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A NEW ALGORITHM FOR FINDING AN L.C.R. SET IN CERTAIN TWO-SIDED CELLS

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Let (W, S) be an irreducible Weyl or affine Weyl group. In 1994, we constructed an algorithm for finding a representative set of left cells (or an l.c.r. set for short) of W in a two-sided cell Ω. Here, we introduce a new simpler algorithm for finding an l.c.r. set of W in Ω when the subset $F(\Omega)$ of Ω is known. We introduce some technical tricks by some examples for applying the algorithm and for finding the set $F(\Omega)$. The resulting set $E(\Omega)$ is useful in verifying a conjecture of Lusztig that any left cell in an affine Weyl group is left-connected.

Let W be an irreducible Weyl or affine Weyl group with S its Coxeter generator set. For a two-sided cell Ω of W (in the sense of [Kazhdan and Lusztig 1979]), we introduced an algorithm for finding an l.c.r. set of W in Ω in [Shi 1994a]. The algorithm has been efficiently applied in many cases; see for example [Chen 2000, Chen and Shi 1998; Rui 1995; Shi 1994a; 1994b; 1998a; 1998b; Shi and Zhang 2008; 2006; Tong 1995; Zhang 1994]. The algorithm consists of three processes $(\boldsymbol{A})$, $(\boldsymbol{B})$, $(\boldsymbol{C})$ on a distinguished set F (see 3.1), where process $(\boldsymbol{C})$ is the most difficult part among the three in which one need to find, for any given $x \in F$, all elements y satisfying y—x, $y < x$, $\mathcal{R}(y) \not\subseteq \mathcal{R}(x)$ and $a(y) = a(x)$ (see Sections 1.1 and 1.3 for the notation). This becomes increasingly difficult as the length of x gets larger.

For any two-sided cell Ω of W, let $F(\Omega)$ be the set of all $w \in \Omega$ such that $a(sw), a(wt) < a(w)$ for any $s \in \mathcal{L}(w)$ and $t \in \mathcal{R}(w)$. We shall introduce a new algorithm for finding an l.c.r. set of W in a two-sided cell Ω, provided that the subset $F(\Omega)$ of Ω is known; see 3.2. The processes in our new algorithm amounts to a mixture of processes $(\boldsymbol{A})$ and $(\boldsymbol{B})$ in the original algorithm, hence avoiding process $(\boldsymbol{C})$.

Theorem 3.5 guarantees that our new algorithm will terminate after finite steps, while Theorem 3.12 shows that the resulting set $E_0(\Omega)$ of Algorithm 3.11 forms an l.c.r. set of W in Ω such that each element of $E_0(\Omega)$ is shortest in the left cell

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of W containing it.

Our new algorithm has been applied successfully for the description of the left cells of a-values 4, 5, 6 in the affine Weyl groups $\widetilde{E}_i$ for $i = 6, 7, 8$; see [Huang 2008; Liu 2007; Shi and Zhang 2006].

To apply our new algorithm, it is desirable to find the subset $F(\Omega)$ explicitly for more two-sided cells Ω in an irreducible Weyl and affine Weyl group.

It is relatively easier to describe the set $F(\Omega)$ when $F(\Omega)$ consists of elements of the form w_J for some $J \subseteq S$, where the subgroup W_J of W generated by J is finite and w_J is the longest element in W_J; see 4.6.

We can also find the sets $F(\Omega)$ for some two-sided cells Ω when Ω contains some elements not of the form w_J with $J \subseteq S$; see 4.1 and 4.7–4.10.

Some technical tricks are needed in applying Algorithm 3.11. We explain by some examples the way to find the set $E_0(\Omega)$ from $F_0(\Omega)$, the set $F_0(\Omega)$ from $F(\Omega)$, the set $F(\Omega)$ from $F(W_{(i)})$ (see 1.3 and 4.7) with $a(\Omega) = i$, and the set $F(W_{(i)})$ from some known conditions (see Section 4).

Lusztig conjectured in [Asai et al. 1983] that any left cell of an affine Weyl group W is left-connected. For a two-sided cell Ω of W, the resulting set $E(\Omega)$ of Algorithm 3.4 is useful in the verification of left-connectedness for a left cell of W; see 3.6–3.7.

This paper is organized as follows. We collect some results on cells of affine Weyl groups W in Section 1 and on the alcove form of elements of W in Section 2. These results are mostly known already except for Proposition 2.3. In Section 3, we introduce a new algorithm for finding an l.c.r. set of W in a two-sided cell Ω of W. Finally, in Section 4, we explain some technical tricks in applying the algorithm.

1. Some results on cells of affine Weyl groups

1.1. Let $W = (W, S)$ be a Coxeter group with S its Coxeter generator set. Let $\leqslant$ be the Bruhat–Chevalley order on W. For $w \in W$, we denote by $\ell(w)$ the length of w. Let $A = \mathbb{Z}[u, u^{-1}]$ be the ring of Laurent polynomials in an indeterminate u with integer coefficients. Let $\mathcal{H}(W)$ be the associated Hecke algebra of W, that is, an associative A-algebra that is free as an A-module with a basis $\{T_w \mid w \in W\}$, subject to the multiplication rule

$$T_x T_y = T_{xy} \quad \text{if } \ell(x) + \ell(y) = \ell(xy),$$
$$(T_s - u^{-1})(T_s + u) = 0 \qquad \text{for any } s \in S.$$

$\mathcal{H}(W)$ has another A-basis $\{C_w \mid w \in W\}$ given by

$$C_w = \sum_{y \leqslant w} u^{\ell(w)-\ell(y)} P_{y,w}(u^{-2}) T_y, \tag{1}$$

where the $P_{y,w} \in \mathbb{Z}[u]$ for $y, w \in W$ are the celebrated *Kazhdan–Lusztig polynomials* satisfying $P_{w,w} = 1$, $P_{y,w} = 0$ if $y \nleqslant w$, and $\deg P_{y,w} \leqslant (1/2)(\ell(w) - \ell(y) - 1)$ if $y < w$; see [Kazhdan and Lusztig 1979]. For $y < w$ in W, let $\mu(w, y) = \mu(y, w)$ be the coefficient of $u^{(1/2)(\ell(w)-\ell(y)-1)}$ in $P_{y,w}$. We write $y—w$ if $\mu(y, w) \neq 0$.

Checking the relation $y—w$ for $y, w \in W$ usually involves a complicated computation of Kazhdan–Lusztig polynomials. But it is easier in some special cases:

(2) If $x, y \in W$ satisfy $y < x$ and $\ell(y) = \ell(x) - 1$, then $y—x$.

1.2. The preorders $\leqslant_L$, $\leqslant_R$, $\leqslant_{LR}$ and the associated equivalence relations $\sim_L$, $\sim_R$, $\sim_{LR}$ on W are defined as in [Kazhdan and Lusztig 1979]. The equivalence classes of W with respect to $\sim_L$ (respectively, $\sim_R$, $\sim_{LR}$) are called *left cells* (respectively, *right cells, two-sided cells*). The preorder $\leqslant_L$ (respectively, $\leqslant_R$, $\leqslant_{LR}$) induces a partial order on the set of left cells (respectively, right cells, two-sided cells) of W.

From now on, we always assume W to be an irreducible Weyl or affine Weyl group unless otherwise specified.

1.3. Lusztig [1985] defined a function a: $W \to \mathbb{N}$ with the following properties:

(1) If $x \leqslant_{LR} y$, then $a(x) \geqslant a(y)$. In particular, if $x \sim_{LR} y$ then $a(x) = a(y)$. So we may define the a-value $a(\Gamma)$ to be $a(x)$ for any $x \in \Gamma$, where Γ is a left, right or two-sided cell of W; see [Lusztig 1985].

(2) Suppose $a(x) = a(y)$. If $x \leqslant_L y$, then $x \sim_L y$; if $x \leqslant_R y$, then $x \sim_R y$. See [Lusztig 1987a].

(3) For any $I \subseteq S$ with $|W_I| < \infty$, we have $a(w_I) = \ell(w_I)$.

For any $w \in W$, set

$$\mathscr{L}(w) = \{s \in S \mid sw < w\} \quad \text{and} \quad \mathscr{R}(w) = \{s \in S \mid ws < w\}.$$

(4) If $x \leqslant_L y$, then $\mathscr{R}(x) \supseteq \mathscr{R}(y)$; if $x \leqslant_R y$, then $\mathscr{L}(x) \supseteq \mathscr{L}(y)$. In particular, if $x \sim_L y$, then $\mathscr{R}(x) = \mathscr{R}(y)$; if $x \sim_R y$, then $\mathscr{L}(x) = \mathscr{L}(y)$. See [Kazhdan and Lusztig 1979, Proposition 2.4].

By the notation $x = y \cdot z$ for $x, y, z \in W$, we mean $x = yz$ and $\ell(x) = \ell(y) + \ell(z)$. In this case, we call x a *left extension* of z and a *right extension* of y; we call z a *left retraction* of x, and y a *right retraction* of x. When $w = x \cdot y \cdot z$, we call w an *extension* of y and call y a *retraction* of w.

(5) If $x = y \cdot z$ then $x \leqslant_L z$ and $x \leqslant_R y$. Hence $a(x) \geqslant a(y), a(z)$ by (2). In particular, if $I \in \{\mathscr{R}(x), \mathscr{L}(x)\}$, then $a(x) \geqslant \ell(w_I)$; see [Lusztig 1985].

Let $W_{(i)} = \{w \in W \mid a(w) = i\}$ for any $i \in \mathbb{N}$. Then by (2), $W_{(i)}$ is a union of some two-sided cells of W.

(6) For any $x \in W$, let $\Sigma(x)$ be the set of all left cells Γ of W such that there exists some $y \in \Gamma$ with $y—x$, $\mathscr{R}(y) \nsubseteq \mathscr{R}(x)$ and $a(y) = a(x)$. Then $x \sim_L y$

in W if and only if $\mathcal{R}(x)=\mathcal{R}(y)$ and $\Sigma(x)=\Sigma(y)$; see [Shi 1994a, Theorem 2.1], [Shi 1994d] and [Shi 1998a, Section 5].

