Pacific Journal of Mathematics

$\mathcal{O}$-OPERATORS ON ASSOCIATIVE ALGEBRAS AND ASSOCIATIVE YANG–BAXTER EQUATIONS

CHENGMING BAI, LI GUO AND XIANG NI

Volume 256 No. 2 April 2012

PACIFIC JOURNAL OF MATHEMATICS
Vol. 256, No. 2, 2012

$\mathcal{O}$-OPERATORS ON ASSOCIATIVE ALGEBRAS AND ASSOCIATIVE YANG–BAXTER EQUATIONS

CHENGMING BAI, LI GUO AND XIANG NI

An $\mathcal{O}$-operator on an associative algebra is a generalization of a Rota–Baxter operator that plays an important role in the Hopf algebra approach of Connes and Kreimer to the renormalization of quantum field theory. It is also the associative analog of an $\mathcal{O}$-operator on a Lie algebra in the study of the classical Yang–Baxter equation. We introduce the concept of an extended $\mathcal{O}$-operator on an associative algebra whose Lie algebra analog has been applied to generalized Lax pairs and PostLie algebras. We study algebraic structures coming from extended $\mathcal{O}$-operators. Continuing the work of Aguiar deriving Rota–Baxter operators from the associative Yang–Baxter equation, we show that its solutions correspond to extended $\mathcal{O}$-operators through a duality. We also establish a relationship of extended $\mathcal{O}$-operators with the generalized associative Yang–Baxter equation.

1. Introduction

1a. ***Motivation.*** The interaction between studies in pure mathematics and mathematical physics has long been a rich source of inspirations that benefited both fields. One such instance can be found in the seminal work of Connes and Kreimer [Connes and Kreimer 2000; Kreimer 1999] on their Hopf algebra approach to the renormalization of quantum field theory. There a curious algebraic identity of linear operators appeared that turned out to be investigated concurrently in the contexts of operads, associative Yang–Baxter equation [Aguiar 2000a; 2000b; 2001], and commutative algebra [Guo and Keigher 2000a; 2000b; Guo 2000], under the name of the Baxter identity (later called the Rota–Baxter identity). It originated in the probability study of G. Baxter [1960] and was influenced by the combinatorial interests of G.-C. Rota [1969a; 1969b; 1995]. Connes and Kreimer's discovery of the connection between Rota–Baxter operators and quantum field theory inspired

C. Bai is supported by NSFC grants 10621101 and 10920161, NKBRPC grant 2006CB805905 and SRFDP grant 200800550015. L. Guo is supported by NSF grant DMS 0505445 and DMS-1001855 and thanks the Chern Institute of Mathematics at Nankai University for hospitality.
Li Guo is the corresponding author.
MSC2010: primary 16W99, 17A30; secondary 57R56.
Keywords: $\mathcal{O}$-operator, Rota–Baxter operator, Yang–Baxter equation, bimodule.

numerous studies to better understand the role played by the Rota–Baxter identity in quantum field theory renormalization, as well as in applying the idea of renormalization to study divergency in mathematics [Ebrahimi-Fard et al. 2004; Ebrahimi-Fard et al. 2006; Guo and Zhang 2008; Manchon and Paycha 2010].

In this paper we consider a generalization of the Rota–Baxter operator in the relative context called the $\mathcal{O}$-operator. It came from another connection between Rota–Baxter operators (on Lie algebras) and mathematical physics. In special cases, the Rota–Baxter identity for Lie algebras coincides with the operator form of the classical Yang–Baxter equation, named after the well-known physicists Yang [1967] and Baxter [1972]. The connection has its origin in the work of Semenov-Tyan-Shanskiĭ [1983] and its extension led to the concept of $\mathcal{O}$-operators [Bai 2007; Bordemann 1990; Kupershmidt 1999]. The relation defining an $\mathcal{O}$-operator was also called the Schouten curvature by Kosmann-Schwarzbach and Magri [1988], and is the algebraic version of the contravariant analog of the Cartan curvature of the Lie algebra-valued one-form on a Lie group.

Back to associative algebras, the first connection between Rota–Baxter operators and an associative analog of the classical Yang–Baxter equation was made by Aguiar [2000a; 2000b], who showed that a solution of the associative Yang–Baxter equation (AYBE) gives rise to a Rota–Baxter operator of weight zero.

Our study of this connection in this paper was motivated by the $\mathcal{O}$-operator approach to the classical Yang–Baxter equation, but we go beyond what was known in the Lie algebra case. On one hand, we generalize the concept of a Rota–Baxter operator to that of an $\mathcal{O}$-operator (of any weight)[1] and further to extended $\mathcal{O}$-operators. On the other hand, we investigate the operator properties of the associative Yang–Baxter equation motivated by the study in the Lie algebra case. Through this approach, we show that the operator property of solutions of the associative Yang–Baxter equation is to a large extent characterized by $\mathcal{O}$-operators. This generalization in the associative context, motivated by Lie algebra studies, has in turn motivated us to establish a similar generalization for Lie algebras and to apply it to generalized Lax pairs, classical Yang–Baxter equations and PostLie algebras [Bai et al. 2010b; 2011; Vallette 2007].

Our approach connects (extended) $\mathcal{O}$-operators to solutions of the AYBE and its generalizations, and therefore [Bai 2010] to the construction of antisymmetric infinitesimal bialgebras and their related Frobenius algebras. The latter plays an important role in topological field theory [Runkel et al. 2007]. In particular, we are able to reverse the connection made by Aguiar and derive, from a Rota–Baxter

[1] In the weight zero case, this has been considered by Uchino [2008] under the name "generalized Rota–Baxter operator". In the general case, the term "relative Rota–Baxter operator" is also used [Bai et al. 2010a].

operator of any weight, a solution of the AYBE and hence give an antisymmetric infinitesimal bialgebra. For further details, see [Bai et al. 2012, Section 4].

1b. *Rota–Baxter algebras and Yang–Baxter equations.*

Notation. In the rest of this paper, $\Bbbk$ denotes a field. By an algebra we mean an associative (not necessarily unitary) $\Bbbk$-algebra, unless otherwise stated.

Definition 1.1. Let R be a $\Bbbk$-algebra and let $\lambda \in \Bbbk$ be given. If a $\Bbbk$-linear map $P : R \to R$ satisfies the *Rota–Baxter relation*

$$P(x)P(y) = P(P(x)y) + P(xP(y)) + \lambda P(xy) \quad \text{for all } x, y \in R, \tag{1-1}$$

then P is called a *Rota–Baxter operator of weight* λ and (R, P) is called a *Rota–Baxter algebra of weight* λ.

For simplicity, we will only discuss the case of Rota–Baxter operators of weight zero in the introduction.

Relation (1-1) still makes sense when R is replaced by a $\Bbbk$-module with any binary operation. If the binary operation is the Lie bracket and if the Lie algebra is equipped with a nondegenerate symmetric invariant bilinear form, then a skew-symmetric solution of the *classical Yang–Baxter equation*

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0. \tag{1-2}$$

is just a Rota–Baxter operator of weight zero. We refer the reader to [Bai 2007; Ebrahimi-Fard 2002; Semenov-Tyan-Shanskiĭ 1983] for further details.

We will consider the following associative analog of the classical Yang–Baxter equation (1-2).

Definition 1.2. Let A be a $\Bbbk$-algebra. An element $r \in A \otimes A$ is called a *solution of the associative Yang–Baxter equation in* A if it satisfies the relation

$$r_{12}r_{13} + r_{13}r_{23} - r_{23}r_{12} = 0, \tag{1-3}$$

called the *associative Yang–Baxter equation (AYBE)*. Here, for $r = \sum_i a_i \otimes b_i \in A \otimes A$, we denote

$$r_{12} = \sum_i a_i \otimes b_i \otimes 1, \quad r_{13} = \sum_i a_i \otimes 1 \otimes b_i, \quad r_{23} = \sum_i 1 \otimes a_i \otimes b_i. \tag{1-4}$$

Both (1-3) and the associative analog

$$r_{13}r_{12} - r_{12}r_{23} + r_{23}r_{13} = 0 \tag{1-5}$$

of (1-2) were introduced by Aguiar [2000a; 2000b; 2001]. In fact, (1-3) is just (1-5) in the opposite algebra [Aguiar 2001]. When r is skew-symmetric it is easy to see that (1-3) comes from (1-5) under the operation $\sigma_{13}(x \otimes y \otimes z) = z \otimes y \otimes x$.

While (1-5) was emphasized by Aguiar in the works above, we will work with (1-3) for notational convenience and to be consistent with some of the earlier works on connections with antisymmetric infinitesimal bialgebras [Bai 2010] and associative D-bialgebras [Zhelyabin 1997].

Theorem 1.3 [Aguiar 2000b]. *Let A be a $\Bbbk$-algebra. If $r = \sum_i a_i \otimes b_i \in A \otimes A$ is a solution of* (1-5) *in A, the map*

$$P : A \to A, \quad x \mapsto \sum\nolimits_i a_i x b_i$$

defines a Rota–Baxter operator of weight zero on A.

The theorem is obtained by replacing the tensor symbols in

$$r_{13}r_{12} - r_{12}r_{23} + r_{23}r_{13} = \sum_{i,j} a_i a_j \otimes b_j \otimes b_i - \sum_{i,j} a_i \otimes b_i a_j \otimes b_j + \sum_{i,j} a_j \otimes a_i \otimes b_i b_j = 0$$

by x and y in A.

1c. $\mathcal{O}$*-operators.* We will introduce an extended $\mathcal{O}$-operator as a generalization of a Rota–Baxter operator and the associative analog of an $\mathcal{O}$-operator on a Lie algebra. We then extend the connections of Rota–Baxter algebras with associative Yang–Baxter equations to those of $\mathcal{O}$-operators. This study is motivated by the relationship between $\mathcal{O}$-operator and the classical Yang–Baxter equation in Lie algebras [Bai 2007; Bai et al. 2010b; Bordemann 1990; Kupershmidt 1999]

Let $(A, \cdot)$ be a $\Bbbk$-algebra. Let (V, ℓ, r) be an A-bimodule, consisting of a compatible pair of a left A-module (V, ℓ) given by $\ell : A \to \mathrm{End}_{\Bbbk}(V)$ and a right A-module (V, r) given by $r : A \to \mathrm{End}_{\Bbbk}(V)$; see Section 2a for the precise definition. Fix a $\kappa \in \Bbbk$. A pair (α, β) of linear maps $\alpha, \beta : V \to A$ is called an *extended $\mathcal{O}$-operator with modification β of mass κ* if

$$\kappa\ell(\beta(u))v = \kappa u r(\beta(v)) \qquad \text{and}$$
$$\alpha(u) \cdot \alpha(v) - \alpha(l(\alpha(u))v + ur(\alpha(v))) = \kappa\beta(u) \cdot \beta(v) \quad \text{for all } u, v \in V.$$

When $\beta = 0$ or $\kappa = 0$, we obtain the concept of an *$\mathcal{O}$-operator* α satisfying

$$\alpha(u) \cdot \alpha(v) = \alpha(\ell(\alpha(u))v) + \alpha(ur(\alpha(v))) \quad \text{for all } u, v \in V. \tag{1-6}$$

When V is taken to be the A-bimodule (A, L, R), where $L, R : A \to \mathrm{End}_{\Bbbk}(A)$ are given by the left and right multiplications, an $\mathcal{O}$-operator $\alpha : V \to A$ of weight zero is just a Rota–Baxter operator of weight zero. To illustrate the close relationship between $\mathcal{O}$-operators and solutions of the AYBE (1-3), we give the following reformulation of a part of Corollary 3.6. See Section 3 for general cases.

Let $\mathbb{k}$ be a field whose characteristic is not 2. Let A be a $\mathbb{k}$-algebra that we for now assume to have finite dimension over $\mathbb{k}$. Let

$$\sigma : A \otimes A \to A \otimes A, a \otimes b \mapsto b \otimes a,$$

be the switch operator and let

$$t : \mathrm{Hom}_{\mathbb{k}}(A^*, A) \to \mathrm{Hom}_{\mathbb{k}}(A^*, A)$$

be the transpose operator. Then the natural bijection

$$\phi : A \otimes A \to \mathrm{Hom}_{\mathbb{k}}(A^*, \mathbb{k}) \otimes A \to \mathrm{Hom}_{\mathbb{k}}(A^*, A)$$

is compatible with the operators σ and t. Let $\mathrm{Sym}^2(A\otimes A)$ and $\mathrm{Alt}^2(A\otimes A)$ (respectively $\mathrm{Hom}_{\mathbb{k}}(A^*, A)_+$ and $\mathrm{Hom}_{\mathbb{k}}(A^*, A)_-$) be the eigenspaces for the eigenvalues 1 and -1 of σ on $A \otimes A$ (respectively of t on $\mathrm{Hom}_{\mathbb{k}}(A^*, A)$). Then we have a commutative diagram of bijective linear maps given by

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\quad\phi\quad} & \mathrm{Hom}_{\mathbb{k}}(A^*, A) \\ \downarrow & & \downarrow \\ \mathrm{Alt}^2(A \otimes A) \oplus \mathrm{Sym}^2(A \otimes A) & \xrightarrow{\phi} & \mathrm{Hom}_{\mathbb{k}}(A^*, A)_- \oplus \mathrm{Hom}_{\mathbb{k}}(A^*, A)_+, \end{array} \tag{1-7}$$

which preserves the factorizations. Define $\mathrm{Hom}_{\mathrm{bim}}(A^*, A)_+$ to be the subset of $\mathrm{Hom}_{\mathbb{k}}(A^*, A)_+$ consisting of A-bimodule homomorphisms from A^* to A, both of which are equipped with the natural A-bimodule structures. Let

$$\mathrm{Sym}^2_{\mathrm{bim}}(A \otimes A) := \phi^{-1}(\mathrm{Hom}_{\mathrm{bim}}(A^*, A)_+) \subseteq \mathrm{Sym}^2(A \otimes A).$$

Then we have this (see Corollary 3.6):

Theorem 1.4. *An element* $r = (r_-, r_+) \in \mathrm{Alt}^2(A\otimes A) \oplus \mathrm{Sym}^2_{\mathrm{bim}}(A\otimes A)$ *is a solution of the AYBE* (1-3) *if and only if the pair* $\phi(r) = (\phi(r)_-, \phi(r)_+) = (\phi(r_-), \phi(r_+))$ *is an extended* $\mathbb{O}$*-operator with modification* $\phi(r_+)$ *of mass* $\kappa = -1$. *In particular, when* r_+ *is zero, an element* $r = (r_-, 0) = r_- \in \mathrm{Alt}^2(A \otimes A)$ *is a solution of the AYBE if and only if the pair* $\phi(r) = (\phi(r)_-, 0) = \phi(r_-)$ *is an* $\mathbb{O}$*-operator of weight zero given by* (1-6) *when* (V, ℓ, r) *is the dual bimodule* (A^*, R^*, L^*) *of* (A, L, R).

