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BOTANY OF IRREDUCIBLE AUTOMORPHISMS OF FREE GROUPS

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We give a classification of iwip (i.e., fully irreducible) outer automorphisms of the free group, by discussing the properties of their attracting and repelling trees.

1. Introduction

An outer automorphism Φ of the free group F_N is *fully irreducible* (abbreviated as *iwip*) if no positive power Φ^n fixes a proper free factor of F_N. Being an iwip is one (in fact the most important) of the analogs for free groups of being pseudo-Anosov for mapping classes of hyperbolic surfaces. Another analog of pseudo-Anosov is the notion of an atoroidal automorphism: an element $\Phi \in \mathrm{Out}(F_N)$ is *atoroidal* or *hyperbolic* if no positive power Φ^n fixes a nontrivial conjugacy class. Bestvina and Feighn [1992] and Brinkmann [2000] proved that Φ is atoroidal if and only if the mapping torus $F_N \rtimes_\Phi \mathbb{Z}$ is Gromov-hyperbolic.

Pseudo-Anosov mapping classes are known to be "generic" elements of the mapping class group (in various senses). Rivin [2008] and Sisto [2011] recently proved that, in the sense of random walks, generic elements of $\mathrm{Out}(F_N)$ are atoroidal iwip automorphisms.

Bestvina and Handel [1992] proved that iwip automorphisms have the key property of being represented by (absolute) train-track maps.

A pseudo-Anosov element f fixes two projective classes of measured foliations $[(\mathscr{F}^+, \mu^+)]$ and $[(\mathscr{F}^-, \mu^-)]$:

$$(\mathscr{F}^+, \mu^+) \cdot f = (\mathscr{F}^+, \lambda\mu^+) \quad \text{and} \quad (\mathscr{F}^-, \mu^-) \cdot f = (\mathscr{F}^-, \lambda^{-1}\mu^-),$$

where $\lambda > 1$ is the expansion factor of f. Alternatively, considering the dual $\mathbb{R}$-trees T^+ and T^-, we get:

$$T^+ \cdot f = \lambda T^+ \quad \text{and} \quad T^- \cdot f = \lambda^{-1} T^-.$$

We now discuss the analogous situation for iwip automorphisms. The group of outer automorphisms $\mathrm{Out}(F_N)$ acts on the *outer space* CV_N and its boundary

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$\partial\mathrm{CV}_N$. Recall that the compactified outer space $\overline{\mathrm{CV}}_N = \mathrm{CV}_N \cup \partial\mathrm{CV}_N$ is made up of (projective classes of) $\mathbb{R}$-trees with an action of F_N by isometries which is minimal and very small. See [Vogtmann 2002] for a survey on outer space. An iwip outer automorphism Φ has north-south dynamics on $\overline{\mathrm{CV}}_N$: it has a unique attracting fixed tree $[T_\Phi]$ and a unique repelling fixed tree $[T_{\Phi^{-1}}]$ in the boundary of outer space (see [Levitt and Lustig 2003]):

$$T_\Phi \cdot \Phi = \lambda_\Phi T_\Phi \text{ and } T_{\Phi^{-1}} \cdot \Phi = \frac{1}{\lambda_{\Phi^{-1}}} T_{\Phi^{-1}},$$

where $\lambda_\Phi > 1$ is the *expansion factor* of Φ (i.e., the exponential growth rate of nonperiodic conjugacy classes).

Contrary to the pseudo-Anosov setting, the expansion factor λ_Φ of Φ is typically different from the expansion factor $\lambda_{\Phi^{-1}}$ of Φ^{-1}. More generally, qualitative properties of the fixed trees T_Φ and $T_{\Phi^{-1}}$ can be fairly different. This is the purpose of this paper to discuss and compare the properties of Φ, T_Φ and $T_{\Phi^{-1}}$.

First, the free group, F_N, may be realized as the fundamental group of a surface S with boundary. It is part of folklore that, if Φ comes from a pseudo-Anosov mapping class on S, then its limit trees T_Φ and $T_{\Phi^{-1}}$ live in the Thurston boundary of Teichmüller space: they are dual to a measured foliation on the surface. Such trees T_Φ and $T_{\Phi^{-1}}$ are called *surface trees* and such an iwip outer automorphism Φ is called *geometric* (in this case S has exactly one boundary component).

The notion of surface trees has been generalized (see for instance [Bestvina 2002]). An $\mathbb{R}$-tree which is transverse to measured foliations on a finite CW-complex is called *geometric*. It may fail to be a surface tree if the complex fails to be a surface.

If Φ does not come from a pseudo-Anosov mapping class and if T_Φ is geometric then Φ is called *parageometric*. For a parageometric iwip Φ, Guirardel [2005] and Handel and Mosher [2007] proved that the repelling tree $T_{\Phi^{-1}}$ is not geometric. So we have that, Φ comes from a pseudo-Anosov mapping class on a surface with boundary if and only if both trees T_Φ and $T_{\Phi^{-1}}$ are geometric. Moreover in this case both trees are indeed surface trees.

In [Coulbois and Hilion 2010] we introduced a second dichotomy for trees in the boundary of outer space with dense orbits. For a tree T, we consider its *limit set* $\Omega \subseteq \overline{T}$ (where $\overline{T}$ is the metric completion of T). The limit set Ω consists of points of $\overline{T}$ with at least two pre-images by the map $\mathcal{Q} : \partial F_N \to \hat{T} = \overline{T} \cup \partial T$ introduced in [Levitt and Lustig 2003]; see Section 4A. We are interested in the two extremal cases: A tree T in the boundary of outer space with dense orbits is of *surface type* if $T \subseteq \Omega$ and T is of *Levitt type* if Ω is totally disconnected. As the terminology suggests, a surface tree is of surface type. Trees of Levitt type where discovered by Levitt [1993].

