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We use surgery along 2-tori embedded in a union of two copies of $T_o^2 \times T_o^2$ to produce a new collection of homotopy 4-spheres.

1. Introduction

With the Poincaré conjecture now established, the attention of many experts is shifting to the 4-dimensional smooth counterpart to the conjecture:

Conjecture (SPC4: the smooth Poincaré conjecture in four dimensions). Let M be a smooth 4-manifold homeomorphic to the 4-sphere S^4. Then M is diffeomorphic to S^4.

The persisting lack of any answer to SPC4 is probably in part due to the wild nature of smooth 4-manifolds in general, which — even restricting our scope to the simply connected setting — have proven exceptionally formidable in terms of constructing any plenary classification scheme. Still, of all simply connected 4-manifolds, the 4-sphere continues to present perhaps the most elusive challenge when it comes to obtaining/finding exotic smooth structures. On the one hand, the literature abounds with *potential* counterexamples to the conjecture; but on the other hand, not one example has yet been verified as exotic. This is, it seems, largely due to the lack of smooth invariants for S^4 (which other exotic 4-manifold constructions have relied upon).

Historically, these exotic constructions of other simply connected 4-manifolds have quite often made use of surgery along 2-tori (generalized logarithmic transformation). This paper highlights the utility of torus surgery in conjunction with SPC4 and (once again) as a potential facet of the classification of smooth 4-manifolds in general. In Section 2 we lay out the background material needed to construct our examples. Section 3 comprises the heart of this work, the production of new homotopy 4-sphere examples (we do not however prove here that our examples are counterexamples to SPC4). These constructions are inspired by an intriguing handlebody presentation of S^4 given by in [Fintushel and Stern 2008] and the role

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of surgery upon 2-tori embedded in $T_o^2 \times T_o^2$ as seen in the "reverse engineering" program of Fintushel, Park, and Stern [Fintushel et al. 2007].

In Section 4 we illustrate a further correspondence between SPC4 and surgery along embedded 2-tori in conjunction with the classic homotopy sphere examples of Cappell and Shaneson [1976a; 1976b] and the recent analysis of these examples by Gompf [2010]. Specifically, their homotopy S^4's can be viewed as the result of surgery along a circle in special mapping tori on T^3, later labeled M_ϕ and referred to as "Cappell–Shaneson mapping tori". Gompf exhibited diffeomorphisms between all members of a certain family of homotopy spheres arising from M_ϕ manifolds by altering the monodromy via surgery along fishtail embedded 2-tori in the T^3 fiber.

Our focus here is on the monodromy changing mechanics of torus surgeries. We exhibit a (perhaps) surprising connection between a subcollection of our surgery manifolds produced from $T_o^2 \times T_o^2$ and the mapping tori M_ϕ of Cappell–Shaneson, but we also show that this approach does not directly imply the trivialization of our general homotopy sphere examples.

2. Background material

Definition: surgery along a torus. Given a 4-manifold M and a torus $T \subset M$ which has a trivial normal bundle $\nu T \subset M$, a surgery (or generalized logarithmic transformation) along T is the process of extracting the interior of a tubular neighborhood of T, and then regluing $T^2 \times D^2$ via some diffeomorphism δ of its boundary. (The restriction on the normal bundle ensures $\nu T \approx T^2 \times D^2$.) Notice that the boundary of $T^2 \times D^2$ is $T^2 \times S^1 \approx T^3$, a three-torus; so diffeomorphisms of the boundary are elements of $GL(3, \mathbb{Z}) \cong \mathit{Diff}(T^3)$. The resulting manifold $M_{\delta,T}$ is given as:

$$M_{\delta,T} = (M \setminus \nu T) \cup_\delta T^2 \times D^2.$$

Due to the handlebody (see [Gompf and Stipsicz 1999; Kirby 1989], for instance) structure of a trivial torus bundle $T^2 \times D^2 = h^0 \cup h_a^1 \cup h_b^1 \cup h^2$, there is a unique way to attach the dualized 3- and 4- handles coming from h_a^1, h_b^1, h^0 to $(M \setminus \nu T) \cup h^2$. Hence, the regluing map δ can be described by the attaching map of the 2-handle. In terms of homology, this gluing of the 2-handle into the boundary of $(M \setminus \nu T)$ — and the surgery itself — depends on a choice of curves along the boundary. Specifically, taking two loops $\{a', b'\}$ which generate $\pi_1(T)$ we push these in νT out to loops a and b on the boundary of $M \setminus \nu T$. If μ is the curve in νT which bounds, then $B = \{[a], [b], [\mu]\}$ forms a basis for $H_1(\partial(M \setminus \nu T); \mathbb{Z}) \cong H_1(T^3; \mathbb{Z}) \cong \mathbb{Z}^3$. The surgery then can be defined by a linear combination in B that gives the attaching curve for ∂D^2, the boundary of the attaching disk of $h^2 \approx D^2 \times D^2$. In sum, the surgery map δ and the resulting manifold $M_{\delta,T}$ are given by the choices a and b

and the map

$$\delta_* : H_1(T^2 \times \partial D^2; \mathbb{Z}) \to H_1(\partial(M \setminus \nu T); \mathbb{Z})$$

such that

$$\delta_*([\partial D^2]) = p\,[\mu] + q\,[a] + r\,[b]\,.$$

Generally, one refers to the above as a (p, q, r)-surgery along T with respect to a, b or a degree p-surgery in the direction $qa + rb$. (From now on we also denote both loops a and their corresponding homology classes $[a]$ by simply a.) For certain simpler situations (like those considered in this paper), one of the last two coefficients will be 0, and we will mimic the notation of Dehn surgery in 3-manifolds by calling this a $(\frac{p}{q})$-surgery with p the coefficient of the meridian.

Reverse engineering and torus surgeries of Fintushel and Stern. Of particular import in this paper, is the approach of [Fintushel and Stern 2008] and [Fintushel et al. 2007] in devising clever ways of discovering *nullhomologous* tori embedded in standard 4-manifolds which are somehow linked to exotic smooth structures on these 4-manifolds.

In the latter paper, Fintushel, Park, and Stern define and implement their reverse engineering process, whereby exotic smooth structures on small Euler characteristic manifolds (such as $\mathbb{CP}^2\#n\overline{\mathbb{CP}}^2$, for $n \leq 8$) can be obtained. In their description, a simply connected manifold such as $M = \mathbb{CP}^2\#n\overline{\mathbb{CP}}^2$ serves as the *target* of the procedure, while a different non-simply connected symplectic manifold — the *model* for M — is actually used as the starting point. The above authors were able to produce an infinite collection of Seiberg–Witten invariant altering surgeries.

For the purposes of this paper, Taubes' result [1994] on Seiberg–Witten invariants and its utilization as in [Fintushel et al. 2007] are not quite applicable. On the other hand, this formulation of *models* is indeed useful for our *target*, S^4. And of the greatest use here is the Fintushel–Park–Stern model for a special target which is not a blow-up of $\mathbb{CP}^2$, the target $S^2 \times S^2$. The model employed in [Fintushel et al. 2007] is a fiber sum along a genus two surface in two copies of $\Sigma_2 \times T^2$, that is $\Sigma_2 \times \Sigma_2$. Also, if each genus-2 surface complement $(\Sigma_2 \times T^2 \setminus \Sigma_2 \times D^2)$ is further decomposed as $T_o^2 \times T_o^2$ (a product of punctured 2-tori), then the (8-many) surgeries leading to a fake $S^2 \times S^2$ can be realized within the four individual $T_o^2 \times T_o^2$ copies.

$\boldsymbol{T_o^2 \times T_o^2}$**.** Now this decomposition of $\Sigma_2 \times \Sigma_2$ as a 4-fold union of $T_o^2 \times T_o^2$'s (equivalently, the complements of a the coordinate axes in copies of $T^4 = T^2 \times T^2$) suggests a key strategy for understanding exotic smooth structures and the related model manifolds might be to focus on this core building block $T_o^2 \times T_o^2$ itself. Actually, Fintushel and Stern have arrived in this situation from an alternate starting point. Pursuing useful nullhomologous tori embedded in standard 4-manifolds, Fintushel and Stern have exhibited a particular manifold-with-boundary (which