Let $\mathcal{M}\mathbb{O}(A^*, A)$ denote the set of extended $\mathbb{O}$-operators (α, β) from A^* to A of mass $\kappa = -1$. Let $\mathbb{O}(A^*, A)$ denote the set of $\mathbb{O}$-operators $\alpha : A^* \to A$ of weight 0. Let $\mathrm{AYB}(A)$ denote the set of solutions of the AYBE (1-3) in A. Let $\mathrm{SAYB}(A)$ denote the set of skew-symmetric solutions of the AYBE (1-3) in A. Then Theorem 1.4 means that the bijection in (1-7) restricts to bijections in the following commutative diagram:

$$
\begin{array}{ccc}
\mathrm{Alt}^2(A\otimes A)\oplus \mathrm{Sym}^2_{\mathrm{bim}}(A\otimes A) & \xrightarrow{\ \phi\ } & \mathrm{Hom}_{\Bbbk}(A^*, A)_- \oplus \mathrm{Hom}_{\mathrm{bim}}(A^*, A)_+ \\
\uparrow & & \uparrow \\
\mathrm{AYB}(A)\bigcap(\mathrm{Alt}^2(A\otimes A)\oplus \mathrm{Sym}^2_{\mathrm{bim}}(A\otimes A)) & \xrightarrow{\ \phi\ } & \mathcal{MO}(A^*, A)\bigcap(\mathrm{Hom}_{\Bbbk}(A^*, A)_- \oplus \mathrm{Hom}_{\mathrm{bim}}(A^*, A)_+) \\
\uparrow & & \uparrow \\
\mathrm{SAYB}(A) & \xrightarrow{\ \phi\ } & \mathcal{O}(A^*, A)\bigcap \mathrm{Hom}_{\Bbbk}(A^*, A)_-
\end{array}
$$

1d. ***Layout of the paper.*** In Section 2, we introduce extended $\mathcal{O}$-operators and study their connection with the associativity of certain products. Section 3 establishes the relationship of extended $\mathcal{O}$-operators with associative and extended associative Yang–Baxter equations. Section 4 introduces the concept of the generalized associative Yang–Baxter equation (GAYBE) and considers its relationship with extended $\mathcal{O}$-operators.

2. $\mathcal{O}$-operators and extended $\mathcal{O}$-operators

We give background notation in Section 2a before introducing the concept of an extended $\mathcal{O}$-operator in Section 2b. We then show in Section 2c and 2d that extended $\mathcal{O}$-operators can be characterized by the associativity of a multiplication derived from this operator.

2a. *Bimodules, A-bimodule $\Bbbk$-algebras and matched pairs of algebras.*

Definition 2.1. Let $(A, \cdot)$ be a $\Bbbk$-algebra.

(i) An A-*bimodule* is a $\Bbbk$-module V and linear maps $\ell, r : A \to \mathrm{End}_{\Bbbk}(V)$ such that (V, ℓ) defines a left A-module, (V, r) defines a right A-module and the two module structures on V are compatible in the sense that

$$(\ell(x)v)r(y) = \ell(x)(vr(y)) \quad \text{for all } x, y \in A, v \in V.$$

If we want more precision, we denote an A-bimodule V by the triple (V, ℓ, r).

(ii) A homomorphism between two A-bimodules (V_1, ℓ_1, r_1) and (V_2, ℓ_2, r_2) is a $\Bbbk$-linear map $g : V_1 \to V_2$ such that

$$g(\ell_1(x)v) = \ell_2(x)g(v) \quad \text{and} \quad g(vr_1(x)) = g(v)r_2(x) \quad \text{for all } x \in A,\ v \in V_1.$$

For a k-algebra A and $x \in A$, define the left and right actions

$$L(x) : A \to A, \quad L(x)y = xy \quad \text{and} \quad R(x) : A \to A,\ yR(x) = yx \quad \text{for all } y \in A.$$

Further define

$$L = L_A : A \to \operatorname{End}_{\Bbbk}(A), \quad x \mapsto L(x) \quad \text{and} \quad R = R_A : A \to \operatorname{End}_{\Bbbk}(A), \quad x \mapsto R(x).$$

Obviously, (A, L, R) is an A-bimodule.

For a $\Bbbk$-module V, let $V^* := \operatorname{Hom}_{\Bbbk}(V, \Bbbk)$ denote the dual $\Bbbk$-module. Denote the usual pairing between V^* and V by

$$\langle\,\cdot\,,\,\cdot\,\rangle : V^* \times V \to \Bbbk, \quad \langle u^*, v\rangle = u^*(v) \quad \text{for all } u^* \in V^* \text{ and } v \in V.$$

Proposition 2.2 [Bai 2010]. *Let A be a $\Bbbk$-algebra and let (V, ℓ, r) be an A-bimodule. Define the linear maps $\ell^*, r^* : A \to \operatorname{End}_{\Bbbk}(V^*)$ by*

$$\langle u^*\ell^*(x), v\rangle = \langle u^*, \ell(x)v\rangle \quad \textit{and} \quad \langle r^*(x)u^*, v\rangle = \langle u^*, vr(x)\rangle$$

for all $x \in A$, $u^* \in V^*$ and $v \in V$. Then (V^*, r^*, ℓ^*) is an A-bimodule, called the dual bimodule of (V, ℓ, r).

Let (A^*, R^*, L^*) denote the dual A-bimodule of the A-bimodule (A, L, R).

We next extend the concept of a bimodule to that of an A-bimodule algebra by replacing the $\Bbbk$-module V by a $\Bbbk$-algebra R.

Definition 2.3. Let $(A, \cdot)$ be a $\Bbbk$-algebra with multiplication $\cdot$ and let $(R, \circ)$ be a $\Bbbk$-algebra with multiplication $\circ$. Let $\ell, r : A \to \operatorname{End}_{\Bbbk}(R)$ be two linear maps. We call R (or the triple (R, ℓ, r) or the quadruple $(R, \circ, \ell, r)$) an *A-bimodule $\Bbbk$-algebra* if (R, ℓ, r) is an A-bimodule that is compatible with the multiplication $\circ$ on R. More precisely, we have, for all $x, y \in A$ and $v, w \in R$

$$\ell(x \cdot y)v = \ell(x)(\ell(y)v), \qquad \ell(x)(v \circ w) = (\ell(x)v) \circ w, \tag{2-1}$$

$$vr(x \cdot y) = (vr(x))r(y), \qquad (v \circ w)r(x) = v \circ (wr(x)), \tag{2-2}$$

$$(\ell(x)v)r(y) = \ell(x)(vr(y)), \qquad (vr(x)) \circ w = v \circ (\ell(x)w). \tag{2-3}$$

Obviously, for any $\Bbbk$-algebra $(A, \cdot)$, the triple $(A, \cdot, L, R)$ is an A-bimodule $\Bbbk$-algebra. An A-bimodule $\Bbbk$-algebra R need not be a left or right A-algebra since we do not assume that $A \cdot 1$ is in the center of R. For example, the A-bimodule $\Bbbk$-algebra (A, L, R) is an A-algebra if and only if A is a commutative ring.

An A-bimodule $\Bbbk$-algebra is a special case of a matched pair as introduced in [Bai 2010]. It is easy to get the following result, which is a generalization of the classical result between bimodule structures on V and semidirect product algebraic structures on $A \oplus V$.

Proposition 2.4. *If $(R, \circ, \ell, r)$ is an A-bimodule $\Bbbk$-algebra, then the direct sum $A \oplus R$ of vector spaces is turned into a $\Bbbk$-algebra (the semidirect sum) by defining multiplication in $A \oplus R$ by*

$$(x_1, v_1) * (x_2, v_2) = (x_1 \cdot x_2, \ell(x_1)v_2 + v_1 r(x_2) + v_1 \circ v_2)$$

for all $x_1, x_2 \in A$ and $v_1, v_2 \in R$.

We denote this algebra by $A \ltimes_{\ell,r} R$ or simply $A \ltimes R$.

2b. *Extended $\mathbb{O}$-operators.* We first define an $\mathbb{O}$-operator before introducing an extended $\mathbb{O}$-operator through an auxiliary operator.

Definition 2.5. Let $(A, \cdot)$ be a $\Bbbk$-algebra and $(R, \circ, \ell, r)$ be an A-bimodule $\Bbbk$-algebra. A linear map $\alpha : R \to A$ is called an *$\mathbb{O}$-operator of weight $\lambda \in \Bbbk$ associated to* $(R, \circ, \ell, r)$ if α satisfies

$$\alpha(u) \cdot \alpha(v) = \alpha(\ell(\alpha(u)v)) + \alpha(ur(\alpha(v))) + \lambda\alpha(u \circ v) \quad \text{for all } u, v \in V. \tag{2-4}$$

Remark 2.6. Under our assumption that $\Bbbk$ is a field, the nonzero weight can be normalized to weight 1. In fact, for a nonzero weight $\lambda \in \Bbbk$, if α is an $\mathbb{O}$-operator of weight λ associated to an A-bimodule $\Bbbk$-algebra $(R, \circ, \ell, r)$, then α is an $\mathbb{O}$-operator of weight 1 associated to $(R, \lambda\circ, \ell, r)$ and α/λ is an $\mathbb{O}$-operator of weight 1 associated to $(R, \circ, \ell, r)$.

When the multiplication on the A-bimodule $\Bbbk$-algebra happens to be trivial, an $\mathbb{O}$-operator is just a generalized Rota–Baxter operator defined in [Uchino 2008]. Further, an $\mathbb{O}$-operator associated to (A, L, R) is just a Rota–Baxter operator on A. An $\mathbb{O}$-operator can be viewed as the relative version of a Rota–Baxter operator in that the domain and range of an $\mathbb{O}$-operator might be different. Thus an $\mathbb{O}$-operator is also called a relative Rota–Baxter operator.

We now further generalize the concept of an $\mathbb{O}$-operator.

Definition 2.7. Let $(A, \cdot)$ be a $\Bbbk$-algebra.

(i) Let $\kappa \in \Bbbk$ and let (V, ℓ, r) be an A-bimodule. A linear map (respectively an A-bimodule homomorphism) $\beta : V \to A$ is called a *balanced linear map of mass κ* (respectively *balanced A-bimodule homomorphism of mass κ*) if

$$\kappa\ell(\beta(u))v = \kappa ur(\beta(v)) \quad \text{for all } u, v \in V. \tag{2-5}$$

(ii) Let $\kappa, \mu \in \Bbbk$ and let $(R, \circ, \ell, r)$ be an A-bimodule $\Bbbk$-algebra. A linear map (respectively an A-bimodule homomorphism) $\beta : R \to A$ is called a *balanced* *linear map of mass* (κ, μ) (respectively a *balanced A-bimodule homomorphism of mass* (κ, μ)) if (2-5) holds and

$$\mu\ell(\beta(u \circ v))w = \mu ur(\beta(v \circ w)) \quad \text{for all } u, v, w \in R. \tag{2-6}$$

Clearly, if $\kappa = 0$ and $\mu = 0$, then (2-5) and (2-6), respectively, impose no restriction. So any A-bimodule homomorphism is balanced of mass $(\kappa, \mu) = (0, 0)$. For a nonzero mass, we have the following examples.

Example 2.8. Let A be a $\Bbbk$-algebra.

(i) The identity map $\beta = \mathrm{id} : (A, L, R) \to A$ is a balanced A-bimodule homomorphism (of any mass (κ, μ)).

(ii) Any A-bimodule homomorphism $\beta : (A, L, R) \to A$ is balanced (of any mass (κ, μ)).

(iii) Let $r \in A \otimes A$ be symmetric. If r regarded as a linear map from (A^*, R^*, L^*) to A is an A-bimodule homomorphism, then r is a balanced A-bimodule homomorphism (of any mass κ). See Lemma 3.2.

We can now introduce our first main concept in this paper.

Definition 2.9. Let $(A, \cdot)$ be a $\Bbbk$-algebra and let $(R, \circ, \ell, r)$ be an A-bimodule $\Bbbk$-algebra.

(i) Let $\lambda, \kappa, \mu \in \Bbbk$. Fix a balanced A-bimodule homomorphism $\beta : (R, \ell, r) \to A$ of mass (κ, μ). A linear map $\alpha : R \to A$ is called an *extended $\mathcal{O}$-operator of weight λ with modification β of mass* (κ, μ) if, for all $u, v \in R$,

$$\alpha(u) \cdot \alpha(v) - \alpha(\ell(\alpha(u))v + ur(\alpha(v)) + \lambda u \circ v) = \kappa\beta(u) \cdot \beta(v) + \mu\beta(u \circ v). \tag{2-7}$$

(ii) We also let (α, β) denote an extended $\mathcal{O}$-operator α with modification β.

(iii) When (V, ℓ, r) is an A-bimodule, we regard V as an A-bimodule $\Bbbk$-algebra with the zero multiplication. Then λ and μ are irrelevant. We then call the pair (α, β) an *extended $\mathcal{O}$-operator with modification β of mass κ*.

We note that, when the modification β is the zero map (and hence κ and μ are irrelevant), then α is the $\mathcal{O}$-operator defined in Definition 2.5.

2c. ***Extended $\mathcal{O}$-operators and associativity.*** The study of classical Yang–Baxter equations often gives rise to the study of additional Lie structures derived from a given Lie algebra [Bai et al. 2010b; Semenov-Tyan-Shanskiĭ 1983]. Similar derived structures in an associative algebra have also appeared in the study of dendriform algebras and Rota–Baxter algebras [Aguiar 2000b; Bai 2010; Loday and Ronco 2004]. Here we study derived structures arising from $\mathcal{O}$-operators.

Let $(A, \cdot)$ be a $\Bbbk$-algebra and $(R, \circ, \ell, r)$ be an A-bimodule $\Bbbk$-algebra. Let $\delta_\pm : R \to A$ be two linear maps and $\lambda \in \Bbbk$. We now consider the associativity of the multiplication

$$u \diamond v := \ell(\delta_+(u))v + ur(\delta_-(v)) + \lambda u \circ v \quad \text{for all } u, v \in R, \tag{2-8}$$

and several other related multiplications. This will be applied in the Section 4.

Let the characteristic of the field $\Bbbk$ be different from 2. Set

$$\alpha := (\delta_+ + \delta_-)/2 \quad \text{and} \quad \beta := (\delta_+ - \delta_-)/2, \tag{2-9}$$

called the *symmetrizer* and *antisymmetrizer* of $\delta_\pm$ respectively. Note that $\delta_\pm$ can be recovered from α and β by $\delta_\pm = \alpha \pm \beta$.

Lemma 2.10. *Let $(A, \cdot)$ be a $\Bbbk$-algebra and $(R, \circ, \ell, r)$ be an A-bimodule $\Bbbk$-algebra. Let $\alpha : R \to A$ be a linear map and let λ be in $\Bbbk$. Then the operation given by*

$$u *_\alpha v := \ell(\alpha(u))v + ur(\alpha(v)) + \lambda u \circ v \quad \textit{for all } u, v \in R \tag{2-10}$$

is associative if and only if

$$\ell(\alpha(u) \cdot \alpha(v) - \alpha(u *_\alpha v))w = ur(\alpha(v) \cdot \alpha(w) - \alpha(v *_\alpha w)) \tag{2-11}$$

for all $u, v, w \in R$.

Proof. It is straightforward to check that, for any $u, v, w \in R$, we have

$$(u *_\alpha v) *_\alpha w - u *_\alpha (v *_\alpha w) = ur(\alpha(v) \cdot \alpha(w) - \alpha(v *_\alpha w)) - \ell(\alpha(u) \cdot \alpha(v) - \alpha(u *_\alpha v))w. \qquad \square$$

Corollary 2.11. *Let $\Bbbk$ be a field of characteristic not equal to 2. Let $(A, \cdot)$ be a $\Bbbk$-algebra and $(R, \circ, \ell, r)$ be an A-bimodule $\Bbbk$-algebra. Let $\delta_\pm : R \to A$ be two linear maps and $\lambda \in \Bbbk$. Let α and β be their symmetrizer and antisymmetrizer defined by* (2-9)*. If β is a balanced linear map of mass $\kappa = 1$, that is,*

$$\ell(\beta(u))v = ur(\beta(v)) \quad \textit{for all } u, v \in R, \tag{2-12}$$

then the operation $\diamond$ in (2-8) *defines an associative product on R if and only if α satisfies* (2-11).

