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**ADDENDUM TO THE ARTICLE
SUPERCONNECTIONS AND PARALLEL TRANSPORT**

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We give here an alternate construction for the previously studied parallel transport associated with a superconnection, having the advantage that it is independent of the way the superconnection splits as a connection part plus a bundle-endomorphism valued form.

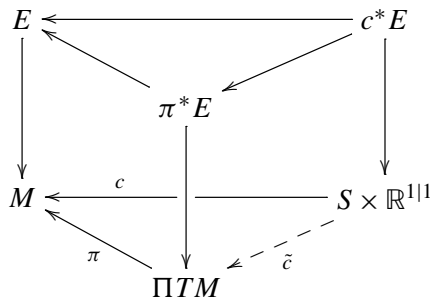
Consider, as in Section 4 of [Dumitrescu 2008] (the paper in the title), a superconnection \mathbb{A} in the sense of Quillen (see [Quillen 1985] and [Berline et al. 1992]) on a $\mathbb{Z}/2$ -graded vector bundle E over a manifold M . That is,

$$\mathbb{A} : \Omega^*(M, E) \rightarrow \Omega^*(M, E)$$

is an odd first-order differential operator satisfying the Leibniz rule

$$\mathbb{A}(\omega \otimes s) = d\omega \otimes s \pm \omega \otimes \mathbb{A}(s),$$

where $\omega \in \Omega^*(M)$ is a differential form on M and $s \in \Gamma(M; E)$ is an arbitrary section of the bundle E over M . For such a superconnection we defined in [Dumitrescu 2008] a notion of parallel transport along (families of) superpaths $c : S \times \mathbb{R}^{1|1} \rightarrow M$ that is compatible under glueing of superpaths. Let us briefly recall this construction. First, we write $\mathbb{A} = \mathbb{A}_1 + A$, where $\mathbb{A}_1 = \nabla$ is the connection part of the superconnection \mathbb{A} and $A \in \Omega^*(M, \text{End } E)^{\text{odd}}$ is the linear part of the superconnection. For an arbitrary superpath c in M , consider the diagram



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where \tilde{c} is a canonical lift (defined in [ibid., Section 4.1]) of the path c to ΠTM , the “odd tangent bundle” of M . Parallel transport along c was defined by *parallel* sections $\psi \in \Gamma(c^*E)$ along c that are solutions to the differential equation

$$(c^*\nabla)_D\psi - (\tilde{c}^*A)\psi = 0.$$

Here $D = \partial_\theta + \theta\partial_t$ denotes the standard (right-invariant) vector field on $\mathbb{R}^{1|1}$ [ibid., Section 2.4].

To describe our alternate construction, we first write $\mathbb{A} = \mathbb{A}_0 + \bar{\mathbb{A}}$, where \mathbb{A}_0 denotes the zero part of the superconnection and $\bar{\mathbb{A}}$ the remaining part. Then we define a connection $\bar{\nabla}$ on the bundle π^*E over ΠTM as follows. For pullback sections $s \in \Gamma(M; E)$ we set

$$\bar{\nabla}_{\mathcal{L}_X}s := \iota_X\bar{\mathbb{A}}s, \quad \bar{\nabla}_{\iota_X}s := 0.$$

Here, for a vector field X on the manifold M , \mathcal{L}_X and ι_X denote the Lie derivative respectively contraction in the X -direction acting as even respectively odd derivations on $\Omega^*(M) = \mathcal{C}^\infty(\Pi TM)$, i.e. as vector fields on ΠTM . For arbitrary sections of π^*E

$$\Gamma(\Pi TM, \pi^*E) = \Omega^*(M) \otimes_{\mathcal{C}^\infty(M)} \Gamma(M, E)$$

we extend the connection $\bar{\nabla}$ by the Leibniz rule

$$\bar{\nabla}_{\mathcal{L}_X}(\omega \otimes s) = \mathcal{L}_X\omega \otimes s \pm \omega \otimes \iota_X\bar{\mathbb{A}}s, \quad \bar{\nabla}_{\iota_X}(\omega \otimes s) = \iota_X\omega \otimes s,$$

whenever $\omega \in \Omega^*(M)$ and $s \in \Gamma(M; E)$. These relations are enough to define a connection $\bar{\nabla}$ on the bundle π^*E over ΠTM since the algebra of vector fields on ΠTM is generated over $\mathcal{C}^\infty(\Pi TM)$ by vector fields of the type \mathcal{L}_X and ι_X , where X denotes an arbitrary vector field on M , i.e.

$$\text{Vect}(\Pi TM) = \mathcal{C}^\infty(\Pi TM)\langle \mathcal{L}_X, \iota_X \mid X \in \text{Vect}(M) \rangle.$$

Parallel transport along a superpath $c : S \times \mathbb{R}^{1|1} \rightarrow M$ is then defined by *parallel sections* $\psi \in \Gamma(c^*E)$ along c which are solutions to the following differential equation

$$(\tilde{c}^*\bar{\nabla})_D\psi - (c^*\mathbb{A}_0)\psi = 0$$

where the lift \tilde{c} of c is defined as before. As in our previous construction, the parallel transport is well-defined [ibid., Proposition 4.2] by this “half-order” differential equation. Moreover, it is compatible under glueing of superpaths; that is, it satisfies properties (i) and (ii) in [ibid., Theorem 4.3]. The advantage of this construction resides in the fact that the parallel transport so defined is invariant under the various ways in which a superconnection can be written as a sum of a connection plus a linear part, as the $\bar{\mathbb{A}}$ part of the superconnection \mathbb{A} which gives rise to the connection $\bar{\nabla}$ is invariant under such splittings.