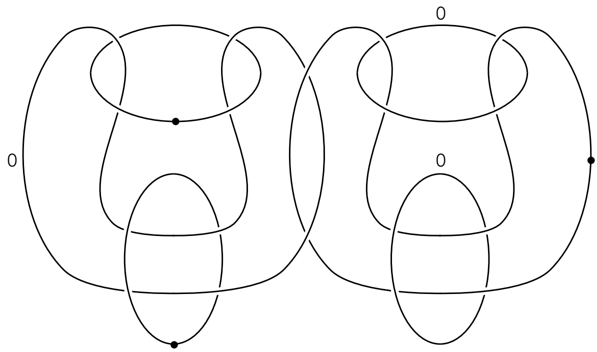

Figure 1. Fintushel and Stern's A manifold.

they denote by A) that itself contains a Bing double of nullhomologous tori, and upon which they build their models. Figure 1 gives a handlebody diagram for A which is equivalent to the one appearing in [Fintushel and Stern 2008].

Two key aspects concerning A now become important for the emphasis of our work. Let $B_T \stackrel{\text{def}}{=}$ the pair of tori mentioned above. Then:

Proposition 2.1 [Fintushel and Stern 2008, Proposition 2]. *The result of* 0*-framed surgery on the pair of tori* $B_T \subset A$ *is* $T_o^2 \times T_o^2$. □

Second, Fintushel and Stern have also made the following observation which is simple to check: If φ is the involution of ∂A which flips the handlebody's boundary about a vertical line through the middle of the diagram above, then $A \cup_\varphi \overline{A} \approx S^4$. Forming this union amounts to gluing in the second copy's 2-handles as 0-framed meridians to the first copy's *1-handles* and then attaching the dualized 1-handles as 3-handles. Essentially, one arrives at the handlebody of Figure 2 union three 3-handles and a 4-handle.

After sliding 2-handles over these 0-framed meridians and canceling pairs of 1-,2-handles, the boundary is explicitly seen as $\#_3 S^1 \times S^2$. Thus, one can add back in the extra 3-handles, cancel the 2-,3-handle pairs, and add the 4-handle to obtain S^4. These two observations above now give the connection between surgery on model manifolds and SPC4, and the way is paved for the main consideration of this section. Overall, this implies:

S^4 *contains four nullhomologous tori*, 0*-framed surgery upon which yields*

$$T_o^2 \times T_o^2 \cup_\varphi \overline{T_o^2 \times T_o^2}.$$

Or dually, starting from the opposite direction:

$T_o^2 \times T_o^2 \cup_\varphi \overline{T_o^2 \times T_o^2}$ *contains four essential tori*, *surgery upon which yields* S^4.

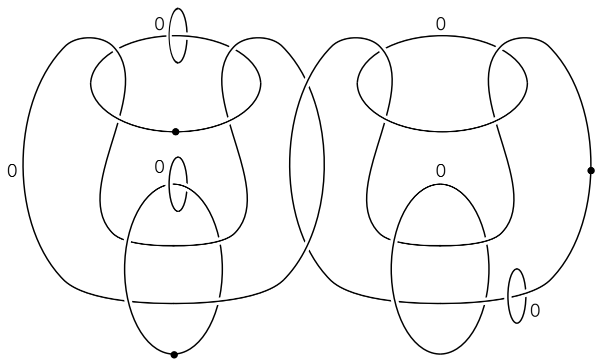

Figure 2. $A \cup_\varphi \overline{A} \setminus$ (3-handles, 4-handle).

This leads one to consider whether *other* surgeries upon tori in

$$T_o^2 \times T_o^2 \cup_\varphi \overline{T_o^2 \times T_o^2}$$

will also produce S^4, or more importantly, whether there are surgeries that might possibly produce an *exotic* S^4. We exhibit below, surgeries which *at least* produce a homotopy S^4 not a priori diffeomorphic to $A \cup_\varphi \overline{A}$.

3. Homotopy 4-spheres from $T_o^2 \times T_o^2$

Constructing a new homotopy 4-sphere. To begin our construction, note that the boundary of $T_o^2 \times T_o^2$ is

$$\partial(T_o^2 \times T_o^2) = T_o^2 \times S^1 \cup S^1 \times T_o^2,$$

where the two boundary terms are not disjoint but overlap in a torus.