Combining together the two sets of properties, we introduced in [Coulbois and Hilion 2010] the following definitions. A tree T in $\partial\mathrm{CV}_N$ with dense orbits is

- a *surface tree* if it is both geometric and of surface type;
- *Levitt* if it is geometric and of Levitt type;
- *pseudo-surface* if it is not geometric and of surface type;
- *pseudo-Levitt* if it is not geometric and of Levitt type

The following theorem is the main result of this paper.

Theorem 5.2. *Let Φ be an iwip outer automorphism of F_N. Let T_Φ and $T_{\Phi^{-1}}$ be its attracting and repelling trees. Then exactly one of the following occurs*

(1) *The trees T_Φ and $T_{\Phi^{-1}}$ are surface trees. Equivalently, Φ is geometric.*

(2) *The tree T_Φ is Levitt (i.e., geometric and of Levitt type), and the tree $T_{\Phi^{-1}}$ is pseudo-surface (i.e., nongeometric and of surface type). Equivalently, Φ is parageometric.*

(3) *The tree $T_{\Phi^{-1}}$ is Levitt (i.e., geometric and of Levitt type), and the tree T_Φ is pseudo-surface (i.e., nongeometric and of surface type). Equivalently, Φ^{-1} is parageometric.*

(4) *The trees T_Φ and $T_{\Phi^{-1}}$ are pseudo-Levitt (nongeometric and of Levitt type).*

Case (1) *corresponds to toroidal iwips whereas cases* (2), (3) *and* (4) *corresponds to atoroidal iwips. In case* (4) *the automorphism Φ is called pseudo-Levitt.*

Gaboriau, Jaeger, Levitt and Lustig [Gaboriau et al. 1998] introduced the notion of an *index* $\mathrm{ind}(\Phi)$, computed from the rank of the fixed subgroup and from the number of attracting fixed points of the automorphisms φ in the outer class Φ. Another index for a tree T in $\overline{\mathrm{CV}}_N$ has been defined and studied by Gaboriau and Levitt [1995]; we call it the *geometric index* $\mathrm{ind}_{\mathrm{geo}}(T)$. Finally in [Coulbois and Hilion 2010] we introduced and studied the *$\mathcal{Q}$-index* $\mathrm{ind}_{\mathcal{Q}}(T)$ of an $\mathbb{R}$-tree T in the boundary of outer space with dense orbits. The two indices $\mathrm{ind}_{\mathrm{geo}}(T)$ and $\mathrm{ind}_{\mathcal{Q}}(T)$ describe qualitative properties of the tree T [Coulbois and Hilion 2010]. We define these indices and recall our botanical classification of trees in Section 4A.

The key to prove Theorem 5.2 is this:

Propositions 4.2 and 4.4. *Let Φ be an iwip outer automorphism of F_N. Let T_Φ and $T_{\Phi^{-1}}$ be its attracting and repelling trees. Replacing Φ by a suitable power, we have*

$$2\,\mathrm{ind}(\Phi) = \mathrm{ind}_{\mathrm{geo}}(T_\Phi) = \mathrm{ind}_{\mathcal{Q}}(T_{\Phi^{-1}}).$$

We prove this proposition in Sections 4B and 4C.

To study limit trees of iwip automorphisms, we need to state that they have the strongest mixing dynamical property, which is called *indecomposability*.

Theorem 2.1. *Let $\Phi \in \mathrm{Out}(F_N)$ be an iwip outer automorphism. The attracting tree T_Φ of Φ is indecomposable.*

The proof of this theorem is quite independent of the rest of the paper and is the purpose of Section 2. The proof relies on a key property of iwip automorphisms: they can be represented by (absolute) train-track maps.

2. Indecomposability of the attracting tree of an iwip automorphism

Following [Guirardel 2008], a (projective class of) $\mathbb{R}$-tree $T \in \overline{\mathrm{CV}}_N$ is *indecomposable* if for all nondegenerate arcs I and J in T, there exists finitely many elements $u_1, \ldots, u_n$ in F_N such that

$$J \subseteq \bigcup_{i=1}^{n} u_i I \tag{2-1}$$

and

$$\forall i = 1, \ldots, n-1, \quad u_i I \cap u_{i+1} I \text{ is a nondegenerate arc.} \tag{2-2}$$

The main purpose of this section is to prove this result:

Theorem 2.1. *Let $\Phi \in \mathrm{Out}(F_N)$ be an iwip outer automorphism. The attracting tree T_Φ of Φ is indecomposable.*

Before proving this theorem in Section 2C, we collect the results we need from [Bestvina and Handel 1992] and [Gaboriau et al. 1998].

2A. ***Train-track representative of Φ.*** The rose R_N is the graph with one vertex $*$ and N edges. Its fundamental group $\pi_1(R_N, *)$ is naturally identified with the free group F_N. A *marked graph* is a finite graph G with a homotopy equivalence $\tau : R_N \to G$. The marking τ induces an isomorphism

$$\tau_* : F_N = \pi_1(R_N, *) \overset{\cong}{\to} \pi_1(G, v_0),$$

where $v_0 = \tau(*)$.

A homotopy equivalence $f : G \to G$ defines an outer automorphism of F_N. Indeed, if a path m from v_0 to $f(v_0)$ is given, $a \mapsto m f(a) m^{-1}$ induces an automorphism φ of $\pi_1(G, v_0)$, and thus of F_N through the marking. Another path m' from v_0 to $f(v_0)$ gives rise to another automorphism φ' of F_N in the same outer class Φ.

A *topological representative* of $\Phi \in \mathrm{Out}(F_N)$ is an homotopy equivalence $f : G \to G$ of a marked graph G, such that

(i) f maps vertices to vertices,

(ii) f is locally injective on any edge, and

(iii) f induces Φ on $F_N \cong \pi_1(G, v_0)$.

