

*Pacific
Journal of
Mathematics*

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Let (W, S) be an irreducible Weyl or affine Weyl group. In 1994, we constructed an algorithm for finding a representative set of left cells (or an l.c.r. set for short) of W in a two-sided cell Ω . Here, we introduce a new simpler algorithm for finding an l.c.r. set of W in Ω when the subset $F(\Omega)$ of Ω is known. We introduce some technical tricks by some examples for applying the algorithm and for finding the set $F(\Omega)$. The resulting set $E(\Omega)$ is useful in verifying a conjecture of Lusztig that any left cell in an affine Weyl group is left-connected.

Let W be an irreducible Weyl or affine Weyl group with S its Coxeter generator set. For a two-sided cell Ω of W (in the sense of [Kazhdan and Lusztig 1979]), we introduced an algorithm for finding an l.c.r. set of W in Ω in [Shi 1994a]. The algorithm has been efficiently applied in many cases; see for example [Chen 2000, Chen and Shi 1998; Rui 1995; Shi 1994a; 1994b; 1998a; 1998b; Shi and Zhang 2008; 2006; Tong 1995; Zhang 1994]. The algorithm consists of three processes (A), (B), (C) on a distinguished set F (see 3.1), where process (C) is the most difficult part among the three in which one need to find, for any given $x \in F$, all elements y satisfying $y \rightarrow x$, $y < x$, $\mathcal{R}(y) \not\subseteq \mathcal{R}(x)$ and $a(y) = a(x)$ (see Sections 1.1 and 1.3 for the notation). This becomes increasingly difficult as the length of x gets larger.

For any two-sided cell Ω of W , let $F(\Omega)$ be the set of all $w \in \Omega$ such that $a(sw), a(wt) < a(w)$ for any $s \in \mathcal{L}(w)$ and $t \in \mathcal{R}(w)$. We shall introduce a new algorithm for finding an l.c.r. set of W in a two-sided cell Ω , provided that the subset $F(\Omega)$ of Ω is known; see 3.2. The processes in our new algorithm amounts to a mixture of processes (A) and (B) in the original algorithm, hence avoiding process (C).

Theorem 3.5 guarantees that our new algorithm will terminate after finite steps, while Theorem 3.12 shows that the resulting set $E_0(\Omega)$ of Algorithm 3.11 forms an l.c.r. set of W in Ω such that each element of $E_0(\Omega)$ is shortest in the left cell

Supported by the NSF of China, the SFUDP of China, PCSIRT, Shanghai Leading Academic Discipline Project (B407), and Program of Shanghai Subject Chief Scientist (11xd1402200).

MSC2010: primary 20F55; secondary 20C08.

Keywords: affine Weyl groups, left cells, two-sided cells, alcove forms, algorithm.

of W containing it.

Our new algorithm has been applied successfully for the description of the left cells of a -values 4, 5, 6 in the affine Weyl groups \tilde{E}_i for $i = 6, 7, 8$; see [Huang 2008; Liu 2007; Shi and Zhang 2006].

To apply our new algorithm, it is desirable to find the subset $F(\Omega)$ explicitly for more two-sided cells Ω in an irreducible Weyl and affine Weyl group.

It is relatively easier to describe the set $F(\Omega)$ when $F(\Omega)$ consists of elements of the form w_J for some $J \subseteq S$, where the subgroup W_J of W generated by J is finite and w_J is the longest element in W_J ; see 4.6.

We can also find the sets $F(\Omega)$ for some two-sided cells Ω when Ω contains some elements not of the form w_J with $J \subseteq S$; see 4.1 and 4.7–4.10.

Some technical tricks are needed in applying Algorithm 3.11. We explain by some examples the way to find the set $E_0(\Omega)$ from $F_0(\Omega)$, the set $F_0(\Omega)$ from $F(\Omega)$, the set $F(\Omega)$ from $F(W_{(i)})$ (see 1.3 and 4.7) with $a(\Omega) = i$, and the set $F(W_{(i)})$ from some known conditions (see Section 4).

Lusztig conjectured in [Asai et al. 1983] that any left cell of an affine Weyl group W is left-connected. For a two-sided cell Ω of W , the resulting set $E(\Omega)$ of Algorithm 3.4 is useful in the verification of left-connectedness for a left cell of W ; see 3.6–3.7.

This paper is organized as follows. We collect some results on cells of affine Weyl groups W in Section 1 and on the alcove form of elements of W in Section 2. These results are mostly known already except for Proposition 2.3. In Section 3, we introduce a new algorithm for finding an l.c.r. set of W in a two-sided cell Ω of W . Finally, in Section 4, we explain some technical tricks in applying the algorithm.

1. Some results on cells of affine Weyl groups

1.1. Let $W = (W, S)$ be a Coxeter group with S its Coxeter generator set. Let \leq be the Bruhat–Chevalley order on W . For $w \in W$, we denote by $\ell(w)$ the length of w . Let $A = \mathbb{Z}[u, u^{-1}]$ be the ring of Laurent polynomials in an indeterminate u with integer coefficients. Let $\mathcal{H}(W)$ be the associated Hecke algebra of W , that is, an associative A -algebra that is free as an A -module with a basis $\{T_w \mid w \in W\}$, subject to the multiplication rule

$$\begin{aligned} T_x T_y &= T_{xy} && \text{if } \ell(x) + \ell(y) = \ell(xy), \\ (T_s - u^{-1})(T_s + u) &= 0 && \text{for any } s \in S. \end{aligned}$$

$\mathcal{H}(W)$ has another A -basis $\{C_w \mid w \in W\}$ given by

$$(1) \quad C_w = \sum_{y \leq w} u^{\ell(w) - \ell(y)} P_{y,w}(u^{-2}) T_y,$$

where the $P_{y,w} \in \mathbb{Z}[u]$ for $y, w \in W$ are the celebrated *Kazhdan–Lusztig polynomials* satisfying $P_{w,w} = 1$, $P_{y,w} = 0$ if $y \not\leq w$, and $\deg P_{y,w} \leq (1/2)(\ell(w) - \ell(y) - 1)$ if $y < w$; see [Kazhdan and Lusztig 1979]. For $y < w$ in W , let $\mu(w, y) = \mu(y, w)$ be the coefficient of $u^{(1/2)(\ell(w) - \ell(y) - 1)}$ in $P_{y,w}$. We write $y \dashrightarrow w$ if $\mu(y, w) \neq 0$.

Checking the relation $y \dashrightarrow w$ for $y, w \in W$ usually involves a complicated computation of Kazhdan–Lusztig polynomials. But it is easier in some special cases:

(2) If $x, y \in W$ satisfy $y < x$ and $\ell(y) = \ell(x) - 1$, then $y \dashrightarrow x$.

1.2. The preorders $\leq_L, \leq_R, \leq_{LR}$ and the associated equivalence relations $\sim_L, \sim_R, \sim_{LR}$ on W are defined as in [Kazhdan and Lusztig 1979]. The equivalence classes of W with respect to \sim_L (respectively, \sim_R, \sim_{LR}) are called *left cells* (respectively, *right cells, two-sided cells*). The preorder \leq_L (respectively, \leq_R, \leq_{LR}) induces a partial order on the set of left cells (respectively, right cells, two-sided cells) of W .

From now on, we always assume W to be an irreducible Weyl or affine Weyl group unless otherwise specified.

1.3. Lusztig [1985] defined a function $a: W \rightarrow \mathbb{N}$ with the following properties:

- (1) If $x \leq_{LR} y$, then $a(x) \geq a(y)$. In particular, if $x \sim_{LR} y$ then $a(x) = a(y)$. So we may define the a -value $a(\Gamma)$ to be $a(x)$ for any $x \in \Gamma$, where Γ is a left, right or two-sided cell of W ; see [Lusztig 1985].
- (2) Suppose $a(x) = a(y)$. If $x \leq_L y$, then $x \sim_L y$; if $x \leq_R y$, then $x \sim_R y$. See [Lusztig 1987a].
- (3) For any $I \subseteq S$ with $|W_I| < \infty$, we have $a(w_I) = \ell(w_I)$.

For any $w \in W$, set

$$\mathcal{L}(w) = \{s \in S \mid sw < w\} \quad \text{and} \quad \mathcal{R}(w) = \{s \in S \mid ws < w\}.$$

- (4) If $x \leq_L y$, then $\mathcal{R}(x) \supseteq \mathcal{R}(y)$; if $x \leq_R y$, then $\mathcal{L}(x) \supseteq \mathcal{L}(y)$. In particular, if $x \sim_L y$, then $\mathcal{R}(x) = \mathcal{R}(y)$; if $x \sim_R y$, then $\mathcal{L}(x) = \mathcal{L}(y)$. See [Kazhdan and Lusztig 1979, Proposition 2.4].