(7) If x—y in W and $s\in S$ satisfy $s\in\mathcal{L}(y)\setminus\mathcal{L}(x)$ (respectively, $s\in\mathcal{R}(y)\setminus\mathcal{R}(x)$), then either $y=s\cdot x$ (respectively, $y=x\cdot s$) or $y<x$; see [Kazhdan and Lusztig 1979, Sections 2.3e and 2.3f]. In particular, we have $\ell(y)\leqslant\ell(x)+1$.

(8) The number of left cells in W is finite; see [Lusztig 1987a, Theorem 2.2].

(9) If a left cell L and a right cell R are in the same two-sided cell of W, then $L\cap R\neq\varnothing$; see [Lusztig 1987b, Section 3.1(k), (l)].

1.4. To each $x\in W$, we denote by $M(x)$ the set of all $y\in W$ such that there is a sequence $x_0=x, x_1, \dots, x_r=y$ in W with some $r\geqslant 0$, where for every $1\leqslant i\leqslant r$, the conditions $x_{i-1}^{-1}x_i\in S$, $\mathcal{R}(x_{i-1})\not\supseteq\mathcal{R}(x_i)$ and $\mathcal{R}(x_{i-1})\not\subseteq\mathcal{R}(x_i)$ are satisfied.

A graph $\mathfrak{M}(x)$ associated to an element $x\in W$ is defined as follows. Its vertex set is $M(x)$, each $y\in M(x)$ is labeled by the set $\mathcal{R}(y)$; its edge set consists of all two-elements subsets $\{y, z\}\subset M(x)$ with $y^{-1}z\in S$, $\mathcal{R}(y)\not\supseteq\mathcal{R}(z)$ and $\mathcal{R}(y)\not\subseteq\mathcal{R}(z)$.

By a *path* in the graph $\mathfrak{M}(x)$, we mean a sequence $z_0, z_1, \dots, z_r$ in $M(x)$ such that $\{z_{i-1}, z_i\}$ is an edge of $\mathfrak{M}(x)$ for any $1\leqslant i\leqslant r$. Two elements $x, x'\in W$ have the same *right generalized τ-invariants*, if for any path $z_0=x, z_1, \dots, z_r$ in $\mathfrak{M}(x)$, there is a path $z_0'=x', z_1', \dots, z_r'$ in $\mathfrak{M}(x')$ with $\mathcal{R}(z_i')=\mathcal{R}(z_i)$ for any $0\leqslant i\leqslant r$, and if the same condition holds when the roles of x and x' are interchanged.

Then the following result is known.

Proposition 1.5 (see [Shi 1990, Section 3]). *Any $x, y\in W$ with $x\sim_L y$ have the same right generalized τ-invariants.*

1.6. For $s, t\in S$ with $m=o(st)>2$, each of the sequences

$$zt, zts, ztst, \dots \quad\text{and}\quad zs, zst, zsts, \dots \qquad \text{(each containing } m-1 \text{ terms)}$$

is called a *right $\{s, t\}$-string*, where $z\in W$ satisfies $\mathcal{R}(z)\cap\{s, t\}=\varnothing$.

Two elements $x, y\in W$ form a *right primitive pair* if there exist two sequences $x_0=x, x_1, \dots, x_r$ and $y_0=y, y_1, \dots, y_r$ in W such that

(a) for each $1\leqslant i\leqslant r$, there exist some $s_i, t_i\in S$ with $o(s_it_i)>2$ such that any of the pairs x_{i-1}, x_i and y_{i-1}, y_i are neighboring terms in a right $\{s_i, t_i\}$-string;

(b) x_i—y_i for all $0\leqslant i\leqslant r$; and

(c) either $\mathcal{R}(x)\not\subseteq\mathcal{R}(y)$ and $\mathcal{R}(y_r)\not\subseteq\mathcal{R}(x_r)$, or $\mathcal{R}(y)\not\subseteq\mathcal{R}(x)$ and $\mathcal{R}(x_r)\not\subseteq\mathcal{R}(y_r)$.

In particular, if $\{w, y\}$ is an edge in a graph $\mathfrak{M}(x)$ for some $x\in W$, then w, y form a right primitive pair by taking $r=0$ in the definition above.

Similarly, we can define a left $\{s, t\}$-string and a left primitive pair.

Proposition 1.7 (see [Shi 1990, Section 3]). *Any right primitive pair $x, y \in W$ satisfies $x \sim_R y$; any left primitive pair $x, y \in W$ satisfies $x \sim_L y$. In particular, the set $M(x)$ is contained in a right cell of W.*

1.8. An affine Weyl group W is a Coxeter group that can be realized geometrically as follows. Let G be a connected, adjoint reductive algebraic group over $\mathbb{C}$. Fix a maximal torus T of G. Let X be the group of characters $T \to \mathbb{C}$ and let $\Phi \subset X$ be the root system with $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$ a choice of simple system. Then $E = X \otimes_{\mathbb{Z}} \mathbb{R}$ is a euclidean space with an inner product $\langle \cdot, \cdot \rangle$ such that the Weyl group (W_0, S_0) of G with respect to T acts naturally on E and preserves its inner product, where S_0 is the set of simple reflections $s_i = s_{\alpha_i}$ for $1 \leq i \leq \ell$. We denote by N the group of all translations T_λ for $\lambda \in X$ on E: T_λ sends x to $x + \lambda$. Then the semidirect product $W = W_0 \ltimes N$ is called *an affine Weyl group*. Let K be the dual of the type of G. Then we define the type of W by $\widetilde{K}$. Sometimes we denote W just by $\widetilde{K}$. There is a canonical homomorphism from W to W_0, sending w to $\overline{w}$.

Let $-\alpha_0$ be the highest short root in Φ. Set $s_0 = s_{\alpha_0} T_{-\alpha_0}$ with s_{α_0} the reflection corresponding to α_0. Then $S = S_0 \cup \{s_0\}$ forms a Coxeter generator set of W.

Theorem 1.9 [Lusztig 1989, Theorem 4.8]. *In the setup of 1.8, there exists a bijection $\boldsymbol{c} : \boldsymbol{u} \mapsto \boldsymbol{c}(\boldsymbol{u})$ from the set of unipotent conjugacy classes in G to the set* Cell W *of two-sided cells in W such that $a(\boldsymbol{c}(\boldsymbol{u})) = \dim \mathfrak{B}_u$, where u is any element in $\boldsymbol{u}$, and* $\dim \mathfrak{B}_u$ *is the dimension of the variety of Borel subgroups of G containing u.*

2. Alcove forms for elements in affine Weyl groups

Keep the setup of 1.8 for an affine Weyl group W.

2.1. The *alcove form* of an element $w \in W$ is, by definition, a Φ-tuple $(k(w;\alpha))_{\alpha \in \Phi}$ over $\mathbb{Z}$, subject to the following conditions:

(a) $k(e;\alpha) = 0$ for any $\alpha \in \Phi$, where e is the identity of W.

(b) For $0 \leqslant i \leqslant l$, we have $k(s_i;\alpha) = 0$ if $\alpha \neq \pm\alpha_i$, and $k(s_i;\alpha) = \mp 1$ if $\alpha = \pm\alpha_i$.

(c) Let $w' = ws_i$ for some $0 \leqslant i \leqslant l$. Then $k(w';\alpha) = k(w;(\alpha)\overline{s}_i) + k(s_i;\alpha)$, where $\overline{s}_i = s_i$ if $1 \leqslant i \leqslant l$, and $\overline{s}_0 = s_{\alpha_0}$; see [Shi 1987, Proposition 4.2].

It is easily checked that $k(w;-\alpha) = -k(w;\alpha)$ for any $\alpha \in \Phi$. Let Φ^+ be the positive system of Φ containing Π. Then the Φ-tuple $(k(w;\alpha))_{\alpha\in\Phi}$ is entirely determined by the Φ^+-tuple $(k(w;\alpha))_{\alpha\in\Phi^+}$. We can identify $(k(w;\alpha))_{\alpha\in\Phi}$ with $(k(w;\alpha))_{\alpha\in\Phi^+}$ and call the latter also the *alcove form* of w.

Recall the definition for a left extension of an element $x \in W$ in 1.3. The following results on the alcove form $(k(w;\alpha))_{\alpha\in\Phi}$ of $w \in W$ are known.

Proposition 2.2 [Shi 1987, Proposition 4.7]. *Let $w = (k(w;\alpha))_{\alpha\in\Phi} \in W$. Write $w = \overline{w} T_\lambda$ with $\overline{w} \in W_0$ and $\lambda \in X = \mathbb{Z}\Phi$.*

(a) *For* $\alpha \in \Phi^+$, *we have* $k(\overline{w};\alpha)=0$ *if* $(\alpha)\overline{w}^{-1}\in\Phi^+$ *and* $k(\overline{w};\alpha)=-1$ *if* $(\alpha)\overline{w}^{-1}\in\Phi^-$.

(b) $\mathcal{R}(w)=\{s_j\in S \mid k(w;\alpha_j)<0\}$.

(c) *Let* $w'=ws_j$ *with* $w\in W$ *and* $0\leqslant j\leqslant l$. *Then for any* $\alpha\in\Phi$, *we have*

$$k(w';\alpha)=k(w;(\alpha)\overline{s}_j)+k(s_j;\alpha).$$

(d) *Let* $w'=(k(w';\alpha))_{\alpha\in\Phi}\in W$. *Then* w' *is a left extension of* w *if and only if the inequalities* $k(w';\alpha)k(w;\alpha)\geqslant 0$ *and* $|k(w';\alpha)|\geqslant|k(w;\alpha)|$ *hold for any* $\alpha\in\Phi$.

The following result is crucial in the proof of Theorem 3.5.

Proposition 2.3. *Let* $x_0,x_1,\ldots,x_r,\ldots$ *be an infinite sequence of elements in* W *such that* x_i *is a right extension of* x_{i-1} *for every* $i\geqslant 1$. *Then there are some* $q>p\geqslant 0$ *such that* x_q *is a left extension of* x_p.

Proof. By Proposition 2.2(a)–(c), we see that there are permutations τ_{ij} with $i,j\geqslant 0$ on the set Φ, satisfying

(i) $(-\alpha)\tau_{ij}=-(\alpha)\tau_{ij}$ for any $\alpha\in\Phi$;

(ii) $|k(x_j;\alpha)|\geqslant|k(x_i;(\alpha)\tau_{ij})|$ for any $\alpha\in\Phi$ and $j>i$; and

(iii) $\tau_{hi}\tau_{ij}=\tau_{hj}$ for any $h,i,j\geqslant 0$.