Let $e_1, \dots, e_p$ be the edges of G (an orientation is arbitrarily given on each edge, and e^{-1} denotes the edge e with the reverse orientation). The *transition matrix* of the map f is the $p \times p$ nonnegative matrix M with (i, j)-entry equal to the number of times the edge e_i occurs in $f(e_j)$ (we say that a path (or an edge) w of a graph G *occurs* in a path u of G if it is w or its inverse w^{-1} is a subpath of u).

A topological representative $f : G \to G$ of Φ is a *train-track map* if, moreover,

(iv) for all $k \in \mathbb{N}$, the restriction of f^k on any edge of G is locally injective, and

(v) any vertex of G has valence at least 3.

According to [Bestvina and Handel 1992, Theorem 1.7], an iwip outer automorphism Φ can be represented by a train-track map, with a primitive transition matrix M (i.e., there exists $k \in \mathbb{N}$ such all the entries of M^k are strictly positive). Thus the Perron–Frobenius theorem applies. In particular, M has a real dominant eigenvalue $\lambda > 1$ associated to a strictly positive eigenvector $u = (u_1, \dots, u_p)$. Indeed, λ is the expansion factor of Φ: $\lambda = \lambda_\Phi$. We turn the graph G to a metric space by assigning the length u_i to the edge e_i (for $i = 1, \dots, p$). Since, with respect to this metric, the length of $f(e_i)$ is λ times the length of e_i, we can assume that, on each edge, f is linear of ratio λ.

We define the set $\mathcal{L}_2(f)$ of paths w of combinatorial length 2 (i.e., $w = ee'$, where e, e' are edges of G, $e^{-1} \neq e'$) which occurs in some $f^k(e_i)$ for some $k \in \mathbb{N}$ and some edge e_i of G:

$$\mathcal{L}_2(f) = \{ee' : \exists e_i \text{ edge of } G,\ \exists k \in \mathbb{N} \text{ such that } ee' \text{ is a subpath of } f^k({e_i}^{\pm 1})\}.$$

Since the transition matrix M is primitive, there exists $k \in \mathbb{N}$ such that for any edge e of G, for any $w \in \mathcal{L}_2(f)$, w occurs in $f^k(e)$.

Let v be a vertex of G. The *Whitehead graph* $\mathcal{W}_v$ of v is the unoriented graph defined as follows:

- The vertices of $\mathcal{W}_v$ are the edges of G with v as terminal vertex.
- There is an edge in $\mathcal{W}_v$ between e and e' if $e'e^{-1} \in \mathcal{L}_2(f)$.

As remarked in [Bestvina et al. 1997, Section 2], if $f : G \to G$ is a train-track representative of an iwip outer automorphism Φ, any vertex of G has a connected Whitehead graph. We summarize the previous discussion:

Proposition 2.2. *Let $\Phi \in \mathrm{Out}(F_N)$ be an iwip outer automorphism. There exists a train-track representative $f : G \to G$ of Φ, with primitive transition matrix M and connected Whitehead graphs of vertices. The edge e_i of G is isometric to the segment $[0, u_i]$, where $u = (u_1, \dots, u_p)$ is a Perron–Frobenius eigenvector of M. The map f is linear of ratio λ on each edge e_i of G.*

Remark 2.3. Let $f : G \to G$ be a train-track map, with primitive transition matrix M and connected Whitehead graphs of vertices. Then for any path $w = ab$ in G of combinatorial length 2, there exist $w_1 = a_1 b_1, \dots, w_q = a_q b_q \in \mathcal{L}_2(f)$ (a, b, a_i, b_i edges of G) such that

- $a_{i+1} = b_i^{-1}$, $i \in \{1, \dots, q-1\}$, and
- $a = a_1$ and $b = b_q$.

2B. ***Construction of T_Φ.*** Let $\Phi \in \mathrm{Out}(F_N)$ be an iwip automorphism, and let T_Φ be its attracting tree. Following [Gaboriau et al. 1998], we recall a concrete construction of the tree T_Φ.

We start with a train-track representative $f : G \to G$ of Φ as in Proposition 2.2. The universal cover $\tilde{G}$ of G is a simplicial tree, equipped with a distance d_0 obtained by lifting the distance on G. The fundamental group F_N acts by deck transformations, and thus by isometries, on $\tilde{G}$. Let $\tilde{f}$ be a lift of f to $\tilde{G}$. This lift $\tilde{f}$ is associated to a unique automorphism φ in the outer class Φ, characterized by

$$\forall u \in F_N, \forall x \in \tilde{G}, \quad \varphi(u)\tilde{f}(x) = \tilde{f}(ux). \tag{2-3}$$

For $x, y \in \tilde{G}$ and $k \in \mathbb{N}$, we define:

$$d_k(x, y) = \frac{d_0(\tilde{f}^k(x), \tilde{f}^k(y))}{\lambda^k}.$$

The sequence of distances d_k is decreasing and converges to a pseudo-distance d_∞ on $\tilde{G}$. Identifying points x, y in $\tilde{G}$ which have distance $d_\infty(x, y)$ equal to 0, we obtain the tree T_Φ. The free group F_N still acts by isometries on T_Φ. The quotient map $p : \tilde{G} \to T_\Phi$ is F_N-equivariant and 1-Lipschitz. Moreover, for any edge e of $\tilde{G}$, for any $k \in \mathbb{N}$, the restriction of p to $f^k(e)$ is an isometry. Through p the map $\tilde{f}$ factors to a homothety H of T_Φ, of ratio λ_Φ:

$$\forall x \in \tilde{G}, \quad H(p(x)) = p(\tilde{f}(x)).$$

Property (2-3) leads to

$$\forall u \in F_N, \forall x \in T_\Phi, \quad \varphi(u)H(x) = H(ux). \tag{2-4}$$

2C. ***Indecomposability of T_Φ.*** We say that a path (or an edge) w of the graph G occurs in a path u of the universal cover $\tilde{G}$ of G if w has a lift $\tilde{w}$ that occurs in u.

