ij} and ∇ denote, respectively, the Hessian matrix and the gradient with respect to $I = df \cdot df$. Then

$$\Phi = \rho C_i\theta_i, \quad A = \rho^2 A_{ij}\theta_i \otimes \theta_j, \quad B = \rho^2 B_{ij}\theta_i \otimes \theta_j.$$

The eigenvalues of (B_{ij}) are called the Möbius principal curvatures of f. Clearly the number of distinct Möbius principal curvatures is the same as that of its distinct Euclidean principal curvatures. Let $\{k_1, \dots, k_n\}$ be the principal curvatures of f, and $\{\lambda_1, \dots, \lambda_n\}$ the corresponding Möbius principal curvatures. Then the curvature sphere of principal curvature k_i is

$$\xi_i = \lambda_i Y + \xi = \Big(\frac{1+|f|^2}{2}k_i + f\cdot e_{n+1}, \frac{1-|f|^2}{2}k_i - f\cdot e_{n+1}, k_i f + e_{n+1}\Big).$$

If $\langle \xi_i, (1, -1, 0, \dots, 0)\rangle = 0$, then $k_i = 0$. This means that the curvature sphere of principal curvature k_i is a hyperplane in $\mathbb{R}^{n+1}$.

The second covariant derivative of B_{ij} is defined by

$$B_{ij,kl}\omega_l = dB_{ij,k} + B_{lj,k}\omega_{li} + B_{il,k}\omega_{lj} + B_{ij,l}\omega_{lk}.$$

We have the Ricci identities

$$B_{ij,kl} - B_{ij,lk} = B_{mj}R_{mikl} + B_{im}R_{mjkl}.$$

The generalized Willmore functional $W(f)$ is the volume functional of the Möbius metric g given by

$$W(f) = \int_{M^n} \rho^n dA = \text{Vol}_g(M).$$

See [Guo et al. 2001; Wang 1998]. A critical point of the Willmore functional is called a Willmore hypersurface.

Theorem 2.3. [*Wang* 1998] *A hypersurface $f : M^n \to \mathbb{R}^{n+1}$ is a Willmore hypersurface if and only if*

$$B_{ij,ij} + B_{ij}B_{jk}B_{ki} + B_{ij}A_{ij} = 0.$$

Using (5) we have

$$B_{ij,j} = -(n-1)C_i.$$

Thus the Euler–Lagrange equation is

$$-(n-1)C_{i,i} + B_{ij}B_{jk}B_{ki} + B_{ij}A_{ij} = 0. \tag{10}$$

3. Some examples of Willmore hypersurfaces in $\mathbb{R}^{n+1}$

Let $M^2(c)$ denote a two-dimensional real space form of curvature c, and suppose $\gamma : I \to M^2(c)$ is a regular curve. For a fixed natural number r, the authors of [Arroyo et al. 2003] have defined the functional

$$F^r(\gamma) = \int_\gamma \kappa^r ds,$$

where s, κ are the arclength parameter, the oriented curvature of γ, respectively. Critical points of $F^r(\gamma)$ are called free r-hyperelastic curves. They also computed the Euler–Lagrange equation for free r-hyperelastic curves. Under suitable boundary conditions, γ is a free r-hyperelastic curve if and only if the following Euler–Lagrange equation holds:

$$r(r-1)\kappa^{r-3}\Big(\kappa\kappa_{ss} + (r-2)\kappa_s^2 + \frac{\kappa^4}{r} + c\frac{\kappa^2}{r-1}\Big) = 0. \tag{11}$$

If $r = 2$, this equation reduces to the classical Euler–Lagrange equation for elastica in two-dimensional space forms. In this section we will construct some Willmore hypersurfaces in $\mathbb{R}^{n+1}$ by free n-elastic curves in space forms.

Example 3.0.1. Let $\gamma : I \to \mathbb{R}^2$ be a regular curve, and s denote the arclength of $\gamma(s)$. we define hypersurface in $\mathbb{R}^{n+1}$

$$f(s, y) = (\gamma(s), y) : I \times \mathbb{R}^{n-1} \to \mathbb{R}^{n+1},$$

where $y : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ is the identity map. Clearly the hypersurface f is a cylinder over the curve $\gamma(s)$ in Euclidean plane $\mathbb{R}^2$.

Theorem 3.1. *The cylinder $f = (\gamma(s), y) : I \times \mathbb{R}^{n-1} \to \mathbb{R}^{n+1}$ as in Example 3.0.1 is a Willmore hypersurface if and only if $\gamma(s)$ is a free n-elastic curve in $\mathbb{R}^2$.*

Proof. The first fundamental form I and the second fundamental form II of the hypersurface f are

$$I = ds^2 + I_{\mathbb{R}^{n-1}} \quad \text{and} \quad II = \kappa ds^2,$$

where κ is the oriented curvature of γ, and $I_{\mathbb{R}^{n-1}}$ is the standard Euclidean metric of $\mathbb{R}^{n-1}$.

Let $\{e_1 = \partial/\partial s, e_2, \dots, e_n\}$ be an orthonormal basis of $T(I \times \mathbb{R}^{n-2})$ with dual basis $\{\omega_1, \dots, \omega_n\}$, which consists of principal vectors, and $\{\omega_{ij}\}$ connection forms with respect to the basis $\{\omega_1, \dots, \omega_n\}$. Since $I \times \mathbb{R}^{n-2}$ is a product manifold, we have

$$\omega_{1i} = 0 \quad \text{for } 2 \le i \le n.$$

Under the orthonormal basis $\{e_1, e_2, \dots, e_n\}$, the coefficients of the second fundamental form of the hypersurface f have the diagonal form

$$(h_{ij}) = \operatorname{diag}(\kappa, 0, \dots, 0).$$

We assume that the hypersurface f is umbilic-free; locally let $\kappa > 0$, so that $\rho = \kappa$. Then the Möbius metric g of the hypersurface f is

$$g = \rho^2 I = \kappa^2 (ds^2 + I_{\mathbb{R}^{n-1}}).$$

We write $\kappa_s = d\kappa/ds$. Since $\{\rho^{-1}e_1, \dots, \rho^{-1}e_n\}$ is an orthonormal basis with respective to g, the coefficients of Möbius invariants of f with respect to the orthonormal basis can be obtained from (9) as follows:

$$\begin{aligned} &C_1 = -\frac{1}{\kappa^2} e_1(\kappa) = -\frac{1}{\kappa^2}\kappa_s, \quad C_2 = \cdots = C_n = 0; \\ &(B_{ij}) = \operatorname{diag}\Big(\frac{n-1}{n}, \frac{-1}{n}, \dots, \frac{-1}{n}\Big), \\ &(A_{ij}) = \operatorname{diag}(a_1, a_2, \dots, a_2), \end{aligned} \tag{12}$$

where

$$a_1 = -\frac{\kappa_{ss}}{\kappa^3} + \frac{3}{2}\frac{(\kappa_s)^2}{\kappa^4} + \frac{2n-1}{2n^2} \quad \text{and} \quad a_2 = -\tfrac{1}{2}\Big(\frac{(\kappa_s)^2}{\kappa^4} + \frac{1}{n^2}\Big).$$

Using (1) and (12), we get that

$$C_{1,1} = \frac{-\kappa_{ss}}{\kappa^3} + 2\frac{(\kappa_s)^2}{\kappa^4} \quad \text{and} \quad C_{i,i} = -C_1^2 = -\frac{(\kappa_s)^2}{\kappa^4} \quad \text{for } 2 \le i \le n. \tag{13}$$

From (12) and (13), we have

$$-(n-1)C_{i,i} + B_{ij}B_{jk}B_{ki} + B_{ij}A_{ij} = \frac{(n-1)^2}{n\kappa^4}\Big(\kappa\kappa_{ss} + (n-2)\kappa_s^2 + \frac{\kappa^4}{n}\Big) \tag{14}$$

From (10), (11) and (14), we finish the proof of Theorem 3.1. □

Example 3.1.1. Let $\gamma : I \to S^2(1) \subset \mathbb{R}^3$ be a regular curve, and s denote the arclength of $\gamma(s)$. We define a hypersurface in $\mathbb{R}^{n+1}$ by

$$f(s, t, y) = (t\gamma(s), y) : I \times \mathbb{R}^+ \times \mathbb{R}^{n-2} \to \mathbb{R}^{n+1},$$

where $y : \mathbb{R}^{n-2} \to \mathbb{R}^{n-2}$ is the identity map and $\mathbb{R}^+ = \{t \mid t > 0\}$. We call the hypersurface f a cone over the curve $\gamma(s)$ in a 2-sphere.

Theorem 3.2. *The cone* $f = (t\gamma(s), y) : I \times \mathbb{R}^+ \times \mathbb{R}^{n-2} \to \mathbb{R}^{n+1}$ *as in Example 3.1.1 is a Willmore hypersurface if and only if* $\gamma(s)$ *is a free n-elastic curve in a* 2*-sphere.*

Proof. The first fundamental form I and the second fundamental form II of the hypersurface f are, respectively,

$$I = t^2 ds^2 + I_{\mathbb{R}^{n-1}} \quad \text{and} \quad II = t\kappa ds^2,$$

where κ is the oriented curvature of γ, and $I_{\mathbb{R}^{n-1}}$ is the standard Euclidean metric of $\mathbb{R}^{n-1}$. Let $\{e_1 = t^{-1}\partial/\partial s, e_2 = \partial/\partial t, \dots, e_n\}$ be an orthonormal basis of $T(I \times \mathbb{R}^+ \times \mathbb{R}^{n-2})$ with dual basis $\{\omega_1, \dots, \omega_n\}$, which consists of principal vectors. Let $\{\omega_{ij}\}$ be connection forms with respect to the basis $\{\omega_1, \omega_2, \dots, \omega_n\}$. Then

$$\omega_{1i} = 0 \quad \text{for } 3 \le i \le n \quad \text{and} \quad \omega_{12} = e_2(\log t^{-1}\kappa)\omega_1.$$

Under the orthonormal basis $\{e_1, e_2, \dots, e_n\}$, the coefficients of the second fundamental form of the hypersurface f have the diagonal form

$$(h_{ij}) = \operatorname{diag}(t^{-1}\kappa, 0, \dots, 0).$$

We assume that the hypersurface f is umbilic-free; locally let $\kappa > 0$, so that $\rho = \kappa/t$. Thus the Möbius metric g of the hypersurface f is

$$g = \rho^2 I = \frac{\kappa^2}{t^2}(t^2 ds^2 + I_{\mathbb{R}^{n-1}}) = \kappa^2(ds^2 + I_{H^{n-1}}),$$

where $I_{H^{n-1}}$ is the standard hyperbolic metric of

$$\mathbb{R}^{n-1}_+ = \{(t, y_2, \dots, y_{n-1}) \mid t > 0\} = \mathbb{R}^+ \times \mathbb{R}^{n-2}.$$

Since $\{\rho^{-1}e_1, \dots, \rho^{-1}e_n\}$ is an orthonormal basis with respect to g, the coefficients of Möbius invariants of f with respect to the orthonormal basis can be obtained as follows using (9):

$$\begin{aligned} &C_1 = -\frac{t}{\kappa^2}e_1(\kappa) = -\frac{1}{\kappa^2}\kappa_s, \quad C_2 = \dots = C_n = 0;\\ &(B_{ij}) = \operatorname{diag}\Big(\frac{n-1}{n}, \frac{-1}{n}, \dots, \frac{-1}{n}\Big),\\ &(A_{ij}) = \operatorname{diag}(a_1, a_2, \dots, a_2), \end{aligned} \tag{15}$$

where

$$a_1 = -\frac{\kappa_{ss}}{\kappa^3} + \frac{3}{2}\frac{(\kappa_s)^2}{\kappa^4} + \frac{1}{2\kappa^2} + \frac{2n-1}{2n^2} \quad \text{and} \quad a_2 = -\frac{1}{2}\Big(\frac{(\kappa_s)^2}{\kappa^4} + \frac{1}{\kappa^2} + \frac{1}{n^2}\Big).$$