In the following, we make use of the same convenient involution

$$\varphi : T_o^2 \times S^1 \cup S^1 \times T_o^2 \to T_o^2 \times S^1 \cup S^1 \times T_o^2$$

which is a flip along the entire boundary. This can be formally defined by

$$\varphi(x) = x^*,$$

where for $x \in T_o^2 \times S^1$, x^* is the corresponding point of $S^1 \times T_o^2$ and conversely. Under this framework, we will prove this:

Theorem 3.1 ($T_o^2 \times T_o^2$ surgery theorem). *For φ as above, there are two lagrangian tori in $T_o^2 \times T_o^2$ and a pair of lagrangian-framed surgeries such that the resulting surgery manifold X' satisfies*

$$X' \cup_\varphi \overline{X}' \cong S^4.$$

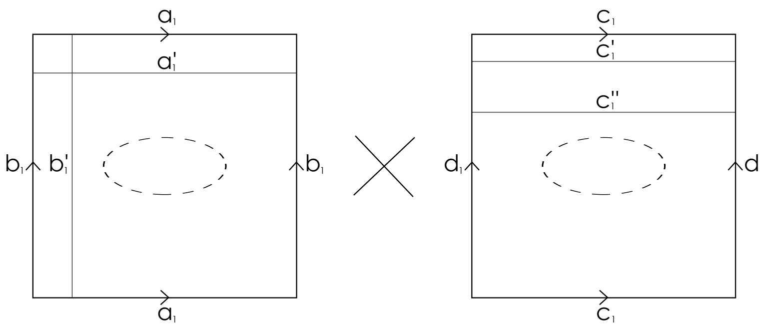

Figure 3. $T_o^2 \times T_o^2$ and lagrangian tori.

Proof. The surgeries in view here are actually performed identically in both copies. For homotopy calculations we appeal to the results of [Baldridge and Kirk 2008], essentially surgering the same pair of tori depicted in their calculation. In order to guarantee the effects of surgeries on π_1, we also utilize a slightly careful description of the torus surgeries. Here label the π_1-generating loops passing through the basepoint (x, y) in the i^{th} punctured torus product by a_i, b_i from one punctured torus factor and c_i, d_i from the other. Similar to [Fintushel et al. 2007] label lagrangian push-offs of these loops by "primes" as in Figure 3.

Set $T_{ac} = a_1' \times c_1'$ and $T_{bc} = b_1' \times c_1''$. Then as in [Baldridge and Kirk 2008, Theorem 2], the complement of the two tori in $T_o^2 \times T_o^2$ has fundamental group generated by a_1, b_1, c_1, d_1 with several relations. These include

$$[a_1, c_1] = 1, \tag{1}$$

$$[b_1, c_1] = 1. \tag{2}$$

In the notation of [Fintushel et al. 2007], the surgery tori, directions, and coefficients selected here are of the form (torus, direction, coefficient). After regluing the two tori along these surgery curves, we have new π_1 relations, again by [Baldridge and Kirk 2008, Theorem 2]:

Surgery	New π_1 relation
$(a_1' \times c_1',\ a_1', -1)$	$[b_1^{-1}, d_1^{-1}] = a_1$
$(b_1' \times c_1'',\ b_1', -1)$	$[a_1^{-1}, d_1] = b_1$

Now to continue the proof of the theorem, we need:

Proposition 3.2. *Each of the loops a_1, b_1, c_1, d_1 are based homotopic to a corresponding loop on the boundary of $T_o^2 \times T_o^2$, in the complement of tori*

$$T_o^2 \times T_o^2 \setminus T_{ac} \cup T_{bc}.$$

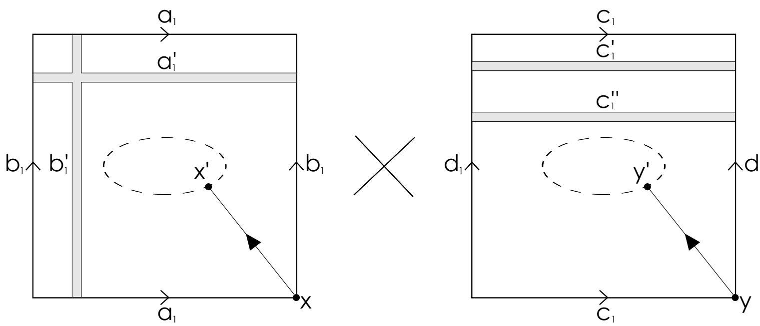

Figure 4. Paths from basepoint (x, y) to the puncture.