Lemma 2.4. *Let I be a nondegenerate arc in T_Φ. There exists an arc I' in $\tilde{G}$ and an integer k such that*

- *$p(I') \subseteq I$, and*
- *any element of $\mathcal{L}_2(f)$ occurs in $H^k(I')$.*

Proof. Let $I \subset T_\Phi$ be a nondegenerate arc. There exists an edge e of $\tilde{G}$ such that $I_0 = p(e) \cap I$ is a nondegenerate arc: $I_0 = [x, y]$. We choose $k_1 \in \mathbb{N}$ such that $d_\infty(H^{k_1}(x), H^{k_1}(y)) > L$ where

$$L = 2\max\{u_i = |e_i| \mid e_i \text{ edge of } G\}.$$

Let x', y' be the points in e such that $p(x') = x$, $p(y') = y$, and let I' be the arc $[x', y']$. Since p maps $f^{k_1}(e)$ isometrically into T_Φ, we obtain that

$$d_0(f^{k_1}(x'), f^{k_1}(y')) \geq L.$$

Hence there exists an edge e' of $\tilde{G}$ contained in $[f^{k_1}(x'), f^{k_1}(y')]$. Moreover, for any $k_2 \in \mathbb{N}$, the path $f^{k_2}(e')$ isometrically injects in $[H^{k_1+k_2}(x), H^{k_1+k_2}(y)]$. We take k_2 big enough so that any path in $\mathcal{L}_2(f)$ occurs in $f^{k_2}(e')$. Then $k = k_1 + k_2$ is suitable. □

Proof of Theorem 2.1. Let I, J be two nontrivial arcs in T_Φ. We have to prove that I and J satisfy properties (2-1) and (2-2). Since H is a homeomorphism, and because of (2-4), we can replace I and J by $H^k(I)$ and $H^k(J)$, accordingly, for some $k \in \mathbb{N}$.

We consider an arc I' in $\tilde{G}$ and an integer $k \in \mathbb{N}$ as given by Lemma 2.4. Let x, y be the endpoints of the arc $H^k(J)$: $H^k(J) = [x, y]$. Let x', y' be points in $\tilde{G}$ such that $p(x') = x$, $p(y') = y$, and let J' be the arc $[x', y']$. According to Remark 2.3, there exist $w_1, \ldots, w_n$ such that

- w_i is a lift of some path in $\mathcal{L}_2(f)$,
- $J' \subseteq \bigcup_{i=1}^n w_i$, and
- $w_i \cap w_{i+1}$ is an edge.

Since Lemma 2.4 ensures that any element of $\mathcal{L}_2(f)$ occurs in $H^k(I')$, we deduce that $H^k(I)$ and $H^k(J)$ satisfy properties (2-1) and (2-2). □

3. Index of an outer automorphism

An automorphism φ of the free group F_N extends to a homeomorphism $\partial\varphi$ of the boundary at infinity ∂F_N. We denote by $\mathrm{Fix}(\varphi)$ the fixed subgroup of φ. It is a finitely generated subgroup of F_N and thus its boundary $\partial\mathrm{Fix}(\varphi)$ naturally embeds in ∂F_N. Elements of $\partial\mathrm{Fix}(\varphi)$ are fixed by $\partial\varphi$ and they are called *singular*. Nonsingular fixed points of $\partial\varphi$ are called *regular*. A fixed point X of $\partial\varphi$ is *attracting* (resp. *repelling*) if it is regular and if there exists an element u in F_N such that $\varphi^n(u)$ (resp. $\varphi^{-n}(u)$) converges to X. The set of fixed points of $\partial\varphi$ is denoted by $\mathrm{Fix}(\partial\varphi)$.

Following Nielsen, fixed points of $\partial\varphi$ have been classified by Gaboriau, Jaeger, Levitt and, Lustig:

Proposition 3.1 [Gaboriau et al. 1998, Proposition 1.1]. *Let φ be an automorphism of the free group F_N, and X a fixed point of $\partial\varphi$. Exactly one of the following occurs*:

(1) *X is in the boundary of the fixed subgroup of φ.*

(2) *X is attracting.*

(3) *X is repelling.* □

We denote by $\mathrm{Att}(\varphi)$ the set of attracting fixed points of $\partial\varphi$. The fixed subgroup $\mathrm{Fix}(\varphi)$ acts on the set $\mathrm{Att}(\varphi)$ of attracting fixed points.

In [Gaboriau et al. 1998] the following *index* of the automorphism φ is defined:

$$\mathrm{ind}(\varphi) = \tfrac{1}{2}\#(\mathrm{Att}(\varphi)/\mathrm{Fix}(\varphi)) + \mathrm{rank}(\mathrm{Fix}(\varphi)) - 1$$

If φ has a trivial fixed subgroup, the above definition is simpler:

$$\mathrm{ind}(\varphi) = \tfrac{1}{2}\#\mathrm{Att}(\varphi) - 1.$$

Let u be an element of F_N and let i_u be the corresponding inner automorphism of F_N:

$$\forall w \in F_N, \quad i_u(w) = uwu^{-1}.$$

The inner automorphism i_u extends to the boundary of F_N as left multiplication by u:

$$\forall X \in \partial F_N, \quad \partial i_u(X) = uX.$$

The group $\mathrm{Inn}(F_N)$ of inner automorphisms of F_N acts by conjugacy on the automorphisms in an outer class Φ. Following Nielsen, two automorphisms, $\varphi, \varphi' \in \Phi$ are *isogredient* if they are conjugated by some inner automorphism i_u:

$$\varphi' = i_u \circ \varphi \circ i_{u^{-1}} = i_{u\varphi(u)^{-1}} \circ \varphi.$$

In this case, the actions of $\partial\varphi$ and $\partial\varphi'$ on ∂F_N are conjugate by the left multiplication by u. In particular, a fixed point X' of $\partial\varphi'$ is a translate $X' = uX$ of a fixed point X of $\partial\varphi$. Two isogredient automorphisms have the same index: this is the index of the isogrediency class. An isogrediency class $[\varphi]$ is *essential* if it has positive index: $\mathrm{ind}([\varphi]) > 0$. We note that essential isogrediency classes are principal in the sense of [Feighn and Handel 2011], but the converse is not true.