Using (1) and (15), we get that

$$C_{1,1} = -\frac{\kappa_{ss}}{\kappa^3} + 2\frac{(\kappa_s)^2}{\kappa^4} \quad \text{and} \quad C_{i,i} = -C_1^2 = -\frac{(\kappa_s)^2}{\kappa^4} \quad \text{for } 2 \le i \le n. \tag{16}$$

From (15) and (16), we have

$$(17)\quad -(n-1)C_{i,i}+B_{ij}B_{jk}B_{ki}+B_{ij}A_{ij} = \frac{(n-1)^2}{n\kappa^4}\Big(\kappa\kappa_{ss}+(n-2)\kappa_s^2+\frac{\kappa^4}{n}+\frac{\kappa^2}{n-1}\Big).$$

From (10), (11) and (17), we finish the proof of Theorem 3.2. □

Example 3.2.1. Let $\mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0\}$. Let $\gamma = (x_1, x_2) : I \to \mathbb{R}^2_+$ be a regular curve, and denote by s the arclength of $\gamma(s)$. We define a hypersurface in $\mathbb{R}^{n+1}$ by

$$f : I \times S^{n-1} \to \mathbb{R}^{n+1}, \quad (x_1, x_2, \theta) \mapsto (x_1, x_2\theta),$$

where $\theta : S^{n-1} \to \mathbb{R}^n$ is a standard immersion of a round sphere. Clearly the hypersurface f is a rotational hypersurface over the curve $\gamma(s)$.

Theorem 3.3 [Arroyo et al. 2003]. *The rotational hypersurface $f = (x_1, x_2\theta) : I \times S^{n-1} \to \mathbb{R}^{n+1}$ as in Example 3.2.1 is a Willmore hypersurface if and only if $\gamma(s)$ is a free n-elastic curve in the Poincare half plane $\mathbb{R}^2_+$ endowed with the hyperbolic metric $ds^2 = x_2^{-2}(dx_1 \cdot dx_1 + dx_2 \cdot dx_2)$.*

For $n = 2$, the theorem was proved in [Langer and Singer 1984a], and it was proved for all other n in [Arroyo et al. 2003]. We will prove it using Möbius invariants.

Proof of Theorem 3.3. Let $\mathbb{R}^3_1$ be the Lorentzian space with the inner product

$$\langle u, u\rangle = -u_1^2 + u_2^2 + u_3^2, \quad \text{where } u = (u_1, u_2, u_3).$$

Let $H^2 = \{u \in \mathbb{R}^3_1 \mid \langle u, u\rangle = -1, u_1 > 0\}$ be hyperbolic space. We define the isometric diffeomorphism

$$\phi : \mathbb{R}^2_+ \to H^2, \quad (x_1, x_2) \mapsto \Big(\frac{1+x_1^2+x_2^2}{2x_2}, \frac{1-x_1^2-x_2^2}{2x_2}, \frac{x_1}{x_2}\Big).$$

Let $\alpha = (x_1'(s), x_2'(s))$ and $\beta = (y_1, y_2)$ be the unit tangent vector field and the normal vector field of the curve γ in $\mathbb{R}^2_+$, respectively. For the curve $\phi(\gamma(s))$ in hyperbolic space H^2, $\phi_*(\alpha)$ and $\phi_*(\beta)$ are, respectively, the unit tangent vector field and the normal vector field of the curve $\phi(\gamma)$. Then

$$(18)\quad \begin{aligned} (\phi(\gamma(s)))' &= \phi_*(\alpha),\\ (\phi_*(\alpha))' &= \phi(\gamma(s)) + \kappa\phi_*(\beta),\\ (\phi_*(\beta))' &= -\kappa\phi_*(\alpha), \end{aligned}$$

where $(\phi(\gamma(s)))'$ denotes $d/ds(\phi(\gamma(s)))$, and κ denotes the oriented curvature of the curve $\phi(\gamma(s))$.

The unit normal vector field of the hypersurface f in $\mathbb{R}^{n+1}$ is $\xi = x_2^{-1}(y_1, y_2\phi)$. Thus the first fundamental form of the hypersurface f in $\mathbb{R}^{n+1}$ is

$$I = df \cdot df = x_2^2(ds^2 + I_{S^{n-1}}),$$

where $I_{S^{n-1}}$ is the standard metric of S^{n-1} and the second fundamental form of the hypersurface f in $\mathbb{R}^{n+1}$ is

$$II = (x_2\kappa - y_2)ds^2 - y_2 I_{s^{n-1}}.$$

The principal curvatures of the hypersurface f in $\mathbb{R}^{n+1}$ are

$$\left\{\frac{\kappa}{x_2} + \frac{-y_2}{x_2^2}, \frac{-y_2}{x_2^2}, \dots, \frac{-y_2}{x_2^2}\right\}. \tag{19}$$

Assume that the hypersurface f is umbilic-free; locally let $\kappa > 0$, so that $\rho = \kappa/x_2$. Thus the Möbius metric of f is

$$g = \rho^2 I = \kappa^2(ds^2 + I_{S^{n-1}}),$$

and Möbius position vector of f is

$$\begin{aligned} Y &= \frac{\kappa^2}{x_2^2}\Big(\frac{1+x_1^2+x_2^2}{2}, \frac{1-x_1^2-x_2^2}{2}, x_1, x_2\theta\Big) \\ &= \frac{\kappa^2}{x_2^2}\Big(\frac{u_1}{u_1+u_2}, \frac{u_2}{u_1+u_2}, \frac{u_3}{u_1+u_2}, \frac{1}{u_1+u_2}\theta\Big) \\ &= \kappa^2(u_1+u_2)(u_1, u_2, u_3, \theta). \end{aligned}$$