By the notation $x = y \cdot z$ for $x, y, z \in W$, we mean $x = yz$ and $\ell(x) = \ell(y) + \ell(z)$. In this case, we call x a *left extension* of z and a *right extension* of y ; we call z a *left retraction* of x , and y a *right retraction* of x . When $w = x \cdot y \cdot z$, we call w an *extension* of y and call y a *retraction* of w .

- (5) If $x = y \cdot z$ then $x \leq_L z$ and $x \leq_R y$. Hence $a(x) \geq a(y), a(z)$ by (2). In particular, if $I \in \{\mathcal{R}(x), \mathcal{L}(x)\}$, then $a(x) \geq \ell(w_I)$; see [Lusztig 1985].

Let $W_{(i)} = \{w \in W \mid a(w) = i\}$ for any $i \in \mathbb{N}$. Then by (2), $W_{(i)}$ is a union of some two-sided cells of W .

- (6) For any $x \in W$, let $\Sigma(x)$ be the set of all left cells Γ of W such that there exists some $y \in \Gamma$ with $y \dashrightarrow x$, $\mathcal{R}(y) \not\subseteq \mathcal{R}(x)$ and $a(y) = a(x)$. Then $x \sim_L y$

in W if and only if $\mathcal{R}(x) = \mathcal{R}(y)$ and $\Sigma(x) = \Sigma(y)$; see [Shi 1994a, Theorem 2.1], [Shi 1994d] and [Shi 1998a, Section 5].

- (7) If $x \dashrightarrow y$ in W and $s \in S$ satisfy $s \in \mathcal{L}(y) \setminus \mathcal{L}(x)$ (respectively, $s \in \mathcal{R}(y) \setminus \mathcal{R}(x)$), then either $y = s \cdot x$ (respectively, $y = x \cdot s$) or $y < x$; see [Kazhdan and Lusztig 1979, Sections 2.3e and 2.3f]. In particular, we have $\ell(y) \leq \ell(x) + 1$.
- (8) The number of left cells in W is finite; see [Lusztig 1987a, Theorem 2.2].
- (9) If a left cell L and a right cell R are in the same two-sided cell of W , then $L \cap R \neq \emptyset$; see [Lusztig 1987b, Section 3.1(k), (l)].

1.4. To each $x \in W$, we denote by $M(x)$ the set of all $y \in W$ such that there is a sequence $x_0 = x, x_1, \dots, x_r = y$ in W with some $r \geq 0$, where for every $1 \leq i \leq r$, the conditions $x_{i-1}^{-1}x_i \in S$, $\mathcal{R}(x_{i-1}) \not\supseteq \mathcal{R}(x_i)$ and $\mathcal{R}(x_{i-1}) \not\subsetneq \mathcal{R}(x_i)$ are satisfied.

A graph $\mathfrak{M}(x)$ associated to an element $x \in W$ is defined as follows. Its vertex set is $M(x)$, each $y \in M(x)$ is labeled by the set $\mathcal{R}(y)$; its edge set consists of all two-elements subsets $\{y, z\} \subset M(x)$ with $y^{-1}z \in S$, $\mathcal{R}(y) \not\supseteq \mathcal{R}(z)$ and $\mathcal{R}(y) \not\subsetneq \mathcal{R}(z)$.

By a *path* in the graph $\mathfrak{M}(x)$, we mean a sequence z_0, z_1, \dots, z_r in $M(x)$ such that $\{z_{i-1}, z_i\}$ is an edge of $\mathfrak{M}(x)$ for any $1 \leq i \leq r$. Two elements $x, x' \in W$ have the same *right generalized τ -invariants*, if for any path $z_0 = x, z_1, \dots, z_r$ in $\mathfrak{M}(x)$, there is a path $z'_0 = x', z'_1, \dots, z'_r$ in $\mathfrak{M}(x')$ with $\mathcal{R}(z'_i) = \mathcal{R}(z_i)$ for any $0 \leq i \leq r$, and if the same condition holds when the roles of x and x' are interchanged.

Then the following result is known.

Proposition 1.5 (see [Shi 1990, Section 3]). *Any $x, y \in W$ with $x \sim_L y$ have the same right generalized τ -invariants.*

1.6. For $s, t \in S$ with $m = o(st) > 2$, each of the sequences

$$zt, zts, ztst, \dots \quad \text{and} \quad zs, zst, zsts, \dots \quad (\text{each containing } m - 1 \text{ terms})$$

is called a *right $\{s, t\}$ -string*, where $z \in W$ satisfies $\mathcal{R}(z) \cap \{s, t\} = \emptyset$.

Two elements $x, y \in W$ form a *right primitive pair* if there exist two sequences $x_0 = x, x_1, \dots, x_r$ and $y_0 = y, y_1, \dots, y_r$ in W such that

- (a) for each $1 \leq i \leq r$, there exist some $s_i, t_i \in S$ with $o(s_i t_i) > 2$ such that any of the pairs x_{i-1}, x_i and y_{i-1}, y_i are neighboring terms in a right $\{s_i, t_i\}$ -string;
- (b) $x_i \dashrightarrow y_i$ for all $0 \leq i \leq r$; and
- (c) either $\mathcal{R}(x) \not\subsetneq \mathcal{R}(y)$ and $\mathcal{R}(y_r) \not\subsetneq \mathcal{R}(x_r)$, or $\mathcal{R}(y) \not\subsetneq \mathcal{R}(x)$ and $\mathcal{R}(x_r) \not\subsetneq \mathcal{R}(y_r)$.

In particular, if $\{w, y\}$ is an edge in a graph $\mathfrak{M}(x)$ for some $x \in W$, then w, y form a right primitive pair by taking $r = 0$ in the definition above.

Similarly, we can define a left $\{s, t\}$ -string and a left primitive pair.

Proposition 1.7 (see [Shi 1990, Section 3]). *Any right primitive pair $x, y \in W$ satisfies $x \sim_R y$; any left primitive pair $x, y \in W$ satisfies $x \sim_L y$. In particular, the set $M(x)$ is contained in a right cell of W .*

1.8. An affine Weyl group W is a Coxeter group that can be realized geometrically as follows. Let G be a connected, adjoint reductive algebraic group over \mathbb{C} . Fix a maximal torus T of G . Let X be the group of characters $T \rightarrow \mathbb{C}$ and let $\Phi \subset X$ be the root system with $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$ a choice of simple system. Then $E = X \otimes_{\mathbb{Z}} \mathbb{R}$ is a euclidean space with an inner product $\langle \cdot, \cdot \rangle$ such that the Weyl group (W_0, S_0) of G with respect to T acts naturally on E and preserves its inner product, where S_0 is the set of simple reflections $s_i = s_{\alpha_i}$ for $1 \leq i \leq \ell$. We denote by N the group of all translations T_λ for $\lambda \in X$ on E : T_λ sends x to $x + \lambda$. Then the semidirect product $W = W_0 \rtimes N$ is called an *affine Weyl group*. Let K be the dual of the type of G . Then we define the type of W by \tilde{K} . Sometimes we denote W just by \tilde{K} . There is a canonical homomorphism from W to W_0 , sending w to \bar{w} .

Let $-\alpha_0$ be the highest short root in Φ . Set $s_0 = s_{\alpha_0} T_{-\alpha_0}$ with s_{α_0} the reflection corresponding to α_0 . Then $S = S_0 \cup \{s_0\}$ forms a Coxeter generator set of W .

Theorem 1.9 [Lusztig 1989, Theorem 4.8]. *In the setup of 1.8, there exists a bijection $\mathbf{c} : \mathbf{u} \mapsto \mathbf{c}(\mathbf{u})$ from the set of unipotent conjugacy classes in G to the set $\text{Cell } W$ of two-sided cells in W such that $a(\mathbf{c}(\mathbf{u})) = \dim \mathfrak{B}_u$, where u is any element in \mathbf{u} , and $\dim \mathfrak{B}_u$ is the dimension of the variety of Borel subgroups of G containing u .*

2. Alcove forms for elements in affine Weyl groups

Keep the setup of 1.8 for an affine Weyl group W .