Proof. The conclusion follows from Lemma 2.10 since in this case, for any $u, v \in R$,

$$u \diamond v = \ell(\delta_+(u))v + ur(\delta_-(v)) + \lambda u \circ v = \ell(\alpha(u))v + ur(\alpha(v)) + \lambda u \circ v. \qquad \square$$

Obviously, if α is an $\mathcal{O}$-operator of weight λ associated to an A-bimodule $\Bbbk$-algebra $(R, \circ, \ell, r)$, then (2-11) holds. Thus the operation on R defined by (2-8) is associative.

Theorem 2.12. *Let $\Bbbk$ have characteristic not equal to 2. Let $(A,\cdot)$ be a $\Bbbk$-algebra and $(R,\circ,\ell,r)$ be an A-bimodule $\Bbbk$-algebra. Let $\delta_\pm : R\to A$ be two linear maps and $\lambda\in\Bbbk$. Let α and β be the symmetrizer and antisymmetrizer of $\delta_\pm$.*

(i) *Suppose that β is a balanced linear map of mass (κ,μ) and α satisfies* (2-7). *Then the product $*_\alpha$ is associative.*

(ii) *Suppose β is a balanced A-bimodule homomorphism of mass $(-1,\pm\lambda)$, that is, β satisfies* (2-5) *with $\kappa=-1$,* (2-6) *with $\mu=\pm\lambda$ and*

$$\beta(\ell(x)u)=x\cdot\beta(u) \quad\text{and}\quad \beta(ur(x))=\beta(u)\cdot x \quad\text{for all } x\in A,\ u\in R. \tag{2-13}$$

Then α is an extended $\mathcal{O}$-operator of weight λ with modification β of mass $(\kappa,\mu)=(-1,\pm\lambda)$ if and only if $\delta_\pm$ is an $\mathcal{O}$-operator of weight 1 associated to a new A-bimodule $\Bbbk$-algebra $(R,\circ_\pm,\ell,r)$:

$$\delta_\pm(u)\cdot\delta_\pm(v)=\delta_\pm(\ell(\delta_\pm(u))v+ur(\delta_\pm(v))+u\circ_\pm v) \quad\text{for all } u,v\in R, \tag{2-14}$$

where the associative products $\circ_\pm=\circ_{\lambda,\beta,\pm}$ on R are defined by

$$u\circ_\pm v=\lambda u\circ v\mp 2\ell(\beta(u))v \quad\text{for all } u,v\in R. \tag{2-15}$$

In item (i) we do not assume that β is an A-bimodule homomorphism. Thus α need not be an extended $\mathcal{O}$-operator.

Proof. (i) The conclusion follows from Lemma 2.10.

(ii) It is straightforward to show that $(R,\circ_\pm,\ell,r)$ equipped with the product $\circ_\pm$ is an A-bimodule $\Bbbk$-algebra. Moreover, for any $u,v\in R$,

$$\begin{aligned}
&(\alpha\pm\beta)(u)\cdot(\alpha\pm\beta)(v)-(\alpha\pm\beta)(\ell((\alpha\pm\beta)(u))v+ur((\alpha\pm\beta)(v))+u\circ_\pm v)\\
&\quad=\alpha(u)\cdot\alpha(v)+\beta(u)\cdot\beta(v)-\alpha(\ell(\alpha(u))v+ur(\alpha(v))+\lambda u\circ v)\mp\lambda\beta(u\circ v)\\
&\qquad\pm(\beta(u)\cdot\alpha(v)-\beta(ur(\alpha(v)))+\alpha(u)\cdot\beta(v)-\beta(\ell(\alpha(u))v)) \quad\text{by (2-12)}\\
&\quad=\alpha(u)\cdot\alpha(v)+\beta(u)\cdot\beta(v)-\alpha(\ell(\alpha(u))v+ur(\alpha(v))+\lambda u\circ v)\mp\lambda\beta(u\circ v)\\
&\qquad\text{by (2-13).}
\end{aligned}$$

Therefore the conclusion holds. □

We close this section with an obvious corollary of Theorem 2.12 by taking $R=V$ with the zero multiplication.

Corollary 2.13. *Let A be a $\Bbbk$-algebra and (V,ℓ,r) be an A-bimodule. Let $\alpha,\beta: V\to A$ be two linear maps such that β is a balanced A-bimodule homomorphism. Then α is an extended $\mathcal{O}$-operator with modification β of mass $\kappa=-1$ if and only if $\alpha\pm\beta$ is an $\mathcal{O}$-operator of weight 1 associated to an A-bimodule $\Bbbk$-algebra $(V,\star_\pm,\ell,r)$, that is, for all $u,v\in V$,*

$$(\alpha\pm\beta)(u)\cdot(\alpha\pm\beta)(v)=(\alpha\pm\beta)(\ell((\alpha\pm\beta)(u))v+ur((\alpha\pm\beta)(v))+u\star_\pm v),$$

where the associative algebra products $\star_\pm$ on V are defined by

$$u \star_\pm v = \mp 2\ell(\beta(u))v \quad \textit{for all } u, v \in V.$$

2d. ***The case of $\mathbb{O}$-operators and Rota–Baxter operators.*** Suppose $(A, \cdot)$ is a $\Bbbk$-algebra. Then $(A, \cdot, L, R)$ is an A-bimodule $\Bbbk$-algebra. Theorem 2.12 can be easily restated in this case. But we are mostly interested in the case of $\mu = 0$ when (2-7) takes the form

$$\alpha(x)\cdot\alpha(y)-\alpha(\alpha(x)\cdot y+x\cdot\alpha(y)+\lambda x\cdot y)=\kappa\beta(x)\cdot\beta(y) \quad \text{for all } x, y\in A. \tag{2-16}$$

We list the following special cases for later reference. If $\lambda = 0$, then (2-16) gives

$$\alpha(x)\cdot\alpha(y)-\alpha(\alpha(x)\cdot y+x\cdot\alpha(y))=\kappa\beta(x)\cdot\beta(y) \quad \text{for all } x, y\in A. \tag{2-17}$$

If in addition, $\beta = \mathrm{id}$, then (2-17) gives

$$\alpha(x)\cdot\alpha(y)-\alpha(\alpha(x)\cdot y+x\cdot\alpha(y))=\kappa x\cdot y \quad \text{for all } x, y\in A. \tag{2-18}$$

If furthermore $\kappa = -1$, then (2-18) becomes

$$\alpha(x)\cdot\alpha(y)-\alpha(\alpha(x)\cdot y+x\cdot\alpha(y))=-x\cdot y \quad \text{for all } x, y\in A. \tag{2-19}$$

By Lemma 2.10 and Theorem 2.12, we reach the following conclusion.

Corollary 2.14. *Let $(A, \cdot)$ be a $\Bbbk$-algebra. Let $\alpha, \beta : A \to A$ be two linear maps and $\lambda \in \Bbbk$.*

(i) *For any $\kappa \in \Bbbk$, let β be balanced of mass $(\kappa, 0)$ and let α be an extended $\mathbb{O}$-operator of weight λ with modification β of mass $(\kappa, \mu) = (\kappa, 0)$, that is, α satisfies* (2-16). *Then the product $*_\alpha$ on A is associative.*

(ii) *If β is an A-bimodule homomorphism, then α and β satisfy* (2-17) *for $\kappa = -1$ if and only if $r_\pm = \alpha \pm \beta$ is an $\mathbb{O}$-operator of weight* 1 *associated to a new A-bimodule $\Bbbk$-algebra $(A, \star_\pm, L, R)$:*

$$r_\pm(x)\cdot r_\pm(y) = r_\pm(r_\pm(x)\cdot y + x\cdot r_\pm(y) + x \star_\pm y) \quad \textit{for all } x, y \in A,$$

where the associative products $\star_\pm$ on A are defined by

$$x \star_\pm y = \mp 2\beta(x)\cdot y \quad \textit{for all } x, y \in A.$$

Let $(A, \cdot)$ be a $\Bbbk$-algebra and let $(A, \cdot, L, R)$ be the corresponding A-bimodule $\Bbbk$-algebra. In this case, $\beta = \mathrm{id}$ clearly satisfies the conditions of Theorem 2.12 and (2-7) takes the form

$$\alpha(x)\cdot\alpha(y)-\alpha(\alpha(x)\cdot y+x\cdot\alpha(y)+\lambda x\cdot y)=\hat\kappa x\cdot y \quad \text{for all } x, y\in A, \tag{2-20}$$

where $\hat\kappa = \kappa + \mu$. Thus we have the following consequence of Theorem 2.12.

Corollary 2.15. *Let* $\hat{\kappa} = -1 \pm \lambda$. *Then* $\alpha : A \to A$ *satisfies* (2-20) *if and only if* $\alpha \pm 1$ *is a Rota–Baxter operator of weight* $\lambda \mp 2$.

When $\lambda = 0$, this fact can be found in [Ebrahimi-Fard 2002]. As noted there, the Lie algebraic version of (2-20) in this case, namely (2-19), is the operator form of the modified classical Yang–Baxter equation [Semenov-Tyan-Shanskiĭ 1983].

3. Extended $\mathbb{O}$-operators and EAYBE

Here we study the relationship between extended $\mathbb{O}$-operators and associative Yang–Baxter equations. We start with introducing various concepts of the associative Yang–Baxter equation (AYBE) in Section 3a. We then establish connections between $\mathbb{O}$-operators in different generalities and solutions of these variations of AYBE in different algebras. The relationship between $\mathbb{O}$-operators on a $\mathbb{k}$-algebra A and solutions of AYBE in A is considered in Section 3b. We then consider in Section 3c the relationship between an extended $\mathbb{O}$-operator and solutions of AYBE and extended AYBE in an extension algebra of A. We finally consider the special case of Frobenius algebras in Section 3d.

3a. ***Extended associative Yang–Baxter equations.*** We define variations of the associative Yang–Baxter equation to be satisfied by two tensors from an algebra. We then study the linear maps from these two tensors in preparation for the relationship between $\mathbb{O}$-operators and solutions of these associative Yang–Baxter equations.

Let A be a $\mathbb{k}$-algebra. Let $r = \sum_i a_i \otimes b_i \in A \otimes A$. We continue to use the notations r_{12}, r_{13} and r_{23} defined in (1-4). We similarly define

$$r_{21} = \sum_i b_i \otimes a_i \otimes 1, \quad r_{31} = \sum_i b_i \otimes 1 \otimes a_i, \quad r_{32} = \sum_i 1 \otimes b_i \otimes a_i.$$

Equip $A \otimes A \otimes A$ with the product of the tensor algebra. In particular,

$$(a_1 \otimes a_2 \otimes a_3)(b_1 \otimes b_2 \otimes b_3) = (a_1 b_1) \otimes (a_2 b_2) \otimes (a_3 b_3) \quad \text{for all } a_i, b_i \in A,\ i = 1, 2, 3.$$

Definition 3.1. Fix $\varepsilon \in \mathbb{k}$.

(i) The equation

$$r_{12} r_{13} + r_{13} r_{23} - r_{23} r_{12} = \varepsilon (r_{13} + r_{31})(r_{23} + r_{32}) \tag{3-1}$$

is called the *extended associative Yang–Baxter equation of mass* ε (or ε-*EAYBE* in short).

(ii) Let A be a $\mathbb{k}$-algebra. An element $r \in A \otimes A$ is called a *solution of the* ε*-EAYBE in* A if it satisfies (3-1).

When $\varepsilon = 0$ or r is skew-symmetric in the sense that $\sigma(r) = -r$ for the switch operator $\sigma : A \otimes A \to A \otimes A$ (and hence $r_{13} = -r_{31}$), then the ε-EAYBE is the same as the AYBE in (1-3):

$$r_{12}r_{13} + r_{13}r_{23} - r_{23}r_{12} = 0. \tag{3-2}$$

Let A be a $\Bbbk$-algebra with finite $\Bbbk$-dimension. For $r \in A \otimes A$, define a linear map $F_r : A^* \to A$ by

$$\langle v, F_r(u)\rangle = \langle u \otimes v, r\rangle \quad \text{for all } u, v \in A^*. \tag{3-3}$$

This defines a bijective linear map $F : A \otimes A \to \mathrm{Hom}_{\Bbbk}(A^*, A)$ and thus allows us to identify r with F_r, which we still denote by r for simplicity of notation. Similarly define a linear map $r^t : A^* \to A$ by

$$\langle u, r^t(v)\rangle = \langle r, u \otimes v\rangle. \tag{3-4}$$

Obviously r is symmetric or skew-symmetric in $A \otimes A$ if and only if, as a linear map, $r = r^t$ or $r = -r^t$, respectively. Suppose that the characteristic of $\Bbbk$ is not 2 and define

$$\alpha = \alpha_r = (r - r^t)/2 \quad \text{and} \quad \beta = \beta_r = (r + r^t)/2, \tag{3-5}$$

called the *skew-symmetric part* and the *symmetric part* of r, respectively. Then $r = \alpha + \beta$ and $r^t = -\alpha + \beta$.

Lemma 3.2. *Let $(A, \cdot)$ be a $\Bbbk$-algebra with finite $\Bbbk$-dimension. Let $s \in A \otimes A$ be symmetric. Then the following conditions are equivalent.*

(i) *s is* invariant, *that is,*

$$(\mathrm{id} \otimes L(x) - R(x) \otimes \mathrm{id})s = 0 \quad \textit{for all } x \in A. \tag{3-6}$$

(ii) *s regarded as a linear map from (A^*, R^*, L^*) to A is balanced, that is,*

$$R^*(s(a^*))b^* = a^*L^*(s(b^*)) \quad \textit{for all } a^*, b^* \in A^*. \tag{3-7}$$

(iii) *s regarded as a linear map from (A^*, R^*, L^*) to A is an A-bimodule homomorphism, that is,*

$$s(R^*(x)a^*) = x \cdot s(a^*), \quad s(a^*L^*(x)) = s(a^*) \cdot x \quad \textit{for all } x \in A, a^* \in A^*. \tag{3-8}$$

Proof. (i) $\Longleftrightarrow$ (ii). Since $s \in A \otimes A$ is symmetric, for any $x \in A$, $a^*, b^* \in A^*$,

$$\begin{aligned}\langle(\mathrm{id} \otimes L(x) - R(x) \otimes \mathrm{id})s, a^* \otimes b^*\rangle &= \langle s, a^* \otimes L^*(x)b^*\rangle - \langle s, R^*(x)a^* \otimes b^*\rangle \\ &= \langle x \cdot s(a^*), b^*\rangle - \langle a^*, s(b^*) \cdot x\rangle \\ &= \langle R^*(s(a^*))b^* - a^*L^*(s(b^*)), x\rangle.\end{aligned}$$

So s is invariant if and only if s regarded as a linear map from (A^*, R^*, L^*) to A is balanced.

(i) $\Longleftrightarrow$ (iii). For any $x \in A$, $a^*, b^* \in A^*$,

$$\begin{aligned}\langle(\mathrm{id}\otimes L(x) - R(x)\otimes \mathrm{id})s, a^*\otimes b^*\rangle &= \langle s, a^*\otimes L^*(x)b^*\rangle - \langle s, R^*(x)a^*\otimes b^*\rangle\\ &= \langle x\cdot s(a^*) - s(R^*(x)a^*), b^*\rangle,\\ \langle(\mathrm{id}\otimes L(x) - R(x)\otimes \mathrm{id})s, a^*\otimes b^*\rangle &= \langle s, a^*\otimes L^*(x)b^*\rangle - \langle s, R^*(x)a^*\otimes b^*\rangle\\ &= \langle s(L^*(x)b^*) - s(b^*)\cdot x, a^*\rangle\end{aligned}$$

by the symmetry of $s \in A\otimes A$. So s is invariant if and only if s regarded as a linear map from (A^*, R^*, L^*) to A is an A-bimodule homomorphism. □

Remark 3.3. The invariant condition in item (i) also arises in the construction of a coboundary antisymmetric infinitesimal bialgebra in the sense of [Bai 2010]; see also [Bai et al. 2012].