From (18) and (19), we can obtain the coefficients of Möbius invariants of f under a local orthonormal basis for g as follows:

$$\begin{aligned} &C_1 = -\kappa_s/\kappa^2, \quad C_2 = \dots = C_n = 0; \\ &(B_{ij}) = \operatorname{diag}\Big(\frac{n-1}{n}, \frac{-1}{n}, \dots, \frac{-1}{n}\Big), \\ &(A_{ij}) = \operatorname{diag}(a_1, a_2, \dots, a_2), \end{aligned} \tag{20}$$

where

$$a_1 = \frac{\kappa_{ss}}{\kappa^3} - \frac{5}{2}\frac{(\kappa_s)^2}{\kappa^4} - \frac{1}{2\kappa^2} + \frac{2n-1}{2n^2} \quad \text{and} \quad a_2 = -\frac{1}{2}\Big(\frac{(\kappa_s)^2}{\kappa^4} - \frac{1}{\kappa^2} + \frac{1}{n^2}\Big).$$

Using (1) and (20), we get that

$$C_{1,1} = -\frac{\kappa_{ss}}{\kappa^3} + 2\frac{(\kappa_s)^2}{\kappa^4} \quad \text{and} \quad C_{i,i} = -C_1^2 = -\frac{(\kappa_s)^2}{\kappa^4} \quad \text{for } 2 \le i \le n. \tag{21}$$

From (20) and (21), we have

$$(22)\quad -(n-1)C_{i,i}+B_{ij}B_{jk}B_{ki}+B_{ij}A_{ij} = \frac{(n-1)^2}{n\kappa^4}\Big(\kappa\kappa_{ss}+(n-2)\kappa_s^2+\frac{\kappa^4}{n}-\frac{\kappa^2}{n-1}\Big).$$

From (10), (11) and (22), we finish the proof of Theorem 3.3. □

4. The proof of Theorem 1.2

Lemma 4.1 [Kobayashi and Nomizu 1963]. *Let (M^n, g) be a Riemannian manifold, $\tilde g$ another Riemannian metric on M^n such that $\tilde g=\rho^2 g$, where ρ is a positive smooth function on M^n. Let $\{e_1,\dots,e_n\}$ be a local orthonormal basis for g with dual basis $\{\omega_1,\dots,\omega_n\}$, and $\{\omega_{ij}\}$ be the connection forms with respect to the basis $\{\omega_1,\dots,\omega_n\}$. Then $\{\tilde e_1=\rho^{-1}e_1,\dots,\tilde e_n=\rho^{-1}e_n\}$ is a local orthonormal basis for $\tilde g$, and $\{\tilde\omega_1=\rho\omega_1,\dots,\tilde\omega_n=\rho\omega_n\}$ is the dual basis.*

If $\{\tilde\omega_{ij}\}$ are the connection forms with respect to the basis $\{\tilde\omega_1,\dots,\tilde\omega_n\}$, then

$$\tilde\omega_{ij}=\omega_{ij}+e_i(\log\rho)\omega_j-e_j(\log\rho)\omega_i \quad \textit{for } 1\le i,j\le n.$$

For $n\ge 3$, let $f: M^n\to\mathbb{R}^{n+1}$ be a Willmore hypersurface with two distinct principal curvatures. We denote by b_1 and b_2 the Möbius principal curvatures, with multiplicity k and $n-k$, respectively. Using (8), we get

$$b_1=\frac{1}{n}\sqrt{\frac{(n-1)(n-k)}{k}} \quad\text{and}\quad b_2=-\frac{1}{n}\sqrt{\frac{(n-1)k}{n-k}}.$$

First we assume that the multiplicities of two principal curvatures are greater than one. We can choose a local orthonormal basis $\{E_1,\dots,E_n\}$ with respect to the Möbius metric g of f such that

$$(B_{ij})=\operatorname{diag}(b_1,\dots,b_1,b_2,\dots,b_2).$$

Using $dB_{ij}+B_{kj}\omega_{ki}+B_{ik}\omega_{kj}=B_{ij,k}\omega_k$, we obtain that

$$(23)\quad \begin{aligned} B_{ij,l}&=0 \quad \text{for } 1\le i,j\le k \text{ and } 1\le l\le n;\\ B_{\alpha\beta,l}&=0 \quad \text{for } k+1\le\alpha,\beta\le n \text{ and } 1\le l\le n.\end{aligned}$$

Since the multiplicities of two principal curvatures are each greater than 1, from (23) we have

$$\begin{aligned} C_j&=B_{ii,j}-B_{ij,i}=0 \qquad \text{for } 1\le i,j\le k \text{ and } i\ne j,\\ C_\alpha&=B_{\beta\beta,\alpha}-B_{\alpha\beta,\beta}=0 \quad \text{for } k+1\le\alpha,\beta\le n \text{ and } \alpha\ne\beta.\end{aligned}$$

Thus the Möbius form Φ vanishes, so f is a Möbius isoparametric hypersurface. In [Li et al. 2002], the authors classified such hypersurfaces with two distinct principal curvatures in S^{n+1}. Using the inverse of the stereographic projection $\sigma : \mathbb{R}^{n+1} \to S^{n+1}$, we have this:

Proposition 4.2 [Li et al. 2002]. *For $n \geq 4$, let $f : M^n \to \mathbb{R}^{n+1}$ be an immersed hypersurface with two distinct principal curvatures. If the multiplicities of two principal curvatures are greater than 1, then f is Möbius equivalent to an open part of one of the following Möbius isoparametric hypersurfaces in $\mathbb{R}^{n+1}$:*

(1) *the image under σ of the standard torus $S^k(r) \times S^{n-k}(\sqrt{1-r^2})$ in S^{n+1} for $1 < k < n-1$,*

(2) *the standard cylinder $S^k(1) \times \mathbb{R}^{n-k} \subset \mathbb{R}^{n+1}$ for $1 < k < n-1$, or*

(3) *the image of $\sigma \circ \tau$ of $S^k(r) \times H^{n-k}(\sqrt{1+r^2})$ in H^{n+1} for $1 < k < n-1$. Here σ and τ are defined in Remark 1.3.1.*

Therefore we have the following results (or see [Guo et al. 2001]).