Since $|\Phi|<\infty$, the permutation group on Φ is finite. So there exists an infinite subsequence $h_1,h_2,\ldots,h_t,\ldots$ of $1,2,3,\ldots$ with $\tau_{0,h_a}=\tau_{0,h_b}$ for any $a,b\geqslant 0$. Hence $|k(x_{h_a};\alpha)|\geqslant|k(x_{h_b};\alpha)|$ for any $a>b\geqslant 0$ and $\alpha\in\Phi$. Then by the finiteness of the set Φ, there exist some $q>p\geqslant 0$ in $\{h_1,h_2,\ldots\}$ such that $|k(x_q;\alpha)|\geqslant|k(x_p;\alpha)|$ and $k(x_q;\alpha)\cdot k(x_p;\alpha)\geqslant 0$ for any $\alpha\in\Phi$. This implies that x_q is a left extension of x_p by Proposition 2.2(d). □

3. A new algorithm for finding an l.c.r. set in a two-sided cell

3.1. Call a nonempty set $F\subseteq W$ *distinguished* if $|\Gamma\cap F|\leqslant 1$ for any left cell Γ of W. Call F *a representative set of left cells (or an l.c.r. set for short) of* W *in a two-sided cell* Ω if $F\subseteq\Omega$ and $|\Gamma\cap F|=1$ for any left cell Γ of W in Ω.

3.2. For any two-sided cell Ω of W, set

$$F(\Omega)=\{z\in\Omega\mid a(sz),a(zt)<a(z)\ \text{ for any } s\in\mathcal{L}(z) \text{ and } t\in\mathcal{R}(z)\},$$
$$E(\Omega)=\{z\in\Omega\mid a(sz)<a(z)\ \text{ for any } s\in\mathcal{L}(z)\}.$$

Clearly, $F(\Omega)\subseteq E(\Omega)$. Also, $w\in F(\Omega)$ if and only if $w^{-1}\in F(\Omega)$. We have the following result.

Lemma 3.3. (1) *Any $w \in \Omega$ has an expression $w = x \cdot z \cdot y$ for some $x, y \in W$ and $z \in F(\Omega)$.*

(2) *An element $w \in \Omega$ is in $E(\Omega)$ if and only if $x = 1$ in any expression of the form $w = x \cdot z \cdot y$ with $z \in F(\Omega)$.*

Proof. If $w \in F(\Omega)$, then take $x = y = 1$ and $z = w$. If $w \notin F(\Omega)$, then by 1.3(2), either $w = s \cdot w'$ for some $s \in \mathscr{L}(w)$, or $w = w' \cdot t$ for some $t \in \mathscr{R}(w)$, where $w' \in \Omega$. By applying induction on $\ell(w)$, we may write $w' = x \cdot z \cdot y$ for some $x, y \in W$ and some $z \in F(\Omega)$. Hence w is equal to either $sx \cdot z \cdot y$ or $x \cdot z \cdot yt$. This implies (1). Then (2) follows by (1) and 1.3(2), (5). □

The following algorithm is for finding the set $E(\Omega)$ from $F(\Omega)$.

Algorithm 3.4. (1) Set $Y_0 = F(\Omega)$.

Let $k \geqslant 0$. Suppose that the set Y_k has been found.

(2) If $Y_k = \varnothing$, then the algorithm terminates;

(3) If $Y_k \neq \varnothing$, then find the set $Y_{k+1} = \{xs \mid x \in Y_k;\ s \in S \setminus \mathscr{R}(x);\ xs \in E(\Omega)\}$.

By Lemma 3.3(2), we have $E(\Omega) = \bigcup_{i \geqslant 0} Y_i$.

Then the following result shows that Algorithm 3.4 must terminate after a finite number of steps, that is, $E(\Omega) = \bigcup_{k=0}^{t} Y_k$ for some $t \in \mathbb{N}$.

Theorem 3.5. *Let Y_j for $j \geqslant 0$ be obtained from the set $F(\Omega)$ by Algorithm 3.4.*

(1) *There exists some $t \in \mathbb{N}$ such that $Y_j \neq \varnothing$ and $Y_h = \varnothing$ for $0 \leqslant j \leqslant t < h$;*

(2) $E(\Omega) = \bigcup_{k=0}^{t} Y_k$.

Proof. It is easily seen that if $Y_i = \varnothing$ for some $i \geqslant 1$, then $Y_j = \varnothing$ for any $j \geqslant i$, or equivalently, if $Y_i \neq \varnothing$ for some $i \geqslant 0$ then $Y_j \neq \varnothing$ for any $0 \leqslant j \leqslant i$. Since $Y_0 \neq \varnothing$, to prove (1) it suffices to prove that there is an integer $i > 0$ such that $Y_i = \varnothing$.

Suppose to the contrary that $Y_i \neq \varnothing$ for any $i \geqslant 0$. By the finiteness for the number of left cells of W in Ω (see 1.3(8)), there are infinite sequences $w_1, w_2, \ldots$ in $E(\Omega)$ and $0 \leqslant i_1 < i_2 < \cdots < i_n < \cdots$ in $\mathbb{N}$ such that

(i) w_{j+1} is a right extension of w_j for any $j \geqslant 1$ (see 1.3);

(ii) $w_1 \sim_L w_2 \sim_L w_3 \sim_L \cdots$;

(iii) $w_j \in Y_{i_j}$ for $j \geqslant 1$.

By (i) and Proposition 2.2 (b)–(c), we see that there are permutations τ_{ij} on Φ, with $i, j \geqslant 1$, such that $|k(w_j; \alpha)| \geqslant |k(w_i; (\alpha)\tau_{ij})|$ for any $\alpha \in \Phi$ and $j > i \geqslant 1$, and such that $\tau_{hi}\tau_{ij} = \tau_{hj}$ for any $h, i, j \geqslant 1$ (see the proof of Proposition 2.3).

Then by Proposition 2.3, there are some $q > p \geqslant 1$ such that w_q is a proper left extension of w_p. Since $w_p \sim_L w_q$ are in Ω, this implies that w_q is not in $E(\Omega)$, a contradiction. This proves (1). Then (2) follows by (1) and the definition of the set $E(\Omega)$. □

Remark 3.6. We say that a subset K of W is *left-connected*, if for any $x, y \in K$, there exists a sequence of elements $x_0 = x, x_1, \ldots, x_r = y$ in K with some $r \geqslant 0$ such that $x_{i-1}x_i^{-1} \in S$ for every $1 \leqslant i \leqslant r$. Lusztig conjectured in [Asai et al. 1983] that any left cell of an affine Weyl group is left-connected. Although it has been verified in many special cases [Shi 1986; 1988; 2008, Shi and Zhang 2008], the conjecture is still open in general. Now the resulting set $E(\Omega)$ of Algorithm 3.4 is useful in dealing with the conjecture. In fact, to verify the left-connectedness for a left cell Γ of W in a two-sided cell Ω, we need only to construct a graph $\mathcal{M}(\Gamma)$ with $\Gamma \cap E(\Omega)$ as its vertex set. We join two vertices $x \neq y$ in $\Gamma \cap E(\Omega)$ with an edge once we find a sequence of elements $x_0 = x, x_1, \ldots, x_r = y$ in Γ with some $r > 0$ such that $x_{i-1}x_i^{-1} \in S$ for any $1 \leqslant i \leqslant r$. Then we complete the proof for Γ being left-connected once the graph $\mathcal{M}(\Gamma)$ we are constructing becomes connected.

Example 3.7. The following example is provided by Q. Huang, one of my Ph.D. students. Let $W = \widetilde{E}_8$ be with $S = \{s_i \mid 0 \leqslant i \leqslant 8\}$ its distinguished generator set such that $o(s_1s_3) = o(s_3s_4) = o(s_2s_4) = o(s_4s_5) = o(s_5s_6) = o(s_6s_7) = o(s_7s_8) = o(s_8s_0) = 3$. Let Ω be the two-sided cell of W containing the element $s_2s_3s_4s_2s_3s_4$. Then we can get the set $E(\Omega)$ by Algorithm 3.4. We observe that the elements $w_1 = s_2s_3s_4s_2s_3s_4 \cdot s_1s_5s_4s_6$ and $w_2 = s_1s_4s_3s_1s_4s_3 \cdot s_2s_4s_5s_4s_6$ and $w_3 = s_3s_5s_4s_3s_5s_4 \cdot s_1s_2s_3s_4s_5s_6$ in $E(\Omega)$ have the same right generalized τ-invariants among themselves and have different right generalized τ-invariants from any other element in $E(\Omega)$. On the other hand, the element $y := s_2 \cdot w_2 = s_3s_1 \cdot w_1$ is a common left extension of both w_1 and w_2. Since $\mathcal{L}(y) = \{s_1, s_2, s_3\}$ and $\mathcal{L}(w_2) = \{s_1, s_3, s_4\}$ and $\mathcal{L}(s_1w_1) = \{s_1, s_2, s_4\}$ and $\mathcal{L}(w_1) = \{s_2, s_3, s_4\}$, the sequence w_1, s_1w_1, y, w_2 is contained in some left cell (say Γ) of W by Proposition 1.7. Also, the element $x := s_4s_2s_1 \cdot w_3 = s_2s_4s_5 \cdot y$ is a common left extension of both w_2 and w_3. Since $\mathcal{L}(w_3) = \{s_3, s_4, s_5\}$ and $\mathcal{L}(s_1w_3) = \{s_1, s_4, s_5\}$ and $\mathcal{L}(s_2s_1w_3) = \{s_1, s_2, s_5\}$ and $\mathcal{L}(x) = \{s_1, s_2, s_4\}$ and $\mathcal{L}(s_5y) = \{s_1, s_2, s_3, s_5\}$ and $\mathcal{L}(s_4s_5y) = \{s_1, s_4\}$, we see that y, s_5y form a left primitive pair and so the sequence $w_2, y, s_5y, s_4s_5y, x, s_2s_1w_3, s_1w_3, w_3$ is contained in Γ by Proposition 1.7. Therefore $\Gamma \cap E(\Omega) = \{w_1, w_2, w_3\}$ by Proposition 1.5. The graph $\mathcal{M}(\Gamma)$ with the vertex set $\{w_1, w_2, w_3\}$ has the edges $\{w_1, w_2\}$ and $\{w_2, w_3\}$; hence it is connected. This implies by Remark 3.6 that Γ is left-connected.