3b. ***Extended $\mathcal{O}$-operators from EAYBE.*** We first state the following special case of Corollary 2.13.

Corollary 3.4. *Let $\Bbbk$ be a field of characteristic not equal to 2. Let A be a $\Bbbk$-algebra with finite $\Bbbk$-dimension and $r \in A\otimes A$. Let α and β be defined by* (3-5). *Suppose β is a balanced A-bimodule homomorphism. These two statements are equivalent*:

(i) *The map α is an extended $\mathcal{O}$-operator with modification β of mass -1:*

$$\alpha(a^*)\cdot\alpha(b^*) - \alpha(R^*(\alpha(a^*))b^* + a^*L^*(\alpha(b^*))) = -\beta(a^*)\cdot\beta(b^*) \tag{3-9}$$

for all $a^*, b^* \in A^*$.

(ii) *The map r (respectively $-r^t$) is an $\mathcal{O}$-operator of weight* 1 *associated to a new A-bimodule $\Bbbk$-algebra $(A^*, \circ_+, R^*, L^*)$ (respectively $(A^*, \circ_-, R^*, L^*)$):*

$$r(a^*)\cdot r(b^*) = r(R^*(r(a^*))b^* + a^*L^*(r(b^*)) + a^*\circ_+ b^*) \tag{3-10}$$

for all $a^*, b^* \in A^*$, (respectively

$$\begin{aligned}&(-r^t)(a^*)\cdot(-r^t)(b^*)\\ &\qquad = (-r^t)(R^*((-r^t)(a^*))b^* + a^*L^*((-r^t)(b^*)) + a^*\circ_- b^*),\end{aligned} \tag{3-11}$$

for all $a^*, b^* \in A^*$), where the associative algebra products $\circ_\pm$ on A^* are defined by

$$a^*\circ_\pm b^* = \mp 2R^*(\beta(a^*))b^* \quad \textit{for all } a^*, b^* \in A^*. \tag{3-12}$$

In the theory of integrable systems [Kosmann-Schwarzbach 1997; Semenov-Tyan-Shanskiĭ 1983], *modified classical Yang–Baxter equation* usually refers to (the Lie algebraic version of) (2-19) and (3-9).

The following theorem establishes a close relationship between extended $\mathbb{O}$-operators on a $\Bbbk$-algebra A and solutions of the AYBE in A.

Theorem 3.5. *Let $\Bbbk$ be a field of characteristic not equal to 2. Let A be a $\Bbbk$-algebra with finite $\Bbbk$-dimension and let $r \in A \otimes A$, which is identified as a linear map from A^* to A.*

(i) *Then r is a solution of the AYBE in A if and only if r satisfies*

$$r(a^*)\cdot r(b^*) = r(R^*(r(a^*))b^* - a^*L^*(r^t(b^*))) \quad \textit{for all } a^*, b^* \in A^*. \tag{3-13}$$

(ii) *Define α and β by* (3-5). *Suppose that the symmetric part β of r is invariant. Then r is a solution of EAYBE of mass $(\kappa+1)/4$:*

$$r_{12}r_{13} + r_{13}r_{23} - r_{23}r_{12} = \tfrac{1}{4}(\kappa+1)(r_{13}+r_{31})(r_{23}+r_{32})$$

if and only if α is an extended $\mathbb{O}$-operator with modification β of mass κ:

$$\alpha(a^*)\cdot\alpha(b^*) - \alpha(R^*(\alpha(a^*))b^* + a^*L^*(\alpha(b^*))) = \kappa\beta(a^*)\cdot\beta(b^*)$$

for all $a^*, b^* \in A^*$.

Proof. (i) Write $r = \sum_{i,j} u_i \otimes v_j$. For any $a^*, b^*, c^* \in A^*$, we have

$$\begin{aligned}
\langle r_{12}\cdot r_{13}, a^*\otimes b^*\otimes c^*\rangle &= \sum_{i,j}\langle u_i\cdot u_j, a^*\rangle\langle v_i, b^*\rangle\langle v_j, c^*\rangle \\
&= \sum_j \langle r^t(b^*)\cdot u_j, a^*\rangle\langle v_j, b^*\rangle = \langle r(a^*L^*(r^t(b^*))), c^*\rangle, \\
\langle r_{13}\cdot r_{23}, a^*\otimes b^*\otimes c^*\rangle &= \sum_{i,j}\langle u_i, a^*\rangle\langle u_j, b^*\rangle\langle v_i\cdot v_j, c^*\rangle \\
&= \sum_j \langle u_j, b^*\rangle\langle r(a^*)\cdot v_j, c^*\rangle = \langle r(a^*)\cdot r(b^*), c^*\rangle, \\
\langle -r_{23}\cdot r_{12}, a^*\otimes b^*\otimes c^*\rangle &= -\sum_{i,j}\langle u_i, a^*\rangle\langle u_j\cdot v_i, b^*\rangle\langle v_j, c^*\rangle \\
&= -\sum_j \langle u_j\cdot r(a^*), b^*\rangle\langle v_j, c^*\rangle = \langle -r(R^*(r(a^*))b^*), c^*\rangle.
\end{aligned}$$

Therefore r is a solution of the AYBE in A if and only if r satisfies (3-13).

(ii) By the proof of item (i), we see that, for any $a^*, b^*, c^* \in A^*$,

$$\begin{aligned}
&\langle \alpha(a^*)\cdot\alpha(b^*) - \alpha(R^*(\alpha(a^*))b^* + a^*L^*(\alpha(b^*))) - \kappa\beta(a^*)\cdot\beta(b^*), c^*\rangle \\
&\quad = \langle \alpha(a^*)\cdot\alpha(b^*) - \alpha(R^*(\alpha(a^*))b^* + a^*L^*(\alpha(b^*))) \\
&\qquad\qquad + \beta(a^*)\cdot\beta(b^*) - (\kappa+1)\beta(a^*)\cdot\beta(b^*), c^*\rangle \\
&\quad = \langle r_{12}\cdot r_{13} + r_{13}\cdot r_{23} - r_{23}\cdot r_{12}, a^*\otimes b^*\otimes c^*\rangle - (\kappa+1)\langle \beta_{13}\cdot\beta_{23}, a^*\otimes b^*\otimes c^*\rangle \\
&\quad = \langle r_{12}\cdot r_{13} + r_{13}\cdot r_{23} - r_{23}\cdot r_{12} - (\kappa+1)\tfrac{1}{2}(r_{13}+r_{31})\cdot\tfrac{1}{2}(r_{23}+r_{32}), a^*\otimes b^*\otimes c^*\rangle.
\end{aligned}$$

So r is a solution of the EAYBE of mass $(\kappa+1)/4$ if and only if α is an extended $\mathcal{O}$-operator with modification β of mass κ. □

In the case when $\kappa = -1$, we have this:

Corollary 3.6. *Let $\Bbbk$ be a field of characteristic not equal to* 2. *Let A be a $\Bbbk$-algebra with finite $\Bbbk$-dimension and let $r \in A\otimes A$. Define α and β by* (3-5).

(i) *If β is invariant, then the following conditions are equivalent.*
   - (a) *r is a solution of the AYBE in A.*
   - (b) *r satisfies* (3-10), *that is, r is an $\mathcal{O}$-operator of weight* 1 *associated to the A-bimodule $\Bbbk$-algebra $(A^*, \circ_+, R^*, L^*)$, where A^* is equipped with the associative algebra structure $\circ_+$ defined by* (3-12). (*With $-r^t$ instead of r, replace* (3-10) *by* (3-11) *and $\circ_+$ with $\circ_-$.*)
   - (c) *α is an extended $\mathcal{O}$-operator with modification β of mass -1.*
   - (d) *For any $a^*, b^* \in A^*$,*

$$(\alpha\pm\beta)(a^* * b^*) = (\alpha\pm\beta)(a^*)\cdot(\alpha\pm\beta)(b^*), \tag{3-14}$$

where

$$a^* * b^* = R^*(r(a^*))b^* - a^*L^*(r^t(b^*)) \quad \textit{for all } a^*, b^* \in A^*.$$

(ii) *If r is skew-symmetric, then r is a solution of the AYBE in A if and only if $r : A^* \to A$ is an $\mathcal{O}$-operator of weight zero.*

Proof. If the symmetric part β of r is invariant, then by Lemma 3.2, for any $a^*, b^* \in A^*$, we have

$$\begin{aligned}
&r(a^*)\cdot r(b^*) - r(R^*(r(a^*))b^* - a^*L^*(r^t(b^*))) \\
&\qquad = r(a^*)\cdot r(b^*) - r(R^*(r(a^*))b^* + a^*L^*(r(b^*)) - 2a^*L^*(\beta(b^*))) \\
&\qquad = r(a^*)\cdot r(b^*) - r(R^*(r(a^*))b^* + a^*L^*(r(b^*)) + a^*\circ_+ b^*),
\end{aligned}$$

where the product $\circ_+$ is defined by (3-12). Therefore by Corollary 3.4, r is a solution of the AYBE if and only if item (b) or (c) holds. Moreover, since for any

$a^*, b^* \in A^*$, we have

$$R^*(r(a^*))b^* - a^*L^*(r^t(b^*)) = R^*(r(a^*))b^* + a^*L^*(r(b^*)) + a^* \circ_+ b^*$$
$$= R^*((-r^t)(a^*))b^* + a^*L^*((-r^t)(b^*)) + a^* \circ_- b^*,$$

(3-14) is just a reformulation of (3-10) and (3-11). So r is a solution of the AYBE if and only if item (c) holds.

(ii) This is the special case of item (i) when $\beta = 0$. □

3c. *EAYBEs from extended $\mathbb{O}$-operators.* We now establish the relationship between an extended $\mathbb{O}$-operator $\alpha : V \to A$ in general and the AYBE and EAYBE. For this purpose we prove that an extended $\mathbb{O}$-operator $\alpha : V \to A$ naturally gives rise to an extended $\mathbb{O}$-operator on a larger associative algebra $\mathscr{A}$ associated to the dual bimodule $(\mathscr{A}^*, R^*_{\mathscr{A}}, L^*_{\mathscr{A}})$. We first introduce some notation.

Definition 3.7. Let A be a $\Bbbk$-algebra and let (V, ℓ, r) be an A-bimodule, both with finite $\Bbbk$-dimension. Let (V^*, r^*, ℓ^*) be the dual A-bimodule and let $\mathscr{A} = A \ltimes_{r^*,\ell^*} V^*$. Identify a linear map $\gamma : V \to A$ as an element in $\mathscr{A} \otimes \mathscr{A}$ through the injective map

$$\text{(3-15)} \qquad \mathrm{Hom}_{\Bbbk}(V, A) \cong A \otimes V^* \hookrightarrow \mathscr{A} \otimes \mathscr{A}.$$

Denote

$$\text{(3-16)} \qquad \tilde{\gamma}_\pm := \gamma \pm \gamma^{21},$$

where $\gamma^{21} = \sigma(\gamma) \in V^* \otimes A \subset \mathscr{A} \otimes \mathscr{A}$ with $\sigma : A \otimes V^* \to V^* \otimes A$, $a \otimes u^* \mapsto u^* \otimes a$ being the switch operator.

Lemma 3.8. *Let A be a $\Bbbk$-algebra and let (V, ℓ, r) be an A-bimodule, both with finite $\Bbbk$-dimension. Suppose that $\beta : V \to A$ is a linear map that is identified as an element in $\mathscr{A} \otimes \mathscr{A}$ by* (3-15)*. Define $\tilde{\beta}_+$ by* (3-16)*. Then $\tilde{\beta}_+$, identified as a linear map from $\mathscr{A}^*$ to $\mathscr{A}$, is a balanced $\mathscr{A}$-bimodule homomorphism from $(\mathscr{A}^*, R^*_{\mathscr{A}}, L^*_{\mathscr{A}})$ to $(\mathscr{A}, L_{\mathscr{A}}, R_{\mathscr{A}})$ if and only if $\beta : V \to A$ is a balanced A-bimodule homomorphism from (V, ℓ, r) to (A, L_A, R_A).*

Proof. For the linear map $\tilde{\beta}_+ : \mathscr{A}^* \to \mathscr{A}$, we have $\tilde{\beta}_+(a^*) = \beta^*(a^*)$ for $a^* \in A^*$ and $\tilde{\beta}_+(u) = \beta(u)$ for $u \in V$, where $\beta^* : A^* \to V^*$ is the dual linear map associated to β given by

$$\langle \beta^*(a^*), v \rangle = \langle a^*, \beta(v) \rangle \quad \text{for all } a^* \in A^*, v \in V.$$

First suppose that $\beta : (V, \ell, r) \to A$ is a balanced A-bimodule homomorphism. Let $b^* \in A^*$ and $v \in V$. Then

$$R^*_{\mathscr{A}}(\tilde{\beta}_+(a^* + u))(b^* + v)$$
$$= R^*_{\mathscr{A}}(\beta^*(a^*))b^* + R^*_{\mathscr{A}}(\beta^*(a^*))v + R^*_{\mathscr{A}}(\beta(u))b^* + R^*_{\mathscr{A}}(\beta(u))v,$$

and

$$(a^*+u)L^*_{\mathscr{A}}(\tilde{\beta}_+(b^*+v))$$
$$= a^*L^*_{\mathscr{A}}(\beta^*(b^*)) + a^*L^*_{\mathscr{A}}(\beta(v)) + uL^*_{\mathscr{A}}(\beta^*(b^*)) + uL^*_{\mathscr{A}}(\beta(v)).$$

On the other hand, for any $x \in A$, $w^* \in V^*$,

$$\begin{aligned}
\langle R^*_{\mathscr{A}}(\beta^*(a^*))b^* - a^*L^*_{\mathscr{A}}(\beta^*(b^*)), x\rangle &= \langle b^*, x\cdot\beta^*(a^*)\rangle - \langle a^*, \beta^*(b^*)\cdot x\rangle = 0,\\
\langle R^*_{\mathscr{A}}(\beta^*(a^*))b^* - a^*L^*_{\mathscr{A}}(\beta^*(b^*)), w^*\rangle &= \langle b^*, w^*\cdot\beta^*(a^*)\rangle - \langle a^*, \beta^*(b^*)\cdot w^*\rangle\\
&= 0,\\
\langle R^*_{\mathscr{A}}(\beta^*(a^*))v - a^*L^*_{\mathscr{A}}(\beta(v)), x\rangle &= \langle v, x\cdot\beta^*(a^*)\rangle - \langle a^*, \beta(v)\cdot x\rangle\\
&= \langle a^*, \beta(vr(x)) - \beta(v)\cdot x\rangle = 0,\\
\langle R^*_{\mathscr{A}}(\beta^*(a^*))v - a^*L^*_{\mathscr{A}}(\beta(v)), w^*\rangle &= \langle v, w^*\cdot\beta^*(a^*)\rangle - \langle a^*, \beta(v)\cdot w^*\rangle = 0,\\
\langle R^*_{\mathscr{A}}(\beta(u))b^* - uL^*_{\mathscr{A}}(\beta^*(b^*)), x\rangle &= \langle b^*, x\cdot\beta(u)\rangle - \langle u, \beta^*(b^*)\cdot x\rangle\\
&= \langle b^*, x\cdot\beta(u) - \beta(l(x)u)\rangle = 0,\\
\langle R^*_{\mathscr{A}}(\beta(u))b^* - uL^*_{\mathscr{A}}(\beta^*(b^*)), w^*\rangle &= \langle b^*, w^*\cdot\beta(u)\rangle - \langle u, \beta^*(b^*)\cdot w^*\rangle = 0,\\
\langle R^*_{\mathscr{A}}(\beta(u))v - uL^*_{\mathscr{A}}(\beta(v)), w^*\rangle &= \langle v, w^*\cdot\beta(u)\rangle - \langle u, \beta(v)\cdot w^*\rangle\\
&= \langle \ell(\beta(u))v - ur(\beta(v)), w^*\rangle = 0,\\
\langle R^*_{\mathscr{A}}(\beta(u))v - uL^*_{\mathscr{A}}(\beta(v)), x\rangle &= \langle v, x\cdot\beta(u)\rangle - \langle u, \beta(v)\cdot x\rangle = 0.
\end{aligned}$$

Therefore, $R^*_{\mathscr{A}}(\tilde{\beta}_+(a^*+u))(b^*+v) = (a^*+u)L^*_{\mathscr{A}}(\tilde{\beta}_+(b^*+v))$. Since $\tilde{\beta}_+ \in \mathscr{A}\otimes\mathscr{A}$ is symmetric, by Lemma 3.2, $\tilde{\beta}_+$ when identified as a linear map from $\mathscr{A}^*$ to $\mathscr{A}$ is a balanced $\mathscr{A}$-bimodule homomorphism from $(\mathscr{A}^*, R^*_{\mathscr{A}}, L^*_{\mathscr{A}})$ to $(\mathscr{A}, L_{\mathscr{A}}, R_{\mathscr{A}})$.