2.1. The *alcove form* of an element $w \in W$ is, by definition, a Φ -tuple $(k(w; \alpha))_{\alpha \in \Phi}$ over \mathbb{Z} , subject to the following conditions:

- (a) $k(e; \alpha) = 0$ for any $\alpha \in \Phi$, where e is the identity of W .
- (b) For $0 \leq i \leq l$, we have $k(s_i; \alpha) = 0$ if $\alpha \neq \pm\alpha_i$, and $k(s_i; \alpha) = \mp 1$ if $\alpha = \pm\alpha_i$.
- (c) Let $w' = ws_i$ for some $0 \leq i \leq l$. Then $k(w'; \alpha) = k(w; (\alpha)\bar{s}_i) + k(s_i; \alpha)$, where $\bar{s}_i = s_i$ if $1 \leq i \leq l$, and $\bar{s}_0 = s_{\alpha_0}$; see [Shi 1987, Proposition 4.2].

It is easily checked that $k(w; -\alpha) = -k(w; \alpha)$ for any $\alpha \in \Phi$. Let Φ^+ be the positive system of Φ containing Π . Then the Φ -tuple $(k(w; \alpha))_{\alpha \in \Phi}$ is entirely determined by the Φ^+ -tuple $(k(w; \alpha))_{\alpha \in \Phi^+}$. We can identify $(k(w; \alpha))_{\alpha \in \Phi}$ with $(k(w; \alpha))_{\alpha \in \Phi^+}$ and call the latter also the *alcove form* of w .

Recall the definition for a left extension of an element $x \in W$ in 1.3. The following results on the alcove form $(k(w; \alpha))_{\alpha \in \Phi}$ of $w \in W$ are known.

Proposition 2.2 [Shi 1987, Proposition 4.7]. *Let $w = (k(w; \alpha))_{\alpha \in \Phi} \in W$. Write $w = \bar{w}T_\lambda$ with $\bar{w} \in W_0$ and $\lambda \in X = \mathbb{Z}\Phi$.*

- (a) For $\alpha \in \Phi^+$, we have $k(\bar{w}; \alpha) = 0$ if $(\alpha)\bar{w}^{-1} \in \Phi^+$ and $k(\bar{w}; \alpha) = -1$ if $(\alpha)\bar{w}^{-1} \in \Phi^-$.
- (b) $\mathcal{R}(w) = \{s_j \in S \mid k(w; \alpha_j) < 0\}$.
- (c) Let $w' = ws_j$ with $w \in W$ and $0 \leq j \leq l$. Then for any $\alpha \in \Phi$, we have
- $$k(w'; \alpha) = k(w; (\alpha)\bar{s}_j) + k(s_j; \alpha).$$
- (d) Let $w' = (k(w'; \alpha))_{\alpha \in \Phi} \in W$. Then w' is a left extension of w if and only if the inequalities $k(w'; \alpha)k(w; \alpha) \geq 0$ and $|k(w'; \alpha)| \geq |k(w; \alpha)|$ hold for any $\alpha \in \Phi$.

The following result is crucial in the proof of Theorem 3.5.

Proposition 2.3. *Let $x_0, x_1, \dots, x_r, \dots$ be an infinite sequence of elements in W such that x_i is a right extension of x_{i-1} for every $i \geq 1$. Then there are some $q > p \geq 0$ such that x_q is a left extension of x_p .*

Proof. By Proposition 2.2(a)–(c), we see that there are permutations τ_{ij} with $i, j \geq 0$ on the set Φ , satisfying

- (i) $(-\alpha)\tau_{ij} = -(\alpha)\tau_{ij}$ for any $\alpha \in \Phi$;
- (ii) $|k(x_j; \alpha)| \geq |k(x_i; (\alpha)\tau_{ij})|$ for any $\alpha \in \Phi$ and $j > i$; and
- (iii) $\tau_{hi}\tau_{ij} = \tau_{hj}$ for any $h, i, j \geq 0$.

Since $|\Phi| < \infty$, the permutation group on Φ is finite. So there exists an infinite subsequence $h_1, h_2, \dots, h_t, \dots$ of $1, 2, 3, \dots$ with $\tau_{0, h_a} = \tau_{0, h_b}$ for any $a, b \geq 0$. Hence $|k(x_{h_a}; \alpha)| \geq |k(x_{h_b}; \alpha)|$ for any $a > b \geq 0$ and $\alpha \in \Phi$. Then by the finiteness of the set Φ , there exist some $q > p \geq 0$ in $\{h_1, h_2, \dots\}$ such that $|k(x_q; \alpha)| \geq |k(x_p; \alpha)|$ and $k(x_q; \alpha) \cdot k(x_p; \alpha) \geq 0$ for any $\alpha \in \Phi$. This implies that x_q is a left extension of x_p by Proposition 2.2(d). \square

3. A new algorithm for finding an l.c.r. set in a two-sided cell

3.1. Call a nonempty set $F \subseteq W$ *distinguished* if $|\Gamma \cap F| \leq 1$ for any left cell Γ of W . Call F a *representative set of left cells* (or an *l.c.r. set for short*) of W in a two-sided cell Ω if $F \subseteq \Omega$ and $|\Gamma \cap F| = 1$ for any left cell Γ of W in Ω .

3.2. For any two-sided cell Ω of W , set

$$F(\Omega) = \{z \in \Omega \mid a(sz), a(zt) < a(z) \text{ for any } s \in \mathcal{L}(z) \text{ and } t \in \mathcal{R}(z)\},$$

$$E(\Omega) = \{z \in \Omega \mid a(sz) < a(z) \text{ for any } s \in \mathcal{L}(z)\}.$$

Clearly, $F(\Omega) \subseteq E(\Omega)$. Also, $w \in F(\Omega)$ if and only if $w^{-1} \in F(\Omega)$. We have the following result.

Lemma 3.3. (1) Any $w \in \Omega$ has an expression $w = x \cdot z \cdot y$ for some $x, y \in W$ and $z \in F(\Omega)$.

(2) An element $w \in \Omega$ is in $E(\Omega)$ if and only if $x = 1$ in any expression of the form $w = x \cdot z \cdot y$ with $z \in F(\Omega)$.

Proof. If $w \in F(\Omega)$, then take $x = y = 1$ and $z = w$. If $w \notin F(\Omega)$, then by 1.3(2), either $w = s \cdot w'$ for some $s \in \mathcal{L}(w)$, or $w = w' \cdot t$ for some $t \in \mathcal{R}(w)$, where $w' \in \Omega$. By applying induction on $\ell(w)$, we may write $w' = x \cdot z \cdot y$ for some $x, y \in W$ and some $z \in F(\Omega)$. Hence w is equal to either $sx \cdot z \cdot y$ or $x \cdot z \cdot yt$. This implies (1). Then (2) follows by (1) and 1.3(2), (5). \square

The following algorithm is for finding the set $E(\Omega)$ from $F(\Omega)$.

Algorithm 3.4. (1) Set $Y_0 = F(\Omega)$.

Let $k \geq 0$. Suppose that the set Y_k has been found.

(2) If $Y_k = \emptyset$, then the algorithm terminates;

(3) If $Y_k \neq \emptyset$, then find the set $Y_{k+1} = \{xs \mid x \in Y_k; s \in S \setminus \mathcal{R}(x); xs \in E(\Omega)\}$.

By Lemma 3.3(2), we have $E(\Omega) = \bigcup_{i \geq 0} Y_i$.

Then the following result shows that Algorithm 3.4 must terminate after a finite number of steps, that is, $E(\Omega) = \bigcup_{k=0}^t Y_k$ for some $t \in \mathbb{N}$.

Theorem 3.5. Let Y_j for $j \geq 0$ be obtained from the set $F(\Omega)$ by Algorithm 3.4.

(1) There exists some $t \in \mathbb{N}$ such that $Y_j \neq \emptyset$ and $Y_h = \emptyset$ for $0 \leq j \leq t < h$;

(2) $E(\Omega) = \bigcup_{k=0}^t Y_k$.

Proof. It is easily seen that if $Y_i = \emptyset$ for some $i \geq 1$, then $Y_j = \emptyset$ for any $j \geq i$, or equivalently, if $Y_i \neq \emptyset$ for some $i \geq 0$ then $Y_j \neq \emptyset$ for any $0 \leq j \leq i$. Since $Y_0 \neq \emptyset$, to prove (1) it suffices to prove that there is an integer $i > 0$ such that $Y_i = \emptyset$.