The *index* of the outer automorphism Φ is the sum, over all essential isogrediency classes of automorphisms φ in the outer class Φ, of their indices, or alternatively:

$$\mathrm{ind}(\Phi) = \sum_{[\varphi]\in\Phi/\mathrm{Inn}(F_N)} \max(0; \mathrm{ind}(\varphi)).$$

We adapt the notion of *forward rotationless outer automorphism* of [Feighn and Handel 2011] to our purpose. We denote by $\mathrm{Per}(\varphi)$ the set of elements of F_N fixed by some positive power of φ:

$$\mathrm{Per}(\varphi) = \bigcup_{n \in \mathbb{N}^*} \mathrm{Fix}(\varphi^n);$$

and by $\mathrm{Per}(\partial\varphi)$ the set of elements of ∂F_N fixed by some positive power of $\partial\varphi$:

$$\mathrm{Per}(\partial\varphi) = \bigcup_{n \in \mathbb{N}^*} \mathrm{Fix}(\partial\varphi^n).$$

Definition 3.2. An outer automorphism $\Phi \in \mathrm{Out}(F_N)$ is FR if:

(FR1) for any automorphism $\varphi \in \Phi$, $\mathrm{Per}(\varphi) = \mathrm{Fix}(\varphi)$ and $\mathrm{Per}(\partial\varphi) = \mathrm{Fix}(\partial\varphi)$, and

(FR2) if ψ is an automorphism in the outer class Φ^n for some $n > 0$, with $\mathrm{ind}(\psi)$ positive, then there exists an automorphism φ in Φ such that $\psi = \varphi^n$.

Proposition 3.3. *Let $\Phi \in \mathrm{Out}(F_N)$. There exists $k \in \mathbb{N}^*$ such that Φ^k is FR.*

Proof. By [Levitt and Lustig 2000, Theorem 1] there exists a power Φ^k with (FR1). An automorphism $\varphi \in \mathrm{Aut}(F_N)$ with positive index $\mathrm{ind}(\varphi) > 0$ is principal in the sense of [Feighn and Handel 2011, Definition 3.1]. Thus our property (FR2) is a consequence of the forward rotationless property of [loc. cit., Definition 3.13]. By [loc. cit., Lemma 4.43] there exists a power $\Phi^{k\ell}$ which is forward rotationless and thus which satisfies (FR2). □

4. Indices

4A. ***Botany of trees.*** We recall in this section the classification of trees in the boundary of outer space, given in [Coulbois and Hilion 2010].

Gaboriau and Levitt [1995] introduced an index for a tree T in $\overline{\mathrm{CV}}_N$, we call it the *geometric index* and denote it by $\mathrm{ind}_{\mathrm{geo}}(T)$. It is defined using the valence of the branch points, of the $\mathbb{R}$-tree T, with an action of the free group by isometries:

$$\mathrm{ind}_{\mathrm{geo}}(T) = \sum_{[P] \in T/F_N} \mathrm{ind}_{\mathrm{geo}}(P).$$

where the local index of a point P in T is

$$\mathrm{ind}_{\mathrm{geo}}(P) = \#(\pi_0(T \smallsetminus \{P\})/\mathrm{Stab}(P)) + 2\,\mathrm{rank}(\mathrm{Stab}(P)) - 2.$$

Gaboriau and Levitt proved that the geometric index of a geometric tree is equal to $2N-2$ and that for any tree in the compactification of outer space $\overline{\mathrm{CV}}_N$ the geometric index is bounded above by $2N-2$. Moreover, they proved that the trees in $\overline{\mathrm{CV}}_N$ with geometric index equal to $2N-2$ are precisely the geometric trees.

If, moreover, T has dense orbits, Levitt and Lustig [2003; 2008] defined the map $\mathfrak{Q} : \partial F_N \to \hat{T}$, characterized as follows:

Proposition 4.1. *Let T be an $\mathbb{R}$-tree in $\overline{\mathrm{CV}}_N$ with dense orbits. There exists a unique map $\mathcal{Q} : \partial F_N \to \hat{T}$ such that for any sequence $(u_n)_{n\in\mathbb{N}}$ of elements of F_N which converges to $X \in \partial F_N$, and any point $P \in T$, if the sequence of points $(u_n P)_{n\in\mathbb{N}}$ converges to a point $Q \in \hat{T}$, then $\mathcal{Q}(X) = Q$. Moreover, $\mathcal{Q}$ is onto.*

Let us consider the case of a tree T dual to a measured foliation $(\mathcal{F}, \mu)$ on a hyperbolic surface S with boundary (T is a surface tree). Let $\tilde{\mathcal{F}}$ be the lift of $\mathcal{F}$ to the universal cover $\tilde{S}$ of S. The boundary at infinity of $\tilde{S}$ is homeomorphic to ∂F_N. On the one hand, a leaf ℓ of $\tilde{\mathcal{F}}$ defines a point in T. On the other hand, the ends of ℓ define points in ∂F_N. The map $\mathcal{Q}$ precisely sends the ends of ℓ to the point in T. The Poincaré–Lefschetz index of the foliation $\mathcal{F}$ can be computed from the cardinal of the fibers of the map $\mathcal{Q}$. This leads to the following definition of the $\mathcal{Q}$-index of an $\mathbb{R}$-tree T in a more general context.