Conversely, if $\tilde{\beta}_+$ identified as a linear map from $\mathscr{A}^*$ to $\mathscr{A}$ is a balanced $\mathscr{A}$-bimodule homomorphism from $(\mathscr{A}^*, R^*_{\mathscr{A}}, L^*_{\mathscr{A}})$ to $(\mathscr{A}, L_{\mathscr{A}}, R_{\mathscr{A}})$, then

$$\begin{aligned}
R^*_{\mathscr{A}}(\tilde{\beta}_+(u))v = uL^*_{\mathscr{A}}(\tilde{\beta}_+(v)) \quad&\Longleftrightarrow\quad \ell(\beta(u))v = ur(\beta(v)),\\
\tilde{\beta}_+(R^*_{\mathscr{A}}(x)v) = x\cdot\tilde{\beta}_+(v) \quad&\Longleftrightarrow\quad \beta(\ell(x)v) = x\cdot\beta(v),\\
\tilde{\beta}_+(uL^*_{\mathscr{A}}(x)) = \tilde{\beta}_+(u)\cdot x \quad&\Longleftrightarrow\quad \beta(ur(x)) = \beta(u)\cdot x
\end{aligned}$$

for any $u, v \in V$, $x \in A$. So $\beta : (V, \ell, r) \to (A, L_A, R_A)$ is a balanced A-bimodule homomorphism. □

Theorem 3.9. *Let A be a $\Bbbk$-algebra and let (V, ℓ, r) be an A-bimodule, both with finite $\Bbbk$-dimension. Let $\alpha, \beta : V \to A$ be two $\Bbbk$-linear maps. Let $\tilde{\alpha}_-$ and $\tilde{\beta}_+$ be defined by* (3-15) *and identified as linear maps from $\mathscr{A}^*$ to $\mathscr{A}$. Then α is an extended 𝒪-operator with modification β of mass κ if and only if $\tilde{\alpha}_-$ is an extended 𝒪-operator with modification $\tilde{\beta}_+$ of mass κ.*

Proof. For any $a^*\in A^*$ and $v\in V$, we have $\tilde{\alpha}_-(a^*)=\alpha^*(a^*)$ and $\tilde{\alpha}_-(v)=-\alpha(v)$, where $\alpha^*:A^*\to V^*$ is the dual linear map of α. Suppose that α is an extended $\mathbb{O}$-operator with modification β of mass κ. Then for any $a^*, b^*\in A^*$ and $u, v\in V$, we have

$$\begin{aligned}&\tilde{\alpha}_-(u+a^*)\cdot\tilde{\alpha}_-(v+b^*)-\tilde{\alpha}_-(R^*_{\mathscr{A}}(\tilde{\alpha}_-(u+a^*))(v+b^*)+(u+a^*)L^*_{\mathscr{A}}(\tilde{\alpha}_-(v+b^*)))\\&=\alpha(u)\cdot\alpha(v)-\alpha(\ell(\alpha(u))v)-\alpha(ur(\alpha(v)))-r^*(\alpha(u))\alpha^*(b^*)+\alpha^*(R^*_{\mathscr{A}}(\alpha(u))b^*)\\&\quad-\alpha^*(uL^*_{\mathscr{A}}(\alpha^*(b^*)))-\alpha^*(a^*)\ell^*(\alpha(v))-\alpha^*(R^*_{\mathscr{A}}(\alpha^*(a^*))v)+\alpha^*(a^*L^*_{\mathscr{A}}(\alpha(v))).\end{aligned}$$

On the other hand, for any $w\in V$ we have

$$\begin{aligned}&\langle -r^*(\alpha(u))\alpha^*(b^*)+\alpha^*(R^*_{\mathscr{A}}(\alpha(u))b^*)-\alpha^*(uL^*_{\mathscr{A}}(\alpha^*(b^*))), w\rangle\\&\qquad=\langle b^*, \alpha(w)\cdot\alpha(u)-\alpha(\ell(\alpha(w))u+wr(\alpha(u))\rangle=\langle b^*, \kappa\beta(w)\cdot\beta(u)\rangle\\&\qquad=\langle b^*, \kappa\beta(wr(\beta(u)))\rangle=\langle\kappa r^*(\beta(u))\beta^*(b^*), w\rangle.\end{aligned}$$

Therefore

$$-r^*(\alpha(u))\alpha^*(b^*)+\alpha^*(R^*_{\mathscr{A}}(\alpha(u))b^*)-\alpha^*(uL^*_{\mathscr{A}}(\alpha^*(b^*)))=\kappa r^*(\beta(u))\beta^*(b^*).$$

Similarly, we have

$$-\alpha^*(a^*)\ell^*(\alpha(v))-\alpha^*(R^*_{\mathscr{A}}(\alpha^*(a^*))v)+\alpha^*(a^*L^*_{\mathscr{A}}(\alpha(v)))=\kappa\beta^*(a^*)\ell^*(\beta(v)).$$

So

$$\begin{aligned}&\tilde{\alpha}_-(u+a^*)\cdot\tilde{\alpha}_-(v+b^*)-\tilde{\alpha}_-(R^*_{\mathscr{A}}(\tilde{\alpha}_-(u+a^*))(v+b^*)\\&\qquad\qquad\qquad\qquad\qquad\qquad+(u+a^*)L^*_{\mathscr{A}}(\tilde{\alpha}_-(v+b^*)))\\&\quad=\kappa\beta(u)\cdot\beta(v)+\kappa r^*(\beta(u))\beta^*(b^*)+\kappa\beta^*(a^*)\ell^*(\beta(v))\\&\quad=\kappa\beta(u)\cdot\beta(v)+\kappa\beta(u)\cdot\beta^*(b^*)+\kappa\beta^*(a^*)\cdot\beta(v)=\kappa\tilde{\beta}_+(u+a^*)\tilde{\beta}_+(v+b^*).\end{aligned}$$

If $\kappa=0$, then this equation implies that $\tilde{\alpha}_-$ is an $\mathbb{O}$-operator of weight zero. If $\kappa\neq 0$, then β is a balanced A-bimodule homomorphism, which, according to Lemma 3.8, implies that $\tilde{\beta}_+$ from $(\mathscr{A}^*, R^*_{\mathscr{A}}, L^*_{\mathscr{A}})$ to $\mathscr{A}$ is a balanced $\mathscr{A}$-bimodule homomorphism. So $\tilde{\alpha}_-$ is an extended $\mathbb{O}$-operator with modification $\tilde{\beta}_+$ of mass κ.

Conversely, suppose $\tilde{\alpha}_-$ is an extended $\mathbb{O}$-operator with modification $\tilde{\beta}_+$ of mass κ. If $\kappa\neq 0$, then $\tilde{\beta}_+$ from $(\mathscr{A}^*, R^*_{\mathscr{A}}, L^*_{\mathscr{A}})$ to $\mathscr{A}$ is a balanced $\mathscr{A}$-bimodule homomorphism, which by Lemma 3.8 implies that β from (V, ℓ, r) to A is a balanced A-bimodule homomorphism. Moreover, for any $u, v\in V$ we have

$$\tilde{\alpha}_-(u)\cdot\tilde{\alpha}_-(v)-\tilde{\alpha}_-(R^*_{\mathscr{A}}(\tilde{\alpha}_-(u))v+uL^*_{\mathscr{A}}(\tilde{\alpha}_-(v)))=\kappa\tilde{\beta}_+(u)\tilde{\beta}_+(v),\tag{3-17}$$

which implies that α is an extended $\mathbb{O}$-operator with modification β of mass κ. If $\kappa=0$, then (3-17) for $\kappa=0$ implies that α is an $\mathbb{O}$-operator of weight zero. □

By Theorem 3.9, the results from the previous sections on $\mathcal{O}$-operators on A can be applied to general $\mathcal{O}$-operators.

Corollary 3.10. *Let A be a $\Bbbk$-algebra and let V be an A-bimodule, both with finite $\Bbbk$-dimension.*

(i) *Suppose the characteristic of the field $\Bbbk$ is not 2. Let $\alpha, \beta : V \to A$ be linear maps that are identified as elements in $(A \ltimes_{r^*,\ell^*} V^*) \otimes (A \ltimes_{r^*,\ell^*} V^*)$. Then α is an extended $\mathcal{O}$-operator with modification β of mass k if and only if $(\alpha - \alpha^{21}) \pm (\beta + \beta^{21})$ is a solution of the EAYBE of mass $(\kappa + 1)/4$ in $A \ltimes_{r^*,\ell^*} V^*$.*

(ii) *Let $\alpha : V \to A$ be a linear map identified as an element in $(A \ltimes_{r^*,\ell^*} V^*) \otimes (A \ltimes_{r^*,\ell^*} V^*)$. Then α is an $\mathcal{O}$-operator of weight zero if and only if $\alpha - \alpha^{21}$ is a skew-symmetric solution of the AYBE in* (3-2) *in $A \ltimes_{r^*,\ell^*} V^*$. In particular, a linear map $P : A \to A$ is a Rota–Baxter operator of weight zero if and only if $r = P - P^{21}$ is a skew-symmetric solution of the AYBE in $A \ltimes_{R^*,L^*} A^*$.*

(iii) *Let $\alpha, \beta : V \to A$ be two linear maps identified as elements in $(A \ltimes_{r^*,\ell^*} V^*) \otimes (A \ltimes_{r^*,\ell^*} V^*)$. Then α is an extended $\mathcal{O}$-operator with modification β of mass -1 if and only if $(\alpha - \alpha^{21}) \pm (\beta + \beta^{21})$ is a solution of the AYBE in $A \ltimes_{r^*,\ell^*} V^*$.*

(iv) *Let $\alpha : A \to A$ be a linear map identified as an element in $(A \ltimes_{R^*,L^*} A^*) \otimes (A \ltimes_{R^*,L^*} A^*)$. Then α satisfies* (2-19) *if and only if $(\alpha - \alpha^{21}) \pm (\mathrm{id} + \mathrm{id}^{21})$ is a solution of the AYBE in $A \ltimes_{R^*,L^*} A^*$.*

(v) *Let $P : A \to A$ be a linear map identified as an element of $A \ltimes_{R^*,L^*} A^*$. Then P is a Rota–Baxter operator of weight $\lambda \neq 0$ if and only if $2/\lambda(P - P^{21}) + 2\,\mathrm{id}$ and $(2/\lambda)(P - P^{21}) - 2\,\mathrm{id}^{21}$ are both solutions of the AYBE in $A \ltimes_{R^*,L^*} A^*$.*

Proof. (i) This follows from Theorem 3.9 and Theorem 3.5.

(ii) This follows from Theorem 3.9 for $\kappa = 0$ (or $\beta = 0$) and Corollary 3.6.

(iii) This follows from Theorem 3.9 for $\kappa = -1$ and Corollary 3.6.

(iv) This follows from item (iii) in the case that $(V, r, \ell) = (A, L, R)$ and $\beta = \mathrm{id}$.

(v) By [Ebrahimi-Fard 2002] (see also the discussion after Corollary 2.15), P is a Rota–Baxter operator of weight $\lambda \neq 0$ if and only if $2P/\lambda + \mathrm{id}$ is an extended $\mathcal{O}$-operator with modification id of mass -1 from (A, L, R) to A, that is, $2P/\lambda + \mathrm{id}$ satisfies (2-19). Then the conclusion follows from item (iv). □

3d. *$\mathcal{O}$-operators and AYBE on Frobenius algebras.* Here we consider the relationship between $\mathcal{O}$-operators and solutions of the AYBE on Frobenius algebras.

Definition 3.11. (i) Let A be a $\Bbbk$-algebra and let $B(\cdot,\cdot) : A \otimes A \to \Bbbk$ be a nondegenerate bilinear form. Let $\varphi : A \to A^*$ denote the induced injective linear map defined by

$$B(x, y) = \langle \varphi(x), y \rangle \quad \text{for all } x, y \in A. \tag{3-18}$$

(ii) A *Frobenius* $\Bbbk$-*algebra* is a $\Bbbk$-algebra $(A,\cdot)$ together with a nondegenerate bilinear form $B(\cdot\,,\cdot): A\otimes A\to \Bbbk$ that is invariant in the sense that

$$B(x\cdot y, z) = B(x, y\cdot z) \quad \text{for all } x, y, z\in A.$$

We use $(A,\cdot\,, B)$ to denote a Frobenius $\Bbbk$-algebra.

(iii) A Frobenius $\Bbbk$-algebra is called *symmetric* if

$$B(x, y) = B(y, x) \quad \text{for all } x, y\in A.$$

(iv) A linear map $\beta : A\to A$ is called *self-adjoint* with respect to a bilinear form B if for any $x, y\in A$, we have $B(\beta(x), y) = B(x, \beta(y))$, and *skew-adjoint* if $B(\beta(x), y) = -B(x, \beta(y))$.

A symmetric Frobenius $\Bbbk$-algebra is also simply called a *symmetric* $\Bbbk$-*algebra* [Brauer and Nesbitt 1937]. We will not use this term to avoid confusion with the symmetrization of the tensor algebra. Frobenius algebras have found applications in broad areas of mathematics and physics. See [Bai 2010; Yamagata 1996] for further details.

It is easy to get the following result.

Proposition 3.12 [Yamagata 1996]. *Let A be a symmetric Frobenius $\Bbbk$-algebra with finite $\Bbbk$-dimension. Then the A-bimodule (A, L, R) is isomorphic to the A-bimodule (A^*, R^*, L^*).*

The following statement gives a class of symmetric Frobenius algebras from symmetric, invariant tensors.