3.8. In the remaining part of the section, we always assume that the set $F(\Omega)$ is known explicitly for a given two-sided cell Ω of W. For any $x \in W$, denote by Γ_x the left cell of W containing x. Take a distinguished subset $F_0(\Omega)$ of $F(\Omega)$ such that any $w \in F_0(\Omega)$ is a shortest element in the left cell Γ_w and that for any $w \in F(\Omega)$, there is some $w' \in F_0(\Omega)$ and some $x \in W$ with $w' \cdot x$ a shortest element in the left cell Γ_w. In particular, when all the elements of $F(\Omega)$ have the same length (and hence each $w \in F(\Omega)$ is a shortest element in the left cell Γ_w according to 4.6–4.7), we can take $F_0(\Omega)$ to be any maximal distinguished subset of $F(\Omega)$.

Lemma 3.9. *Assume that the set $F_0(\Omega)$ has been chosen for a two-sided cell Ω of W. Then any left cell Γ in Ω contains a shortest element w that has an expression of the form $w = z \cdot y$ for some $z \in F_0(\Omega)$ and $y \in W$.*

Proof. Let Γ and Γ' be two left cells of W in Ω and let $x \in \Gamma$. We see by 1.3(9) that there exists a sequence $x_0 = x, x_1, \ldots, x_r$ in Ω with some $r \geqslant 0$ such that $x_r \in \Gamma'$ and that x_{i-1}—x_i and $\mathcal{R}(x_{i-1}) \not\supseteq \mathcal{R}(x_i)$ for every $1 \leqslant i \leqslant r$. We claim that for any $x' \in \Gamma$, there exists a sequence $x'_0 = x', x'_1, \ldots, x'_r$ such that x'_{i-1}—x'_i and $x'_i \sim_L x_i$ for every $1 \leqslant i \leqslant r$. To show the claim, it is enough to consider the case where $r = 1$ and x'—x with $\mathcal{L}(x') \not\supseteq \mathcal{L}(x)$. Take $s \in \mathcal{R}(x_1) \setminus \mathcal{R}(x)$ and $t \in \mathcal{L}(x) \setminus \mathcal{L}(x')$. Then we have $a_{x_1} \neq 0 \neq b_x$ in the expressions $C_x C_s = \sum_z a_z C_z$ and $C_t C_{x'} = \sum_y b_y C_y$, where $a_z, b_y \in A$. By the positivity of the coefficients of a_z, b_y in u [Lusztig 1985, Section 3.1], we see that $c_{x_1} \neq 0$ in the expression $C_t C_{x'} C_s = \sum_v c_v C_v$ ($c_v \in A$). By the multiplicative associativity of $\mathcal{H}$, this implies that there exists some $x'_1 \in W$ with $d_{x'_1} \neq 0 \neq f_{x_1}$ in the expressions $C_{x'} C_s = \sum_{y'} d_{y'} C_{y'}$ and $C_t C_{x'_1} = \sum_{v'} f_{v'} C_{v'}$, where $d_{y'}, f_{v'} \in A$. By 1.3(1), we get $a(x') = a(x) = a(x_1) \geqslant a(x'_1) \geqslant a(x')$ and hence $a(x'_1) = a(x_1)$. So $x'_1 \sim_L x_1$ by 1.3(2) and the fact $x_1 \leqslant_L x'_1$. The claim is proved.

Now we are ready to show our result. Take a shortest element w' in Γ. Then $w' \in E(\Omega)$. There exist some $z \in F(\Omega)$ and $y \in W$ with $w' = z \cdot y$ by Lemma 3.3(2). By the construction of the set $F_0(\Omega)$, there exist some $z' \in F_0(\Omega)$ and $y' \in W$ such that $z' \cdot y' \sim_L z$ and $\ell(z' \cdot y') \leqslant \ell(z)$. Let $y = s_1 s_2 \cdots s_r$ be a reduced expression of y with $s_i \in S$ and let $z_i = z s_1 s_2 \cdots s_i$ for any $0 \leqslant i \leqslant r$. Then the sequence $z_0 = z, z_1, \ldots, z_r = w'$ is in Ω with z_{i-1}—z_i and $s_i \in \mathcal{R}(z_i) \setminus \mathcal{R}(z_{i-1})$. By the claim above, there exists some sequence $x_0 = z' \cdot y', x_1, \ldots, x_r$ in Ω such that x_{i-1}—x_i and $x_i \sim_L z_i$ for any $1 \leqslant i \leqslant r$. By 1.3(7), we have $\ell(x_r) \leqslant \ell(x_0) + r \leqslant \ell(z) + r = \ell(z \cdot y) = \ell(w')$. Since w' is shortest in Γ and $w' \sim_L x_r$, this forces $\ell(x_r) = \ell(w')$. Hence $\ell(z' \cdot y') = \ell(z)$ and $x_i = x_{i-1} \cdot s_i$ for any $1 \leqslant i \leqslant r$; in particular, $x_r = z' \cdot y' \cdot y$, which is a required element w in the lemma. □

3.10. For any left cell Γ in a two-sided cell Ω of W, let $n(\Gamma)$ be the length of a shortest element in Γ. Then $n(\Gamma)$ is also the smallest number of $\ell(z \cdot y)$ as $z \cdot y$ ranges over all such expressions that $z \in F_0(\Omega)$ and $y \in W$ and $z \cdot y \in \Gamma$ by Lemma 3.9. Let $n(\Omega)$ be the length of a shortest element in Ω.

By modifying Algorithm 3.4, we get the another algorithm such that the resulting set forms an l.c.r. set of W in Ω (see Theorem 3.12):

Algorithm 3.11. (1) Let $X_0 = \{w \in F_0(\Omega) \mid \ell(w) = n(\Omega)\}$.
For $k \geqslant 0$, suppose that the set X_k has been found.

(2) If $X_k = \varnothing$, then our algorithm terminates;

(3) If $X_k \neq \varnothing$, then find the set

$$X'_{k+1} = \{xs \mid x \in X_k,\, s \in S \setminus \mathcal{R}(x),\, xs \in E(\Omega)\} \cup \{w \in F_0(\Omega) \mid \ell(w) = n(\Omega) + k + 1\}.$$

Then take a maximal subset X_{k+1} in X'_{k+1} such that $\bigcup_{i=0}^{k+1} X_i$ is distinguished whenever $X'_{k+1} \neq \varnothing$.

Theorem 3.12. *Let* $E_0(\Omega) := \bigcup_{k \geqslant 0} X_k$.

(1) $E_0(\Omega) \subseteq E(\Omega)$.

(2) *The set* $E_0(\Omega)$ *forms an l.c.r. set of* W *in* Ω.

(3) *Any* $w \in E_0(\Omega)$ *satisfies* $\ell(w) = n(\Gamma_w)$.

Proof. Assertion (1) and the distinguishedness of $E_0(\Omega)$ follows by the construction of the set $E_0(\Omega)$. So for the assertions (2)–(3), it is enough to prove that $\Gamma \cap E_0(\Omega)$ contains an element w with $\ell(w) = n(\Gamma)$ for any left cell Γ of W in Ω.

By Lemma 3.9, there exists some $w' \in \Gamma$ with $\ell(w') = n(\Gamma)$ (hence $w' \in E(\Omega)$) and $w' = x \cdot y$ for some $x \in F_0(\Omega)$ and $y \in W$. We want to find some $w \in E_0(\Omega) \cap \Gamma$ with $\ell(w) = n(\Gamma)$. Apply induction on $n(\Gamma) \geqslant n(\Omega)$ (see 3.10). If $n(\Gamma) = n(\Omega)$ then there exists some $w \in X_0 \cap \Gamma \subseteq E_0(\Omega) \cap \Gamma$ by the construction of the set $F_0(\Omega)$ and Algorithm 3.11. Now assume $n(\Gamma) > n(\Omega)$. Let $k = n(\Gamma) - n(\Omega)$. If $w' \in F_0(\Omega)$ then we can find some $w \in X_k \cap \Gamma \subseteq E_0(\Omega) \cap \Gamma$ by Algorithm 3.11. If $w' = x \cdot y \notin F_0(\Omega)$, that is, $\ell(y) > 0$, take any $s \in \mathcal{R}(y)$; then $z := w's \in E(\Omega)$.

We claim that z is a shortest element in the left cell Γ_z, for, otherwise, there would exist some $z' \in \Gamma_z$ with $\ell(z') < \ell(z)$. By 1.3(6), there is some $w'' \in \Gamma$ with w''—z' by the facts that w'—z (by 1.1(2)) and $z \sim_L z'$ and $\mathcal{R}(w') \nsubseteq \mathcal{R}(z)$. Since $s \in \mathcal{R}(w') \setminus \mathcal{R}(z)$, we have $s \in \mathcal{R}(w'') \setminus \mathcal{R}(z')$ by 1.3(4). Hence $\ell(w'') \leqslant \ell(z') + 1 \leqslant \ell(z) < \ell(w')$ by 1.3(7), contradicting the assumption of $\ell(w') = n(\Gamma)$. The claim is proved.

Since $\ell(z) < \ell(w')$, we have $n(\Gamma_z) < n(\Gamma)$. By the induction hypothesis, there exists some $z_0 \in E_0(\Omega) \cap \Gamma_z$ with $\ell(z_0) = n(\Gamma_z) = \ell(z)$. By the same argument as above with z_0 in the place of z', there exists some $w_0 \in \Gamma$ with w_0—z_0 and $s \in \mathcal{R}(w_0) \setminus \mathcal{R}(z_0)$ and $\ell(w_0) \leqslant \ell(z_0) + 1 = \ell(z) + 1 = \ell(w')$. By the assumption of $\ell(w') = n(\Gamma)$, we have $\ell(w_0) = \ell(z_0) + 1$ and $n(\Gamma) = n(\Gamma_z) + 1$. Hence $w_0 = z_0 \cdot s \in X'_k$ by 1.3(7). By the construction of the set X_k in Algorithm 3.11 and the fact $n(\Gamma) = \ell(w_0)$, there must exist some element in the set $X_k \cap \Gamma$ (and hence in $E_0(\Omega) \cap \Gamma$). So our result follows by induction. □

Remark 3.13. (1) By Theorem 3.5, there is some $t_0 \leqslant t$ with $E_0(\Omega) = \bigcup_{k=0}^{t_0} X_k$, where t is given as in Theorem 3.5(1).