Proof. For $a_1' \times y$, $b_1' \times y$, etc. push the corresponding point x or y along a straight linear path to x' or y'. This can be done in such a way that νT_{ac} and νT_{bc} are avoided, as in Figure 4. □

With this proposition, in π_1 of the union of surgery manifolds $X' \cup_\varphi \overline{X}'$ we have the relations $a_1 \sim c_2$, $b_1 \sim d_2$, $a_2 \sim c_1$, and $b_2 \sim d_1$. Hence, applying the equivalent of (2) to the second copy we also obtain

(3) $$[b_2, c_2] = [d_1, a_1] = 1$$

and from the first and second pair of surgeries

(4) $$[b_1^{-1}, d_1^{-1}] = a_1,$$

(5) $$[a_1^{-1}, d_1] = b_1,$$

(6) $$[b_2^{-1}, d_2^{-1}] = a_2 = c_1 = [d_1^{-1}, b_1^{-1}],$$

(7) $$[a_2^{-1}, d_2] = b_2 = d_1 = [c_1^{-1}, b_1]..$$

Using (3) together with (5) implies $b_1 = 1$, and then (4) and (6) in turn give $a_1, c_1 = 1$. Finally, (7) gives $d_1 = 1$. After the four surgeries in the union $T_o^2 \times T_o^2 \cup_\varphi \overline{T_o^2 \times T_o^2}$, we obtain the simply connected manifold $\mathcal{S}' \overset{\text{def}}{=} X' \cup_\varphi \overline{X}'$. Since $\mathcal{S}'$ also has $\chi = 2$, it is therefore homeomorphic to S^4. □

Families of homotopy 4-spheres. By the choice of surgeries (in fact *either* of the -1 or $+1$ surgeries works so that X' as depicted above is only one such possible choice of surgery manifolds; it is not yet known whether these are pairwise diffeomorphic). However, any such X' is distinct from A (see Section 4), hence it is not obvious that $\mathcal{S}'$ is standard S^4. Now if one is willing to sacrifice the benefit of having a symplectic surgery manifold like X', allowing a greater freedom in surgeries can still yield a homotopy S^4.

Theorem 3.3 (main theorem). *For $m, n \in \mathbb{Z}$, Let $X_{m,n}$ denote the result of performing the $(\frac{m}{1})$- and $(\frac{n}{1})$-surgeries on T_{ac}, T_{bc} and in the directions a, b respectively. Then the 4-manifold*

$$\mathcal{S}_{(m,n,m',n')} \stackrel{\text{def}}{=} X_{m,n} \cup_\varphi \overline{X_{m',n'}}$$

is homeomorphic to S^4 for all $m, n, m', n' \in \mathbb{Z}$.

Proof. Replace the relations such as (5) above with $[a_1^{-1}, d_1]^n = b_1$, etc. Again this gives $b_1 = 1$ and in turn all three of the other generators are trivial as before. □

Remark 3.4. In the following section we will see that even the nonsymplectic surgery manifolds $X_{m,n}$ above, when also both $m, n \neq 0$, are distinct from A, for slightly more subtle reasons (see Proposition 4.1).

Furthermore, we have described these surgeries and surgery coefficients from the starting point and point of view of $T_o^2 \times T_o^2 \cup_\varphi \overline{T_o^2 \times T_o^2}$. However, our specific pairs of tori, $a_i \times c_i$ and $b_i \times c_i$, were precisely those producing A (see [Fintushel and Stern 2008]) as well, and if W_T is a tubular neighborhood of the union of the four surgery tori, then

$$T_o^2 \times T_o^2 \cup_\varphi \overline{T_o^2 \times T_o^2} \setminus W_T = A \cup_\varphi \overline{A} \setminus W_T = S^4 \setminus W_T.$$

Hence, some regluing of four tori embedded in S^4 gives the manifold $\mathcal{S}_{(m,n,m',n')}$, and this proves:

Proposition 3.5. *The manifolds $\mathcal{S}_{(m,n,m',n')}$, obtained above by surgery along null-homologous tori in S^4, form a four-parameter family of homotopy four-spheres.* □