Corollary 3.13. *Let $(A,\cdot)$ be a $\Bbbk$-algebra with finite $\Bbbk$-dimension. Let $s\in A\otimes A$ be symmetric and invariant. Suppose that s regarded as a linear map from $A^*\to A$ is invertible. Then $s^{-1}: A\to A^*$ regarded as a bilinear form $B(\cdot\,,\cdot): A\otimes A\to\Bbbk$ on A through* (3-18) *for $\varphi = s^{-1}$ is symmetric, nondegenerate and invariant. Thus $(A,\cdot\,, B)$ is a symmetric Frobenius algebra.*

Proof. Since s is symmetric and s regarded as a linear map from A^* to A is invertible, $B(\cdot\,,\cdot)$ is symmetric and nondegenerate. On the other hand, since s is invariant, (3-7) holds by Lemma 3.2. Thus, for any $x, y, z\in A$ and $a^* = s^{-1}(x)$, $b^* = s^{-1}(y)$ and $c^* = s^{-1}(z)$, we have

$$\begin{aligned} B(x\cdot y, z) &= \langle c^*, s(a^*)\cdot s(b^*)\rangle = \langle c^* L^*(s(a^*)), b^*\rangle \\ &= \langle R^*(s(c^*))a^*, b^*\rangle = \langle a^*, s(b^*)\cdot s(c^*)\rangle = B(x, y\cdot z), \end{aligned}$$

that is, $B(\cdot\,,\cdot)$ is invariant. So the conclusion follows. □

Lemma 3.14. *Let $(A,\cdot\,, B)$ be a symmetric Frobenius $\Bbbk$-algebra with finite $\Bbbk$-dimension. Suppose that $\beta : A\to A$ is an endomorphism of A that is self-adjoint with respect to B. Then $\tilde{\beta} = \beta\varphi^{-1} : A^*\to A$ regarded as an element of $A\otimes A$*

is symmetric, where $\varphi : A \to A^*$ *is defined by* (3-18)*. Moreover,* β *is a balanced* A*-bimodule homomorphism if and only if* $\tilde{\beta}$ *is.*

Proof. Since β is self-adjoint with respect to B, it is easy to show that $\tilde{\beta}$ regarded as an element of $A \otimes A$ is symmetric. Moreover, for any $a^*, b^* \in A^*$, $z \in A$ and $x = \varphi^{-1}(a^*)$, $y = \varphi^{-1}(b^*)$, we have

$$\begin{aligned}\langle R^*(\tilde{\beta}(a^*))b^*, z\rangle &= \langle R^*(\beta(x))\varphi(y), z\rangle = B(y, z \cdot \beta(x)),\\ \langle a^* L^*(\tilde{\beta}(b^*)), z\rangle &= \langle \varphi(x) L^*(\beta(y)), z\rangle = B(x, \beta(y) \cdot z)\\ &= B(\beta(y), z \cdot x) = B(y, \beta(z \cdot x)).\end{aligned}$$

Thus $\tilde{\beta}$ satisfies (3-7) if and only if $\beta(z \cdot x) = z \cdot \beta(x)$ for any $x, z \in A$. On the other hand,

$$\begin{aligned}\langle R^*(\tilde{\beta}(a^*))b^*, z\rangle &= \langle R^*(\beta(x))\varphi(y), z\rangle = B(y, z \cdot \beta(x))\\ &= B(\beta(x), y \cdot z) = B(x, \beta(y \cdot z)),\\ \langle a^* L^*(\tilde{\beta}(b^*)), z\rangle &= \langle \varphi(x) L^*(\beta(y)), z\rangle = B(x, \beta(y) \cdot z).\end{aligned}$$

Therefore, $\tilde{\beta}$ satisfies (3-7) if and only if $\beta(y \cdot z) = \beta(y) \cdot z$ for any $y, z \in A$. Hence β is an A-bimodule homomorphism if and only if $\tilde{\beta}$ is. □

If $\beta = \mathrm{id}$, the lemma above states that $\varphi^{-1} : A^* \to A$ is a balanced A-bimodule homomorphism.

Corollary 3.15. *Let* $(A, \cdot, B)$ *be a symmetric Frobenius* $\Bbbk$*-algebra of finite* $\Bbbk$*-dimension and let* $\varphi : A \to A^*$ *be the linear map defined by* (3-18)*. Suppose* $\beta \in A \otimes A$ *is symmetric. Then* β *regarded as a linear map from* (A^*, R^*, L^*) *to* A *is a balanced* A*-bimodule homomorphism if and only if* $\hat{\beta} = \beta\varphi : A \to A$ *is a balanced* A*-bimodule homomorphism.*

Proof. In fact, $\hat{\beta} = \beta\varphi$ is self-adjoint with respect to $B(\,\cdot\,,\,\cdot\,)$ since for any $x, y \in A$,

$$\begin{aligned}\langle \beta, \varphi(x) \otimes \varphi(y)\rangle = \langle \beta, \varphi(y) \otimes \varphi(x)\rangle &\iff \langle \beta(\varphi(x)), \varphi(y)\rangle = \langle \beta(\varphi(y)), \varphi(x)\rangle\\ &\iff B(\hat{\beta}(x), y) = B(\hat{\beta}(y), x).\end{aligned}$$

So the conclusion follows from Lemma 3.14. □

Theorem 3.16. *Let* $\Bbbk$ *be a field of characteristic not equal to* 2*. Let* $(A, \cdot, B)$ *be a symmetric Frobenius algebra of finite* $\Bbbk$*-dimension. Suppose that* α *and* β *are two endomorphisms of* A *and that* β *is self-adjoint with respect to* B*.*

(i) α *is an extended* $\mathcal{O}$*-operator with modification* β *of mass* κ *if and only if* $\tilde{\alpha} := \alpha \circ \varphi^{-1} : A^* \to A$ *is an extended* $\mathcal{O}$*-operator with modification* $\tilde{\beta} := \beta \circ \varphi^{-1} : A^* \to A$ *of mass* κ*, where the linear map* $\varphi : A \to A^*$ *is defined by* (3-18)*.*

(ii) *Suppose that in addition, α is skew-adjoint with respect to B. Then $\tilde{\alpha}$ regarded as an element of $A \otimes A$ is skew-symmetric and*

(a) *$r_{\pm} = \tilde{\alpha} \pm \tilde{\beta}$ regarded as an element of $A \otimes A$ is a solution of the EAYBE of mass $(\kappa+1)/4$ if and only if α is an extended $\mathbb{O}$-operator with modification β of mass k;*

(b) *if $\kappa = -1$, then $r_{\pm} = \tilde{\alpha} \pm \tilde{\beta}$ regarded as an element of $A \otimes A$ is a solution of the AYBE if and only if α is an extended $\mathbb{O}$-operator with modification β of mass -1; and*

(c) *if $\kappa = 0$, then $\tilde{\alpha}$ regarded as an element of $A \otimes A$ is a solution of the AYBE if and only if α is a Rota–Baxter operator of weight zero.*

Proof. (i) Since B is symmetric and invariant, for any $x, y, z \in A$, we have

$$B(x \cdot y, z) = B(x, y \cdot z) \Longleftrightarrow \langle \varphi(x \cdot y), z \rangle = \langle \varphi(x), y \cdot z \rangle \Longleftrightarrow \varphi(xR(y)) = \varphi(x)L^*(y), \tag{3-19}$$

$$B(z, x \cdot y) = B(y \cdot z, x) \Longleftrightarrow \langle \varphi(z), x \cdot y \rangle = \langle \varphi(y \cdot z), x \rangle \Longleftrightarrow R^*(y)\varphi(z) = \varphi(L(y)z). \tag{3-20}$$

On the other hand, since φ is invertible, for any $a^*, b^* \in A^*$, there exist $x, y \in A$ such that $\varphi(x) = a^*$, $\varphi(y) = b^*$. So according to (3-19) and (3-20), the equation

$$\tilde{\alpha}(a^*) \cdot \tilde{\alpha}(b^*) - \tilde{\alpha}(\varphi(\tilde{\alpha}(a^*) \cdot \varphi^{-1}(b^*) + \varphi^{-1}(a^*) \cdot \tilde{\alpha}(b^*))) = \kappa \tilde{\beta}(a^*) \cdot \tilde{\beta}(b^*),$$

is equivalent to

$$\tilde{\alpha}(a^*) \cdot \tilde{\alpha}(b^*) - \tilde{\alpha}(R^*(\tilde{\alpha}(a^*))b^* + a^* L^*(\tilde{\alpha}(b^*))) = \kappa \tilde{\beta}(a^*) \cdot \tilde{\beta}(b^*).$$

By Lemma 3.14, $\beta : A \to A$ is a balanced A-bimodule homomorphism if and only if $\tilde{\beta} : A^* \to A$ is. So α is an extended $\mathbb{O}$-operator with modification β of mass κ if and only if $\tilde{\alpha}$ is an extended $\mathbb{O}$-operator with modification $\tilde{\beta}$ of mass κ.

(i) If α is skew-adjoint with respect to B, then

$$\langle \alpha(x), \varphi(y) \rangle + \langle \varphi(x), \alpha(y) \rangle = 0 \quad \text{for all } x, y \in A.$$

Hence $\langle \tilde{\alpha}(a^*), b^* \rangle + \langle a^*, \tilde{\alpha}(b^*) \rangle = 0$ for any $a^*, b^* \in A^*$. So $\tilde{\alpha}$ regarded as an element of $A \otimes A$ is skew-symmetric.

By Theorem 3.5, item (a) holds. By Corollary 3.6, items (b) and (c) hold. □

Corollary 3.17. *Let $\mathbb{k}$ be a field of characteristic not equal to 2. Let A be a $\mathbb{k}$-algebra of finite $\mathbb{k}$-dimension and let $r \in A \otimes A$. Define $\alpha, \beta \in A \otimes A$ by* (3-5)*. Then $r = \alpha + \beta$. Let $B : A \otimes A \to \mathbb{k}$ be a nondegenerate symmetric and invariant bilinear form. Define the linear map $\varphi : A \to A^*$ by* (3-18).

(i) *Suppose that* $\beta \in A \otimes A$ *is invariant. Then* r *is a solution of the EAYBE of mass* $(\kappa+1)/4$ *if and only if* $\hat{\alpha} = \alpha\varphi : A \to A$ *is an extended* $\mathbb{O}$*-operator with modification* $\hat{\beta} = \beta\varphi : A \to A$ *of mass* k.

(ii) *Suppose that* $\beta \in A \otimes A$ *is invariant. Then* r *is a solution of the AYBE if and only if* $\hat{\alpha} = \alpha\varphi : A \to A$ *is an extended* $\mathbb{O}$*-operator with modification* $\hat{\beta} = \beta\varphi : A \to A$ *of mass* -1. *If in addition,* $\beta = 0$, *that is,* r *is skew-symmetric, then* r *is a solution of the AYBE if and only if* $\hat{\alpha} = \hat{r} = r\varphi : A \to A$ *is a Rota–Baxter operator of weight zero.*

Proof. By the proof of Corollary 3.15, we show that $\hat{\beta} = \beta\varphi$ is self-adjoint with respect to $B(\cdot,\cdot)$ since $\beta \in A \otimes A$ is symmetric. Similarly, since $\alpha \in A \otimes A$ is skew-symmetric, $\hat{\alpha} = \alpha\varphi$ is skew-adjoint with respect to $B(\cdot,\cdot)$. So the conclusion follows from Theorem 3.16. □

4. Extended $\mathbb{O}$-operators and the generalized associative Yang–Baxter equation

We define the generalized associative Yang–Baxter equation and study its relationship with extended $\mathbb{O}$-operators.

4a. ***Generalized associative Yang–Baxter equation.*** We adapt the same notation as in Definition 3.1.

The following proposition (also see [Aguiar 2000a, Proposition 5.1]) is related to the construction of variations of bialgebras under the names of associative D-bialgebras [Zhelyabin 1997], balanced infinitesimal bialgebras (in the opposite algebras) [Aguiar 2001] and antisymmetric infinitesimal bialgebras [Bai 2010].

Proposition 4.1 [Aguiar 2000a; 2001; Bai 2010]. *Let* A *be a* $\mathbb{k}$*-algebra with finite* $\mathbb{k}$*-dimension and let* $r \in A \otimes A$. *Define* $\Delta : A \to A \otimes A$ *by*

(4-1) $$\Delta(x) = (\mathrm{id} \otimes L(x) - R(x) \otimes \mathrm{id})r \quad \text{for all } x \in A.$$

Then

(4-2) $$\Delta^* : A^* \otimes A^* \hookrightarrow (A \otimes A)^* \to A^*$$

defines an associative multiplication on A^* *if and only if* r *is a solution of the equation*

(4-3) $$(\mathrm{id} \otimes \mathrm{id} \otimes L(x) - R(x) \otimes \mathrm{id} \otimes \mathrm{id})(r_{12}r_{13} + r_{13}r_{23} - r_{23}r_{12}) = 0 \quad \text{for all } x \in A.$$

Definition 4.2. Let A be a $\mathbb{k}$-algebra. Equation (4-3) is called the *generalized associative Yang–Baxter equation (GAYBE)*. An element $r \in A \otimes A$ satisfying (4-3) is called a *solution of the GAYBE in* A.

Lemma 4.3. *Let $(A, \cdot)$ be a $\Bbbk$-algebra with finite $\Bbbk$-dimension. Let $r \in A \otimes A$. The multiplication $*$ on A^* defined by* (4-2) *is also given by*

$$a^* * b^* = R^*(r(a^*))b^* - L^*(r^t(b^*))a^* \quad \textit{for all } a^*, b^* \in A^*. \tag{4-4}$$

Proof. Let $\{e_1, \dots, e_n\}$ be a basis of A and $\{e_1^*, \dots, e_n^*\}$ be its dual basis. Suppose that $r = \sum_{i,j} a_{i,j} e_i \otimes e_j$ and $e_i \cdot e_j = \sum_k c_{i,j}^k e_k$. Then for any k and l, we have

$$\begin{aligned} e_k^* * e_l^* &= \sum_s \langle e_k^* \otimes e_l^*, \Delta(e_s)\rangle e_s^* \\ &= \sum_s \langle e_k^* \otimes e_l^*, (\mathrm{id} \otimes L(e_s) - R(e_s) \otimes \mathrm{id})r\rangle e_s^* \\ &= \sum_{s,t} (a_{k,t} c_{s,t}^l - c_{t,s}^k a_{t,l}) e_s^* = R^*(r(e_k^*))e_l^* - L^*(r^t(e_l^*))e_k^*. \end{aligned}$$ □

This lemma suggests that we apply the approach considered in Section 2b. More precisely, we take the A-bimodule $\Bbbk$-algebra $(R, \circ, \ell, r)$ to be (A^*, R^*, L^*) with the zero multiplication and set

$$\delta_+ = r \quad \text{and} \quad \delta_- = -r^t. \tag{4-5}$$

Assume that $\Bbbk$ has characteristic not equal to 2 and define

$$\alpha = (r - r^t)/2 \quad \text{and} \quad \beta = (r + r^t)/2, \tag{4-6}$$

that is, α and β are the *skew-symmetric part* and the *symmetric part* of r. So $r = \alpha + \beta$ and $r^t = -\alpha + \beta$.