Let T be an $\mathbb{R}$-tree in $\overline{\mathrm{CV}}_N$ with dense orbits. The $\mathcal{Q}$-index of the tree T is defined by

$$\mathrm{ind}_{\mathcal{Q}}(T) = \sum_{[P]\in\hat{T}/F_N} \max(0; \mathrm{ind}_{\mathcal{Q}}(P)),$$

where the local index of a point P in T is

$$\mathrm{ind}_{\mathcal{Q}}(P) = \#(\mathcal{Q}_r^{-1}(P)/\mathrm{Stab}(P)) + 2\,\mathrm{rank}(\mathrm{Stab}(P)) - 2$$

with $\mathcal{Q}_r^{-1}(P) = \mathcal{Q}^{-1}(P) \smallsetminus \partial\mathrm{Stab}(P)$ the regular fiber of P.

Levitt and Lustig [2003] proved that points in ∂T have exactly one pre-image by $\mathcal{Q}$. Thus, only points in $\overline{T}$ contribute to the $\mathcal{Q}$-index of T.

We proved in [Coulbois and Hilion 2010] that the $\mathcal{Q}$-index of an $\mathbb{R}$-tree in the boundary of outer space with dense orbits is bounded above by $2N-2$. And it is equal to $2N-2$ if and only if it is of surface type.

The botanical classification in [Coulbois and Hilion 2010] of a tree T with a minimal very small indecomposable action of F_N by isometries is as follows:

	geometric $\mathrm{ind}_{\mathrm{geo}}(T) = 2N-2$	not geometric $\mathrm{ind}_{\mathrm{geo}}(T) < 2N-2$
Surface type: $\mathrm{ind}_{\mathcal{Q}}(T) = 2N-2$	surface	pseudo-surface
Levitt type: $\mathrm{ind}_{\mathcal{Q}}(T) < 2N-2$	Levitt	pseudo-Levitt

The following remark is not necessary for the sequel of the paper, but may help the reader's intuition.

Remark. In [Coulbois et al. 2008a; 2008b], in collaboration with Lustig, we defined and studied the dual lamination of an $\mathbb{R}$-tree T with dense orbits:

$$L(T) = \{(X, Y) \in \partial^2 F_N \mid \mathcal{Q}(X) = \mathcal{Q}(Y)\}.$$

The $\mathcal{Q}$-index of T can be interpreted as the index of this dual lamination.

Using the dual lamination, with Lustig [Coulbois et al. 2009], we defined the compact heart $K_A \subseteq \overline{T}$ (for a basis A of F_N). We proved that the tree T is completely encoded by a system of partial isometries $S_A = (K_A, A)$. We also proved that the tree T is geometric if and only if the compact heart K_A is a finite tree (that is to say the convex hull of finitely many points). In [Coulbois and Hilion 2010] we used the Rips machine on the system of isometries S_A to get the bound on the $\mathcal{Q}$-index of T. In particular, an indecomposable tree T is of Levitt type if and only if the Rips machine never halts.

4B. ***Geometric index.*** As in Section 2B, an iwip outer automorphism Φ has an expansion factor $\lambda_\Phi > 1$, an attracting $\mathbb{R}$-tree T_Φ in $\partial\mathrm{CV}_N$. For each automorphism φ in the outer class Φ there is a homothety H of the metric completion $\bar{T}_\Phi$, of ratio λ_Φ, such that

$$\forall P \in \overline{T}_\Phi,\ \forall u \in F_N, \quad H(uP) = \varphi(u)H(P). \tag{4-1}$$

In addition, the action of Φ on the compactification of Culler and Vogtmann's outer space has north-south dynamics and the projective class of T_Φ is the attracting fixed point [Levitt and Lustig 2003]. Of course the attracting trees of Φ and Φ^n ($n > 0$) are equal.

For the attracting tree T_Φ of the iwip outer automorphism Φ, the geometric index is well understood.

Proposition 4.2 [Gaboriau et al. 1998, Section 4]. *Let Ψ be an iwip outer automorphism. There exists a power $\Phi = \Psi^k$ ($k > 0$) of Ψ such that*

$$2\,\mathrm{ind}(\Phi) = \mathrm{ind}_{\mathrm{geo}}(T_\Phi),$$

where T_Φ is the attracting tree of Φ (and of Ψ). $\square$

4C. ***$\mathcal{Q}$-index.*** Let Φ be an iwip outer automorphism of F_N. Let T_Φ be its attracting tree. The action of F_N on T_Φ has dense orbits.

Let φ an automorphism in the outer class Φ. The homothety H associated to φ extends continuously to an homeomorphism of the boundary at infinity of T_Φ which we still denote by H. We get from Proposition 4.1 and identity (4-1):

$$\forall X \in \partial F_N, \quad \mathcal{Q}(\partial\varphi(X)) = H(\mathcal{Q}(X)). \tag{4-2}$$

We are going to prove that the $\mathcal{Q}$-index of T_Φ is twice the index of Φ^{-1}. As mentioned in the introduction for geometric automorphisms both these numbers are equal to $2N-2$ and thus we restrict to the study of nongeometric automorphisms. For the rest of this section we assume that Φ is nongeometric. This will be used in two ways:

- The action of F_N on T_Φ is free.
- For any φ in the outer class Φ, all the fixed points of φ in ∂F_N are regular.

Let C_H be the center of the homothety H. The following Lemma is essentially contained in [Gaboriau et al. 1998], although the map $\mathcal{Q}$ is not used there.

Lemma 4.3. *Let $\Phi \in \mathrm{Out}(F_N)$ be a FR nongeometric iwip outer automorphism. Let T_Φ be the attracting tree of Φ. Let $\varphi \in \Phi$ be an automorphism in the outer class Φ, and let H be the homothety of T_Φ associated to φ, with C_H its center. The $\mathcal{Q}$-fiber of C_H is the set of repelling points of φ.*

Proof. Let $X \in \partial F_N$ be a repelling point of $\partial\varphi$. By definition there exists an element $u \in F_N$ such that the sequence $(\varphi^{-n}(u))_n$ converges towards X. By (4-1),

$$\varphi^{-n}(u)C_H = \varphi^{-n}(u)H^{-n}(C_H) = H^{-n}(uC_H).$$

The homothety H^{-1} is strictly contracting and therefore the sequence of points $(\varphi^{-n}(u)C_H)_n$ converges towards C_H. By Proposition 4.1 we get that $\mathcal{Q}(X) = C_H$.