Suppose to the contrary that $Y_i \neq \emptyset$ for any $i \geq 0$. By the finiteness for the number of left cells of W in Ω (see 1.3(8)), there are infinite sequences w_1, w_2, \dots in $E(\Omega)$ and $0 \leq i_1 < i_2 < \dots < i_n < \dots$ in \mathbb{N} such that

(i) w_{j+1} is a right extension of w_j for any $j \geq 1$ (see 1.3);

(ii) $w_1 \sim_L w_2 \sim_L w_3 \sim_L \dots$;

(iii) $w_j \in Y_{i_j}$ for $j \geq 1$.

By (i) and Proposition 2.2 (b)–(c), we see that there are permutations τ_{ij} on Φ , with $i, j \geq 1$, such that $|k(w_j; \alpha)| \geq |k(w_i; (\alpha)\tau_{ij})|$ for any $\alpha \in \Phi$ and $j > i \geq 1$, and such that $\tau_{hi}\tau_{ij} = \tau_{hj}$ for any $h, i, j \geq 1$ (see the proof of Proposition 2.3).

Then by Proposition 2.3, there are some $q > p \geq 1$ such that w_q is a proper left extension of w_p . Since $w_p \sim_L w_q$ are in Ω , this implies that w_q is not in $E(\Omega)$,

a contradiction. This proves (1). Then (2) follows by (1) and the definition of the set $E(\Omega)$. □

Remark 3.6. We say that a subset K of W is *left-connected*, if for any $x, y \in K$, there exists a sequence of elements $x_0 = x, x_1, \dots, x_r = y$ in K with some $r \geq 0$ such that $x_{i-1}x_i^{-1} \in S$ for every $1 \leq i \leq r$. Lusztig conjectured in [Asai et al. 1983] that any left cell of an affine Weyl group is left-connected. Although it has been verified in many special cases [Shi 1986; 1988; 2008, Shi and Zhang 2008], the conjecture is still open in general. Now the resulting set $E(\Omega)$ of Algorithm 3.4 is useful in dealing with the conjecture. In fact, to verify the left-connectedness for a left cell Γ of W in a two-sided cell Ω , we need only to construct a graph $\mathcal{M}(\Gamma)$ with $\Gamma \cap E(\Omega)$ as its vertex set. We join two vertices $x \neq y$ in $\Gamma \cap E(\Omega)$ with an edge once we find a sequence of elements $x_0 = x, x_1, \dots, x_r = y$ in Γ with some $r > 0$ such that $x_{i-1}x_i^{-1} \in S$ for any $1 \leq i \leq r$. Then we complete the proof for Γ being left-connected once the graph $\mathcal{M}(\Gamma)$ we are constructing becomes connected.

Example 3.7. The following example is provided by Q. Huang, one of my Ph.D. students. Let $W = \tilde{E}_8$ be with $S = \{s_i \mid 0 \leq i \leq 8\}$ its distinguished generator set such that $o(s_1s_3) = o(s_3s_4) = o(s_2s_4) = o(s_4s_5) = o(s_5s_6) = o(s_6s_7) = o(s_7s_8) = o(s_8s_0) = 3$. Let Ω be the two-sided cell of W containing the element $s_2s_3s_4s_2s_3s_4$. Then we can get the set $E(\Omega)$ by Algorithm 3.4. We observe that the elements $w_1 = s_2s_3s_4s_2s_3s_4 \cdot s_1s_5s_4s_6$ and $w_2 = s_1s_4s_3s_1s_4s_3 \cdot s_2s_4s_5s_4s_6$ and $w_3 = s_3s_5s_4s_3s_5s_4 \cdot s_1s_2s_3s_4s_5s_6$ in $E(\Omega)$ have the same right generalized τ -invariants among themselves and have different right generalized τ -invariants from any other element in $E(\Omega)$. On the other hand, the element $y := s_2 \cdot w_2 = s_3s_1 \cdot w_1$ is a common left extension of both w_1 and w_2 . Since $\mathcal{L}(y) = \{s_1, s_2, s_3\}$ and $\mathcal{L}(w_2) = \{s_1, s_3, s_4\}$ and $\mathcal{L}(s_1w_1) = \{s_1, s_2, s_4\}$ and $\mathcal{L}(w_1) = \{s_2, s_3, s_4\}$, the sequence w_1, s_1w_1, y, w_2 is contained in some left cell (say Γ) of W by Proposition 1.7. Also, the element $x := s_4s_2s_1 \cdot w_3 = s_2s_4s_5 \cdot y$ is a common left extension of both w_2 and w_3 . Since $\mathcal{L}(w_3) = \{s_3, s_4, s_5\}$ and $\mathcal{L}(s_1w_3) = \{s_1, s_4, s_5\}$ and $\mathcal{L}(s_2s_1w_3) = \{s_1, s_2, s_5\}$ and $\mathcal{L}(x) = \{s_1, s_2, s_4\}$ and $\mathcal{L}(s_5y) = \{s_1, s_2, s_3, s_5\}$ and $\mathcal{L}(s_4s_5y) = \{s_1, s_4\}$, we see that y, s_5y form a left primitive pair and so the sequence $w_2, y, s_5y, s_4s_5y, x, s_2s_1w_3, s_1w_3, w_3$ is contained in Γ by Proposition 1.7. Therefore $\Gamma \cap E(\Omega) = \{w_1, w_2, w_3\}$ by Proposition 1.5. The graph $\mathcal{M}(\Gamma)$ with the vertex set $\{w_1, w_2, w_3\}$ has the edges $\{w_1, w_2\}$ and $\{w_2, w_3\}$; hence it is connected. This implies by Remark 3.6 that Γ is left-connected.

3.8. In the remaining part of the section, we always assume that the set $F(\Omega)$ is known explicitly for a given two-sided cell Ω of W . For any $x \in W$, denote by Γ_x the left cell of W containing x . Take a distinguished subset $F_0(\Omega)$ of $F(\Omega)$ such that any $w \in F_0(\Omega)$ is a shortest element in the left cell Γ_w and that for any $w \in F(\Omega)$, there is some $w' \in F_0(\Omega)$ and some $x \in W$ with $w' \cdot x$ a shortest

element in the left cell Γ_w . In particular, when all the elements of $F(\Omega)$ have the same length (and hence each $w \in F(\Omega)$ is a shortest element in the left cell Γ_w according to 4.6–4.7), we can take $F_0(\Omega)$ to be any maximal distinguished subset of $F(\Omega)$.

Lemma 3.9. *Assume that the set $F_0(\Omega)$ has been chosen for a two-sided cell Ω of W . Then any left cell Γ in Ω contains a shortest element w that has an expression of the form $w = z \cdot y$ for some $z \in F_0(\Omega)$ and $y \in W$.*

Proof. Let Γ and Γ' be two left cells of W in Ω and let $x \in \Gamma$. We see by 1.3(9) that there exists a sequence $x_0 = x, x_1, \dots, x_r$ in Ω with some $r \geq 0$ such that $x_r \in \Gamma'$ and that $x_{i-1} \rightarrow x_i$ and $\mathcal{R}(x_{i-1}) \not\subseteq \mathcal{R}(x_i)$ for every $1 \leq i \leq r$. We claim that for any $x' \in \Gamma$, there exists a sequence $x'_0 = x', x'_1, \dots, x'_r$ such that $x'_{i-1} \rightarrow x'_i$ and $x'_i \sim_L x_i$ for every $1 \leq i \leq r$. To show the claim, it is enough to consider the case where $r = 1$ and $x' \rightarrow x$ with $\mathcal{L}(x') \not\subseteq \mathcal{L}(x)$. Take $s \in \mathcal{R}(x_1) \setminus \mathcal{R}(x)$ and $t \in \mathcal{L}(x) \setminus \mathcal{L}(x')$. Then we have $a_{x_1} \neq 0 \neq b_x$ in the expressions $C_x C_s = \sum_z a_z C_z$ and $C_t C_{x'} = \sum_y b_y C_y$, where $a_z, b_y \in A$. By the positivity of the coefficients of a_z, b_y in u [Lusztig 1985, Section 3.1], we see that $c_{x_1} \neq 0$ in the expression $C_t C_{x'} C_s = \sum_v c_v C_v$ ($c_v \in A$). By the multiplicative associativity of \mathcal{H} , this implies that there exists some $x'_1 \in W$ with $d_{x'_1} \neq 0 \neq f_x$ in the expressions $C_{x'} C_s = \sum_{y'} d_{y'} C_{y'}$ and $C_t C_{x'_1} = \sum_{v'} f_{v'} C_{v'}$, where $d_{y'}, f_{v'} \in A$. By 1.3(1), we get $a(x') = a(x) = a(x_1) \geq a(x'_1) \geq a(x')$ and hence $a(x'_1) = a(x_1)$. So $x'_1 \sim_L x_1$ by 1.3(2) and the fact $x_1 \leq_L x'_1$. The claim is proved.