Proposition 4.4. *Let $\Bbbk$ have characteristic not equal to 2. Let $(A, \cdot)$ be a $\Bbbk$-algebra with finite $\Bbbk$-dimension and $r \in A \otimes A$. Let α and β be given by* (4-6). *Suppose that β is a balanced A-bimodule homomorphism, that is, β satisfies* (3-6). *If α is an extended $\mathbb{O}$-operator with modification β of any mass $\kappa \in \Bbbk$, then the product defined by* (4-4) *defines a $\Bbbk$-algebra structure on A^* and r is a solution of the GAYBE.*

Proof. By applying Theorem 2.12 to the A-bimodule $\Bbbk$-algebra $(R, \circ, \ell, r)$ we constructed before the proposition, we see that the product defined by (4-4) is associative. Then r is a solution of the GAYBE by Lemma 4.3. □

Corollary 4.5. *Under the assumptions of Proposition 4.4, a solution of the EAYBE of any mass $\kappa \in \Bbbk$ is also a solution of the GAYBE.*

Proof. Let r be a solution of the EAYBE of mass κ. Define α and β by (4-6). Then by Theorem 3.5, α is an extended $\mathbb{O}$-operator with modification β of mass $4\kappa - 1$. Hence by Proposition 4.4, r is a solution of the GAYBE. □

4b. *GAYBE and extended $\mathcal{O}$-operators.* We now consider the operator form of GAYBE with emphasis on its relationship with extended $\mathcal{O}$-operators.

Lemma 4.6. *Let A be a $\Bbbk$-algebra and (V, ℓ, r) be a bimodule. Let $\alpha : V \to A$ be a linear map. Then the product*

$$u *_\alpha v := \ell(\alpha(u))v + ur(\alpha(v)) \quad \textit{for all } u, v \in V, \tag{4-7}$$

defines a $\Bbbk$-algebra structure on V if and only if

$$\ell\big(\alpha(u)\cdot\alpha(v) - \alpha(u *_\alpha v)\big)w = ur\big(\alpha(v)\cdot\alpha(w) - \alpha(v *_\alpha w)\big) \quad \textit{for all } u, v \in V. \tag{4-8}$$

Proof. It follows from Lemma 2.10 by setting $(R, \ell, r) = (V, \ell, r)$ and $\lambda = 0$. □

Theorem 4.7. *Let A be a $\Bbbk$-algebra and (V, ℓ, r) be an A-bimodule, both of finite dimension over $\Bbbk$. Let $\alpha : V \to A$ be a linear map. Using the same notation as in Definition 3.7, $\tilde{\alpha}_-$ identified as an element of $\mathcal{A} \otimes \mathcal{A}$ is a skew-symmetric solution of the GAYBE* (4-3) *if and only if* (4-8) *and the equations*

$$\alpha(u)\cdot\alpha(\ell(x)v) - \alpha(u *_\alpha (\ell(x)v)) = \alpha(ur(x))\cdot\alpha(v) - \alpha((ur(x)) *_\alpha v), \tag{4-9}$$

$$\alpha(u)\cdot\alpha(vr(x)) - \alpha(u *_\alpha (vr(x))) = (\alpha(u)\cdot\alpha(v))\cdot x - \alpha(u *_\alpha v)\cdot x, \tag{4-10}$$

$$\alpha(\ell(x)u)\cdot\alpha(v) - \alpha((\ell(x)u) *_\alpha v) = x\cdot(\alpha(u)\cdot\alpha(v)) - x\cdot\alpha(u *_\alpha v) \tag{4-11}$$

hold for any $u, v \in V$, $x \in A$.

Proof. By Proposition 4.1, Lemma 4.3 and Lemma 4.6, we see that $\tilde{\alpha}_- \in \mathcal{A} \otimes \mathcal{A}$ is a skew-symmetric solution of the GAYBE (4-3) if and only if for any $u, v, w \in V$ and $a^*, b^*, c^* \in A^*$,

$$\begin{aligned}
&R^*_{\mathcal{A}}\big(\tilde{\alpha}_-(u+a^*)\cdot\tilde{\alpha}_-(v+b^*) - \tilde{\alpha}_-(R^*_{\mathcal{A}}(\tilde{\alpha}_-(u+a^*))(v+b^*) \\
&\qquad\qquad + (u+a^*)L^*_{\mathcal{A}}(\tilde{\alpha}_-(v+b^*)))\big)(w+c^*) \\
&\quad = (u+a^*)L^*_{\mathcal{A}}\big(\tilde{\alpha}_-(v+b^*)\cdot\tilde{\alpha}_-(w+c^*) - \tilde{\alpha}_-(R^*_{\mathcal{A}}(\tilde{\alpha}_-(v+b^*))(w+c^*) \\
&\qquad\qquad + (v+b^*)L^*_{\mathcal{A}}(\tilde{\alpha}_-(w+c^*)))\big),
\end{aligned}$$

By the proof of Theorem 3.9, the equation above is equivalent to

$$\begin{aligned}
&R^*_{\mathcal{A}}\big(\alpha(u)\cdot\alpha(v) - \alpha(\ell(\alpha(u))v) - \alpha(ur(\alpha(v))) - r^*(\alpha(u))\alpha^*(b^*) + \alpha^*(R^*_{\mathcal{A}}(\alpha(u))b^*) \\
&\qquad - \alpha^*(uL^*_{\mathcal{A}}(\alpha^*(b^*))) - \alpha^*(a^*)\ell^*(\alpha(v)) - \alpha^*(R^*_{\mathcal{A}}(\alpha^*(a^*))v) + \alpha^*(a^*L^*_{\mathcal{A}}(\alpha(v)))\big)w \\
&\ + R^*_{\mathcal{A}}\big(\alpha(u)\cdot\alpha(v) - \alpha(\ell(\alpha(u))v) - \alpha(ur(\alpha(v))) - r^*(\alpha(u))\alpha^*(b^*) + \alpha^*(R^*_{\mathcal{A}}(\alpha(u))b^*) \\
&\qquad - \alpha^*(uL^*_{\mathcal{A}}(\alpha^*(b^*))) - \alpha^*(a^*)\ell^*(\alpha(v)) - \alpha^*(R^*_{\mathcal{A}}(\alpha^*(a^*))v) + \alpha^*(a^*L^*_{\mathcal{A}}(\alpha(v)))\big)c^* \\
&= uL^*_{\mathcal{A}}\big(\alpha(v)\cdot\alpha(w) - \alpha(\ell(\alpha(v))w) - \alpha(vr(\alpha(w))) - r^*(\alpha(v))\alpha^*(c^*) + \alpha^*(R^*_{\mathcal{A}}(\alpha(v))c^*) \\
&\qquad -\alpha^*(vL^*_{\mathcal{A}}(\alpha^*(c^*))) - \alpha^*(b^*)\ell^*(\alpha(w)) - \alpha^*(R^*_{\mathcal{A}}(\alpha^*(b^*))w) + \alpha^*(b^*L^*_{\mathcal{A}}(\alpha(w)))\big) \\
&\ + a^*L^*_{\mathcal{A}}(\alpha(v)\cdot\alpha(w) - \alpha(\ell(\alpha(v))w) - \alpha(vr(\alpha(w))) - r^*(\alpha(v))\alpha^*(c^*) + \alpha^*(R^*_{\mathcal{A}}(\alpha(v))c^*) \\
&\qquad - \alpha^*(vL^*_{\mathcal{A}}(\alpha^*(c^*))) - \alpha^*(b^*)\ell^*(\alpha(w)) - \alpha^*(R^*_{\mathcal{A}}(\alpha^*(b^*))w) + \alpha^*(b^*L^*_{\mathcal{A}}(\alpha(w)))).
\end{aligned}$$

By suitable choices of $u, v, w \in V$ and $a^*, b^*, c^* \in A^*$, we find that this equation holds if and only if the following equations hold:

$$R^*_{\mathscr{A}}(\alpha(u)\cdot\alpha(v)-\alpha(\ell(\alpha(u))v+ur(\alpha(v))))w = uL^*_{\mathscr{A}}(\alpha(v)\cdot\alpha(w)-\alpha(\ell(\alpha(v))w+vr(\alpha(w)))) \quad \text{(take } a^*=b^*=c^*=0\text{)}, \tag{4-12}$$

$$R^*_{\mathscr{A}}(-r^*(\alpha(u))\alpha^*(b^*)+\alpha^*(R^*_{\mathscr{A}}(\alpha(u))b^*)-\alpha^*(uL^*_{\mathscr{A}}(\alpha^*(b^*))))w = uL^*_{\mathscr{A}}(-\alpha^*(b^*)\ell^*(\alpha(w))-\alpha^*(R^*_{\mathscr{A}}(\alpha^*(b^*))w)+\alpha^*(b^*L^*_{\mathscr{A}}(\alpha(w)))) \quad \text{(take } v=a^*=c^*=0\text{)}, \tag{4-13}$$

$$R^*_{\mathscr{A}}(-\alpha^*(a^*)\ell^*(\alpha(v))-\alpha^*(R^*_{\mathscr{A}}(\alpha^*(a^*))v)+\alpha^*(a^*L^*_{\mathscr{A}}(\alpha(v))))w = a^*L^*_{\mathscr{A}}(\alpha(v)\cdot\alpha(w)-\alpha(\ell(\alpha(v))w)-\alpha(vr(\alpha(w)))) \quad \text{(take } u=b^*=c^*=0\text{)}, \tag{4-14}$$

$$R^*_{\mathscr{A}}(\alpha(u)\cdot\alpha(v)-\alpha(\ell(\alpha(u))v)-\alpha(ur(\alpha(v))))c^* = uL^*_{\mathscr{A}}(-r^*(\alpha(v))\alpha^*(c^*)+\alpha^*(R^*_{\mathscr{A}}(\alpha(v))c^*)-\alpha^*(vL^*_{\mathscr{A}}(\alpha^*(c^*)))) \quad \text{(take } w=a^*=b^*=0\text{)}, \tag{4-15}$$

$$R^*_{\mathscr{A}}(-r^*(\alpha(u))\alpha^*(b^*)+\alpha^*(R^*_{\mathscr{A}}(\alpha(u))b^*)-\alpha^*(uL^*_{\mathscr{A}}(\alpha^*(b^*))))c^*=0 \quad \text{(take } v=w=a^*=0\text{)}, \tag{4-16}$$

$$R^*_{\mathscr{A}}(-\alpha^*(a^*)\ell^*(\alpha(v))-\alpha^*(R^*_{\mathscr{A}}(\alpha^*(a^*))v)+\alpha^*(a^*L^*_{\mathscr{A}}(\alpha(v))))c^* = a^*L^*_{\mathscr{A}}(-r^*(\alpha(v))\alpha^*(c^*)+\alpha^*(R^*_{\mathscr{A}}(\alpha(v))c^*)-\alpha^*(vL^*_{\mathscr{A}}(\alpha^*(c^*)))) \quad \text{(take } u=w=b^*=0\text{)}, \tag{4-17}$$

$$a^*L^*_{\mathscr{A}}(-\alpha^*(b^*)\ell^*(\alpha(w))-\alpha^*(R^*_{\mathscr{A}}(\alpha^*(b^*))w)+\alpha^*(b^*L^*_{\mathscr{A}}(\alpha(w))))=0 \quad \text{(take } u=v=c^*=0\text{)}. \tag{4-18}$$

Thus we just need to prove

(i) (4-12) $\Longleftrightarrow$ (4-8),	(ii) (4-13) $\Longleftrightarrow$ (4-9),
(iii) (4-14) $\Longleftrightarrow$ (4-10),	(iv) (4-15) $\Longleftrightarrow$ (4-11),
(v) both sides of (4-17) equal zero,	(vi) (4-16) and (4-18) hold.

The proofs of these statements are similar. So we just prove that (4-13) holds if and only if (4-9) holds. Let *LHS* and *RHS* denote the left-hand side and right-hand side of (4-13). Then for any $x \in A$ and $s^* \in V^*$, we have

$$\langle LHS, s^*\rangle = \langle RHS, s^*\rangle = 0.$$

Further

$$\begin{aligned}\langle LHS, x\rangle &= \langle w, -r^*(x)(r^*(\alpha(u))\alpha^*(b^*)) \\ &\qquad + r^*(x)\alpha^*(R^*_{\mathscr{A}}(\alpha(u))b^*) - r^*(x)\alpha^*(uL^*_{\mathscr{A}}(\alpha^*(b^*)))\rangle \\ &= \langle -\alpha((wr(x))r(\alpha(u))) + \alpha(wr(x))\cdot\alpha(u), b^*\rangle \\ &\qquad - \langle \alpha^*(b^*)\cdot\alpha(wr(x)), u\rangle \\ &= \langle -\alpha((wr(x))r(\alpha(u))) + \alpha(wr(x))\cdot\alpha(u) - \alpha(\ell(\alpha(wr(x)))u), b^*\rangle, \\ \langle RHS, x\rangle &= \langle u, -(\alpha^*(b^*)\ell^*(\alpha(w)))\ell^*(x) \\ &\qquad - \alpha^*(R^*_{\mathscr{A}}(\alpha^*(b^*))w)\ell^*(x) + \alpha^*(b^*L^*_{\mathscr{A}}(\alpha(w)))\ell^*(x)\rangle \\ &= \langle -\alpha(\ell(\alpha(w))(\ell(x)u)), b^*\rangle \\ &\qquad - \langle \alpha(\ell(x)u)\cdot\alpha^*(b^*), w\rangle + \langle \alpha(w)\cdot\alpha(\ell(x)w), b^*\rangle \\ &= \langle -\alpha(\ell(\alpha(w))(\ell(x)u)) - \alpha(wr(\alpha(\ell(x)u))) + \alpha(w)\cdot\alpha(\ell(x)u), b^*\rangle.\end{aligned}$$

So (4-13) holds if and only if (4-9) holds. □

Equations (4-9)–(4-11) in Theorem 4.7 can be regarded as an operator form of GAYBE. To get a more manageable form, we restrict below to the case of extended $\mathcal{O}$-operators.