Proposition 4.3. *For $n \geq 4$, let $f : M^n \to \mathbb{R}^{n+1}$ be a Willmore hypersurface with two distinct principal curvatures. If the multiplicity of two principal curvatures are greater than 1, then f is Möbius equivalent to an open part of the image of σ of the standard torus $S^k((n-k)/n) \times S^{n-k}(k/n)$ in S^{n+1} for $1 < k < n-1$.*

Next we assume that one of principal curvatures is simple, and $\{k_1, k_2, \dots, k_2\}$ are the principal curvatures. We can choose a local orthonormal basis $\{E_1, \dots, E_n\}$ with respect to the Möbius metric g such that $(B_{ij}) = \operatorname{diag}(\lambda, \mu, \dots, \mu)$. From (8), we can assume that

$$(B_{ij}) = \operatorname{diag}\Big(\frac{n-1}{n}, \frac{-1}{n}, \dots, \frac{-1}{n}\Big).$$

In this section we make fix the range of indices according to the convention

$$1 \leq i, j, k \leq n \quad \text{and} \quad 2 \leq \alpha, \beta, \gamma \leq n.$$

Since $B_{\alpha\beta} = n^{-1}\delta_{\alpha\beta}$, we can choose another local orthonormal basis $\{E_1, \dots, E_n\}$ with respect to the Möbius metric g such that

$$(B_{ij}) = \operatorname{diag}\Big(\frac{n-1}{n}, \frac{-1}{n}, \dots, \frac{-1}{n}\Big) \quad \text{and} \quad (A_{ij}) = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1n} \\ A_{21} & a_2 & 0 & \cdots & 0 \\ A_{31} & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{n1} & 0 & 0 & \cdots & a_n \end{pmatrix}.$$

Let $\{\omega_1, \dots, \omega_n\}$ be the dual basis, and $\{\omega_{ij}\}$ the connection forms.

Lemma 4.4. *For $n \geq 3$, let $f : M^n \to \mathbb{R}^{n+1}$ be a Willmore hypersurface. If f has two distinct principal curvatures, and one of the principal curvatures is simple,* *then we can choose a local orthonormal basis* $\{E_1, \ldots, E_n\}$ *with respect to the Möbius metric g such that*

$$(B_{ij}) = \mathrm{diag}\Big(\frac{n-1}{n}, \frac{-1}{n}, \ldots, \frac{-1}{n}\Big), \quad (A_{ij}) = \mathrm{diag}(a_1, a_2, \ldots, a_2),$$
$$C_2 = \cdots = C_n = 0, \quad R_{1\alpha 1\alpha} - C_{1,1} + C_1^2 = 0,$$
$$B_{1\alpha,\alpha} = -C_1, C_{\alpha,\alpha} = -C_1^2, \quad C_{\alpha,\beta} = 0, \quad \alpha \neq \beta.$$

Moreover, *the distribution* $\mathrm{span}\{E_2, \ldots, E_n\}$ *is integrable.*

Proof of Lemma 4.4. Using $dB_{ij} + B_{kj}\omega_{ki} + B_{ik}\omega_{kj} = B_{ij,k}\omega_k$ and (5), we get

$$\begin{aligned} B_{1\alpha,\alpha} &= -C_1, \quad \text{otherwise } B_{ij,k} = 0; \\ \omega_{1\alpha} &= -C_1\omega_\alpha, \quad C_\alpha = 0. \end{aligned} \tag{24}$$

Thus $d\omega_1 = 0$ and the distribution $\mathrm{span}\{E_2, \ldots, E_n\}$ is integrable.

Using $dC_i + C_k\omega_{ki} = C_{i,k}\omega_k$ and (24), we can obtain

$$C_{\alpha,\alpha} = -C_1^2, \quad C_{\alpha,k} = 0 \quad \text{for } \alpha \neq k. \tag{25}$$

From (24),

$$d\omega_{1\alpha} = -dC_1 \wedge \omega_\alpha - C_1 d\omega_\alpha = -dC_1 \wedge \omega_\alpha - C_1^2\omega_1 \wedge \omega_\alpha - C_1\omega_\gamma \wedge \omega_{\gamma\alpha},$$

and from $d\omega_{1\alpha} - \omega_{1j} \wedge \omega_{j\alpha} = -\frac{1}{2}R_{1\alpha kl}\omega_k \wedge \omega_l$, we get that

$$R_{1\alpha 1\alpha} = C_{1,1} - C_1^2, \quad R_{1\alpha\beta\alpha} - C_{1,\beta} = 0. \tag{26}$$

Since $R_{1\alpha 1\alpha} = -(n-1)/n^2 + a_1 + a_\alpha = C_{1,1} - C_1^2$ and $R_{1\alpha\beta\alpha} = A_{1\beta}$ for $\alpha \neq \beta$; thus we have

$$a_2 = a_3 = \cdots = a_n \quad \text{and} \quad A_{1\beta} = C_{1,\beta}. \tag{27}$$

Now we assume that f is a Willmore hypersurface, using (10), that is,

$$-(n-1)C_{i,i} + b_i^3 + b_i a_i = 0,$$

and (25) and (26), we get that

$$a_1 - a_2 = nC_{1,1} - n(n-1)C_1^2 - \frac{n-2}{n},$$
$$a_1 + a_2 = C_{1,1} - C_1^2 + \frac{n-1}{n^2}.$$

Thus we have

$$\begin{aligned} a_1 &= \frac{n+1}{2}C_{1,1} - \frac{n^2-n+1}{2}C_1^2 - \frac{n^2-3n+1}{2n^2}, \\ a_2 &= -\frac{n-1}{2}C_{1,1} + \frac{n^2-n-1}{2}C_1^2 + \frac{n^2-n-1}{2n^2}. \end{aligned} \tag{28}$$

Using $dA_{\alpha\beta}+A_{k\beta}\omega_{k\alpha}+A_{\alpha k}\omega_{k\beta}=A_{\alpha\beta,k}\omega_k$ and $C_\alpha=0$ we get that

$$\begin{aligned}&E_\beta(a_2)=A_{\alpha\alpha,\beta}=A_{\alpha\beta,\alpha}=-A_{1\beta}C_1 \quad \text{for } \alpha\neq\beta,\\ &A_{\alpha\beta,1}=0 \quad \text{and} \quad A_{\alpha\beta,\gamma}=0 \quad \text{for } \alpha\neq\beta,\ \beta\neq\gamma,\ \alpha\neq\gamma,\\ &E_1(a_2)=A_{\alpha\alpha,1}=A_{\beta\beta,1}, \quad A_{1\alpha,\alpha}=A_{\alpha\alpha,1}-\frac{C_1}{n}=A_{\beta\beta,1}-\frac{C_1}{n}=A_{1\beta,\beta}.\end{aligned} \tag{29}$$

