ADDENDUM TO THE ARTICLE
SUPERCONNECTIONS AND PARALLEL TRANSPORT

FLORENI DUMITRESCU
ADDENDUM TO THE ARTICLE
SUPERCONNECTIONS AND PARALLEL TRANSPORT

FLORIN DUMITRESCU

Volume 236:2 (2008), 307–332

We give here an alternate construction for the previously studied parallel transport associated with a superconnection, having the advantage that it is independent of the way the superconnection splits as a connection part plus a bundle-endomorphism valued form.

Consider, as in Section 4 of [Dumitrescu 2008] (the paper in the title), a superconnection $\mathcal{A}$ in the sense of Quillen (see [Quillen 1985] and [Berline et al. 1992]) on a $\mathbb{Z}/2$-graded vector bundle $E$ over a manifold $M$. That is,

$$\mathcal{A} : \Omega^*(M, E) \to \Omega^*(M, E)$$

is an odd first-order differential operator satisfying the Leibniz rule

$$\mathcal{A}(\omega \otimes s) = d\omega \otimes s \pm \omega \otimes \mathcal{A}(s),$$

where $\omega \in \Omega^*(M)$ is a differential form on $M$ and $s \in \Gamma(M; E)$ is an arbitrary section of the bundle $E$ over $M$. For such a superconnection we defined in [Dumitrescu 2008] a notion of parallel transport along (families of) superpaths $c : S \times [1, 1] \to M$ that is compatible under glueing of superpaths. Let us briefly recall this construction. First, we write $\mathcal{A} = A_1 + A$, where $A_1 = \nabla$ is the connection part of the superconnection $\mathcal{A}$ and $A \in \Omega^*(M, \text{End } E)^{\text{odd}}$ is the linear part of the superconnection. For an arbitrary superpath $c$ in $M$, consider the diagram

\[\begin{array}{ccc}
E & \xleftarrow{\pi^*E} & c^*E \\
\downarrow & & \downarrow \\
M & \xleftarrow{\pi} & S \times [1, 1] \\
\downarrow & & \downarrow \\
\Pi TM & \xleftarrow{\tilde{c}} & \tilde{c}
\end{array}\]

MSC2010: primary 53C05, 55N15; secondary 81T60.

Keywords: superconnections, parallel transport.
where $\tilde{c}$ is a canonical lift (defined in [ibid., Section 4.1]) of the path $c$ to $\Pi TM$, the “odd tangent bundle” of $M$. Parallel transport along $c$ was defined by parallel sections $\psi \in \Gamma(c^*E)$ along $c$ that are solutions to the differential equation

$$(c^*\nabla)_D \psi - (\tilde{c}^* A) \psi = 0.$$ 

Here $D = \partial_\theta + \theta \partial_t$ denotes the standard (right-invariant) vector field on $\mathbb{R}^{1|1}$ [ibid., Section 2.4].

To describe our alternate construction, we first write $A = A_0 + \bar{A}$, where $A_0$ denotes the zero part of the superconnection and $\bar{A}$ the remaining part. Then we define a connection $\bar{\nabla}$ on the bundle $\pi^*E$ over $\Pi TM$ as follows. For pullback sections $s \in \Gamma(M; E)$ we set

$$\bar{\nabla}_{\xi_X}s := \iota_X\bar{A}s, \quad \bar{\nabla}_s := 0.$$ 

Here, for a vector field $X$ on the manifold $M$, $\xi_X$ and $\iota_X$ denote the Lie derivative respectively contraction in the $X$-direction acting as even respectively odd derivations on $\Omega^*(M) = \mathcal{C}^\infty(\Pi TM)$, i.e. as vector fields on $\Pi TM$. For arbitrary sections of $\pi^*E$

$$\Gamma(\Pi TM, \pi^*E) = \Omega^*(M) \otimes \mathcal{C}^\infty(M) \Gamma(M, E)$$

we extend the connection $\bar{\nabla}$ by the Leibniz rule

$$\bar{\nabla}_{\xi_X} (\omega \otimes s) = \mathcal{L}_X \omega \otimes s \pm \omega \otimes \iota_X \bar{A}s, \quad \bar{\nabla}_s (\omega \otimes s) = \iota_X \omega \otimes s,$$

whenever $\omega \in \Omega^*(M)$ and $s \in \Gamma(M; E)$. These relations are enough to define a connection $\bar{\nabla}$ on the bundle $\pi^*E$ over $\Pi TM$ since the algebra of vector fields on $\Pi TM$ is generated over $\mathcal{C}^\infty(\Pi TM)$ by vector fields of the type $\mathcal{L}_X$ and $\iota_X$, where $X$ denotes an arbitrary vector field on $M$, i.e.

$$\text{Vect}(\Pi TM) = \mathcal{C}^\infty(\Pi TM) \langle \mathcal{L}_X, \iota_X \mid X \in \text{Vect}(M) \rangle.$$ 