Corollary 4.8. *Let $(A, \cdot)$ be a $\Bbbk$-algebra with finite $\Bbbk$-dimension.*

(i) *Let $(R, \circ, \ell, r)$ be an A-bimodule $\Bbbk$-algebra with finite $\Bbbk$-dimension. Let $\alpha, \beta : R \to A$ be two linear maps such that α is an extended $\mathcal{O}$-operator of weight λ with modification β of mass (κ, μ), that is, β is an A-bimodule homomorphism and the conditions* (2-5) *and* (2-6) *in Definition 2.7 hold, and α and β satisfy* (2-7)*. Then $\alpha - \alpha^{21}$, when identified as an element of $(A \ltimes_{r^*,\ell^*} R^*)\otimes(A \ltimes_{r^*,\ell^*} R^*)$, is a skew-symmetric solution of the GAYBE* (4-3) *if and only if*

$$\lambda\ell(\alpha(u\circ v))w = \lambda ur(\alpha(v\circ w)) \quad \text{for all } u, v, w \in R, \tag{4-19}$$

$$\lambda\alpha(u(vr(x))) = \lambda\alpha(u\circ v)\cdot x \quad \text{for all } u, v \in R, x \in A, \tag{4-20}$$

$$\lambda\alpha((\ell(x)u)\circ v) = \lambda x\cdot\alpha(u\circ v) \quad \text{for all } u, v \in R, x \in A. \tag{4-21}$$

In particular, when $\lambda = 0$, that is, α is an extended $\mathcal{O}$-operator of weight zero with modification β of mass (κ, μ), then $\alpha - \alpha^{21}$ identified as an element of $(A \ltimes_{r^*,\ell^*} R^*)\otimes(A \ltimes_{r^*,\ell^*} R^*)$ is a skew-symmetric solution of the GAYBE (4-3)*.*

(ii) *Let $(R, \circ, \ell, r)$ be an A-bimodule $\Bbbk$-algebra with finite $\Bbbk$-dimension. Let $\alpha : R \to A$ be an $\mathcal{O}$-operator of weight λ. Then $\alpha - \alpha^{21}$ identified as an element of $(A \ltimes_{r^*,\ell^*} R^*) \otimes (A \ltimes_{r^*,\ell^*} R^*)$ is a skew-symmetric solution of the GAYBE if and only if* (4-19)–(4-21) *hold.*

(iii) *Let (V, ℓ, r) be a bimodule of A with finite $\Bbbk$-dimension. Let $\alpha, \beta : V \to A$ be two linear maps such that α is an extended $\mathcal{O}$-operator with modification β of mass κ. Then $\alpha - \alpha^{21}$ identified as an element of $(A \ltimes_{r^*,\ell^*} V^*) \otimes (A \ltimes_{r^*,\ell^*} V^*)$ is a skew-symmetric solution of the GAYBE.*

(iv) *Let $\alpha : A \to A$ be a linear endomorphism of A. Suppose that α satisfies* (2-18). *Then $\alpha - \alpha^{21}$ identified as an element of $(A \ltimes_{R^*,L^*} A^*) \otimes (A \ltimes_{R^*,L^*} A^*)$ is a skew-symmetric solution of the GAYBE.*

(v) *Let $(R, \circ, \ell, r)$ be an A-bimodule $\Bbbk$-algebra of finite $\Bbbk$-dimension. Let $\alpha, \beta : R \to A$ be two linear maps such that α is an extended $\mathcal{O}$-operator with modification β of mass $(\kappa, \mu) = (0, \mu)$, that is, β is an A-bimodule homomorphism and the condition* (2-6) *in Definition 2.7 holds, and α and β satisfy*

$$\alpha(u) \cdot \alpha(v) - \alpha(\ell(\alpha(u))v + ur(\alpha(v))) = \mu\beta(u \circ v) \quad \textit{for all } u, v \in R.$$

Then $\alpha - \alpha^{21}$ identified as an element of $(A \ltimes_{r^*,\ell^*} R^*) \otimes (A \ltimes_{r^*,\ell^*} R^*)$ is a skew-symmetric solution of the GAYBE.

Proof. (i) Since α is an extended $\mathcal{O}$-operator of weight λ with modification β of mass (κ, μ), by Theorem 4.7, $\alpha - \alpha^{21}$ identified as an element of $(A \ltimes_{r^*,\ell^*} R^*) \otimes (A \ltimes_{r^*,\ell^*} R^*)$ is a skew-symmetric solution of the GAYBE (4-3) if and only if

$$\text{(4-22)} \quad -\lambda\ell(\alpha(u \circ v))w + \kappa\ell(\beta(u) \cdot \beta(v))w + \mu\ell(\beta(u \circ v))w = -\lambda ur(\alpha(v \circ w)) + \kappa ur(\beta(v) \cdot \beta(w)) + \mu ur(\beta(v \circ w)),$$

$$\text{(4-23)} \quad -\lambda\alpha((ur(x)) \circ v) + \kappa\beta(ur(x)) \cdot \beta(v) + \mu\beta((ur(x)) \circ v) = -\lambda\alpha(u \circ (l(x)v)) + \kappa\beta(u) \cdot \beta(\ell(x)v) + \mu\beta(u \circ (\ell(x)v)),$$

$$\text{(4-24)} \quad -\lambda\alpha(u \circ (vr(x))) + \kappa\beta(u) \cdot \beta(vr(x)) + \mu\beta(u \circ (vr(x))) = -\lambda\alpha(u \circ v) \cdot x + \kappa(\beta(u) \cdot \beta(v)) \cdot x + \mu\beta(u \circ v) \cdot x,$$

$$\text{(4-25)} \quad -\lambda\alpha((\ell(x)u) \circ v) + \kappa\beta(\ell(x)u) \cdot \beta(v) + \mu\beta((\ell(x)u) \circ v) = -\lambda x \cdot \alpha(u \circ v) + \kappa x \cdot (\beta(u) \cdot \beta(v)) + \mu x \cdot \beta(u \circ v)$$

for any $u, v \in R$, $x \in A$. Since β is an A-bimodule homomorphism and the conditions (2-5) and (2-6) in Definition 2.7 hold, we have (4-19) holds if and only if (4-22) holds, (4-20) holds if and only if (4-24) holds, (4-21) holds if and only if (4-25) holds and (4-23) holds automatically.

(ii) This follows from item (i) by setting $\kappa = \mu = 0$.

(iii) This follows from item (i) by setting $\lambda = \mu = 0$.

(iv) This follows from item (iii) for $(V, \ell, r) = (A, L, R)$ and $\beta = \mathrm{id}$.

(v) This follows from item (i) by setting $\lambda = \kappa = 0$. □

References

[Aguiar 2000a] M. Aguiar, "Infinitesimal Hopf algebras", pp. 1–29 in *New trends in Hopf algebra theory* (La Falda, 1999), edited by N. Andruskiewitsch et al., Contemp. Math. **267**, Amer. Math. Soc., Providence, RI, 2000. MR 2001k:16066 Zbl 0982.16028

[Aguiar 2000b] M. Aguiar, "Pre-Poisson algebras", *Letters Math. Phys.* **54**:4 (2000), 263–277. MR 2002k:17041 Zbl 1032.17038

[Aguiar 2001] M. Aguiar, "On the associative analog of Lie bialgebras", *J. Algebra* **244**:2 (2001), 492–532. MR 2003c:17035 Zbl 0991.16033

[Bai 2007] C. Bai, "A unified algebraic approach to the classical Yang–Baxter equation", *J. Phys. A* **40**:36 (2007), 11073–11082. MR 2009b:17014 Zbl 1118.17008

[Bai 2010] C. Bai, "Double constructions of Frobenius algebras, Connes cocycles and their duality", *J. Noncommut. Geom.* **4**:4 (2010), 475–530. MR 2012b:17045 Zbl 05797230

[Bai et al. 2010a] C. Bai, L. Guo, and X. Ni, "𝒪-operators on associative algebras and dendriform algebras", preprint, 2010. arXiv 1003.2432

[Bai et al. 2010b] C. Bai, L. Guo, and X. Ni, "Nonabelian generalized Lax pairs, the classical Yang–Baxter equation and PostLie algebras", *Comm. Math. Phys.* **297** (2010), 553–596. MR 2011i:17032 Zbl 1206.17020

[Bai et al. 2011] C. Bai, L. Guo, and X. Ni, "Generalizations of the classical Yang–Baxter equation and 𝒪-operators", *J. Math. Phys.* **52**:6 (2011), 063515. MR 2841771

[Bai et al. 2012] C. Bai, L. Guo, and X. Ni, "𝒪-operators on associative algebras, associative Yang–Baxter equations and dendriform algebras: a survey", pp. 10–51 in *Quantized algebra and physics* (Tianjin, 2009), edited by M.-L. Ge et al., Nankai Series Pure Appl. Math. Theor. Phys. **8**, World Scientific, Singapore, 2012.

[Baxter 1960] G. Baxter, "An analytic problem whose solution follows from a simple algebraic identity", *Pacific J. Math.* **10**:3 (1960), 731–742. MR 22 #9990 Zbl 0095.12705

[Baxter 1972] R. J. Baxter, "One-dimensional anisotropic Heisenberg chain", *Ann. Physics* **70**:2 (1972), 323–337. MR 45 #8147

[Bordemann 1990] M. Bordemann, "Generalized Lax pairs, the modified classical Yang–Baxter equation, and affine geometry of Lie groups", *Comm. Math. Phys.* **135**:1 (1990), 201–216. MR 91k:58049 Zbl 0714.58025

[Brauer and Nesbitt 1937] R. Brauer and C. Nesbitt, "On the regular representations of algebras", *Proc. Nat. Acad. Sci. USA* **23**:4 (1937), 236–240. JFM 63.0091.06

[Connes and Kreimer 2000] A. Connes and D. Kreimer, "Renormalization in quantum field theory and the Riemann–Hilbert problem, I: The Hopf algebra structure of graphs and the main theorem", *Comm. Math. Phys.* **210**:1 (2000), 249–273. MR 2002f:81070 Zbl 1032.81026

[Ebrahimi-Fard 2002] K. Ebrahimi-Fard, "Loday-type algebras and the Rota–Baxter relation", *Lett. Math. Phys.* **61**:2 (2002), 139–147. MR 2004b:17003 Zbl 1035.17001

[Ebrahimi-Fard et al. 2004] K. Ebrahimi-Fard, L. Guo, and D. Kreimer, "Spitzer's identity and the algebraic Birkhoff decomposition in pQFT", *J. Phys. A* **37**:45 (2004), 11037–11052. MR 2006b:81332 Zbl 1062.81113

[Ebrahimi-Fard et al. 2006] K. Ebrahimi-Fard, L. Guo, and D. Manchon, "Birkhoff type decompositions and the Baker–Campbell–Hausdorff recursion", *Comm. Math. Phys.* **267**:3 (2006), 821–845. MR 2008c:17020 Zbl 1188.17020

[Guo 2000] L. Guo, "Properties of free Baxter algebras", *Adv. Math.* **151**:2 (2000), 346–374. MR 2001f:16048 Zbl 0964.16028

[Guo and Keigher 2000a] L. Guo and W. Keigher, "Baxter algebras and shuffle products", *Adv. Math.* **150**:1 (2000), 117–149. MR 2001g:05015 Zbl 0947.16013

[Guo and Keigher 2000b] L. Guo and W. Keigher, "On free Baxter algebras: completions and the internal construction", *Adv. Math.* **151**:1 (2000), 101–127. MR 2001c:16046 Zbl 0964.16027

[Guo and Zhang 2008] L. Guo and B. Zhang, "Renormalization of multiple zeta values", *J. Algebra* **319**:9 (2008), 3770–3809. MR 2009b:11155 Zbl 1165.11071

[Kosmann-Schwarzbach 1997] Y. Kosmann-Schwarzbach, "Lie bialgebras, Poisson Lie groups and dressing transformations", pp. 104–170 in *Integrability of nonlinear systems* (Pondicherry, 1996), edited by Y. Kosmann-Schwarzbach et al., Lecture Notes in Phys. **495**, Springer, Berlin, 1997. MR 99m:58092 Zbl 1078.37517

[Kosmann-Schwarzbach and Magri 1988] Y. Kosmann-Schwarzbach and F. Magri, "Poisson–Lie groups and complete integrability, I: Drinfeld bialgebras, dual extensions and their canonical representations", *Ann. Inst. H. Poincaré Phys. Théor.* **49**:4 (1988), 433–460. MR 91a:17018 Zbl 0667.16005

[Kreimer 1999] D. Kreimer, "Chen's iterated integral represents the operator product expansion", *Adv. Theor. Math. Phys.* **3**:3 (1999), 627–670. MR 2003b:81128 Zbl 0971.81093 arXiv hep-th/9901099

[Kupershmidt 1999] B. A. Kupershmidt, "What a classical r-matrix really is", *J. Nonlinear Math. Phys.* **6**:4 (1999), 448–488. MR 2001a:17023 Zbl 1015.17015

[Loday and Ronco 2004] J.-L. Loday and M. Ronco, "Trialgebras and families of polytopes", pp. 369–398 in *Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory*, edited by P. Goerss and S. Priddy, Contemp. Math. **346**, Amer. Math. Soc., Providence, RI, 2004. MR 2006e:18016 Zbl 1065.18007 arXiv math/0205043

[Manchon and Paycha 2010] D. Manchon and S. Paycha, "Nested sums of symbols and renormalized multiple zeta values", *Int. Math. Res. Not.* **2010**:24 (2010), 4628–4697. MR 2012a:81184 Zbl 1206.11108

[Rota 1969a] G.-C. Rota, "Baxter algebras and combinatorial identities, I", *Bull. Amer. Math. Soc.* **75**:2 (1969), 325–329. MR 39 #5387 Zbl 0192.33801

[Rota 1969b] G.-C. Rota, "Baxter algebras and combinatorial identities, II", *Bull. Amer. Math. Soc.* **75**:2 (1969), 330–334. MR 39 #5387 Zbl 0319.05008

[Rota 1995] G.-C. Rota, "Baxter operators: An introduction", pp. 504–512 in *Gian-Carlo Rota on combinatorics*, edited by J. P. S. Kung, Birkhäuser, Boston, 1995. MR 1392973 Zbl 0841.01031

[Runkel et al. 2007] I. Runkel, J. Fjelstad, J. Fuchs, and C. Schweigert, "Topological and conformal field theory as Frobenius algebras", pp. 225–247 in *Categories in algebra, geometry and mathematical physics*, edited by A. Davydov et al., Contemp. Math. **431**, Amer. Math. Soc., Providence, RI, 2007. MR 2009d:81324 Zbl 1154.18006 arXiv math/0512076

[Semenov-Tyan-Shanskiĭ 1983] M. A. Semenov-Tyan-Shanskiĭ, "What is a classical r-matrix?", *Funktsional. Anal. i Prilozhen.* **17**:4 (1983), 17–33. In Russian; translated in *Funct. Anal. Appl.* **17**:4 (1983), 259–272. MR 85i:58061 Zbl 0535.58031

[Uchino 2008] K. Uchino, "Quantum analogy of Poisson geometry, related dendriform algebras and Rota–Baxter operators", *Lett. Math. Phys.* **85**:2-3 (2008), 91–109. MR 2010a:17003 Zbl 05544981

[Vallette 2007] B. Vallette, "Homology of generalized partition posets", *J. Pure Appl. Algebra* **208**:2 (2007), 699–725. MR 2007m:18010 Zbl 1109.18002

[Yamagata 1996] K. Yamagata, "Frobenius algebras", pp. 841–887 in *Handbook of algebra*, vol. 1, edited by M. Hazewinkel, North-Holland, Amsterdam, 1996. MR 97k:16022 Zbl 0879.16008

[Yang 1967] C. N. Yang, "Some exact results for the many-body problem in one dimension with repulsive delta-function interaction", *Phys. Rev. Lett.* **19**:23 (1967), 1312–1315. MR 41 #6480 Zbl 0152.46301

[Zhelyabin 1997] V. N. Zhelyabin, "Jordan bialgebras and their relation to Lie bialgebras", *Algebra i Logika* **36**:1 (1997), 3–25. In Russian; translated in *Algebra and Logic* **36**:1 (1997), 1–15. MR 98d:17036 Zbl 0935.17014

Received April 14, 2011. Revised August 5, 2011.

CHENGMING BAI
CHERN INSTITUTE OF MATHEMATICS AND LPMC
NANKAI UNIVERSITY
TIANJIN 300071
CHINA
baicm@nankai.edu.cn

LI GUO
SCHOOL OF MATHEMATICS AND STATISTICS
LANZHOU UNIVERSITY
LANZHOU, GANSU 730000
CHINA
and
DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
RUTGERS UNIVERSITY
216 SMITH HALL
101 WARREN STREET
NEWARK, NJ 07102
UNITED STATES
liguo@rutgers.edu

XIANG NI
DEPARTMENT OF MATHEMATICS
CALTECH
PASADENA, CA 91125
UNITED STATES
xni@caltech.edu

PACIFIC JOURNAL OF MATHEMATICS

http://pacificmath.org

Founded in 1951 by
E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

See inside back cover or pacificmath.org for submission instructions.

The subscription price for 2012 is US $420/year for the electronic version, and $485/year for print and electronic.
Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. Prior back issues are obtainable from Periodicals Service Company, 11 Main Street, Germantown, NY 12526-5635. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and the Science Citation Index.

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

PACIFIC JOURNAL OF MATHEMATICS

Volume 256 No. 2 April 2012

0030-8730(201204)256:2;1-8