Similarly we have

$$\begin{aligned}A_{1\alpha,k}&=E_k(A_{1\alpha})+A_{1\beta}\omega_{\beta\alpha}(E_k) \quad \text{for } k\neq\alpha,\\ A_{1\alpha,\alpha}&=E_\alpha(A_{1\alpha})-(a_1-a_2)C_1+A_{1\beta}\omega_{\beta\alpha}(E_\alpha),\end{aligned} \tag{30}$$

and

$$\begin{aligned}C_{1,\alpha k}&=E_k(C_{1,\alpha})+C_{1,\beta}\omega_{\beta\alpha}(E_k) \quad \text{for } k\neq\alpha,\\ C_{1,\alpha\alpha}&=E_\alpha(C_{1,\alpha})-(C_{1,1}-C_{\alpha,\alpha})C_1+C_{1,\beta}\omega_{\beta\alpha}(E_\alpha),\end{aligned} \tag{31}$$

Since $A_{1\alpha}=C_{1,\alpha}$, from (30) and (31) we get

$$A_{1\alpha,k}=C_{1,\alpha k} \quad \text{for } k\neq\alpha \quad \text{and} \quad C_{1,\alpha\beta}=A_{1\alpha,\beta}=0 \quad \text{for } \alpha\neq\beta. \tag{32}$$

From (28), (29), (32) and $dC_{1,1}+C_{k,1}\omega_{k1}+C_{1,k}\omega_{k1}=C_{1,1k}\omega_k$ we get that

$$C_{1,1\alpha}=A_{1\alpha,1}=(2n-1)C_{1,\alpha}C_1.$$

From (29) and $dA_{1\alpha,\beta}+A_{k\alpha,\beta}\omega_{k1}+A_{1k,\beta}\omega_{k\alpha}+A_{1\alpha,k}\omega_{k\beta}=A_{1\alpha,\beta k}\omega_k$ we get that

$$A_{1\alpha,\beta 1}=0 \quad \text{for } \alpha\neq\beta. \tag{33}$$

Similarly we have

$$A_{1\alpha,1\beta}=-(2n-1)C_{1,\alpha}C_{1,\beta} \quad \text{for } \alpha\neq\beta. \tag{34}$$

Using (33), (34) and the Ricci identity we get that

$$-(2n-1)C_{1,\alpha}C_{1,\beta}=A_{1\alpha,1\beta}-A_{1\alpha,\beta 1}=A_{1\beta}R_{\beta\alpha 1\beta}=-A_{1\beta}A_{1\alpha}=-C_{1,\alpha}C_{1,\beta}.$$

Thus there exist at least $n-2$ coefficients in $\{C_{1,\alpha}\}$ such that $C_{1,\alpha}=0$. If there exists a $C_{1,\alpha}\neq 0$, we can assume that

$$C_{1,2}\neq 0,\ C_{1,3}=\cdots=C_{1,n}=0.$$

Thus

$$A_{12}\neq 0,\ A_{1,3}=\cdots=A_{1,n}=0.$$

From $dA_{1\alpha}+A_{k\alpha}\omega_{k1}+A_{1k}\omega_{k\alpha}=A_{1\alpha,k}\omega_k$, we have

$$\begin{aligned}A_{1\alpha,k}&=E_k(A_{1\alpha})+(a_1-a_2)\omega_{1\alpha}(E_k)+A_{12}\omega_{2\alpha}(E_k),\\ A_{1\alpha,\alpha}&=-(a_1-a_2)C_1+A_{12}\omega_{2\alpha}(E_\alpha) \quad \text{for } \alpha\neq 2.\end{aligned} \tag{35}$$

From (24), (29) and (35) we get that

$$\omega_{2\alpha}(E_1)=0,\quad \omega_{2\alpha}(E_2)=0 \quad\text{and}\quad \omega_{2\alpha}(E_\beta)=0 \quad\text{for } \alpha\neq\beta.$$

Thus we assume that

$$\omega_{2\alpha}=\psi\omega_\alpha \quad\text{for } \alpha\neq 2, \tag{36}$$

where ψ is local function on M^n.

From (24) and (36) we obtain

$$[E_1, E_2]=C_1E_2.$$

Using (24), (36) and $d\omega_{2\alpha}-\omega_{2m}\wedge\omega_{k\alpha}=-\frac{1}{2}R_{2\alpha kl}\omega_k\wedge\omega_l$ we derive that

$$E_1(\psi)=\psi C_1-A_{12} \quad\text{and}\quad E_2(\psi)=-\psi^2-C_1^2-R_{2\alpha 2\alpha}. \tag{37}$$

From (29) and (35), we derive that

$$E_2(A_{12})=A_{12}\psi,\quad E_1(A_{12})=A_{12,1}=A_{11,2}=(2n-1)C_1C_{1,2}. \tag{38}$$

From (30), (38) and $[E_1, E_2](A_{12})=C_1A_{12}$, we get that

$$-2nC_{1,2}^2=0.$$

This is a contradiction, so

$$C_{1,2}=C_{1,3}=\cdots=C_{1,n}=0$$

and

$$A_{12}=A_{13}=\cdots=A_{1n}=0.$$

From (29) and (30) we get that

$$E_1(a_2)=A_{\alpha\alpha,1}=\Big(a_2-a_1+\frac{1}{n}\Big)C_1 \quad\text{and}\quad E_\alpha(a_1)=E_\alpha(a_2)=0. \tag{39}$$

From (24), (25), (26) and (27), we finish the proof of Lemma 4.4. □

Now we choose the local orthonormal basis $\{E_1,\dots,E_n\}$ as in Lemma 4.4, which consists of principal vectors. Then $\{Y, N, Y_1,\dots,Y_n,\xi\}$ forms a moving frame in $\mathbb{R}_1^{n+3}$ along M^n. We define

$$F=-\frac{1}{n}Y+\xi,\quad X_1=-C_1Y-Y_1,\quad P=-a_2Y+N+C_1X_1+\frac{1}{n}F.$$

Clearly F is the curvature sphere of principal curvature k_2 of multiplicity $n-1$.