Denote by δ the de Rham differential on ΠTM . If ω is a function on ΠTM , the 1-form $\delta\omega$ on ΠTM evaluated on the standard odd vector field d on ΠTM gives

$$(\delta\omega)(d) = d\omega,$$

the exterior derivative of ω , understood as a function on ΠTM . Therefore we have

$$\bar{\nabla}_d s = \bar{\mathbb{A}}s,$$

for any s a section of the bundle E over M . We remark that the connection $\bar{\nabla}$ is torsion free in the odd directions, i.e.,

$$[\bar{\nabla}_{\iota_X}, \bar{\nabla}_{\iota_Y}] = \bar{\nabla}_{[\iota_X, \iota_Y]}$$

(and both sides are of course equal to zero). Here X and Y denote arbitrary vector fields on the manifold M .

Remarks. (1) The two constructions of parallel transport associated to a superconnection presented above coincide when the superconnection on the bundle E over M reduces to an ordinary connection (has no linear part). When the manifold M is just a point, a graded vector bundle with superconnection reduces to a $\mathbb{Z}/2$ -vector space V together with an odd endomorphism A ($= \mathbb{A}_0$) of V . In this situation the two constructions of parallel transport also coincide, giving rise to the supergroup homomorphism of [Stolz and Teichner 2004, Example 3.2.9]:

$$\mathbb{R}^{1|1} \ni (t, \theta) \longmapsto e^{-tA^2 + \theta A} \in GL(V),$$

encoding the solutions to the half-order differential equation $D\psi = A\psi$.

(2) The superconnection can be *recovered* from its associated parallel transport, as was the case with our previous construction. First, one recovers the zero part \mathbb{A}_0 of the superconnection \mathbb{A} by considering constant superpaths in M . One then recovers $\bar{\mathbb{A}}$ by looking at parallel transport along the superpath given by

$$\mathbb{R}^{1|1} \times \Pi TM \rightarrow \mathbb{R}^{0|1} \times \Pi TM \rightarrow M,$$

where the first map is the obvious projection and the second map is the standard superpoint evaluation map. The lift of such a superpath to ΠTM is given by the composition

$$\mathbb{R}^{1|1} \times \Pi TM \rightarrow \mathbb{R}^{0|1} \times \Pi TM \rightarrow \Pi TM,$$

where the first map is the projection as before and the second map expresses the flow of the vector field d on ΠTM (since $d^2 = 0$, the flow of d is given by an $\mathbb{R}^{0|1}$ -action). Given that the push-forward of the vector field D along the projection map $\mathbb{R}^{1|1} \rightarrow \mathbb{R}^{0|1}$ is the vector field d on $\mathbb{R}^{0|1}$ and that $\bar{\nabla}_d s = \bar{\mathbb{A}}s$, the parallel transport equation recovers $\bar{\mathbb{A}}$. Compare with Section 4.4 of [Dumitrescu 2008],

where we first obtained the connection part by taking an inverse adiabatic limit and afterwards the *linear* part of the superconnection.

Acknowledgements

The alternate construction presented here is a mere continuation of an idea of Stephan Stolz to interpret a Quillen superconnection on a bundle E over M as a connection on the pullback bundle π^*E over ΠTM . I would like to thank Peter Teichner for suggesting I write up this addendum.

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On slim double Lie groupoids	1
NICOLAS ANDRUSKIEWITSCH, JESUS OCHOA ARANGO and ALEJANDRO TIRABOSCHI	
Topological classification of quasitoric manifolds with second Betti number 2	19
SUYOUNG CHOI, SEONJEONG PARK and DONG YOUP SUH	
Refined Kato inequalities for harmonic fields on Kähler manifolds	51
DANIEL CIBOTARU and PENG ZHU	
Deformation retracts to the fat diagonal and applications to the existence of peak solutions of nonlinear elliptic equations	67
E. NORMAN DANCER, JONATHAN HILLMAN and ANGELA PISTOIA	
Descent for differential Galois theory of difference equations: confluence and q -dependence	79
LUCIA DI VIZIO and CHARLOTTE HARDOUIN	
Modulation and natural valued quiver of an algebra	105
FANG LI	
Willmore hypersurfaces with two distinct principal curvatures in \mathbb{R}^{n+1}	129
TONGZHU LI	
Variational inequality for conditional pressure on a Borel subset	151
YUAN LI, ERCAI CHEN and WEN-CHIAO CHENG	
New homotopy 4-spheres	165
DANIEL NASH	
Combinatorial constructions of three-dimensional small covers	177
YASUZO NISHIMURA	
On a theorem of Paul Yang on negatively pinched bisectonal curvature	201
AERYEONG SEO	
Orders of elements in finite quotients of Kleinian groups	211
PETER B. SHALEN	
A new algorithm for finding an l.c.r. set in certain two-sided cells	235
JIAN-YI SHI	
Addendum to the article Superconnections and parallel transport	253
FLORIN DUMITRESCU	



0030-8730(201203)256:1;1-A