(2) In the case where all the elements in $F_0(\Omega)$ have the same length, we can take $X_0 = F_0(\Omega)$. This is so for most of the cases we have encountered while applying Algorithm 3.11.

4. Some applications of Algorithm 3.11

Example 4.1. Let $W = \widetilde{C}_4$ be with $S = \{s_0, s_1, s_2, s_3, s_4\}$ its Coxeter generator set, where $o(s_0s_1) = o(s_3s_4) = 4$ and $o(s_1s_2) = o(s_2s_3) = 3$. In the subsequent discussion, we abbreviate the notation by writing s_i as $\mathbf{i}$ for $0 \leqslant i \leqslant 4$.

The set $W_{(5)}$ is a single two-sided cell of W by Theorem 1.9. Let $x_1 = \mathbf{01013}$, $x_2 = \mathbf{01014}$, $x_3 = \mathbf{1210124}$, $y_1 = \mathbf{34341}$, $y_2 = \mathbf{34340}$, $y_3 = \mathbf{3234320}$. Then $F(W_{(5)}) = \{x_i, y_j \mid 1 \leqslant i, j \leqslant 3\}$. Since $x_3 \sim_L x_2\mathbf{21}$ and $y_3 \sim_L y_2\mathbf{23}$ with $\ell(x_3) = \ell(x_2\mathbf{21})$ and $\ell(y_3) = \ell(y_2\mathbf{23})$, we can take $F_0(W_{(5)}) = \{x_1, x_2, y_1, y_2\}$ by 3.8.

By applying Algorithm 3.11, we get the following:

$X_0 = F_0(W_{(5)})$.

$X_1 = X_1' = \{x_1\mathbf{2}, x_2\mathbf{2}, y_1\mathbf{2}, y_2\mathbf{2}\}$.

$X_2 = X_2' = \{x_1\mathbf{21}, x_1\mathbf{23}, x_2\mathbf{21}, x_2\mathbf{23}, y_1\mathbf{23}, y_1\mathbf{21}, y_2\mathbf{23}, y_2\mathbf{21}\}$.

$X_3' = \{x_1\mathbf{210}, x_1\mathbf{213}, x_1\mathbf{234}, x_2\mathbf{210}, x_2\mathbf{213}, x_2\mathbf{234}, y_1\mathbf{234}, y_1\mathbf{231}, y_1\mathbf{210}, y_2\mathbf{234}, y_2\mathbf{231}, y_2\mathbf{210}\}$,

$X_3 = \{x_1\mathbf{210}, x_1\mathbf{213}, x_1\mathbf{234}, x_2\mathbf{213}, x_2\mathbf{234}, y_1\mathbf{234}, y_1\mathbf{231}, y_1\mathbf{210}, y_2\mathbf{231}, y_2\mathbf{210}\}$
since $x_2\mathbf{210} \sim_L x_2\mathbf{2}$ and $y_2\mathbf{234} \sim_L y_2\mathbf{2}$.

$X_4' = \{x_1\mathbf{2101}, x_1\mathbf{2310}, x_1\mathbf{2134}, x_1\mathbf{2343}, x_2\mathbf{2103}, x_2\mathbf{2132}, x_2\mathbf{2134}, y_1\mathbf{2343}, y_1\mathbf{2134}, y_1\mathbf{2310}, y_1\mathbf{2101}, y_2\mathbf{2341}, y_2\mathbf{2312}, y_2\mathbf{2310}\}$,

$X_4 = \{x_1\mathbf{2101}, x_1\mathbf{2310}, x_1\mathbf{2134}, x_1\mathbf{2343}, x_2\mathbf{2132}, x_2\mathbf{2134}, y_1\mathbf{2343}, y_1\mathbf{2134}, y_1\mathbf{2310}, y_1\mathbf{2101}, y_2\mathbf{2312}, y_2\mathbf{2310}\}$
since $x_2\mathbf{2103} \sim_L x_2\mathbf{23}$ and $y_2\mathbf{2341} \sim_L y_2\mathbf{21}$.

$X_5' = \{x_1\mathbf{23104}, x_1\mathbf{21343}, x_2\mathbf{21032}, x_2\mathbf{21034}, x_2\mathbf{21324}, y_1\mathbf{21340}, y_1\mathbf{23101}, y_2\mathbf{23412}, y_2\mathbf{23410}, y_2\mathbf{23120}\}$,

$X_5 = \{x_1\mathbf{23104}, x_1\mathbf{21343}, x_2\mathbf{21034}, x_2\mathbf{21324}, y_1\mathbf{21340}, y_1\mathbf{23101}, y_2\mathbf{23410}, y_2\mathbf{23120}\}$
since $x_2\mathbf{21032} \sim_L x_1\mathbf{23}$ and $y_2\mathbf{23412} \sim_L y_1\mathbf{21}$.

$X_6' = \{x_1\mathbf{231043}, x_1\mathbf{213432}, x_2\mathbf{210324}, x_2\mathbf{213243}, y_1\mathbf{213401}, y_1\mathbf{231012}, y_2\mathbf{234120}, y_2\mathbf{231201}\}$,

$X_6 = \{x_1\mathbf{231043}, x_1\mathbf{213432}, x_2\mathbf{213243}, y_1\mathbf{213401}, y_1\mathbf{231012}, y_2\mathbf{231201}\}$
since $x_2\mathbf{210324} \sim_L x_1\mathbf{234}$ and $y_2\mathbf{234120} \sim_L y_1\mathbf{210}$.

$X_7' = \{x_1\mathbf{2310432}, x_2\mathbf{2132434}, y_1\mathbf{2134012}, y_2\mathbf{2312010}\}$.

$X_7 = \{x_1\mathbf{2310432}, y_1\mathbf{2134012}\}$ since $x_2\mathbf{2132434} \sim_L y_1\mathbf{2343}$ and $y_2\mathbf{2312010} \sim_L x_1\mathbf{2101}$.

$X_8 = X_8' = \{x_1\mathbf{23104321}, y_1\mathbf{21340123}\}$.

$X_9 = X_9' = \{x_1\mathbf{231043210}, y_1\mathbf{213401234}\}$.

Since $X_{10} = X'_{10} = \varnothing$, we see by Theorem 3.12 that $E_0 = \bigcup_{i=0}^{9} X_i$ forms an l.c.r. set of W in $W_{(5)}$ with $|X| = 56$.

4.2. The most technical part in applying Algorithm 3.11 is to determine whether or not the element xs is in $E(\Omega)$ for any given $x \in X_k$ and $s \in S \setminus \mathcal{R}(x)$, that is, to check the equation $a(xs) = a(x)$ and the inequality $a(rxs) < a(xs)$ for any $r \in \mathcal{L}(xs)$.

4.3. Checking the equation $a(xs) = a(x)$ amounts to determining the value $a(xs)$. The relation $a(xs) \geqslant a(x)$ holds in general by 1.3(5).

It would be helpful to find all the graphs $\mathfrak{M}(x)$ and $\mathfrak{M}(xs)$ for any $x \in X_k$ and any $s \in S \setminus \mathcal{R}(x)$.

These graphs could be worked out efficiently by computer program. In the case when the graph $\mathfrak{M}(x)$ is larger or even infinite, one need only to work out a local part $\mathfrak{M}$ of $\mathfrak{M}(x)$ around the vertex x. It depends on the actual size of $\mathfrak{M}$. Usually, we take $\mathfrak{M}$ to be a connected subgraph with vertex set $M \subseteq M(x)$ satisfying that

$(*)$ the condition $\Gamma \cap M(x) \neq \varnothing$ implies $\Gamma \cap M \neq \varnothing$ for any left cell Γ of W.

Call a subgraph $\mathfrak{M}$ of $\mathfrak{M}(x)$ *representative* if the vertex set M of $\mathfrak{M}$ satisfies condition $(*)$.

Checking that a subgraph $\mathfrak{M}$ is representative in $\mathfrak{M}(x)$ is an easy matter: One need only check if there always exists some $z_0 \in M$ satisfying $z_0 \sim_L z$ for any $y \in M$ and any $z \in M(x)$ with $\{y, z\}$ an edge of $\mathfrak{M}(x)$.

For any $x \in W$, the following method is efficient for finding the value $a(x)$ in the case where a direct computation for $a(x)$ is difficult (for example, when $\ell(x)$ is larger). One may try to find a sequence $x_0 = x, x_1, \ldots, x_r$ in W such that for every $1 \leqslant i \leqslant r$, the element $x_i s_i$ is in $M(x_{i-1})$ with $\{x_i, x_i s_i\}$ a right primitive pair for some $s_i \in S$ and such that the computation for the value $a(x_r)$ is much easier than that for $a(x)$ (for example, this is the case when $w_J \in M(x_r)$ for some $J \subset S$). In this case, we have $a(x) = a(x_r)$ by repeatedly applying Proposition 1.7.

In practice, we often choose such a sequence $x_0 = x, x_1, \ldots, x_r$ with $\ell(x_r)$ much smaller than $\ell(x_0)$ since the value $a(z)$ can generally be computed relatively more easily when $\ell(z)$ is getting smaller.

When W is a finite Weyl group, one can easily get the value $a(x)$ from the value $a(w_0 x)$ by Theorem 1.9 and by the knowledge of the special unipotent classes of the corresponding reductive algebraic group, where w_0 is the longest element of W; see [Kazhdan and Lusztig 1979, Section 3.3].

4.4. For any $x \in X_k$ and any $s \in S \setminus \mathcal{R}(x)$ with $a(xs) = a(x)$, checking the inequality $a(rxs) < a(xs)$ for any $r \in \mathcal{L}(xs)$ amounts to checking if we always have $y = 1$ in any expression of the form $xs = y \cdot w \cdot z$ with $w \in F(\Omega)$ and $y, z \in W$. The latter can proceed efficiently in terms of alcoves forms of elements once the set $F(\Omega)$ is given explicitly.