Let $K=2a_2+C_1^2+1/n^2$. By direct computations, we have

$$\begin{aligned}&\langle F, X_1\rangle=0, \qquad \langle F, P\rangle=0,\quad \langle X_1, P\rangle=0,\\ &\langle F, F\rangle=\langle X_1, X_1\rangle=1,\quad \langle P, P\rangle=-K.\end{aligned} \tag{40}$$

From Lemma 4.4, (39) and the structure equations of f we derive that

$$\begin{aligned} E_1(F) &= X_1, & E_\alpha(F) &= 0, \\ E_1(X_1) &= P - F, & E_\alpha(X_1) &= 0, \\ E_1(P) &= C_1 P + K X_1, & E_\alpha(P) &= 0. \end{aligned} \tag{41}$$

Thus subspace $V = \text{span}\{F, X_1, P\}$ is fixed along M^n. From (39) we get that

$$E_1(K) = 2C_1 K \quad \text{and} \quad E_\alpha(K) = 0. \tag{42}$$

Using theory of linear first order differential equations for K, formula (42) implies that $K \equiv 0$ or $K \neq 0$ on an open subset $U \subset M^n$. Therefore we have to consider three cases that $K = 0$ on M^n, that $K < 0$ on M^n, and that $K > 0$ on M^n.

Theorem 1.2 is proved in the next three propositions, one for each case.

Proposition 4.5. *For $n \geq 3$, let $f : M^n \to \mathbb{R}^{n+1}$ be a Willmore hypersurface with two distinct principal curvatures, of which one is simple. Under the local orthonormal basis $\{E_1, \ldots, E_n\}$, if $K = 2a_2 + C_1^2 + 1/n^2 = 0$, then f is Möbius equivalent to a cylinder over a free n-elastic curve in a Euclidean 2-plane.*

Proof. Since $K = 0$, we have $\langle P, P\rangle = 0$. From (41), we know that P is of fixed direction. From (40), up to a Möbius transformation we can write

$$\begin{aligned} P &= \nu(1, -1, 0, \ldots, 0) \quad \text{for } \nu \in C^\infty(U), \\ V &= \text{span}\{F, X_1, P\} \\ &= \text{span}\{(1, -1, 0, \ldots, 0), (0, 0, 1, 0, \ldots, 0), (0, 0, 0, 1, 0, \ldots, 0)\} \\ &= \mathbb{R}^3_0. \end{aligned}$$

Since $f : M^n \to \mathbb{R}^{n+1}$ has principal curvatures $(k_1, k_2, \ldots, k_2)$ and

$$\langle P, F\rangle = \langle (1, -1, 0, \ldots, 0), F\rangle = 0 \quad \text{and} \quad \langle X_1, P\rangle = 0,$$

we have

$$k_2 = 0 \quad \text{and} \quad C_1\rho + E_1(\rho) = 0, \quad \text{that is, } E_1(\log\rho) = -C_1. \tag{43}$$

From definition of F, X_1 and P, we get that $Y_\alpha \perp V$; thus $\langle P, Y_\alpha\rangle = 0$, and

$$E_\alpha(\rho) = 0, \quad \text{that is, } E_\alpha(\log\rho) = 0. \tag{44}$$

Let $\{e_i = \rho E_i, 1 \leq i \leq n\}$; then $\{e_1, \ldots, e_n\}$ is a local orthonormal basis with respect to the first fundamental form $df \cdot df$. Let $\{\tilde\omega_1, \ldots, \tilde\omega_n\}$ be the dual basis and $\{\tilde\omega_{ij}\}$ connection forms with respect to the basis $\{\tilde\omega_1, \ldots, \tilde\omega_n\}$. Then from Lemma 4.1, (24), (43) and (44) we get

$$\tilde\omega_{1\alpha} = 0.$$

Therefore hypersurface $f : M^n \to \mathbb{R}^{n+1}$ is Möbius equivalent to the hypersurface given by Example 3.0.1. Since f is a Willmore hypersurface, from Theorem 3.1 we finish the proof. □

Proposition 4.6. *For $n \geq 3$, let $f : M^n \to \mathbb{R}^{n+1}$ be a Willmore hypersurface with two distinct principal curvatures, one of which is simple. Under the local orthonormal basis $\{E_1, \ldots, E_n\}$, if $K = 2a_2 + C_1^2 + 1/n^2 < 0$, then f is Möbius equivalent to a cone over a free n-elastic curve in a 2-sphere.*

Proof. Since $K < 0$, we know $\langle P, P\rangle$ is positive. From (40), up to a Möbius transformation we can write

$$\begin{aligned} V &= \operatorname{span}\{F, X_1, P\} \\ &= \operatorname{span}\{(0, 0, 1, 0, \ldots, 0), (0, 0, 0, 1, 0, \ldots, 0), (0, 0, 0, 0, 1, 0, \ldots, 0)\} \\ &= \mathbb{R}^3. \end{aligned}$$

Thus

$$e = (1, -1, 0, \ldots, 0) \perp V.$$

Since $f : M^n \to \mathbb{R}^{n+1}$ has principal curvatures $(k_1, k_2, \ldots, k_2)$ and

$$\langle e, F\rangle = \langle e, X_1\rangle = 0,$$

we have

$$k_2 = 0 \quad \text{and} \quad C_1\rho + E_1(\rho) = 0, \quad \text{that is, } E_1(\log\rho) = -C_1. \tag{45}$$

Setting

$$T = -a_2 Y - N + C_1 Y_1 - \frac{1}{n}\xi, \quad \overline{P} = \frac{P}{\sqrt{-K}}, \quad \theta = \frac{T}{\sqrt{-K}},$$

we have

$$\begin{aligned} &\langle \overline{P}, \overline{P}\rangle = 1, \qquad \langle\theta, \theta\rangle = -1, \\ &\theta \perp V = \mathbb{R}^3, \quad \langle\theta, Y_\alpha\rangle = 0 \quad \text{for } \theta \in \mathbb{R}^n_1. \end{aligned} \tag{46}$$