4. Further analysis and final remarks

Now of course $T_o^2 \times T_o^2$ is nothing other than the 4-torus, viewed as $T^2 \times T^2$, with its coordinate axis 2-tori $\nu_{TT} \stackrel{\text{def}}{=} \nu(S_a^1 \times S_b^1 \cup S_c^1 \times S_d^1)$ deleted. Recombining the surgery manifolds $X_{m,n}$ with ν_{TT} then gives the result of performing the same pair of surgeries in T^4. Two consequences emerge from this. First:

Proposition 4.1. *The Fintushel-Stern manifold A and the surgery manifolds $X_{m,n}$ satisfy*:

(1) $A = X_{m,0} = X_{0,n}$ *for all $m, n \in \mathbb{Z}$.*

(2) $X_{m,n} \ncong A$ *if both $m, n \in \mathbb{Z} \neq 0$.*

Proof. We prove the above by recasting the pair of surgeries in $T^4 = T_o^2 \times T_o^2 \cup \nu_{TT}$ as surgeries in $T^4 = T^3 \times S^1$. A similar trick was used already by Akhmedov, Baykur, and Park [Akhmedov et al. 2008]. In particular, note that regluings of

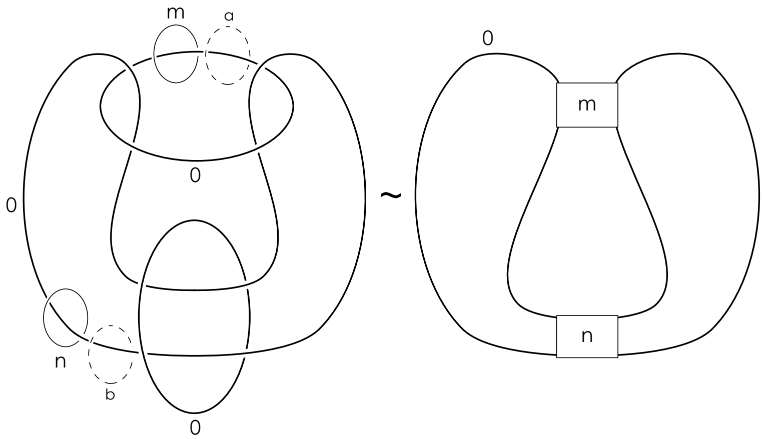

Figure 5. Two surgeries in T^3 (left) produce $Y_{m,n}$. (right).

both torus surgeries $(a' \times c',\ a', m)$ and $(b' \times c'',\ b', n)$ are trivial on the c-factor, that is, the surgery maps are equivalent to

$$(\text{Dehn-surgery on a loop in } T^3) \times Id|_{S^1}$$

in T^4 viewed as $T^3 \times S^1 = (a \times b \times d) \times c$. We can then fully depict the surgery manifolds (union ν_{TT}) by taking the cartesian product of S^1 and the resulting 3-manifold, $Y_{m,n}$, obtained from T^3 after the pair of Dehn-surgeries. This is depicted in Figure 5, left, where Dehn surgery is performed along push-offs of two of the meridians to the 0-framed Borromean link.

Since all of the link coefficients are integral, we can consider $Y_{m,n}$ as the boundary of some 4-manifold, say $U_{m,n}$. After sliding one of $U_{m,n}$'s 0-framed 2-handles over and then off of the m- and n-framed components and then removing hopf pairs, we obtain the diagram on the right in Figure 5. That diagram (viewed again as a 3-manifold surgery diagram) with $m = 0$ or $n = 0$ is of course $S^2 \times S^1$. Hence, for any such (m, n) pair,

$$Y_{m,n} \times S^1 \setminus \nu_{TT} = S^2 \times S^1 \times S^1 \setminus \nu_{TT} = A;$$

see [Fintushel and Stern 2008, Lemma 1], for instance. On the other hand, $Y_{m,n} \ncong S^2 \times S^1$ for any choice of a nonzero pair (m, n) since in that case $Y_{m,n}$ is not the unknot. □

Second, by gluing ν_{TT} onto any of the surgery manifolds of Theorem 3.1, we can recast the union as a T^3-bundle over S^1, that is, a mapping torus of the form

$$M_\phi \overset{\text{def}}{=} \frac{I \times T^3}{(0, x) \sim (1, \phi(x))}$$

for some diffeomorphism $\phi : T^3 \to T^3$. For instance, $\nu_{TT} \cup T_o^2 \times T_o^2 = T^4 = T^3 \times S^1 = M_I$, for I the identity map. Furthermore, $A \cup \nu_{TT} \approx S^2 \times T^2$, so A does not correspond to a T^3-bundle over S^1.