4.5. To find X_{k+1} from the set $\left(\bigcup_{i=0}^{k} X_i\right) \cup X'_{k+1}$, we need to determine whether or not two concerning elements x, y, with at least one of them in X'_{k+1}, are in the same left cell of W.

By Propositions 1.5 and 1.7, this can proceed either by comparing their right generalized τ-invariants or with the aid of right primitive pairs.

Suppose that we have all the graphs $\mathfrak{M}(x)$ (or their representative subgraphs) with x ranging over $\left(\bigcup_{i=0}^{k} X_i\right) \cup X'_{k+1}$. These data will help us in determining if two elements (say x, y) so obtained are in the same left cell: We have $x \sim_L y$ only if x and y have the same right generalized τ-invariants, while 1.3(6) provides a complete invariant for the relation $\sim_L$.

4.6. The most interesting for our algorithm is when $F(\Omega) = \{w_J \in \Omega \mid J \subseteq S\} \neq \varnothing$. In this case, $F(\Omega)$ is distinguished and all the elements in $F(\Omega)$ have the same length; hence $F_0(\Omega) = F(\Omega)$ by 3.8. The following are some known cases (not exhaustive) for $F(\Omega)$ of such a form:

(1) Ω is the lowest two-sided cell of W under the partial order $\leqslant_{LR}$; see [Shi 1988, Section 1.1].

(2) Ω consists of fully commutative elements (for example, the case when the Coxeter graph of W contains no subgraph of type D_4, $\widetilde{B}_3$ or $\widetilde{F}_4$, and Ω contains a fully commutative element); see [Shi 2003, Theorem 3.4 and Section 3.5].

(3) W is of simply laced type and $a(\Omega) \leqslant 6$; see [Shi 2008, Theorem B].

(4) W is of type $\widetilde{A}_{n-1}$ with $n > 1$ and Ω corresponds to a partition

$$\lambda = (\lambda_1, \ldots, \lambda_r, 1, \ldots, 1)$$

of n with $\lambda_r + 1 \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_r > 1$; see [Shi 1994c, Theorem 3.1].

(5) W is of type $\widetilde{C}_l$ with $l > 1$ and $a(\Omega) = (l-1)^2 + 1$.

(6) W is of type $\widetilde{B}_l$ with $l > 2$ and $a(\Omega) = l(l-1)$.

4.7. We can describe the set $F(\Omega)$ for some two-sided cell Ω of W even when $F(\Omega)$ does not consist of elements of the form w_J, $J \subseteq S$. For example, when $W = \widetilde{D}_4$, the set $W_{(7)} = \{z \in W \mid a(z) = 7\}$ forms a single two-sided cell but contains no element of the form w_J for $J \subset S$. Let s_0, s_1, s_2, s_3, s_4 be the Coxeter generator set of W with s_2 corresponding to the branching node of its Coxeter graph. Then

$$F(W_{(7)}) = \{s_i s_2 s_k s_i s_2 s_i s_j s_2 s_i \mid i, j, k \in \{0, 1, 3, 4\} \text{ distinct}\};$$

see [Du 1990, Theorem 4.6].

It is desirable to find the sets $F(\Omega)$ for more two-sided cells Ω of W in order to apply Algorithm 3.11.

Some more technical tricks are needed to apply the algorithm. For example, when the set $W_{(i)} = \bigcup_{j=1}^{r} \Omega_j$ for some $i \in \mathbb{N}$ is a union of two-sided cells Ω_j with some $r > 1$, sometimes we know the set

$$F(W_{(i)}) := \{x \in W_{(i)} \mid a(tx) < i \text{ and } a(xs) < i \text{ for all } t \in \mathcal{L}(x), s \in \mathcal{R}(x)\}$$

but not the sets $F(\Omega_j)$ individually. Let us explain it by some examples.

Examples 4.8. Let $W = \widetilde{C}_4$ with $S = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}\}$ be as in Example 4.1.

(a) The set $W_{(3)}$ is a union of two two-sided cells (say $\Omega_{3,1}$ and $\Omega_{3,2}$) of W by Theorem 1.9. We have $F(W_{(3)}) = \{\mathbf{121}, \mathbf{232}, \mathbf{024}\}$ and $F_0(\Omega_{3,i}) = F(\Omega_{3,i})$. At moment, we don't know what the set $F_0(\Omega_{3,i})$ is for any $i = 1, 2$. So we have to assume $X_0 = \{\mathbf{121}, \mathbf{232}, \mathbf{024}\}$ in applying Algorithm 3.11 to find an l.c.r. set for each of the $\Omega_{3,i}$, $i = 1, 2$. We get

$$\begin{aligned}
X_1' &= \{\mathbf{1213}, \mathbf{1210}, \mathbf{2324}, \mathbf{2321}, \mathbf{0241}, \mathbf{0243}\},\\
X_1 &= \{\mathbf{1213}, \mathbf{1210}, \mathbf{2324}, \mathbf{0241}, \mathbf{0243}\} && \text{since } \mathbf{2321} \sim_L \mathbf{1213}.\\
X_2' &= \{\mathbf{12134}, \mathbf{12130}, \mathbf{12101}, \mathbf{23241}, \mathbf{23243}, \mathbf{02413}, \mathbf{02410}, \mathbf{02434}\},\\
X_2 &= \{\mathbf{12134}, \mathbf{12130}, \mathbf{12101}, \mathbf{23243}, \mathbf{02413}, \mathbf{02410}, \mathbf{02434}\} && \text{since } \mathbf{23241} \sim_L \mathbf{12134}.\\
X_3 &= X_3' = \{\mathbf{121343}, \mathbf{121340}, \mathbf{121301}, \mathbf{024132}, \mathbf{024103}, \mathbf{024341}\}.\\
X_4 &= X_4' = \{\mathbf{1213432}, \mathbf{1213430}, \mathbf{1213014}, \mathbf{1213012}, \mathbf{0241324}, \mathbf{0241320}, \mathbf{0241034}\}.\\
X_5 &= X_5' = \{\mathbf{12134320}, \mathbf{12130142}, \mathbf{12130143}, \mathbf{02413243}, \mathbf{02413201}\}.\\
X_6 &= X_6' = \{\mathbf{121343201}, \mathbf{121301423}, \mathbf{121301432}, \mathbf{024132434}, \mathbf{024132010}\}.\\
X_7 &= X_7' = \{\mathbf{1213432010}, \mathbf{1213014234}\}.\\
X_8 &= X_8' = \varnothing.
\end{aligned}$$

We call a subset K of W *right-connected* if, for any pair $x, y \in K$, there is a sequence $x_0 = x, x_1, \ldots, x_r = y$ in K with some $r \geqslant 0$ such that $x_i^{-1}x_{i-1} \in S$ for every $1 \leqslant i \leqslant r$.

By 1.3(2), we see that for any $i \geqslant 0$ with $W_{(i)} \neq \varnothing$, any nonempty right-connected subset of $W_{(i)}$ is contained in a right cell of W and hence also in a two-sided cell of W.

Assume $\mathbf{121} \in \Omega_{3,1}$. Let

$$\begin{aligned}
E_1 = \{&\mathbf{121}, \mathbf{232}, \mathbf{1213}, \mathbf{1210}, \mathbf{2324}, \mathbf{12134}, \mathbf{12130}, \mathbf{12101}, \mathbf{23243}, \mathbf{121343}, \mathbf{121340}, \mathbf{121301},\\
&\mathbf{1213432}, \mathbf{1213430}, \mathbf{1213014}, \mathbf{1213012}, \mathbf{12134320}, \mathbf{12130142}, \mathbf{12130143},\\
&\mathbf{121343201}, \mathbf{121301423}, \mathbf{121301432}, \mathbf{1213432010}, \mathbf{1213014234}\},\\
E_2 = \{&\mathbf{024}, \mathbf{0241}, \mathbf{0243}, \mathbf{02413}, \mathbf{02410}, \mathbf{02434}, \mathbf{024132}, \mathbf{024103}, \mathbf{024341}, \mathbf{0241324}, \mathbf{0241320},\\
&\mathbf{0241034}, \mathbf{02413243}, \mathbf{02413201}, \mathbf{024132434}, \mathbf{024132010}\}.
\end{aligned}$$

Then $\mathbf{121} \in E_1$ and $E := \bigcup_{i=1}^{2} E_0(\Omega_{3,i}) = \bigcup_{k=0}^{7} X_k = E_1 \cup E_2$. We see that E_2 is a maximal right-connected subset of the set E. Also, $E' := E_1 \cup \{\mathbf{2321}\}$ is a union of two right-connected subsets with $\mathbf{1213} \sim_L \mathbf{2321}$ such that $\mathbf{1213}$ and $\mathbf{2321}$ belong to different right-connected subsets of E'. This implies that $E_0(\Omega_{3,1}) = E_1$ and $E_0(\Omega_{3,2}) = E_2$ by 1.3(2) and by the fact that $W_{(3)} = \bigcup_{i=1}^{2} \Omega_{3,i}$.

(b) The set $W_{(4)}$ is a union of two two-sided cells (say $\Omega_{4,1}$ and $\Omega_{4,2}$) of W by Theorem 1.9. Let $x_1 = \mathbf{0101}$, $x_2 = \mathbf{1214}$, $x_3 = \mathbf{121012}$, $y_1 = \mathbf{3434}$, $y_2 = \mathbf{2320}$, $y_3 = \mathbf{232432}$. Then $F(W_{(4)}) = \{x_i, y_j \mid 1 \leqslant i, j \leqslant 3\}$. Since $x_3 \sim_L x_1\mathbf{21}$ and $y_3 \sim_L y_1\mathbf{23}$ with $\ell(x_1\mathbf{21}) = \ell(x_3)$ and $\ell(y_1\mathbf{23}) = \ell(y_3)$, we can take $\bigcup_{i=1}^{2} F_0(\Omega_{4,i}) = \{x_1, x_2, y_1, y_2\}$ by 3.8. Again, we don't know yet what the set $F_0(\Omega_{4,i})$ is for any $i = 1, 2$. We assume $X_0 = \{x_1, x_2, y_1, y_2\}$ in applying Algorithm 3.11. Then

$X_1 = X_1' = \{x_1\mathbf{2}, x_2\mathbf{3}, x_2\mathbf{0}, y_1\mathbf{2}, y_2\mathbf{1}, y_2\mathbf{4}\}$.