Parallel transport along a superpath $c : S \times [\mathbb{R}^{1|1}] \to M$ is then defined by parallel sections $\psi \in \Gamma(c^*E)$ along $c$ which are solutions to the following differential equation

$$(\tilde{c}^*\bar{\nabla})_D \psi - (c^*A_0) \psi = 0$$

where the lift $\tilde{c}$ of $c$ is defined as before. As in our previous construction, the parallel transport is well-defined [ibid., Proposition 4.2] by this “half-order” differential equation. Moreover, it is compatible under gluing of superpaths; that is, it satisfies properties (i) and (ii) in [ibid., Theorem 4.3]. The advantage of this construction resides in the fact that the parallel transport so defined is invariant under the various ways in which a superconnection can be written as a sum of a connection plus a linear part, as the $\bar{A}$ part of the superconnection $A$ which gives rise to the connection $\bar{\nabla}$ is invariant under such splittings.
Denote by $\delta$ the de Rham differential on $\Pi TM$. If $\omega$ is a function on $\Pi TM$, the 1-form $\delta \omega$ on $\Pi TM$ evaluated on the standard odd vector field $d$ on $\Pi TM$ gives

$$(\delta \omega)(d) = d \omega,$$

the exterior derivative of $\omega$, understood as a function on $\Pi TM$. Therefore we have

$$\bar{\nabla}_d s = \bar{\mathbb{A}} s,$$

for any $s$ a section of the bundle $E$ over $M$. We remark that the connection $\bar{\nabla}$ is torsion free in the odd directions, i.e.,

$$[\bar{\nabla}_{iX}, \bar{\nabla}_{iY}] = \bar{\nabla}_{[iX,iY]},$$

(and both sides are of course equal to zero). Here $X$ and $Y$ denote arbitrary vector fields on the manifold $M$.

**Remarks.** (1) The two constructions of parallel transport associated to a superconnection presented above coincide when the superconnection on the bundle $E$ over $M$ reduces to an ordinary connection (has no linear part). When the manifold $M$ is just a point, a graded vector bundle with superconnection reduces to a $\mathbb{Z}/2$-vector space $V$ together with an odd endomorphism $A (= A_0)$ of $V$. In this situation the two constructions of parallel transport also coincide, giving rise to the supergroup homomorphism of [Stolz and Teichner 2004, Example 3.2.9]:

$$\mathbb{R}^{1|1} \ni (t, \theta) \mapsto e^{-tA^2 + \theta A} \in GL(V),$$

encoding the solutions to the half-order differential equation $D \psi = A \psi$.

(2) The superconnection can be recovered from its associated parallel transport, as was the case with our previous construction. First, one recovers the zero part $A_0$ of the superconnection $\mathbb{A}$ by considering constant superpaths in $M$. One then recovers $\bar{\mathbb{A}}$ by looking at parallel transport along the superpath given by

$$\mathbb{R}^{1|1} \times \Pi TM \to \mathbb{R}^{0|1} \times \Pi TM \to M,$$

where the first map is the obvious projection and the second map is the standard superpoint evaluation map. The lift of such a superpath to $\Pi TM$ is given by the composition

$$\mathbb{R}^{1|1} \times \Pi TM \to \mathbb{R}^{0|1} \times \Pi TM \to \Pi TM,$$

where the first map is the projection as before and the second map expresses the flow of the vector field $d$ on $\Pi TM$ (since $d^2 = 0$, the flow of $d$ is given by an $\mathbb{R}^{0|1}$-action). Given that the push-forward of the vector field $D$ along the projection map $\mathbb{R}^{1|1} \to \mathbb{R}^{0|1}$ is the vector field $d$ on $\mathbb{R}^{0|1}$ and that $\bar{\nabla}_d s = \bar{\mathbb{A}} s$, the parallel transport equation recovers $\bar{\mathbb{A}}$. Compare with Section 4.4 of [Dumitrescu 2008].
where we first obtained the connection part by taking an inverse adiabatic limit and afterwards the linear part of the superconnection.

Acknowledgements

The alternate construction presented here is a mere continuation of an idea of Stephan Stolz to interpret a Quillen superconnection on a bundle $E$ over $M$ as a connection on the pullback bundle $\pi^*E$ over $\Pi TM$. I would like to thank Peter Teichner for suggesting I write up this addendum.

References


Received January 2, 2011.

FLORIN DUMITRESCU

INSTITUTE OF MATHEMATICS OF THE ROMANIAN ACADEMY “SIMION STOILOW”

21 CALEA GRIVITEI
010702 BUCHAREST
ROMANIA

florinndo@gmail.com
On slim double Lie groupoids
Nicolas Andruskiewitsch, Jesus Ochoa Arango and Alejandro Tiraboschi

Topological classification of quasitoric manifolds with second Betti number 2
Suyoung Choi, Seonjeong Park and Dong Youp Suh

Refined Kato inequalities for harmonic fields on Kähler manifolds
Daniel Cibotaru and Peng Zhu

Deformation retracts to the fat diagonal and applications to the existence of peak solutions of nonlinear elliptic equations
E. Norman Dancer, Jonathan Hillman and Angela Pistoia

Descent for differential Galois theory of difference equations: confluence and $q$-dependence
Lucia Di Vizio and Charlotte Hardouin

Modulation and natural valued quiver of an algebra
Fang Li

Willmore hypersurfaces with two distinct principal curvatures in $\mathbb{R}^{n+1}$
Tongzhu Li

Variational inequality for conditional pressure on a Borel subset
Yuan Li, Ercai Chen and Wen-Chiao Cheng

New homotopy 4-spheres
Daniel Nash

Combinatorial constructions of three-dimensional small covers
Yasuzo Nishimura

On a theorem of Paul Yang on negatively pinched bisectional curvature
Aeryeong Seo

Orders of elements in finite quotients of Kleinian groups
Peter B. Shalen

A new algorithm for finding an l.c.r. set in certain two-sided cells
Jian-yi Shi

Addendum to the article Superconnections and parallel transport
Florin Dumitrescu