Now mapping tori such as these are precisely the kind that arise in the classic homotopy 4-sphere construction of Cappell and Shaneson [1976a; 1976b]. However, such monodromies ϕ obtained here from $T_o^2 \times T_o^2$ are *not* restricted to $SL(3; \mathbb{Z})$ and do not satisfy the additional condition $\det(\phi - I) = \pm 1$ of [Cappell and Shaneson 1976a], so surgery along the 0-section in any of our mapping tori will not produce one of their homotopy spheres directly.

Theorem 4.2 [Nash 2010]. *Any Cappell–Shaneson mapping torus M_ϕ can be obtained by some sequence of surgeries along 2-tori in the fiber of the trivial bundle $T^4 = T^3 \times S^1$.* □

We contend however that Theorem 4.2 is still not enough to immediately trivialize even one of the examples $\mathcal{S}_{(m,n,m',n')}$ by relating these spheres to any of those within the Cappell–Shaneson collection that are now known to be standard, most recently due to Akbulut [2010] and then later Gompf [2010] (which depends on a result from [Akbulut and Kirby 1979]).

The correspondences and differences can be seen as follows: Performing $(\frac{1}{q})$-surgeries along product 2-tori embedded in the T^3-fiber of any mapping torus M_A alters its monodromy by left multiplication with the surgery matrix (as in [Gompf 2010], and one dimension lower in [Gompf and Stipsicz 1999, Example 8.2.4]). Unlike Proposition 4.1 above, this time we factor the trivial fibration $T^4 = T^3 \times S^1$ as $(a \times b \times c) \times S_d^1$. Recall that the surgeries on $T_o^2 \times T_o^2$ producing the manifolds whose union is a homotopy S^4, say for instance $X_{1,1}$, have surgery curves $\mu + a$ or $\mu + b$, respectively (in the basis $\{a, b, \mu\}$, μ the meridian of the torus). Hence, back within the mapping-torus framework a $+1$-surgery along each of these two tori in these directions would give monodromy-multiplying matrices

$$R_{12} \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_{21} \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

respectively (now in the basis $\{a, b, c\}$).

The $X_{1,1}$ surgery manifolds then translate to mapping tori $M_{R_{12}R_{21}I} = M_A$ where

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

However, the full range of Cappell–Shaneson mapping tori M_ϕ also involve surgering in the direction of the third T^3 basis factor, so for instance, $b_1 = 1$ in the

Cappell–Shaneson case vs. $b_1(X_{m,n}) = 2$ here. Moreover, any $X_{m,n}$ with either m or $n \neq \pm 1$ no longer even gives a T^3-bundle over S^1 when ν_{TT} is added back in: In the case of a $(\pm\frac{1}{1})$-surgery, the diffeomorphism of the surgery torus "lines up" with a diffeomorphism of a fiber torus, but in a general $(\frac{m}{1})$-surgery ($m \neq \pm 1$) this correspondence fails. Thus in general, the complement of ν_{TT} in a true Cappell–Shaneson M_ϕ is *not* an $X_{m,n}$ surgery manifold.

One single surgery of the $X_{m,n}$ type is enough to derail $T_o^2 \times T_o^2$ from the Cappell–Shaneson track. Note that the surgery is still reversible. The point is that it is *not* reversible or achievable by torus surgeries obtained from product-framed $(\frac{1}{q})$-surgeries on product tori in the T^3 fiber — the sort used in Theorem 4.2.

Conclusion. The combined results above should indicate that once again $T_o^2 \times T_o^2$ itself remains an important component to a diverse range of 4-manifold constructions, surgeries along tori playing a role in each case. In fact, a slight alteration of the gluing φ in the $T_o^2 \times T_o^2$ unions above into a fixed-point-free involution of $\partial(T_o^2 \times T_o^2) = T_o^2 \times S^1 \cup S^1 \times T_o^2$ allows for the construction of a fixed-point-free involution on the resulting homotopy sphere. From this, homotopy $\mathbb{RP}^4$'s can then be constructed (given that two identical pairs of surgeries were performed) — again with $T_o^2 \times T_o^2$ playing the role of the fundamental piece to the construction.