Now we are ready to show our result. Take a shortest element w' in Γ . Then $w' \in E(\Omega)$. There exist some $z \in F(\Omega)$ and $y \in W$ with $w' = z \cdot y$ by Lemma 3.3(2). By the construction of the set $F_0(\Omega)$, there exist some $z' \in F_0(\Omega)$ and $y' \in W$ such that $z' \cdot y' \sim_L z$ and $\ell(z' \cdot y') \leq \ell(z)$. Let $y = s_1 s_2 \cdots s_r$ be a reduced expression of y with $s_i \in S$ and let $z_i = z s_1 s_2 \cdots s_i$ for any $0 \leq i \leq r$. Then the sequence $z_0 = z, z_1, \dots, z_r = w'$ is in Ω with $z_{i-1} \rightarrow z_i$ and $s_i \in \mathcal{R}(z_i) \setminus \mathcal{R}(z_{i-1})$. By the claim above, there exists some sequence $x_0 = z' \cdot y', x_1, \dots, x_r$ in Ω such that $x_{i-1} \rightarrow x_i$ and $x_i \sim_L z_i$ for any $1 \leq i \leq r$. By 1.3(7), we have $\ell(x_r) \leq \ell(x_0) + r \leq \ell(z) + r = \ell(z \cdot y) = \ell(w')$. Since w' is shortest in Γ and $w' \sim_L x_r$, this forces $\ell(x_r) = \ell(w')$. Hence $\ell(z' \cdot y') = \ell(z)$ and $x_i = x_{i-1} \cdot s_i$ for any $1 \leq i \leq r$; in particular, $x_r = z' \cdot y' \cdot y$, which is a required element w in the lemma. \square

3.10. For any left cell Γ in a two-sided cell Ω of W , let $n(\Gamma)$ be the length of a shortest element in Γ . Then $n(\Gamma)$ is also the smallest number of $\ell(z \cdot y)$ as $z \cdot y$ ranges over all such expressions that $z \in F_0(\Omega)$ and $y \in W$ and $z \cdot y \in \Gamma$ by Lemma 3.9. Let $n(\Omega)$ be the length of a shortest element in Ω .

By modifying Algorithm 3.4, we get the another algorithm such that the resulting set forms an l.c.r. set of W in Ω (see Theorem 3.12):

Algorithm 3.11. (1) Let $X_0 = \{w \in F_0(\Omega) \mid \ell(w) = n(\Omega)\}$.

For $k \geq 0$, suppose that the set X_k has been found.

(2) If $X_k = \emptyset$, then our algorithm terminates;

(3) If $X_k \neq \emptyset$, then find the set

$$X'_{k+1} = \{xs \mid x \in X_k, s \in S \setminus \mathcal{R}(x), xs \in E(\Omega)\} \cup \{w \in F_0(\Omega) \mid \ell(w) = n(\Omega) + k + 1\}.$$

Then take a maximal subset X_{k+1} in X'_{k+1} such that $\bigcup_{i=0}^{k+1} X_i$ is distinguished whenever $X'_{k+1} \neq \emptyset$.

Theorem 3.12. Let $E_0(\Omega) := \bigcup_{k \geq 0} X_k$.

(1) $E_0(\Omega) \subseteq E(\Omega)$.

(2) The set $E_0(\Omega)$ forms an l.c.r. set of W in Ω .

(3) Any $w \in E_0(\Omega)$ satisfies $\ell(w) = n(\Gamma_w)$.

Proof. Assertion (1) and the distinguishedness of $E_0(\Omega)$ follows by the construction of the set $E_0(\Omega)$. So for the assertions (2)–(3), it is enough to prove that $\Gamma \cap E_0(\Omega)$ contains an element w with $\ell(w) = n(\Gamma)$ for any left cell Γ of W in Ω .

By Lemma 3.9, there exists some $w' \in \Gamma$ with $\ell(w') = n(\Gamma)$ (hence $w' \in E(\Omega)$) and $w' = x \cdot y$ for some $x \in F_0(\Omega)$ and $y \in W$. We want to find some $w \in E_0(\Omega) \cap \Gamma$ with $\ell(w) = n(\Gamma)$. Apply induction on $n(\Gamma) \geq n(\Omega)$ (see 3.10). If $n(\Gamma) = n(\Omega)$ then there exists some $w \in X_0 \cap \Gamma \subseteq E_0(\Omega) \cap \Gamma$ by the construction of the set $F_0(\Omega)$ and Algorithm 3.11. Now assume $n(\Gamma) > n(\Omega)$. Let $k = n(\Gamma) - n(\Omega)$. If $w' \in F_0(\Omega)$ then we can find some $w \in X_k \cap \Gamma \subseteq E_0(\Omega) \cap \Gamma$ by Algorithm 3.11. If $w' = x \cdot y \notin F_0(\Omega)$, that is, $\ell(y) > 0$, take any $s \in \mathcal{R}(y)$; then $z := w's \in E(\Omega)$.

We claim that z is a shortest element in the left cell Γ_z , for, otherwise, there would exist some $z' \in \Gamma_z$ with $\ell(z') < \ell(z)$. By 1.3(6), there is some $w'' \in \Gamma$ with $w'' \rightarrow z'$ by the facts that $w' \rightarrow z$ (by 1.1(2)) and $z \sim_L z'$ and $\mathcal{R}(w') \not\subseteq \mathcal{R}(z)$. Since $s \in \mathcal{R}(w') \setminus \mathcal{R}(z)$, we have $s \in \mathcal{R}(w'') \setminus \mathcal{R}(z')$ by 1.3(4). Hence $\ell(w'') \leq \ell(z') + 1 \leq \ell(z) < \ell(w')$ by 1.3(7), contradicting the assumption of $\ell(w') = n(\Gamma)$. The claim is proved.

Since $\ell(z) < \ell(w')$, we have $n(\Gamma_z) < n(\Gamma)$. By the induction hypothesis, there exists some $z_0 \in E_0(\Omega) \cap \Gamma_z$ with $\ell(z_0) = n(\Gamma_z) = \ell(z)$. By the same argument as above with z_0 in the place of z' , there exists some $w_0 \in \Gamma$ with $w_0 \rightarrow z_0$ and $s \in \mathcal{R}(w_0) \setminus \mathcal{R}(z_0)$ and $\ell(w_0) \leq \ell(z_0) + 1 = \ell(z) + 1 = \ell(w')$. By the assumption of $\ell(w') = n(\Gamma)$, we have $\ell(w_0) = \ell(z_0) + 1$ and $n(\Gamma) = n(\Gamma_z) + 1$. Hence $w_0 = z_0 \cdot s \in X'_k$ by 1.3(7). By the construction of the set X_k in Algorithm 3.11 and the fact $n(\Gamma) = \ell(w_0)$, there must exist some element in the set $X_k \cap \Gamma$ (and hence in $E_0(\Omega) \cap \Gamma$). So our result follows by induction. □

Remark 3.13. (1) By Theorem 3.5, there is some $t_0 \leq t$ with $E_0(\Omega) = \bigcup_{k=0}^{t_0} X_k$, where t is given as in Theorem 3.5(1).

(2) In the case where all the elements in $F_0(\Omega)$ have the same length, we can take $X_0 = F_0(\Omega)$. This is so for most of the cases we have encountered while applying Algorithm 3.11.

4. Some applications of Algorithm 3.11

Example 4.1. Let $W = \tilde{C}_4$ be with $S = \{s_0, s_1, s_2, s_3, s_4\}$ its Coxeter generator set, where $o(s_0s_1) = o(s_3s_4) = 4$ and $o(s_1s_2) = o(s_2s_3) = 3$. In the subsequent discussion, we abbreviate the notation by writing s_i as i for $0 \leq i \leq 4$.