$X_2 = X_2' = \{x_1\mathbf{21}, x_1\mathbf{23}, x_2\mathbf{30}, x_2\mathbf{32}, x_2\mathbf{34}, x_2\mathbf{01}, y_1\mathbf{23}, y_1\mathbf{21}, y_2\mathbf{10}, y_2\mathbf{12}, y_2\mathbf{14}, y_2\mathbf{43}\}$.

$X_3' = \{x_1\mathbf{210}, x_1\mathbf{213}, x_1\mathbf{234}, x_2\mathbf{301}, x_2\mathbf{324}, x_2\mathbf{304}, y_1\mathbf{234}, y_1\mathbf{213}, y_1\mathbf{210}, y_2\mathbf{143}, y_2\mathbf{102}, y_2\mathbf{104}\}$,

$X_3 = \{x_1\mathbf{213}, x_1\mathbf{234}, x_2\mathbf{301}, x_2\mathbf{324}, x_2\mathbf{304}, y_1\mathbf{213}, y_1\mathbf{210}, y_2\mathbf{143}, y_2\mathbf{102}, y_2\mathbf{104}\}$

since $x_1\mathbf{210} \sim_L x_1\mathbf{2}$ and $y_1\mathbf{234} \sim_L y_1\mathbf{2}$.

$X_4' = \{x_1\mathbf{2103}, x_1\mathbf{2132}, x_1\mathbf{2134}, x_1\mathbf{2343}, x_2\mathbf{3012}, x_2\mathbf{3014}, x_2\mathbf{3243}, y_1\mathbf{2101}, y_1\mathbf{2103}, y_1\mathbf{2132}, y_1\mathbf{2134}, y_2\mathbf{1043}, y_2\mathbf{1432}, y_2\mathbf{1021}\}$,

$X_4 = \{x_1\mathbf{2132}, x_1\mathbf{2134}, x_1\mathbf{2343}, x_2\mathbf{3012}, x_2\mathbf{3014}, x_2\mathbf{3243}, y_1\mathbf{2101}, y_1\mathbf{2103}, y_1\mathbf{2132}, y_2\mathbf{1043}, y_2\mathbf{1432}, y_2\mathbf{1021}\}$

since $x_1\mathbf{2103} \sim_L x_1\mathbf{23}$ and $y_1\mathbf{2134} \sim_L y_1\mathbf{21}$.

$X_5' = \{x_1\mathbf{21034}, x_1\mathbf{21324}, x_1\mathbf{21343}, x_1\mathbf{23432}, x_2\mathbf{30124}, y_1\mathbf{21012}, y_1\mathbf{21013}, y_1\mathbf{21032}, y_1\mathbf{21034}, y_2\mathbf{10432}\}$,

$X_5 = \{x_1\mathbf{21324}, x_1\mathbf{21343}, x_1\mathbf{23432}, x_2\mathbf{30124}, y_1\mathbf{21012}, y_1\mathbf{21013}, y_1\mathbf{21032}, y_2\mathbf{10432}\}$

since $x_1\mathbf{21034} \sim_L x_1\mathbf{234}$ and $y_1\mathbf{21034} \sim_L y_1\mathbf{210}$.

$X_6' = \{x_1\mathbf{210343}, x_1\mathbf{213243}, x_1\mathbf{213432}, x_1\mathbf{234321}, x_2\mathbf{301243}, y_1\mathbf{210123}, y_1\mathbf{210132}, y_1\mathbf{210134}, y_1\mathbf{210321}, y_2\mathbf{104321}\}$,

$X_6 = \{x_1\mathbf{213243}, x_1\mathbf{213432}, x_1\mathbf{234321}, x_2\mathbf{301243}, y_1\mathbf{210123}, y_1\mathbf{210132}, y_1\mathbf{210321}, y_2\mathbf{104321}\}$

since $x_1\mathbf{210343} \sim_L x_1\mathbf{2343}$ and $y_1\mathbf{210134} \sim_L y_1\mathbf{2101}$.

$X_7' = \{x_1\mathbf{2103432}, x_1\mathbf{2134321}, x_1\mathbf{2343210}, x_2\mathbf{3012434}, y_1\mathbf{2341012}, y_1\mathbf{2310123}, y_1\mathbf{2101234}, y_2\mathbf{1432010}\}$,

$X_7 = \{x_1\mathbf{2343210}, x_2\mathbf{3012434}, y_1\mathbf{2101234}, y_2\mathbf{1432010}\}$ since $x_1\mathbf{2103432} \sim_L x_1\mathbf{23432}$ and $x_1\mathbf{2134321} \sim_L y_1\mathbf{2132}$ and $y_1\mathbf{2341012} \sim_L y_1\mathbf{21012}$ and $y_1\mathbf{2310123} \sim_L x_1\mathbf{2312}$.

$X_8 = X_8' = \varnothing$.

Assume $x_1 \in \Omega_{4,1}$. Then 1.3(2) gives $E_0(\Omega_{4,1}) = E_{11} \cup E_{12}$ and $E_0(\Omega_{4,2}) = E_{21} \cup E_{22}$, where

$$E_{11} = \{x_1, x_1\mathbf{2}, x_1\mathbf{21}, x_1\mathbf{23}, x_1\mathbf{213}, x_1\mathbf{234}, x_1\mathbf{2132}, x_1\mathbf{2134}, x_1\mathbf{2343}, x_1\mathbf{21324}, x_1\mathbf{21343}, x_1\mathbf{23432}, x_1\mathbf{213243}, x_1\mathbf{213432}, x_1\mathbf{234321}, x_1\mathbf{2343210}\},$$

$$E_{12} = \{y_1, y_1\mathbf{2}, y_1\mathbf{21}, y_1\mathbf{23}, y_1\mathbf{210}, y_1\mathbf{213}, y_1\mathbf{2101}, y_1\mathbf{2103}, y_1\mathbf{2132}, y_1\mathbf{21012}, y_1\mathbf{21013}, y_1\mathbf{21032}, y_1\mathbf{210123}, y_1\mathbf{210132}, y_1\mathbf{210321}, y_1\mathbf{2101234}\},$$

$$E_{21} = \{x_2, x_2\mathbf{3}, x_2\mathbf{0}, x_2\mathbf{30}, x_2\mathbf{32}, x_2\mathbf{34}, x_2\mathbf{01}, x_2\mathbf{301}, x_2\mathbf{324}, x_2\mathbf{304}, x_2\mathbf{3012}, x_2\mathbf{3014}, x_2\mathbf{3243}, x_2\mathbf{30124}, x_2\mathbf{301243}, x_2\mathbf{3012434}\},$$

$$E_{22} = \{y_2, y_2\mathbf{1}, y_2\mathbf{4}, y_2\mathbf{10}, y_2\mathbf{14}, y_2\mathbf{12}, y_2\mathbf{43}, y_2\mathbf{102}, y_2\mathbf{104}, y_2\mathbf{143}, y_2\mathbf{1043}, y_2\mathbf{1432}, y_2\mathbf{1021}, y_2\mathbf{10432}, y_2\mathbf{104321}, y_2\mathbf{1043210}\}$$

by the following facts:

(i) Each of E_{ij} for $i, j = 1, 2$ is a maximal right-connected set in $\bigcup_{i,j=1}^{2} E_{ij}$.

(ii) $y_2\mathbf{12} \sim_R y_2\mathbf{124} \sim_L x_2$ and $y_2\mathbf{12} \in E_{22}$ and $x_2 \in E_{21}$.

(iii) $\{x_1\mathbf{213423}, x_1\mathbf{2134232}\}$ forms a right primitive pair.

(iv) $x_1\mathbf{213423} \in E_{11}$ and $y_1 \in E_{12}$ and $x_1\mathbf{2134232} \sim_L \mathbf{234232} \sim_R \mathbf{2342324} \sim_R \mathbf{234234} \sim_L y_1$.

(v) $W_{(4)}$ is a union of two two-sided cells of W.

4.9. Assume that W is an irreducible finite or affine Coxeter group of simply laced type. We see by [Shi 2008, Lemma 6.1] that if $w \in W$ satisfies $a(w) \geqslant 6$ and $a(tw), a(ws) < a(w)$ for any $t \in J := \mathscr{L}(w)$ and $s \in I := \mathscr{R}(w)$, then we have $\ell(w_J), \ell(w_I) \geqslant 6$. This fact will help us to find the set $F(W_{(7)})$. Actually, all the elements of the form w_J with $J \subseteq S$ and $\ell(w_J) = 7$ should be in $F(W_{(7)})$, while all the other elements w of $F(W_{(7)})$ should satisfy $\ell(w_J) = \ell(w_I) = 6$ and $a(tw), a(ws) < a(w) = 7$ for any $t \in J := \mathscr{L}(w)$ and $s \in I := \mathscr{R}(w)$. The set $F(W_{(k)})$ for $k > 7$ can be described similarly but with more cases.