Conversely let $X \in \mathcal{Q}^{-1}(C_H)$ be a point in the $\mathcal{Q}$-fiber of C_H. Using the identity (4-2), $\partial\varphi(X)$ is also in the $\mathcal{Q}$-fiber. The $\mathcal{Q}$-fiber is finite by [Coulbois and Hilion 2010, Corollary 5.4], X is a periodic point of $\partial\varphi$. Since Φ satisfies property (FR1), X is a fixed point of $\partial\varphi$. From [Gaboriau et al. 1998, Lemma 3.5], attracting fixed points of $\partial\varphi$ are mapped by $\mathcal{Q}$ to points in the boundary at infinity ∂T_Φ. Thus X has to be a repelling fixed point of $\partial\varphi$. □

Proposition 4.4. *Let $\Phi \in \mathrm{Out}(F_N)$ be a FR nongeometric iwip outer automorphism. Let T_Φ be the attracting tree of Φ. Then*

$$2\,\mathrm{ind}(\Phi^{-1}) = \mathrm{ind}_{\mathcal{Q}}(T_\Phi).$$

Proof. To each automorphism φ in the outer class Φ is associated a homothety H of T_Φ and the center C_H of this homothety. As the action of F_N on T_Φ is free, two automorphisms are isogredient if and only if the corresponding centers are in the same F_N-orbit.

The index of Φ^{-1} is the sum over all essential isogrediency classes of automorphism φ^{-1} in Φ^{-1} of the index of φ^{-1}. For each of these automorphisms the index $2\,\mathrm{ind}(\varphi^{-1})$ is equal by Lemma 4.3 to the contribution $\#\mathcal{Q}^{-1}(C_H)$ of the orbit of C_H to the $\mathcal{Q}$ index of T_Φ.

Conversely, let now P be a point in $\overline{T}_\Phi$ with at least three elements in its $\mathcal{Q}$-fiber. Let φ be an automorphism in Φ and let H be the homothety of T_Φ associated to φ. For any integer n, the $\mathcal{Q}$-fiber $\mathcal{Q}^{-1}(H^n(P)) = \partial\varphi^n(\mathcal{Q}^{-1}(P))$ of $H^n(P)$ also has at least three elements. By [Coulbois and Hilion 2010, Theorem 5.3] there are finitely many orbits of such points in T_Φ and thus we can assume that $H^n(P) = wP$ for some $w \in F_N$ and some integer $n > 0$. Then P is the center of the homothety

$w^{-1}H^n$ associated to $i_{w^{-1}} \circ \varphi^n$. Since Φ satisfies property (FR2), P is the center of a homothety uH associated to $i_u \circ \varphi$ for some $u \in F_N$. This concludes the proof of the equality of the indices. □

This proposition can alternatively be deduced from the techniques of [Handel and Mosher 2011].

5. Botanical classification of irreducible automorphisms

Theorem 5.1. *Let Φ be an iwip outer automorphism of F_N. Let T_Φ and $T_{\Phi^{-1}}$ be its attracting and repelling trees. Then, the $\mathcal{Q}$-index of the attracting tree is equal to the geometric index of the repelling tree*:

$$\operatorname{ind}_{\mathcal{Q}}(T_\Phi) = \operatorname{ind}_{\text{geo}}(T_{\Phi^{-1}}).$$

Proof. First, if Φ is geometric, then the trees T_Φ and $T_{\Phi^{-1}}$ have maximal geometric indices $2N-2$. On the other hand the trees T_Φ and $T_{\Phi^{-1}}$ are surface trees and thus their $\mathcal{Q}$-indices are also maximal:

$$\operatorname{ind}_{\text{geo}}(T_\Phi) = \operatorname{ind}_{\mathcal{Q}}(T_\Phi) = \operatorname{ind}_{\text{geo}}(T_{\Phi^{-1}}) = \operatorname{ind}_{\mathcal{Q}}(T_{\Phi^{-1}}) = 2N-2.$$

We now assume that Φ is not geometric and we can apply Propositions 4.2 and 4.4 to get the desired equality. □

From Theorem 5.1 and from the characterization of geometric and surface-type trees by the maximality of the indices we get

Theorem 5.2. *Let Φ be an iwip outer automorphism of F_N. Let T_Φ and $T_{\Phi^{-1}}$ be its attracting and repelling trees. Then exactly one of the following occurs*:

(1) *T_Φ and $T_{\Phi^{-1}}$ are surface trees.*

(2) *T_Φ is Levitt and $T_{\Phi^{-1}}$ is pseudo-surface.*

(3) *$T_{\Phi^{-1}}$ is Levitt and T_Φ is pseudo-surface.*

(4) *T_Φ and $T_{\Phi^{-1}}$ are pseudo-Levitt.*

Proof. The trees T_Φ and $T_{\Phi^{-1}}$ are indecomposable by Theorem 2.1 and thus they are either of surface type or of Levitt type by [Coulbois and Hilion 2010, Proposition 5.14]. Recall, from [Gaboriau and Levitt 1995] (see also [Coulbois and Hilion 2010, Theorem 5.9] or [Coulbois et al. 2009, Corollary 6.1]) that T_Φ is geometric if and only if its geometric index is maximal:

$$\operatorname{ind}_{\text{geo}}(T_\Phi) = 2N-2.$$

From [Coulbois and Hilion 2010, Theorem 5.10], T_Φ is of surface type if and only if its $\mathcal{Q}$-index is maximal:

$$\operatorname{ind}_{\mathcal{Q}}(T_\Phi) = 2N-2.$$

The theorem now follows from Theorem 5.1. □

Let $\Phi \in \mathrm{Out}(F_N)$ be an iwip outer automorphism.