The set $W_{(5)}$ is a single two-sided cell of W by Theorem 1.9. Let $x_1 = 01013$, $x_2 = 01014$, $x_3 = 1210124$, $y_1 = 34341$, $y_2 = 34340$, $y_3 = 3234320$. Then $F(W_{(5)}) = \{x_i, y_j \mid 1 \leq i, j \leq 3\}$. Since $x_3 \sim_L x_221$ and $y_3 \sim_L y_223$ with $\ell(x_3) = \ell(x_221)$ and $\ell(y_3) = \ell(y_223)$, we can take $F_0(W_{(5)}) = \{x_1, x_2, y_1, y_2\}$ by 3.8.

By applying Algorithm 3.11, we get the following:

$$X_0 = F_0(W_{(5)}).$$

$$X_1 = X'_1 = \{x_12, x_22, y_12, y_22\}.$$

$$X_2 = X'_2 = \{x_121, x_123, x_221, x_223, y_123, y_121, y_223, y_221\}.$$

$$X'_3 = \{x_1210, x_1213, x_1234, x_2210, x_2213, x_2234, y_1234, y_1231, y_1210, y_2234, y_2231, y_2210\},$$

$$X_3 = \{x_1210, x_1213, x_1234, x_2213, x_2234, y_1234, y_1231, y_1210, y_2231, y_2210\}$$

$$\text{since } x_2210 \sim_L x_22 \text{ and } y_2234 \sim_L y_22.$$

$$X'_4 = \{x_12101, x_12310, x_12134, x_12343, x_22103, x_22132, x_22134, y_12343,$$

$$y_12134, y_12310, y_12101, y_22341, y_22312, y_22310\},$$

$$X_4 = \{x_12101, x_12310, x_12134, x_12343, x_22132, x_22134,$$

$$y_12343, y_12134, y_12310, y_12101, y_22312, y_22310\}$$

$$\text{since } x_22103 \sim_L x_223 \text{ and } y_22341 \sim_L y_221.$$

$$X'_5 = \{x_123104, x_121343, x_221032, x_221034, x_221324, y_121340, y_123101,$$

$$y_223412, y_223410, y_223120\},$$

$$X_5 = \{x_123104, x_121343, x_221034, x_221324, y_121340, y_123101, y_223410, y_223120\}$$

$$\text{since } x_221032 \sim_L x_123 \text{ and } y_223412 \sim_L y_121.$$

$$X'_6 = \{x_1231043, x_1213432, x_2210324, x_2213243, y_1213401, y_1231012, y_2234120, y_2231201\},$$

$$X_6 = \{x_1231043, x_1213432, x_2213243, y_1213401, y_1231012, y_2231201\}$$

$$\text{since } x_2210324 \sim_L x_1234 \text{ and } y_2234120 \sim_L y_1210.$$

$$X'_7 = \{x_12310432, x_22132434, y_12134012, y_22312010\}.$$

$$X_7 = \{x_12310432, y_12134012\}$$

$$\text{since } x_22132434 \sim_L y_12343 \text{ and } y_22312010 \sim_L x_12101.$$

$$X_8 = X'_8 = \{x_123104321, y_121340123\}.$$

$$X_9 = X'_9 = \{x_1231043210, y_1213401234\}.$$

Since $X_{10} = X'_{10} = \emptyset$, we see by Theorem 3.12 that $E_0 = \bigcup_{i=0}^9 X_i$ forms an l.c.r. set of W in $W_{(5)}$ with $|X| = 56$.

4.2. The most technical part in applying Algorithm 3.11 is to determine whether or not the element xs is in $E(\Omega)$ for any given $x \in X_k$ and $s \in S \setminus \mathcal{R}(x)$, that is, to check the equation $a(xs) = a(x)$ and the inequality $a(rxs) < a(xs)$ for any $r \in \mathcal{L}(xs)$.

4.3. Checking the equation $a(xs) = a(x)$ amounts to determining the value $a(xs)$. The relation $a(xs) \geq a(x)$ holds in general by 1.3(5).

It would be helpful to find all the graphs $\mathfrak{M}(x)$ and $\mathfrak{M}(xs)$ for any $x \in X_k$ and any $s \in S \setminus \mathcal{R}(x)$.

These graphs could be worked out efficiently by computer program. In the case when the graph $\mathfrak{M}(x)$ is larger or even infinite, one need only to work out a local part \mathfrak{M} of $\mathfrak{M}(x)$ around the vertex x . It depends on the actual size of \mathfrak{M} . Usually, we take \mathfrak{M} to be a connected subgraph with vertex set $M \subseteq M(x)$ satisfying that

(*) the condition $\Gamma \cap M(x) \neq \emptyset$ implies $\Gamma \cap M \neq \emptyset$ for any left cell Γ of W .

Call a subgraph \mathfrak{M} of $\mathfrak{M}(x)$ *representative* if the vertex set M of \mathfrak{M} satisfies condition (*).

Checking that a subgraph \mathfrak{M} is representative in $\mathfrak{M}(x)$ is an easy matter: One need only check if there always exists some $z_0 \in M$ satisfying $z_0 \sim_L z$ for any $y \in M$ and any $z \in M(x)$ with $\{y, z\}$ an edge of $\mathfrak{M}(x)$.

For any $x \in W$, the following method is efficient for finding the value $a(x)$ in the case where a direct computation for $a(x)$ is difficult (for example, when $\ell(x)$ is larger). One may try to find a sequence $x_0 = x, x_1, \dots, x_r$ in W such that for every $1 \leq i \leq r$, the element $x_i s_i$ is in $M(x_{i-1})$ with $\{x_i, x_i s_i\}$ a right primitive pair for some $s_i \in S$ and such that the computation for the value $a(x_r)$ is much easier than that for $a(x)$ (for example, this is the case when $w_J \in M(x_r)$ for some $J \subset S$). In this case, we have $a(x) = a(x_r)$ by repeatedly applying Proposition 1.7.

In practice, we often choose such a sequence $x_0 = x, x_1, \dots, x_r$ with $\ell(x_r)$ much smaller than $\ell(x_0)$ since the value $a(z)$ can generally be computed relatively more easily when $\ell(z)$ is getting smaller.

When W is a finite Weyl group, one can easily get the value $a(x)$ from the value $a(w_0 x)$ by Theorem 1.9 and by the knowledge of the special unipotent classes of the corresponding reductive algebraic group, where w_0 is the longest element of W ; see [Kazhdan and Lusztig 1979, Section 3.3].

4.4. For any $x \in X_k$ and any $s \in S \setminus \mathcal{R}(x)$ with $a(xs) = a(x)$, checking the inequality $a(rxs) < a(xs)$ for any $r \in \mathcal{L}(xs)$ amounts to checking if we always have $y = 1$ in any expression of the form $xs = y \cdot w \cdot z$ with $w \in F(\Omega)$ and $y, z \in W$. The latter

can proceed efficiently in terms of alcoves forms of elements once the set $F(\Omega)$ is given explicitly.

4.5. To find X_{k+1} from the set $(\bigcup_{i=0}^k X_i) \cup X'_{k+1}$, we need to determine whether or not two concerning elements x, y , with at least one of them in X'_{k+1} , are in the same left cell of W .

By Propositions 1.5 and 1.7, this can proceed either by comparing their right generalized τ -invariants or with the aid of right primitive pairs.

Suppose that we have all the graphs $\mathfrak{M}(x)$ (or their representative subgraphs) with x ranging over $(\bigcup_{i=0}^k X_i) \cup X'_{k+1}$. These data will help us in determining if two elements (say x, y) so obtained are in the same left cell: We have $x \sim_L y$ only if x and y have the same right generalized τ -invariants, while 1.3(6) provides a complete invariant for the relation \sim_L .

4.6. The most interesting for our algorithm is when $F(\Omega) = \{w_J \in \Omega \mid J \subseteq S\} \neq \emptyset$. In this case, $F(\Omega)$ is distinguished and all the elements in $F(\Omega)$ have the same length; hence $F_0(\Omega) = F(\Omega)$ by 3.8. The following are some known cases (not exhaustive) for $F(\Omega)$ of such a form:

- (1) Ω is the lowest two-sided cell of W under the partial order \leq_{LR} ; see [Shi 1988, Section 1.1].
- (2) Ω consists of fully commutative elements (for example, the case when the Coxeter graph of W contains no subgraph of type D_4, \tilde{B}_3 or \tilde{F}_4 , and Ω contains a fully commutative element); see [Shi 2003, Theorem 3.4 and Section 3.5].
- (3) W is of simply laced type and $a(\Omega) \leq 6$; see [Shi 2008, Theorem B].
- (4) W is of type \tilde{A}_{n-1} with $n > 1$ and Ω corresponds to a partition

$$\lambda = (\lambda_1, \dots, \lambda_r, 1, \dots, 1)$$

of n with $\lambda_r + 1 \geq \lambda_1 \geq \dots \geq \lambda_r > 1$; see [Shi 1994c, Theorem 3.1].