Example 4.10. Let $W = \widetilde{E}_6$ be with $S = \{s_i \mid 0 \leqslant i \leqslant 6\}$ its Coxeter generator set, where $o(s_1s_3) = o(s_3s_4) = o(s_4s_2) = o(s_2s_0) = o(s_4s_5) = o(s_5s_6) = 3$. Then the set $W_{(7)}$ is a single two-sided cell of $\widetilde{E}_6$ by Theorem 1.9. Denote s_i simply by $\mathbf{i}$, $0 \leqslant i \leqslant 6$. By the facts mentioned in 4.9, we get

$$F(W_{(7)}) = \{w_{\mathbf{1346}}, w_{\mathbf{1340}}, w_{\mathbf{0246}}, w_{\mathbf{0241}}, w_{\mathbf{4561}}, w_{\mathbf{4560}}, w_{\mathbf{2346}}, w_{\mathbf{2451}}, w_{\mathbf{3450}}, w_{\mathbf{13562}}, w_{\mathbf{13560}}, w_{\mathbf{13025}}, w_{\mathbf{13026}}, w_{\mathbf{02561}}, w_{\mathbf{02563}}, w_{\mathbf{243}} \cdot \mathbf{543}, w_{\mathbf{243}} \cdot \mathbf{542}, w_{\mathbf{345}} \cdot \mathbf{243}, \mathbf{2031} \cdot w_{\mathbf{342}}, \mathbf{5631} \cdot w_{\mathbf{345}}, \mathbf{5620} \cdot w_{\mathbf{245}}, w_{\mathbf{245}} \cdot \mathbf{345}, w_{\mathbf{345}} \cdot \mathbf{245}, w_{\mathbf{245}} \cdot \mathbf{342}, w_{\mathbf{342}} \cdot \mathbf{1302}, w_{\mathbf{345}} \cdot \mathbf{1365}, w_{\mathbf{245}} \cdot \mathbf{0265}\}.$$

The set $F_0(W_{(7)})$ is obtained from $F(W_{(7)})$ by removing the last nine elements since $w_{\mathbf{243}} \cdot \mathbf{543} \sim_L w_{\mathbf{245}} \cdot \mathbf{345} \sim_L \mathbf{5631} \cdot w_{\mathbf{345}}$ and $w_{\mathbf{243}} \cdot \mathbf{542} \sim_L w_{\mathbf{345}} \cdot \mathbf{245} \sim_L \mathbf{5620} \cdot w_{\mathbf{245}}$ and $w_{\mathbf{345}} \cdot \mathbf{243} \sim_L w_{\mathbf{245}} \cdot \mathbf{342} \sim_L \mathbf{2031} \cdot w_{\mathbf{342}}$ and $w_{\mathbf{342}} \cdot \mathbf{1302} \sim_L w_{\mathbf{1340}} \cdot \mathbf{20}$ and $w_{\mathbf{345}} \cdot \mathbf{1365} \sim_L w_{\mathbf{4561}} \cdot \mathbf{31}$ and $w_{\mathbf{245}} \cdot \mathbf{0265} \sim_L w_{\mathbf{0246}} \cdot \mathbf{56}$.

Remark 4.11. In each of Examples 4.1, 4.8 and 4.10, the related set $F(W_{(k)})$ is given at the beginning. Since it is the set of all two-sided minimal elements w of $W_{(k)}$ (that is, $w \in W_{(k)}$ but $sw, wt \notin W_{(k)}$ for any $s \in \mathscr{L}(w)$ and $t \in \mathscr{R}(w)$), $F(W_{(k)})$ can be found easily because it was described explicitly for all the sets $W_{(k)}$ for $k \in \mathbb{N}$ of the group $\widetilde{C}_4$ and for the set $W_{(7)}$ of $\widetilde{E}_6$; see [Shi 1998b; Shi and Zhang 2006]. In general, without knowing the set $W_{(k)}$ in advance, the set $F(W_{(k)})$ for $k \in \mathbb{N}$ of any Weyl or affine Weyl group W can be found recurrently as follows. By Theorem 1.9 and the knowledge of unipotent conjugacy classes of reductive algebraic groups [Carter 1985], we can get the set $E(W) := \{i \in \mathbb{N} \mid W_{(i)} \neq \varnothing\}$. For any $k \in E(W)$, suppose that the sets $F(W_{(h)})$ for $h < k$ have been found already. Then the set $W_{<k} := \{w \in W \mid a(w) < k\} = \bigcup_{h<k} W_{(h)}$ can be described explicitly by Algorithm 3.11 together with some other techniques. Find the set $E_{\geqslant k}$ of all two-sided minimal elements of $W \setminus W_{<k}$, which is finite by Theorem 1.9 and by the fact $E_{\geqslant k} \subseteq \bigcup_{\Omega \in \mathrm{Cell}(W)} F(\Omega)$. One can determine the set $F(W_{(k)}) = \{w \in E_{\geqslant k} \mid a(w) = k\}$ by computing the a-values of elements in $E_{\geqslant k}$.

References

[Asai et al. 1983] T. Asai et al., *Open problems in algebraic groups: Conference on Algebraic Groups and Their Representations* (Katata, 1983), edited by R. Hotta and N. Kawanaka, Tohoku University, Sendai, 1983.

[Carter 1985] R. W. Carter, *Finite groups of Lie type: conjugacy classes and complex characters*, Wiley, New York, 1985. MR 87d:20060 Zbl 0567.20023

[Chen 2000] Y. Chen, "Left cells in the Weyl group of type E_8", *J. Algebra* **231**:2 (2000), 805–830. MR 2001k:20081 Zbl 0968.20022

[Chen and Shi 1998] Y. Chen and J.-Y. Shi, "Left cells in the Weyl group of type E_7", *Comm. Algebra* **26**:11 (1998), 3837–3852. MR 99h:20065 Zbl 0916.20027

[Du 1990] J. Du, "Cells in the affine Weyl group of type $\widetilde{D}_4$", *J. Algebra* **128**:2 (1990), 384–404. MR 91a:20052 Zbl 0734.20023

[Huang 2008] Q. Huang, "Left cells of a-values 5, 6 in the affine Weyl group $\widetilde{E}_8$", Master's thesis, East China Normal University, Shanghai, 2008.

[Kazhdan and Lusztig 1979] D. Kazhdan and G. Lusztig, "Representations of Coxeter groups and Hecke algebras", *Invent. Math.* **53**:2 (1979), 165–184. MR 81j:20066 Zbl 0499.20035

[Liu 2007] Y. R. Liu, "Left cells of a-values 5, 6 in the affine Weyl group $\widetilde{E}_7$", Master's thesis, East China Normal University, Shanghai, 2007.

[Lusztig 1985] G. Lusztig, "Cells in affine Weyl groups", pp. 255–287 in *Algebraic groups and related topics* (Kyoto/Nagoya, 1983), edited by R. Hotta, Adv. Stud. Pure Math. **6**, North-Holland, Amsterdam, 1985. MR 87h:20074 Zbl 0569.20032

[Lusztig 1987a] G. Lusztig, "Cells in affine Weyl groups, II", *J. Algebra* **109**:2 (1987), 536–548. MR 88m:20103a Zbl 0625.20032

[Lusztig 1987b] G. Lusztig, "Leading coefficients of character values of Hecke algebras", pp. 235–262 in *The Arcata Conference on Representations of Finite Groups* (Arcata, CA, 1986), edited by P. Fong, Proc. Sympos. Pure Math. **47**, Part 2, Amer. Math. Soc., Providence, RI, 1987. MR 89b:20087 Zbl 0657.20037

[Lusztig 1989] G. Lusztig, "Cells in affine Weyl groups, IV", *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **36**:2 (1989), 297–328. MR 90k:20068 Zbl 0688.20020

[Rui 1995] H. B. Rui, "Left cells in affine Weyl groups of types other than $\widetilde{A}_n$ and $\widetilde{G}_2$", *J. Algebra* **175**:2 (1995), 732–756. MR 97a:20070 Zbl 0831.20050

[Shi 1986] J.-Y. Shi, *The Kazhdan–Lusztig cells in certain affine Weyl groups*, Lecture Notes in Mathematics **1179**, Springer, Berlin, 1986. MR 87i:20074 Zbl 0582.20030

[Shi 1987] J.-Y. Shi, "Alcoves corresponding to an affine Weyl group", *J. London Math. Soc.* (2) **35**:1 (1987), 42–55. MR 88g:20103a Zbl 0681.20032

[Shi 1988] J.-Y. Shi, "A two-sided cell in an affine Weyl group, II", *J. London Math. Soc.* (2) **37**:2 (1988), 253–264. MR 89a:20055 Zbl 0613.20026

[Shi 1990] J.-Y. Shi, "A survey on the cell theory of affine Weyl groups", pp. 79–98 in *Advances in science of China: Mathematics*, vol. 3, edited by C. Gu and Y. Wang, Science Press, Beijing, 1990. MR 93f:20063

[Shi 1994a] J.-Y. Shi, "Left cells in affine Weyl groups", *Tohoku Math. J.* (2) **46**:1 (1994), 105–124. MR 94k:20077 Zbl 0798.20040

[Shi 1994b] J.-Y. Shi, "Left cells in the affine Weyl group $W_a(\widetilde{D}_4)$", *Osaka J. Math.* **31**:1 (1994), 27–50. MR 94k:20075 Zbl 0816.20040

[Shi 1994c] J.-Y. Shi, "Some results relating two presentations of certain affine Weyl groups", *J. Algebra* **163**:1 (1994), 235–257. MR 94k:20076 Zbl 0818.20047

[Shi 1994d] J.-Y. Shi, "The verification of a conjecture on left cells of certain Coxeter groups", *Hiroshima Math. J.* **24**:3 (1994), 627–646. MR 95k:20065 Zbl 0831.20051

[Shi 1998a] J.-Y. Shi, "Left cells in the affine Weyl group of type $\widetilde{F}_4$", *J. Algebra* **200**:1 (1998), 173–206. MR 99b:20069 Zbl 0907.20040

[Shi 1998b] J.-Y. Shi, "Left cells in the affine Weyl group of type $\widetilde{C}_4$", *J. Algebra* **202**:2 (1998), 745–776. MR 99d:20065 Zbl 0912.20039

[Shi 2003] J.-Y. Shi, "Fully commutative elements and Kazhdan–Lusztig cells in the finite and affine Coxeter groups", *Proc. Amer. Math. Soc.* **131**:11 (2003), 3371–3378. MR 2004d:20044 Zbl 1064.20043

[Shi 2008] J.-Y. Shi, "Left-connectedness of some left cells in certain Coxeter groups of simply-laced type", *J. Algebra* **319**:6 (2008), 2410–2433. MR 2009d:20094 Zbl 1142.20024

[Shi and Zhang 2006] J.-Y. Shi and X.-G. Zhang, "Left cells in the affine Weyl group $\widetilde{E}_6$", preprint, 2006.

[Shi and Zhang 2008] J.-Y. Shi and X.-G. Zhang, "Left cells with a-value 4 in the affine Weyl groups $\widetilde{E}_i$ $(i = 6, 7, 8)$", *Comm. Algebra* **36**:9 (2008), 3317–3346. MR 2009g:20090 Zbl 1173.20030

[Tong 1995] C. Q. Tong, "Left cells in Weyl group of type E_6", *Comm. Algebra* **23**:13 (1995), 5031–5047. MR 96k:20078 Zbl 0836.20057

[Zhang 1994] X.-F. Zhang, "Cell decomposition in the affine Weyl group $W_a(\widetilde{B}_4)$", *Comm. Algebra* **22**:6 (1994), 1955–1974. MR 95a:20052 Zbl 0802.20038

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