The outer automorphism Φ is *geometric* if both its attracting and repelling trees T_Φ and $T_{\Phi^{-1}}$ are geometric. This is equivalent to saying that Φ is induced by a pseudo-Anosov homeomorphism of a surface with boundary; see [Guirardel 2005] and [Handel and Mosher 2007]. This is case (1) of Theorem 5.2.

The outer automorphism Φ is *parageometric* if its attracting tree T_Φ is geometric but its repelling tree $T_{\Phi^{-1}}$ is not. This is case (2) of Theorem 5.2.

The outer automorphism Φ is *pseudo-Levitt* if both its attracting and repelling trees are not geometric. This is case (4) of Theorem 5.2

We now bring expansion factors into play. An iwip outer automorphism Φ of F_N has an expansion factor $\lambda_\Phi > 1$: it is the exponential growth rate of (nonfixed) conjugacy classes under iteration of Φ.

If Φ is geometric, the expansion factor of Φ is equal to the expansion factor of the associated pseudo-Anosov mapping class and thus $\lambda_\Phi = \lambda_{\Phi^{-1}}$.

Handel and Mosher [2007] proved that if Φ is a parageometric outer automorphism of F_N then $\lambda_\Phi > \lambda_{\Phi^{-1}}$ (see also [Behrstock et al. 2010]). Examples are also given by Gautero [2007].

For pseudo-Levitt outer automorphisms of F_N nothing can be said on the comparison of the expansion factors of the automorphism and its inverse. On one hand, Handel and Mosher [2007, Introduction] gave an explicit example of a nongeometric automorphism with $\lambda_\Phi = \lambda_{\Phi^{-1}}$: thus this automorphism is pseudo-Levitt. On the other hand, there are examples of pseudo-Levitt automorphisms with $\lambda_\Phi > \lambda_{\Phi^{-1}}$. Let $\varphi \in \mathrm{Aut}(F_3)$ be the automorphism such that

$$\varphi: \begin{array}{rcl} a & \mapsto & b \\ b & \mapsto & ac \\ c & \mapsto & a \end{array} \qquad \text{and} \qquad \varphi^{-1}: \begin{array}{rcl} a & \mapsto & c \\ b & \mapsto & a \\ c & \mapsto & c^{-1}b \end{array}$$

Let Φ be its outer class. Then Φ^6 is FR, has index $\mathrm{ind}(\Phi^6) = \frac{3}{2} < 2$. The expansion factor is $\lambda_\Phi \simeq 1{,}3247$. The outer automorphism Φ^{-3} is FR, has index $\mathrm{ind}(\Phi^{-3}) = \frac{1}{2} < 2$. The expansion factor is $\lambda_{\Phi^{-1}} \simeq 1{,}4655 > \lambda_\Phi$. The computation of these two indices can be achieved using the algorithm of [Jullian 2009].

Now that we have classified outer automorphisms of F_N into four categories, questions of genericity naturally arise. In particular, is a generic outer automorphism of F_N iwip, pseudo-Levitt and with distinct expansion factors? This was suggested in [Handel and Mosher 2007], in particular for statistical genericity: given a set of generators of $\mathrm{Out}(F_N)$ and considering the word metric associated to it, is it the case that

$$\lim_{k\to\infty} \frac{\#(\text{pseudo-Levitt iwip with } \lambda_\Phi \neq \lambda_{\Phi^{-1}}) \cap B(k))}{\# B(k)} = 1,$$

where $B(k)$ is the ball of radius k, centered at 1, in $\mathrm{Out}(F_N)$?

5A. *Botanical memo.* In this section we give a glossary of our classification of automorphisms for the working mathematician.

For a FR iwip outer automorphism Φ of F_N, we used 6 indices which are related in the following way:

$$\begin{aligned} 2\,\mathrm{ind}(\Phi) &= \mathrm{ind}_{\mathrm{geo}}(T_\Phi) &&= \mathrm{ind}_{\mathcal{Q}}(T_{\Phi^{-1}}), \\ 2\,\mathrm{ind}(\Phi^{-1}) &= \mathrm{ind}_{\mathrm{geo}}(T_{\Phi^{-1}}) &&= \mathrm{ind}_{\mathcal{Q}}(T_\Phi). \end{aligned}$$

All these indices are bounded above by $2N-2$. We sum up our Theorem 5.2 in the following table.

Automorphisms		Trees		Indices
Φ geometric $\Updownarrow$ Φ^{-1} geometric	$\Leftrightarrow$	T_Φ and $T_{\Phi^{-1}}$ geometric $\Updownarrow$ T_Φ surface $\Updownarrow$ $T_{\Phi^{-1}}$ surface	$\Leftrightarrow$	$\mathrm{ind}(\Phi) = \mathrm{ind}(\Phi^{-1}) = N-1$
Φ parageometric	$\Leftrightarrow$	$\begin{cases} T_\Phi \text{ geometric} \\ \text{and} \\ T_{\Phi^{-1}} \text{ nongeometric} \end{cases}$ $\Updownarrow$ T_Φ Levitt $\Updownarrow$ $T_{\Phi^{-1}}$ pseudo-surface	$\Leftrightarrow$	$\begin{cases} \mathrm{ind}(\Phi) = N-1 \\ \text{and} \\ \mathrm{ind}(\Phi^{-1}) < N-1 \end{cases}$
Φ pseudo-Levitt $\Updownarrow$ Φ^{-1} pseudo-Levitt	$\Leftrightarrow$	T_Φ, $T_{\Phi^{-1}}$ nongeometric $\Updownarrow$ T_Φ pseudo-Levitt $\Updownarrow$ $T_{\Phi^{-1}}$ pseudo-Levitt	$\Leftrightarrow$	$\begin{cases} \mathrm{ind}(\Phi) < N-1 \\ \text{and} \\ \mathrm{ind}(\Phi^{-1}) < N-1 \end{cases}$

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