- (5) W is of type \tilde{C}_l with $l > 1$ and $a(\Omega) = (l - 1)^2 + 1$.
- (6) W is of type \tilde{B}_l with $l > 2$ and $a(\Omega) = l(l - 1)$.

4.7. We can describe the set $F(\Omega)$ for some two-sided cell Ω of W even when $F(\Omega)$ does not consist of elements of the form $w_J, J \subseteq S$. For example, when $W = \tilde{D}_4$, the set $W_{(7)} = \{z \in W \mid a(z) = 7\}$ forms a single two-sided cell but contains no element of the form w_J for $J \subset S$. Let s_0, s_1, s_2, s_3, s_4 be the Coxeter generator set of W with s_2 corresponding to the branching node of its Coxeter graph. Then

$$F(W_{(7)}) = \{s_i s_2 s_k s_i s_2 s_i s_j s_2 s_i \mid i, j, k \in \{0, 1, 3, 4\} \text{ distinct}\};$$

see [Du 1990, Theorem 4.6].

It is desirable to find the sets $F(\Omega)$ for more two-sided cells Ω of W in order to apply Algorithm 3.11.

Some more technical tricks are needed to apply the algorithm. For example, when the set $W_{(i)} = \bigcup_{j=1}^r \Omega_j$ for some $i \in \mathbb{N}$ is a union of two-sided cells Ω_j with some $r > 1$, sometimes we know the set

$$F(W_{(i)}) := \{x \in W_{(i)} \mid a(tx) < i \text{ and } a(xs) < i \text{ for all } t \in \mathcal{L}(x), s \in \mathcal{R}(x)\}$$

but not the sets $F(\Omega_j)$ individually. Let us explain it by some examples.

Examples 4.8. Let $W = \tilde{C}_4$ with $S = \{0, 1, 2, 3, 4\}$ be as in Example 4.1.

(a) The set $W_{(3)}$ is a union of two two-sided cells (say $\Omega_{3,1}$ and $\Omega_{3,2}$) of W by Theorem 1.9. We have $F(W_{(3)}) = \{121, 232, 024\}$ and $F_0(\Omega_{3,i}) = F(\Omega_{3,i})$. At moment, we don't know what the set $F_0(\Omega_{3,i})$ is for any $i = 1, 2$. So we have to assume $X_0 = \{121, 232, 024\}$ in applying Algorithm 3.11 to find an l.c.r. set for each of the $\Omega_{3,i}, i = 1, 2$. We get

$$\begin{aligned} X'_1 &= \{1213, 1210, 2324, 2321, 0241, 0243\}, \\ X_1 &= \{1213, 1210, 2324, 0241, 0243\} && \text{since } 2321 \sim_L 1213. \\ X'_2 &= \{12134, 12130, 12101, 23241, 23243, 02413, 02410, 02434\}, \\ X_2 &= \{12134, 12130, 12101, 23243, 02413, 02410, 02434\} && \text{since } 23241 \sim_L 12134. \\ X_3 &= X'_3 = \{121343, 121340, 121301, 024132, 024103, 024341\}. \\ X_4 &= X'_4 = \{1213432, 1213430, 1213014, 1213012, 0241324, 0241320, 0241034\}. \\ X_5 &= X'_5 = \{12134320, 12130142, 12130143, 02413243, 02413201\}. \\ X_6 &= X'_6 = \{121343201, 121301423, 121301432, 024132434, 024132010\}. \\ X_7 &= X'_7 = \{1213432010, 1213014234\}. \\ X_8 &= X'_8 = \emptyset. \end{aligned}$$

We call a subset K of W *right-connected* if, for any pair $x, y \in K$, there is a sequence $x_0 = x, x_1, \dots, x_r = y$ in K with some $r \geq 0$ such that $x_i^{-1}x_{i-1} \in S$ for every $1 \leq i \leq r$.

By 1.3(2), we see that for any $i \geq 0$ with $W_{(i)} \neq \emptyset$, any nonempty right-connected subset of $W_{(i)}$ is contained in a right cell of W and hence also in a two-sided cell of W .

Assume $121 \in \Omega_{3,1}$. Let

$$\begin{aligned} E_1 &= \{121, 232, 1213, 1210, 2324, 12134, 12130, 12101, 23243, 121343, 121340, 121301, \\ &\quad 1213432, 1213430, 1213014, 1213012, 12134320, 12130142, 12130143, \\ &\quad 121343201, 121301423, 121301432, 1213432010, 1213014234\}, \\ E_2 &= \{024, 0241, 0243, 02413, 02410, 02434, 024132, 024103, 024341, 0241324, 0241320, \\ &\quad 0241034, 02413243, 02413201, 024132434, 024132010\}. \end{aligned}$$

Then $121 \in E_1$ and $E := \bigcup_{i=1}^2 E_0(\Omega_{3,i}) = \bigcup_{k=0}^7 X_k = E_1 \cup E_2$. We see that E_2 is a maximal right-connected subset of the set E . Also, $E' := E_1 \cup \{2321\}$ is a union of two right-connected subsets with $1213 \sim_L 2321$ such that 1213 and 2321 belong to different right-connected subsets of E' . This implies that $E_0(\Omega_{3,1}) = E_1$ and $E_0(\Omega_{3,2}) = E_2$ by 1.3(2) and by the fact that $W_{(3)} = \bigcup_{i=1}^2 \Omega_{3,i}$.

(b) The set $W_{(4)}$ is a union of two two-sided cells (say $\Omega_{4,1}$ and $\Omega_{4,2}$) of W by Theorem 1.9. Let $x_1 = 0101, x_2 = 1214, x_3 = 121012, y_1 = 3434, y_2 = 2320, y_3 = 232432$. Then $F(W_{(4)}) = \{x_i, y_j \mid 1 \leq i, j \leq 3\}$. Since $x_3 \sim_L x_121$ and $y_3 \sim_L y_123$ with $\ell(x_121) = \ell(x_3)$ and $\ell(y_123) = \ell(y_3)$, we can take $\bigcup_{i=1}^2 F_0(\Omega_{4,i}) = \{x_1, x_2, y_1, y_2\}$ by 3.8. Again, we don't know yet what the set $F_0(\Omega_{4,i})$ is for any $i = 1, 2$. We assume $X_0 = \{x_1, x_2, y_1, y_2\}$ in applying Algorithm 3.11. Then

$$\begin{aligned}
 X_1 &= X'_1 = \{x_12, x_23, x_20, y_12, y_21, y_24\}. \\
 X_2 &= X'_2 = \{x_121, x_123, x_230, x_232, x_234, x_201, y_123, y_121, y_210, y_212, y_214, y_243\}. \\
 X'_3 &= \{x_1210, x_1213, x_1234, x_2301, x_2324, x_2304, y_1234, y_1213, y_1210, y_2143, y_2102, y_2104\}, \\
 X_3 &= \{x_1213, x_1234, x_2301, x_2324, x_2304, y_1213, y_1210, y_2143, y_2102, y_2104\} \\
 &\qquad\qquad\qquad \text{since } x_1210 \sim_L x_12 \text{ and } y_1234 \sim_L y_12. \\
 X'_4 &= \{x_12103, x_12132, x_12134, x_12343, x_23012, x_23014, x_23243, y_12101, y_12103, y_12132, \\
 &\qquad\qquad\qquad y_12134, y_21043, y_21432, y_21021\}, \\
 X_4 &= \{x_12132, x_12134, x_12343, x_23012, x_23014, x_23243, y_12101, y_12103, \\
 &\qquad\qquad\qquad y_12132, y_21043, y_21432, y_21021\} \\
 &\qquad\qquad\qquad \text{since } x_12103 \sim_L x_123 \text{ and } y_12134 \sim_L y_121. \\
 X'_5 &= \{x_121034, x_121324, x_121343, x_123432, x_230124, y_121012, y_121013, \\
 &\qquad\qquad\qquad y_121032, y_121034, y_210432\}, \\
 X_5 &= \{x_121324, x_121343, x_123432, x_230124, y_121012, y_121013, y_121032, y_210432\} \\
 &\qquad\qquad\qquad \text{since } x_121034 \sim_L x_1234 \text{ and } y_121034 \sim_L y_1210. \\
 X'_6 &= \{x_1210343, x_1213243, x_1213432, x_1234321, x_2301243, y_1210123, y_1210132, \\
 &\qquad\qquad\qquad y_1210134, y_1210321, y_2104321\}, \\
 X_6 &= \{x_1213243, x_1213432, x_1234321, x_2301243, y_1210123, y_1210132, y_1210321, y_2104321\} \\
 &\qquad\qquad\qquad \text{since } x_1210343 \sim_L x_12343 \text{ and } y_1210134 \sim_L y_12101. \\
 X'_7 &= \{x_12103432, x_12134321, x_12343210, x_23012434, y_12341012, y_12310123, \\
 &\qquad\qquad\qquad y_12101234, y_21432010\}, \\
 X_7 &= \{x_12343210, x_23012434, y_12101234, y_21432010\} \qquad \text{since } x_12103432 \sim_L x_123432 \\
 &\qquad\qquad\qquad \text{and } x_12134321 \sim_L y_12132 \text{ and } y_12341012 \sim_L y_121012 \text{ and } y_12310123 \sim_L x_12312. \\
 X_8 &= X'_8 = \emptyset.
 \end{aligned}$$