From Lemma 4.4, (39), (41) and the structure equations of f, we derive that

$$E_1(\theta) = 0 \quad \text{and} \quad E_\alpha(\theta) = \sqrt{-K}\, Y_\alpha. \tag{47}$$

Since $P + T = -KY$, we have

$$Y = \frac{1}{\sqrt{-K}}(\overline{P}, \theta) \in \mathbb{R}^{n+3}_1 = \mathbb{R}^3 \times \mathbb{R}^n_1.$$

Since the distribution $\operatorname{span}\{E_2, \ldots, E_n\}$ is integrable from (41), (46) and (47), the map Y factors through a conformal diffeomorphism θ from the space of leaves V

of this foliation to H^{n-1}. Thus

$$\overline{P} : I \to S^2 \subset \mathbb{R}^3 \quad \text{and} \quad \theta : H^{n-1} \to \mathbb{R}^n_1.$$

From (9), we get $\rho^2 = k_1^2$. Since $k_2 = 0$, we may assume that $k_1 > 0$, and $\rho = k_1$. Using Lemma 4.1, (9) and (45), the Möbius metric of f is

$$g = \langle dY, dY \rangle = k_1^2 (ds^2 + I_{H^{n-1}}).$$

and under the local orthonormal basis $\{E_1, \ldots, E_n\}$, the Möbius second fundamental form of f is

$$(B_{ij}) = \operatorname{diag}\Big(\frac{n-1}{n}, \frac{-1}{n}, \ldots, \frac{-1}{n}\Big).$$

From Theorem 2.2, we know that the hypersurface $f : M^n \to \mathbb{R}^{n+1}$ is Möbius equivalent to the hypersurface given by Example 3.1.1. Since f is a Willmore hypersurface, the claim follows from Theorem 3.2. □

Proposition 4.7. *For $n \geq 3$, let $f : M^n \to \mathbb{R}^{n+1}$ be a Willmore hypersurface with two distinct principal curvatures, on which is simple. Under the local orthonormal basis $\{E_1, \ldots, E_n\}$, if $K = 2a_2 + C_1^2 + 1/n^2 > 0$, then f is Möbius equivalent to a rotational hypersurface over a free n-elastic curve in a hyperbolic 2-plane.*

Proof. Since $K > 0$, we know $\langle P, P \rangle$ is negative. From (40), up to a Möbius transformation we can write

$$\begin{aligned} V &= \operatorname{span}\{F, X_1, P\} \\ &= \operatorname{span}\{(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), (0, 0, 1, 0, \ldots, 0)\} \\ &= \mathbb{R}^3_1. \end{aligned}$$

Thus $e = (1, -1, 0, \ldots, 0) \in V$, and from $\langle e, Y_\alpha \rangle = 0$ we get that

$$E_\alpha(\rho) = 0.$$

setting

$$T = -a_2 Y - N + C_1 Y_1 - \frac{1}{n}\xi, \quad \overline{P} = \frac{P}{\sqrt{K}}, \quad \theta = \frac{T}{\sqrt{K}}.$$

Then

$$\begin{aligned} &\langle \overline{P}, \overline{P} \rangle = -1, \quad \langle \theta, \theta \rangle = 1, \\ &\theta \perp V = \mathbb{R}^3_1, \quad \langle \theta, Y_\alpha \rangle = 0, \quad \theta \in \mathbb{R}^n. \end{aligned} \tag{48}$$

From Lemma 4.4, (39), (41) and the structure equations of f we derive that

$$E_1(\theta) = 0 \quad \text{and} \quad E_\alpha(\theta) = -\sqrt{K}\, Y_\alpha. \tag{49}$$

Since $P+T=-KY$, we have

$$Y=\frac{1}{-\sqrt{K}}(\overline{P},\theta)\in\mathbb{R}^{n+3}_1=\mathbb{R}^3_1\times\mathbb{R}^n.$$

Since the distribution span$\{E_2,\dots,E_n\}$ is integrable from (41), (48) and (49), the map Y factors through a conformal diffeomorphism θ from the space of leaves V of this foliation to S^{n-1}. Thus

$$\overline{P}:I\to H^2\subset\mathbb{R}^3_1\quad\text{and}\quad\theta:S^{n-1}\to\mathbb{R}^n.$$

Write $\overline{P}=(u_1,u_2,u_3)\in H^2$; then

$$Y=\frac{u_1+u_2}{-\sqrt{K}}\Big(\frac{u_1}{u_1+u_2},\frac{u_2}{u_1+u_2},\frac{u_3}{u_1+u_2},\frac{1}{u_1+u_2}\theta\Big).$$

Then the hypersurface $f:I\times S^{n-1}\to\mathbb{R}^{n+1}$ is

$$f=\Big(\frac{u_3}{u_1+u_2},\frac{1}{u_1+u_2}\theta\Big).$$

Using $\phi^{-1}:H^2\to\mathbb{R}^2_+$, we know that the hypersurface $f:M^n\to\mathbb{R}^{n+1}$ is Möbius equivalent to the hypersurface given by Example 3.2.1. Since f is a Willmore hypersurface, the claim follows from Theorem 3.3. □

Theorem 1.2 follows from Propositions 4.5, 4.6 and 4.7. □

If the hypersurface $f:M^n\to\mathbb{R}^{n+1}$ with $n\ge 4$ and without umbilical point is conformally flat, then f has two distinct principal curvatures, one of which is simple. Therefore Corollary 1.3 is proved by Theorem 1.2.

Remark. The circle $S^1(\sqrt{(n-1)/n})$ with radius $\sqrt{(n-1)/n}$ is a closed free n-elastic curve with constant oriented curvature in the Poincare half plane $\mathbb{R}^2_+$. The rotational hypersurface over the circle $S^1(\sqrt{(n-1)/n})$ is the image of σ of the standard torus $S^1(\sqrt{(n-1)/n})\times S^{n-1}(\sqrt{1/n})$.

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