Finally, despite the correlations between the two realms, it does not appear that any of the homotopy spheres $\mathcal{S}_{(m,n,m',n')}$ (parameters in $\mathbb{Z}^{\neq 0}$) actually relate directly to any member of the Cappell–Shaneson collection (do they even contain fibered 2-spheres?), nor does it seem that Gompf's trick of fishtail surgery [Gompf 2010] would help in trivializing them.

An earlier draft of this paper asked whether the spheres $\mathcal{S}_{(m,n,m'n')}$ are standard. Since then, Selman Akbulut [2011] has answered the question affirmatively. However, it is still unknown if gluing two surgery manifolds $X_{p,q}$ via some alternate diffeomorphism of the boundary would also produce S^4.

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References

[Akbulut 2010] S. Akbulut, "Cappell–Shaneson homotopy spheres are standard", *Ann. of Math.* (2) **171**:3 (2010), 2171–2175. MR 2011i:57024 Zbl 1216.57017

[Akbulut 2011] S. Akbulut, "Nash homotopy spheres are standard", preprint, 2011. arXiv 1102.2683

[Akbulut and Kirby 1979] S. Akbulut and R. Kirby, "An exotic involution of S^4", *Topology* **18**:1 (1979), 75–81. MR 80m:57006 Zbl 0465.57013

[Akhmedov et al. 2008] A. Akhmedov, R. İ. Baykur, and B. D. Park, "Constructing infinitely many smooth structures on small 4-manifolds", *J. Topol.* **1**:2 (2008), 409–428. MR 2010c:57040 Zbl 1146.57041

[Baldridge and Kirk 2008] S. Baldridge and P. Kirk, "A symplectic manifold homeomorphic but not diffeomorphic to $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$", *Geom. Topol.* **12**:2 (2008), 919–940. MR 2009e:57042 Zbl 1152.57026

[Cappell and Shaneson 1976a] S. E. Cappell and J. L. Shaneson, "Some new four-manifolds", *Ann. of Math.* (2) **104**:1 (1976), 61–72. MR 54 #6167 Zbl 0345.57003

[Cappell and Shaneson 1976b] S. E. Cappell and J. L. Shaneson, "There exist inequivalent knots with the same complement", *Ann. of Math.* (2) **103**:2 (1976), 349–353. MR 54 #1238 Zbl 0338.57008

[Fintushel and Stern 2008] R. Fintushel and R. J. Stern, "Surgery on nullhomologous tori and simply connected 4-manifolds with $b^+ = 1$", *J. Topol.* **1**:1 (2008), 1–15. MR 2008k:57050 Zbl 1148.57037

[Fintushel et al. 2007] R. Fintushel, B. D. Park, and R. J. Stern, "Reverse engineering small 4-manifolds", *Algebr. Geom. Topol.* **7** (2007), 2103–2116. MR 2009h:57044 Zbl 1142.57018

[Gompf 2010] R. E. Gompf, "More Cappell–Shaneson spheres are standard", *Algebr. Geom. Topol.* **10**:3 (2010), 1665–1681. MR 2011g:57034 Zbl 05769280

[Gompf and Stipsicz 1999] R. E. Gompf and A. I. Stipsicz, *4-manifolds and Kirby calculus*, Graduate Studies in Math. **20**, American Mathematical Society, Providence, RI, 1999. MR 2000h:57038 Zbl 0933.57020

[Kirby 1989] R. C. Kirby, *The topology of 4-manifolds*, Lecture Notes in Math. **1374**, Springer, Berlin, 1989. MR 90j:57012 Zbl 0668.57001

[Nash 2010] D. Nash, *Homotopy 4-spheres and surgery on 2-tori*, Ph.D. thesis, University of California, Irvine, 2010. MR 2753205

[Taubes 1994] C. H. Taubes, "The Seiberg–Witten invariants and symplectic forms", *Math. Res. Lett.* **1**:6 (1994), 809–822. MR 95j:57039 Zbl 0853.57019

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