Assume $x_1 \in \Omega_{4,1}$. Then 1.3(2) gives $E_0(\Omega_{4,1}) = E_{11} \cup E_{12}$ and $E_0(\Omega_{4,2}) = E_{21} \cup E_{22}$, where

$$\begin{aligned}
E_{11} &= \{x_1, x_{12}, x_{121}, x_{123}, x_{1213}, x_{1234}, x_{12132}, x_{12134}, x_{12343}, x_{121324}, \\
&\quad x_{121343}, x_{123432}, x_{1213243}, x_{1213432}, x_{1234321}, x_{12343210}\}, \\
E_{12} &= \{y_1, y_{12}, y_{121}, y_{123}, y_{1210}, y_{1213}, y_{12101}, y_{12103}, y_{12132}, y_{121012}, \\
&\quad y_{121013}, y_{121032}, y_{1210123}, y_{1210132}, y_{1210321}, y_{12101234}\}, \\
E_{21} &= \{x_2, x_{23}, x_{20}, x_{230}, x_{232}, x_{234}, x_{201}, x_{2301}, x_{2324}, x_{2304}, x_{23012}, x_{23014}, \\
&\quad x_{23243}, x_{230124}, x_{2301243}, x_{23012434}\}, \\
E_{22} &= \{y_2, y_{21}, y_{24}, y_{210}, y_{214}, y_{212}, y_{243}, y_{2102}, y_{2104}, y_{2143}, y_{21043}, y_{21432}, \\
&\quad y_{21021}, y_{210432}, y_{2104321}, y_{21043210}\}
\end{aligned}$$

by the following facts:

- (i) Each of E_{ij} for $i, j = 1, 2$ is a maximal right-connected set in $\bigcup_{i,j=1}^2 E_{ij}$.
- (ii) $y_{212} \sim_R y_{2124} \sim_L x_2$ and $y_{212} \in E_{22}$ and $x_2 \in E_{21}$.
- (iii) $\{x_{1213423}, x_{12134232}\}$ forms a right primitive pair.
- (iv) $x_{1213423} \in E_{11}$ and $y_1 \in E_{12}$ and $x_{12134232} \sim_L 234232 \sim_R 2342324 \sim_R 234234 \sim_L y_1$.
- (v) $W_{(4)}$ is a union of two two-sided cells of W .

4.9. Assume that W is an irreducible finite or affine Coxeter group of simply laced type. We see by [Shi 2008, Lemma 6.1] that if $w \in W$ satisfies $a(w) \geq 6$ and $a(tw), a(ws) < a(w)$ for any $t \in J := \mathcal{L}(w)$ and $s \in I := \mathcal{R}(w)$, then we have $\ell(w_J), \ell(w_I) \geq 6$. This fact will help us to find the set $F(W_{(7)})$. Actually, all the elements of the form w_J with $J \subseteq S$ and $\ell(w_J) = 7$ should be in $F(W_{(7)})$, while all the other elements w of $F(W_{(7)})$ should satisfy $\ell(w_J) = \ell(w_I) = 6$ and $a(tw), a(ws) < a(w) = 7$ for any $t \in J := \mathcal{L}(w)$ and $s \in I := \mathcal{R}(w)$. The set $F(W_{(k)})$ for $k > 7$ can be described similarly but with more cases.

Example 4.10. Let $W = \tilde{E}_6$ be with $S = \{s_i \mid 0 \leq i \leq 6\}$ its Coxeter generator set, where $o(s_1s_3) = o(s_3s_4) = o(s_4s_2) = o(s_2s_0) = o(s_4s_5) = o(s_5s_6) = 3$. Then the set $W_{(7)}$ is a single two-sided cell of \tilde{E}_6 by Theorem 1.9. Denote s_i simply by i , $0 \leq i \leq 6$. By the facts mentioned in 4.9, we get

$$\begin{aligned}
F(W_{(7)}) &= \{w_{1346}, w_{1340}, w_{0246}, w_{0241}, w_{4561}, w_{4560}, w_{2346}, w_{2451}, w_{3450}, \\
&\quad w_{13562}, w_{13560}, w_{13025}, w_{13026}, w_{02561}, w_{02563}, \\
&\quad w_{243} \cdot 543, w_{243} \cdot 542, w_{345} \cdot 243, 2031 \cdot w_{342}, 5631 \cdot w_{345}, 5620 \cdot w_{245}, \\
&\quad w_{245} \cdot 345, w_{345} \cdot 245, w_{245} \cdot 342, w_{342} \cdot 1302, w_{345} \cdot 1365, w_{245} \cdot 0265\}.
\end{aligned}$$

The set $F_0(W_{(7)})$ is obtained from $F(W_{(7)})$ by removing the last nine elements since $w_{243} \cdot 543 \sim_L w_{245} \cdot 345 \sim_L 5631 \cdot w_{345}$ and $w_{243} \cdot 542 \sim_L w_{345} \cdot 245 \sim_L 5620 \cdot w_{245}$ and $w_{345} \cdot 243 \sim_L w_{245} \cdot 342 \sim_L 2031 \cdot w_{342}$ and $w_{342} \cdot 1302 \sim_L w_{1340} \cdot 20$ and $w_{345} \cdot 1365 \sim_L w_{4561} \cdot 31$ and $w_{245} \cdot 0265 \sim_L w_{0246} \cdot 56$.

Remark 4.11. In each of Examples 4.1, 4.8 and 4.10, the related set $F(W_{(k)})$ is given at the beginning. Since it is the set of all two-sided minimal elements w of $W_{(k)}$ (that is, $w \in W_{(k)}$ but $sw, wt \notin W_{(k)}$ for any $s \in \mathcal{L}(w)$ and $t \in \mathcal{R}(w)$), $F(W_{(k)})$ can be found easily because it was described explicitly for all the sets $W_{(k)}$ for $k \in \mathbb{N}$ of the group \tilde{C}_4 and for the set $W_{(7)}$ of \tilde{E}_6 ; see [Shi 1998b; Shi and Zhang 2006]. In general, without knowing the set $W_{(k)}$ in advance, the set $F(W_{(k)})$ for $k \in \mathbb{N}$ of any Weyl or affine Weyl group W can be found recurrently as follows. By Theorem 1.9 and the knowledge of unipotent conjugacy classes of reductive algebraic groups [Carter 1985], we can get the set $E(W) := \{i \in \mathbb{N} \mid W_{(i)} \neq \emptyset\}$. For any $k \in E(W)$, suppose that the sets $F(W_{(h)})$ for $h < k$ have been found already. Then the set $W_{<k} := \{w \in W \mid a(w) < k\} = \bigcup_{h < k} W_{(h)}$ can be described explicitly by Algorithm 3.11 together with some other techniques. Find the set $E_{\geq k}$ of all two-sided minimal elements of $W \setminus W_{<k}$, which is finite by Theorem 1.9 and by the fact $E_{\geq k} \subseteq \bigcup_{\Omega \in \text{Cell}(W)} F(\Omega)$. One can determine the set $F(W_{(k)}) = \{w \in E_{\geq k} \mid a(w) = k\}$ by computing the a -values of elements in $E_{\geq k}$.

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Received April 9, 2011. Revised December 7, 2011.

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS

at the University of California, Berkeley 94720-3840

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Volume 256 No. 1 March 2012

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0030-8730(201203)256:1;1-A